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New Values for Top Entails

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Abstract. The game of Top Entails introduces the curious theory of entailing moves. In *Winning Ways*, simple positions are analysed and stacks of one and three coins are shown to be loony positions, since any move leaving such a stack is a losing move, regardless of other stacks that might also be present. We analyse all stacks of up to 600,000 coins and find three more loony stacks: those having 2403, 2505 and 33,243 coins.

The game of Top Entails is introduced in *Winning Ways* [Berlekamp et al. 1982, p. 376]. It is a nim-like, impartial game played with stacks of coins and the following two types of move. A player may either (1) split just one stack into two strictly smaller stacks; or (2) remove just one coin from just one stack; in this case the opponent is *entailed* to reply with a move (of either type) in the resulting stack.

An unentailed stack of zero coins can never really occur in this game, since neither type of move can create one. Nevertheless, we can posit one as a hypothetical object; indeed, it is clear that it has value $*0 = 0$, since it hardly matters whether there is such a stack on the table or not!

What about an entailed stack of zero coins? Such a position can certainly occur, by an entailing (type 2) move from any stack containing just one coin. Since the next player, $\mathcal{N}$, is now forced to move in an empty stack, the game ends instantly with a win for the previous player, $\mathcal{P}$. Indeed, this is the only way a game can actually end (short of the usual situation of players calculating the value of the position, agreeing on the winner, picking up their coins and going home). So what value z do we assign to the entailed zero-stack? Since the rest of the stacks on the table, G, constitute an impartial game, they must sum to some nimber $*n$; but $G + z$ can be seen to be 0 (a $\mathcal{P}$-position) for any value of n. So it must be right to think of z as having whatever nim-value $*n$ seems convenient, or equivalently to ascribe it the set of all nim-values: $\{*0, *1, *2, \ldots\}$.

On the other hand, as soon as a player creates a stack of size 1, she has lost. Since the stack has just been created, her opponent can not possibly be entailed to move elsewhere, and may reply by leaving the entailed zero-stack z. Creating a stack of size 1 is thus a *loony* move. Since no nimber can be added to loony to get the $\mathcal{P}$-position 0, we can identify loony with $\{\,\}$, the set of no nim-values. Naturally, then, the set z of all nim-values must be called *sunny*.

In general, a stack will have a set of nim-values $\{a_0, a_1, \dots\}$, of which only the smallest, a_0, is relevant if the stack is unentailed; if the stack is entailed, all are relevant. (Above we saw that an unentailed zero-stack had value 0, which is indeed the smallest nimber in the sunny set.) The complete set of nim-values for a position G consists of *all nimbers not among the relevant values* for options of G. This fairly simple rule enables us to compute new values. A little work by hand produces a table, the first 120 lines of which are reproduced in Table 1. The entries are of course sets of nimbers, but stars have been suppressed. An arrow indicates that all larger nimbers are included in the set.

For this paper, a program was written in C to compute game values. It computed 600,000 values in approximately 30 CPU hours on a Sun workstation. For the first 38,000 values, the complete set of nim-values was returned, exactly as in the above table; thereafter only the smallest member of the set, that is, the unentailed value of the stack.

The file of 38,000 values was searched for the first appearance of each integer. This is a rather arbitrary measure, because an integer n can appear in this table in two different ways, which were not distinguished: either n is in the set of nim values but some larger m is not; or else $n-1$ is the largest value to be absent (and so n begins the tail). The number 14 is the smallest to make its appearance in the first of these ways. The point of interest is that all values between 2^k+1 and 2^{k+1} tend to appear rather soon after 2^k appears as an unentailed value. (The value 16 first appears as the unentailed value for a 94-stack; 17 through 32 enter the table between lines 94 and 184; values larger than 32 do not appear until the unentailed 32 finally appears with the 534-stack.) This is not surprising, as this means that the k-th bit has now come into play in the nim-additions occasioned by stack-splitting moves. Indeed, a little work with the above table shows that if 2^k is the unentailed value for the T-stack, then 2^k+1 in general appears on line $T+2$, 2^k+2 on line $T+4$, 2^k+3 on line $T+6$, 2^k+4 on line $T+8$, and 2^k+5 on line $T+12$.

Here is the list of the smallest stack to have each power of 2 as its unentailed value:

power of two	1	2	4	8	16	32	64	128	356	512	1024
stack size	4	6	12	32	94	534	2556	8062	35138	119094	293692

In *Winning Ways*, the only known loony values are for stacks of one and three coins. Among the first 600,000 values there are three more loony values, namely at 2403, 2505 and 33,243 coins.

The output enables us easily to determine the strategy for playing from a stack of 2403 coins, and we show it in Table 2. The value of this stack is loony because it is possible to move from it to any nim-value at all. For instance,

0: 0 →	40: 1 4 8 9 12 →	80: 3 4 6 15 →
1: ∅	41: 6 7 11	81: 12 13
2: 0 →	42: 2 5 8 10 12 →	82: 2 5 7 9 14 →
3: ∅	43: 9	83: 13
4: 1 →	44: 3 6 7 8 11 13 →	84: 4 11 12 16 →
5: 0	45: 0 2 4 10 12	85: 2 7 14
6: 2 →	46: 5 7 8 9 13 →	86: 13 15 →
7: 1	47: 3 6 10 11	87: 1 4 6 8
8: 3 →	48: 4 8 12 →	88: 11 14 →
9: 0 2	49: 2 7 9 10	89: 5 13
10: 1 4 →	50: 11 →	90: 8 15 →
11: 3	51: 3 4 6 8 9	91: 3 4 6 11
12: 4 →	52: 2 12 →	92: 12 14 16 →
13: 0 2	53: 5 7 10 11	93: 0 13 15
14: 3 5 →	54: 3 4 8 9 12 13 15 →	94: 16 →
15: 4	55: 6 14	95: 1 4 12 14
16: 2 6 →	56: 5 8 10 11 12 13 16 →	96: 2 9 17 →
17: 5	57: 0 9 14 15	97: 5 7 13 15 16
18: 1 3 4 7 →	58: 6 10 11 16 →	98: 1 4 8 12 18 →
19: 6	59: 1 2 4 9 12 13 14	99: 6 10 14 17
20: 2 5 8 →	60: 5 7 8 15 →	100: 2 5 11 12 13 16 19 →
21: 0 1 7	61: 10 11 13 14	101: 18
22: 3 4 6 8 →	62: 1 4 9 12 16 →	102: 6 10 14 16 17 20 →
23: 2 5	63: 14 15	103: 0 9 11 19
24: 7 →	64: 5 10 11 12 13 16 →	104: 7 15 16 18 20 →
25: 3	65: 4 6 8	105: 1 10 17
26: 1 4 6 8 →	66: 2 12 14 →	106: 4 8 9 12 16 19 21 →
27: 7	67: 0 10 11 13	107: 0 7 14 17 18 20
28: 3 5 8 →	68: 8 12 15 →	108: 2 5 8 10 12 15 16 21 →
29: 4 6	69: 1 3 6 10 11 14	109: 3 4 13 18 19
30: 2 7 →	70: 12 13 16 →	110: 11 15 16 17 20 →
31: 0 5	71: 0 7 8 9 14 15	111: 5 7 12 13 18
32: 8 →	72: 10 16 →	112: 15 16 19 →
33: 1 6	73: 1 4 6 12 13	113: 3 6 9 11 14 17 18
34: 5 9 →	74: 2 3 8 11 14 →	114: 5 12 16 19 →
35: 0 7 8	75: 7 13	115: 7 10 13 18
36: 1 6 10 →	76: 9 12 15 →	116: 4 6 16 17 20 21 23 →
37: 4 5 9	77: 3 4 6 11 14	117: 9 12 19 22
38: 7 8 11 →	78: 5 12 16 →	118: 2 7 16 17 18 20 21 24 →
39: 10	79: 9 10 13 14	119: 13 15 22 23

Table 1. Distribution of unentailed nim-values for stacks of up to 600,000 coins.

n	x	$v(x)$	$v(\bar{x})$	n	x	$v(x)$	$v(\bar{x})$	n	x	$v(x)$	$v(\bar{x})$
0	75	7	7	22	47	3	21	44	79	9	37
1	462	2	3	23	123	7	16	45	17	5	40
2	84	4	6	24	132	11	19	46	28	3	45
3	811	38	37	25	43	9	16	47	20	2	45
4	847	33	37	26	61	10	16	48	5	0	48
5	146	11	14	27	349	11	16	49	344	25	40
6	555	14	8	28	111	5	25	50	14†	3	49
7	26†	1	6	29	63	14	19	51	19	6	53
8	13	0	8	30	182	16	14	52	59	1	53
9	907	32	41	31	225	15	16	53	163†	1	52
10	2	0	10	32	68	8	4	54	8	6	48
11	1176	7	12	33	496	17	48	55	192†	3	52
12	913	44	32	34	37	4	38	56	365†	17	73
13	33	1	12	35	101	18	49	57	70	12	53
14	942	46	32	36	10	1	37	58	1097†	25	35
15	9	0	15	37	6	2	39	59	32	8	51
16	233	27	11	38	66	2	36	60	72	10	54
17	7	1	16	39	55	6	33	61	50†	11	54
18	533	7	21	40	49	2	42	62	90†	8	54
19	16	2	17	41	334	16	57	≥ 63	†		
20	34	5	17	42	88	11	33				
21	12	4	17	43	30	2	41				

Table 2. How to play when a position contains a stack of 2403 coins.

to move to $*2$, separate off a stack of $x = 84$ coins, leaving another stack of $\bar{x} = 2403 - x = 2319$ coins. The first of these stacks has the unentailed value $v(84) = *4$, the second $v(2319) = *6$. A † next to a number indicates that an entailing move is also possible. Thus to move to $*7$, either split into stacks of 26 and 2377, or make the entailing move by removing one coin. There may be other possibilities as well; 26 is the option with smallest x.

The histograms in Figure 1 show the distributions of unentailed nim-values among the first $100,000$ and the first $600,000$ stacks. It will be noted that values congruent to 0 or to 1 modulo a large power of 2 are particularly common, followed by a big drop and a general tapering off before the next large power of 2.

The program used to generate the above data is readily adaptable to any *octal* game (*Winning Ways*, chapter 4). For instance, it was used to generate 40 million values for the game called .611 … . (This game is easy to analyse computationally because it has a particularly strong division into rare and common values). Plans are made to attack various outstanding octal games in the near future.

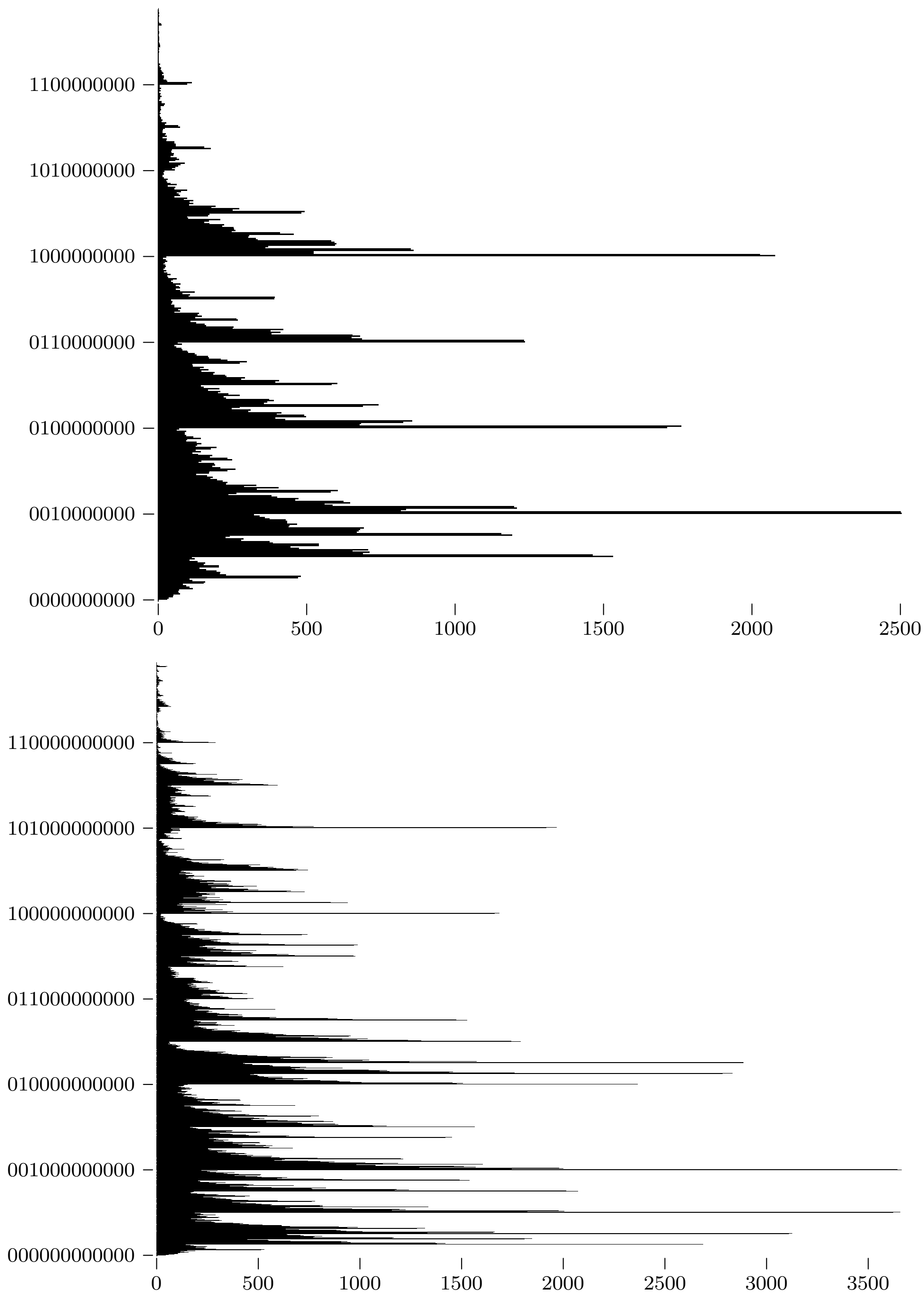

Figure 1. Distribution of unentailed nim-values for stacks of up to 100,000 coins.

At the MSRI meeting, John Conway proposed that an effort should be made to devise some game with entailing moves that is non-trivial, but (unlike Top Entails) susceptible to a complete analysis. All obvious suggestions which have been tried turn out to lead to a sequence of values of very small period, hence are not very interesting.

Reference

[Berlekamp et al. 1982] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways For Your Mathematical Plays*, Academic Press, London, 1982.

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