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Take-Away Games

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Abstract. Several authors have considered take-away games where the players alternately remove a positive number of counters from a single pile, the player removing the last counter being the winner. On his initial move, the first player may remove any number of counters, so long as he leaves at least one. On each subsequent move, a player may remove at most $f(n)$ counters, where n is the number of counters removed by his opponent on the preceding move. We prove various results (improving all previously known results) about the sequence of losing positions when f is a linear function.

1. Introduction

Several works, including [Berlekamp et al. 1982; Epp and Ferguson 1980; Schwenk 1970], have studied take-away games where the players alternately remove a positive number of counters from a single pile, the player removing the last counter being the winner. On his initial move, the first player may remove any number of counters, so long as he leaves at least one. On each subsequent move, a player may remove at most $f(n)$ counters, where n is the number of counters removed by his opponent on the preceding move. Thus, any mapping f from the positive integers to themselves defines such a take-away game.

Epp and Ferguson [1980] considered the case where f is nondecreasing and $f(1) \geq 1$. For any such f, let $H_1 = 1$, $H_2, \ldots$ be the sizes of the initial pile from which the first player has no winning strategy; we call these the *losing positions* (these are the $\mathcal{P}$-positions of [Berlekamp et al. 1982], the $\mathcal{P}$ standing for a Previous-player win). We will study this sequence of losing positions. We begin with a result from [Epp and Ferguson 1980]:

Theorem 1.1. *If $f(H_j) \geq H_j$, then $H_{j+1} = H_j + H_l$, where*

$$H_l = \min_{i \leq j} \{H_i | f(H_i) \geq H_j\}.$$

If $f(H_j) < H_j$, the sequence of losing positions is finite and H_j is the final term.

PROOF. Assume $f(H_j) \geq H_j$; then $H_l = \min_{i \leq j}\{H_i : f(H_i) \geq H_j\}$ exists. For any losing position $H_i < H_l$, we have $f(H_i) < H_j$, so from an initial pile of size $H_j + H_i$ the first player can remove H_i counters and win (since this leaves the second player a pile of size H_j from which he cannot remove all counters, so he is in a losing position).

Now let $x < H_l$ be a winning position. Given a pile of size $H_j + x$, the first player can employ a winning strategy for a pile of size x whose final move removes y counters, where $f(y) < H_j$; this leaves the second player with a pile of size H_j from which he cannot remove all counters, and the first player wins. (He can always arrange for y to satisfy this property because, if the last move y of a winning strategy for x satisfies instead $f(y) \geq H_j$, then $y < H_l$ cannot be a losing position, and consideration of a winning strategy for y leads to a smaller final move.)

Finally, from a pile of size $H_j + H_l$, if the first player takes at least H_l counters the second player takes the rest and wins; if the first player takes fewer than H_l counters, we fall into the preceding paragraph's situation, with roles reversed. This proves the first statement of the theorem.

If $f(H_j) < H_j$, suppose we had $H_{j+1} = H_j + x$ for some x. As above, x cannot be any H_i, since the first player wins from $H_j + H_i$ by removing H_i counters (since $f(H_i) \leq f(H_j) < H_j$). Since $x < H_{j+1}$, x must be a winning position; thus, the first player can win from $H_j + x$ by employing a winning strategy for x whose final move is y, where $f(y) < H_j$. Thus H_{j+1} is not a losing position, which is a contradiction, so in fact there is no H_{j+1}. □

A natural question is whether the sequence of losing positions eventually becomes a simple linear recursion $H_{i+1} = H_i + H_{i-k}$ for sufficiently large i. For the functions $f(n) = cn$, where $c \geq 1$, the answer is yes, as proved in [Schwenk 1970]; a positive answer for the functions $f(n) = cn - 1$, where $c \geq 2$, was claimed in [Berlekamp et al. 1982].

In this paper we modify the methods of [Schwenk 1970] to give a positive answer for functions $f(n) = cn - d$, where $c - 1 \geq d \geq 0$. Using entirely different methods we give a positive answer for functions $f(n) = cn + d$, where $c \geq 1$ and $d > 0$. We also derive results describing k, the degree of the recursion, in terms of c (our results do not depend on d); these are the first results about k that have been found. For certain small values of c we sharpen the general results to find k exactly. We also present algorithms for computing k and prove that they are valid. We have implemented these algorithms for several linear functions f, and we make several conjectures based on this data (in particular, about the dependence of k on d).

2. The Functions $f(n) = cn - d$

This section is devoted to the proof of the following result.

THEOREM 2.1. *In the game associated to the function $f(n) = cn - d$, where $c - 1 \geq d \geq 0$, there is a nonnegative integer k such that the sequence of losing positions H_i satisfies $H_{i+1} = H_i + H_{i-k}$ for all sufficiently large i.*

Note that $f(n) \geq n$ for all n, so Theorem 1.1 implies that the sequence of losing positions is infinite. The theorem follows from the following two lemmas.

LEMMA 2.2. *If $H_i \leq f(H_j)$, then $H_{i+1} \leq f(H_{j+1})$.*

PROOF. Decreasing j if necessary, we may assume $H_{i+1} = H_i + H_j$. Also, $H_{j+1} = H_j + H_l$ where $f(H_l) \geq H_j$. Thus,

$$\begin{aligned} f(H_{j+1}) &= cH_{j+1} - d = cH_j + cH_l - d \\ &\geq cH_j + H_j = f(H_j) + d + H_j \geq H_i + d + H_j = H_{i+1} + d, \end{aligned}$$

and the lemma is proved. □

LEMMA 2.3. *There exists a positive integer r such that $f(H_{n-r}) < H_n$ for all $n > r$.*

PROOF. For any integer $i \geq 1$ we have $H_{i+1} = H_i + H_j$, where $cH_j - d = f(H_j) \geq H_i$; thus,

$$\frac{H_{i+1}}{H_i} = 1 + \frac{H_j}{H_i} \geq 1 + \frac{1}{c}.$$

Let r be any integer for which $(1 + 1/c)^r > c$; then, for any $n > r$,

$$\frac{H_n}{H_{n-r}} = \prod_{i=n-r}^{n-1} \frac{H_{i+1}}{H_i} \geq \left(1 + \frac{1}{c}\right)^r > c.$$

Thus, $f(H_{n-r}) \leq cH_{n-r} < H_n$. □

THEOREM 2.1. Each H_{i+1} equals $H_i + H_j$ for some $j \leq i$; Lemma 2.2 implies that the sequence of differences $i - j$ is nondecreasing, and Lemma 2.3 implies that this sequence is bounded above by $r - 1$, so together they show that the sequence must be constant for sufficiently large i. This limiting value is the k described in the theorem. □

3. The Functions $f(n) = cn + d$

The proof that the sequence of losing positions for the take-away game asociated to $f(n) = cn - d$ eventually satisfies a simple recursion was quite simple: the sequence of differences $i - j$ (where $H_{i+1} = H_i + H_j$) was shown to be nondecreasing and bounded. The sequence of differences for $f(n) = cn + d$, however, is generally not monotonic, so one cannot hope for such a simple proof in this case.

THEOREM 3.1. *In the take-away game associated to the function $f(n) = cn + d$, where $c \geq 1$ and $d > 0$, there is a nonnegative integer k such that the sequence of losing positions H_i satisfies $H_{i+1} = H_i + H_{i-k}$ for all sufficiently large i.*

Proof. First, any function f as in the theorem satisfies $f(n) \geq n$ for all n, so Theorem 1.1 implies that the sequence of losing positions is infinite. Now, each H_{i+1} equals $H_i + H_j$ for some $j \leq i$; we will show that the sequence of differences $i - j$ is nonincreasing for sufficiently large i, which implies that the sequence is eventually constant, finishing the proof.

Define $J_0 = \{1\}$, and inductively

$$J_{i+1} = \{\alpha : H_\alpha = H_{\alpha-1} + H_r \text{ for } r \in J_i\}.$$

Let r_i and s_i be the minimal and maximal elements of J_i, respectively. We claim that every J_i is a finite nonempty set of consecutive integers, $J_i = \{r_i, r_i + 1, \ldots, s_i\}$, and that $r_{i+1} = s_i + 1$; thus the sets $J_0, J_1, \ldots$ partition the positive integers into intervals. The proof of this claim is inductive: we have

$$J_1 = \{\alpha : H_\alpha = H_{\alpha-1} + 1\} = \{\alpha : H_{\alpha-1} \leq f(1)\} = \{2, 3, \ldots, s_2\}$$

and, for $i \geq 1$,

$$\begin{aligned} J_{i+1} &= \{\alpha : f(H_{r-1}) < H_{\alpha-1} \leq f(H_r), r \in J_i\} \\ &= \{\alpha : f(H_{r_i-1}) < H_{\alpha-1} \leq f(H_{s_i})\}, \end{aligned}$$

so J_{i+1} is a finite set of consecutive integers whose least element is $s_i + 1$; thus the claim is proved. Now, every positive integer j is in some J_i, and we define $\psi(j) = i$. Then ψ is a nondecreasing function.

Suppose that, for each $\alpha \in J_i$ (where $i > 0$, so that $\alpha > 1$), the unique r satisfying $H_\alpha = H_{\alpha-1} + H_r$ also satisfies $H_\alpha \geq f(H_r) - d\varepsilon$. Then, for any $\beta \in J_{i+1}$, we have $H_\beta = H_{\beta-1} + H_\alpha$ for some $\alpha \in J_i$, so $H_\alpha \geq f(H_r) - d\varepsilon$. Thus $f(H_{\alpha-1}) < H_{\beta-1} \leq f(H_\alpha)$. Now

$$\begin{aligned} H_\beta - f(H_\alpha) &= H_{\beta-1} + H_\alpha - f(H_\alpha) = H_{\beta-1} + H_\alpha - cH_\alpha - d \\ &> f(H_{\alpha-1}) + H_\alpha - cH_\alpha - d = cH_{\alpha-1} + H_\alpha - cH_\alpha = H_\alpha - cH_r \\ &= H_\alpha + d - f(H_r) \geq d - d\varepsilon, \end{aligned}$$

so $H_\beta > f(H_\alpha) - d(\varepsilon - 1)$.

Put $\varepsilon = (c + d - 2)/d$. Note that $J_1 = \{2, 3, \ldots, \lfloor c + d \rfloor + 1\}$, where $\lfloor c + d \rfloor$ denotes the greatest integer not exceeding $c + d$. For any $\alpha \in J_1$, we have $H_\alpha = H_{\alpha-1} + H_1$, and

$$f(H_1) - d\varepsilon = c + d - d\varepsilon = 2 \leq H_\alpha.$$

By the above paragraph, for each $\alpha \in J_i$, the unique r satisfying $H_\alpha = H_{\alpha-1} + H_r$ also satisfies $H_\alpha \geq f(H_r) - d(\varepsilon + 1 - i)$.

For any $j > 1$ such that $i = \psi(j) \geq \max\{2, 1 + \varepsilon\}$, we have $H_j = H_{j-1} + H_m$ where $m \in J_{i-1}$. Since $i \geq 1$, we have $H_m = H_{m-1} + H_t$, where

$$H_m \geq f(H_t) - d(\varepsilon + 2 - i) \geq f(H_t) - d.$$

Now, since $f(H_{m-1}) < H_{j-1} \le f(H_m)$, we get

$$\begin{aligned} H_j - f(H_m) &= H_{j-1} + H_m - cH_m - d \\ &> f(H_{m-1}) - d + H_m - cH_m \\ &= cH_{m-1} + H_m - cH_m = H_m - cH_t = H_m - f(H_t) + d \ge 0, \end{aligned}$$

so $H_j > f(H_m)$. Thus, if $H_{j+1} = H_j + H_l$, we have $f(H_l) \ge H_j$, so $l > m$, and thus the difference $j - l \le (j-1) - m$.

We have now shown that, for sufficiently large j, the sequence of differences $j - r$, where $H_{j+1} = H_j + H_r$, is nonincreasing; since each such difference is a nonnegative integer, this implies that the sequence is eventually constant, which completes the proof. □

4. The Degree of the Recursion

The theorems of the previous two sections imply that, if $f(n) = cn + d$ where $c \ge 1$ and $d \ge 1 - c$, the sequence of losing positions H_i for the corresponding take-away game satisfies $H_{i+1} = H_i + H_{i-k}$ for all sufficiently large i. In this section we derive some general results about k. Our methods use only the ultimate behavior of the sequence, namely that $H_{i+1} = H_i + H_{i-k}$ for sufficiently large i, and disregard the early behavior.

Theorem 4.1. *If $f(n) = cn + d$ where $c > 1$ and $d \ge 1 - c$, and the sequence of losing positions for the corresponding take-away game satisfies the recursion $H_{i+1} = H_i + H_{i-k}$ for all sufficiently large i, then*

$$\frac{\log(c-1)}{\log c - \log(c-1)} \le k \le \frac{\log c}{\log(c+1) - \log c}.$$

Proof. We will only consider i that are "sufficiently large", so we may assume that $H_{i+1} = H_i + H_{i-k}$ for all i under consideration. The characteristic polynomial for this recursion relation is $g(x) = x^{k+1} - x^k - 1$. By Descartes' rule of signs, g has exactly one positive real root, which we will denote by r. Note that $g(t) \to +\infty$ as $t \to +\infty$, where t is a real variable; thus, for $t > r$ we must have $g(t) > 0$. Now, any complex root z of this polynomial satisfies

$$1 = |z^{k+1} - z^k| = |z|^k \cdot |z - 1| \ge |z|^k \cdot (|z| - 1) = |z|^{k+1} - |z|^k,$$

or equivalently $g(|z|) \le 0$. Thus, $|z| \le r$. If $|z| = r$ then $g(|z|) = 0$, so we have equality in the displayed equation above, implying that $|z - 1| = |z| - 1$; by the equality criteria for the triangle inequality, this last equality implies that z is a nonnegative real number, so $z = r$. Thus, any complex root of g other than r has absolute value less than r. Since r is necessarily a simple root of g, it follows by a standard result about linear recursion relations that

$$\lim_{i\to\infty} \frac{H_{i+1}}{H_i} = r.$$

Since $cH_{i-1} + d < H_{i+k} \le cH_i + d$, we have

$$\frac{H_{i-1}}{H_{i+k}} < \frac{1 - \ d/H_{i+k}}{c} \le \frac{H_i}{H_{i+k}};$$

taking the limit as $i \to \infty$, we see that

$$\frac{1}{r^{k+1}} \le \frac{1}{c} \le \frac{1}{r^k},$$

or, equivalently, $r^k \le c \le r^{k+1}$. Note that $r > 1$, since $f(1) < 0$; now, since $r^k = 1/(r-1)$, we have

$$\begin{aligned}
&r^k \le c \le r^{k+1}\\
&\qquad \Leftrightarrow 1/(r-1) \le c \le r/(r-1)\\
&\qquad \Leftrightarrow 1/c \le r-1 \quad \text{and} \quad (c-1)r \le c\\
&\qquad \Leftrightarrow (c+1)/c \le r \le c/(c-1)\\
&\qquad \Leftrightarrow g\Big(\frac{c+1}{c}\Big) \le 0 \le g\Big(\frac{c}{c-1}\Big)\\
&\qquad \Leftrightarrow \Big(\frac{c+1}{c}\Big)^k \Big(\frac{1}{c}\Big) \le 1 \le \Big(\frac{c}{c-1}\Big)^k \Big(\frac{1}{c-1}\Big)\\
&\qquad \Leftrightarrow k\Big(\log(c+1) - \log c\Big) - \log c \le 0 \le k\Big(\log c - \log(c-1)\Big) - \log(c-1)\\
&\qquad \Leftrightarrow \frac{\log(c-1)}{\log c - \log(c-1)} \le k \le \frac{\log c}{\log(c+1) - \log c},
\end{aligned}$$

which completes the proof. □

Define $\chi(c) = (\log c)/\big(\log(c+1) - \log c\big)$ for $c > 0$. Then the result of the above theorem is that $\chi(c-1) \le k \le \chi(c)$.

Lemma 4.2. *The function $\chi(c)$ is increasing for $c > 1$.*

Proof. Since

$$\chi(c) = \frac{1}{\log(c+1)/\log c \ - 1}$$

for $c > 1$, this function is increasing if and only if

$$\phi(c) = \frac{\log(c+1)}{\log c}$$

is decreasing. But its derivative satisfies

$$\phi'(c) = \frac{(\log c)/(c+1) - \big(\log(c+1)\big)/c}{(\log c)^2} = \frac{c\log c - (c+1)\log(c+1)}{c(c+1)(\log c)^2} < 0,$$

so indeed ϕ is decreasing, hence χ is increasing for $c > 1$. □

Corollary 4.3. *For any integers c_1, c_2, d_1, d_2 such that $c_1 - 1 \geq c_2 \geq 2$ and $d_1 \geq 1 - c_1$, $d_2 \geq 1 - c_2$, we have $k_1 > k_2$, where the losing positions of the take-away game associated to the function $f(n) = c_j n + d_j$ satisfy $H_{i+1} = H_i + H_{i-k_j}$ for sufficiently large i.*

Proof. We have $k_2 \leq \chi(c_2) \leq \chi(c_1 - 1) \leq k_1$; if $k_2 = k_1$ every equality would hold, so $\chi(c_2) = k_2$ would be an integer and $1 + 1/\chi(c_2) = \log(c_2+1)/\log c_2$ would be a rational number, implying that $c_2 + 1$ is a rational power of c_2, which cannot hold in light of the unique prime factorization theorem for the integers. □

When $d_1 = d_2 = 0$, this corollary becomes:

Corollary 4.4. *For any integers $c_1 > c_2 \geq 2$, the degree of the recursion satisfied by the ultimate losing positions for $f(n) = c_1 n$ is greater than the corresponding degree for $f(n) = c_2 n$.*

References [Berlekamp et al. 1982; Schwenk 1970; Whinihan 1963] asked for results about these degrees; the above corollary and the preceding theorem are the first such results.

Theorem 4.1 gives an interval, in terms of c, in which the degrees corresponding to all $f(n) = cn + d$ lie; the length of this interval, $\chi(c) - \chi(c-1)$, is asymptotic to $\log c$ as $c \to \infty$, since $\chi(c)$ is asymptotic to $c \log c$.

5. Special Values of c

In this section we derive sharp results about the degree corresponding to $f(n) = cn + d$ when c has certain special values.

Proposition 5.1. *For any positive real number d, the sequence of losing positions for the take-away game corresponding to $f(n) = n + d$ satisfies $H_{i+1} = 2H_i$ for all sufficiently large i.*

Note that this could be stated: if $c = 1$ the degree is 0.

Proof. Since the H_i form an increasing sequence of positive integers, there is some $H_i > 2d$. Then $H_{i+1} = H_i + H_r$, where $H_r + d \geq H_i$. Thus $H_i + d \leq H_r + 2d < H_r + H_i = H_{i+1}$, so $H_{i+2} = 2H_{i+1}$. Then $f(H_{i+1}) < H_{i+2}$, so $H_{i+3} = 2H_{i+2}$, and so on. □

Proposition 5.2. *For any positive real number d, the sequence of losing positions for the take-away game corresponding to $f(n) = 2n + d$ satisfies $H_{i+1} = H_i + H_{i-1}$ for all sufficiently large i.*

This could be stated: if $c = 2$ the degree is 1.

Proof. By Theorem 3.1, we know that $H_{i+1} = H_i + H_{i-k}$ for all sufficiently large i. Theorem 4.1 implies that

$$k \leq \frac{\log 2}{\log 3 - \log 2} < 2,$$

so $k = 0$ or $k = 1$. For any i, we have $H_{i+1} = H_i + H_r$, so

$$f(H_i) > 2H_i \geq H_i + H_r = H_{i+1}.$$

Thus, we cannot have $k = 0$, so $k = 1$. □

6. How to Compute the Degrees

In this section we derive theoretical results which provide algorithms for computing the degrees; in the next section we present data generated using these algorithms, and make several conjectures based on it.

PROPOSITION 6.1. *Suppose* $f(n) = cn - d$, *where* $c - 1 \geq d \geq 0$. *If* $H_{j+1} = H_j + H_{j-k}$ *for some* j, *and* $cH_{j+i-1-k} < H_{j+i}$ *for* $1 \leq i \leq k+1$, *then* $H_{r+1} = H_r + H_{r-k}$ *for all* $r \geq j$.

PROOF. By Lemma 2.2, $f(H_{r-k-1}) < H_r \leq f(H_{r-k})$ implies $H_{r+1} \leq f(H_{r-k+1})$. Thus, since f is increasing, $H_r \leq f(H_{r-k})$ implies $H_{r+1} \leq f(H_{r-k+1})$. Since $H_j \leq f(H_{j-k})$, we have $H_r \leq f(H_{r-k})$ for all $r \geq j$. Now, since $cH_{j+i-1-k} < H_{j+i}$, we have

$$f(H_{j+i-1-k}) \leq cH_{j+i-1-k} < H_{j+i} \leq f(H_{j+i-k}),$$

so $H_{j+i+1} = H_{j+i} + H_{j+i-k}$ for $1 \leq i \leq k+1$. In particular, $H_{j+2} = H_{j+1} + H_{j+1-k}$ and $H_{j+k+2} = H_{j+k+1} + H_{j+1}$. Thus

$$H_{j+k+2} = H_{j+k+1} + H_{j+1} > cH_j + cH_{j-k} = cH_{j+1}.$$

We have shown that $H_{j+2} = H_{j+1} + H_{j+1-k}$ and $cH_{j+1} < H_{j+k+2}$; thus, for $J = j+1$ we have $H_{J+1} = H_J + H_{J-k}$ and, for $i = 1, 2, \ldots, k+1$, we have $cH_{J+i-1-k} < H_{J+i}$. So, by induction, for all $r \geq j$ and all $1 \leq i \leq k+1$ we have $H_{r+1} = H_r + H_{r-k}$ and $cH_{r+i-1-k} < H_{r+i}$. □

This suggests the following algorithm.

ALGORITHM 6.2. For $f(n) = cn - d$, where $c - 1 \geq d \geq 0$, compute successive terms H_i, by putting $H_1 = 1$ and

$$H_{i+1} = H_i + \min_{j \leq i} \{H_j : f(H_j) \geq H_i\}.$$

Stop when an integer j is found for which $H_{j+1} = H_j + H_{j-k}$ and $cH_{j+i-1-k} < H_{j+i}$ for each $1 \leq i \leq k+1$, and output k.

We do not know, a priori, whether this algorithm will terminate. If it does, it will output the integer k for which $H_{r+1} = H_r + H_{r-k}$ for sufficiently large r. We implemented this algorithm and applied it for all pairs of integers (c, d) such that $90 \geq c > d > 0$; it terminated rather quickly in every case. Our data is considered in the next section.

Our algorithm for functions $f(n) = cn + d$ is a bit more complicated. We begin with a definition: For a given function $f(n) = cn + d$ with $c > 1, d > 0$, and any integer $j > 1$, let $\delta(j)$ be the unique positive integer such that $H_j = H_{j-1} + H_{\delta(j)}$.

LEMMA 6.3. *Suppose $f(n) = cn + d$, where $c > 1$ and $d > 0$. If j is an integer such that $\delta(j) > 1$ and that each $i = \delta(j), \delta(j)+1, \ldots, j-1$ satisfies $H_i \geq cH_{\delta(i)}$, the sequence of differences $\bigl(i - \delta(i)\bigr)_{i \geq j}$ is nonincreasing.*

PROOF. Put $m = \delta(j)$; then

$$\begin{aligned} H_j - cH_{\delta(j)} &= H_{j-1} - (c-1)H_m \\ &> f(H_{m-1}) - (c-1)H_m = cH_{m-1} + d - cH_m + H_m \\ &= d + H_m - cH_{\delta(m)} \geq d, \end{aligned}$$

so $H_j > f(H_{\delta(j)}) > cH_{\delta(j)}$. Now, $H_{j+1} = H_j + H_{\delta(j+1)}$ implies that $f(H_{\delta(j+1)}) \geq H_j$, so $\delta(j+1) > \delta(j)$. Since $\delta(j+1) \leq j$, we know that $H_i \geq cH_{\delta(i)}$ for each $i = \delta(j+1), \ldots, j$, so our hypotheses are satisfied if we replace j by $j+1$. By induction, our hypotheses are satisfied if we replace j by any larger integer r. As above, this implies that $\delta(r+1) > \delta(r)$, so $r - \delta(r) \geq r + 1 - \delta(r+1)$ for any $r \geq j$, which is what we are trying to show. □

The proof of Theorem 3.1 shows that there do exist integers j satisfying the hypotheses of the above lemma.

PROPOSITION 6.4. *For f as in the preceding lemma, suppose that the sequence of differences $\bigl(i - \delta(i)\bigr)_{i \geq j}$ is nonincreasing. If $H_j = H_{j-1} + H_{j-1-k}$ and also $H_{j+i} \leq cH_{j+i-k}$ for each $i = 0, 1, \ldots, k$, then $H_{r+1} = H_r + H_{r-k}$ for all $r \geq j-1$.*

PROOF. First, $H_{j+i} \leq cH_{j+i-k}$ implies that $H_{j+i} < f(H_{j+i-k})$; but $H_{j+i+1} = H_{j+i} + H_{\delta(j+i+1)}$ implies that $H_{j+i} > f(H_{\delta(j+i+1)-1})$, so $\delta(j+i+1) \leq j+i-k$. Since the sequence of differences is nonincreasing,

$$k + 1 = j - \delta(j) \geq j + i + 1 - \delta(j+i+1),$$

so $\delta(j+i+1) \geq j+i-k$. Thus, $\delta(j+i+1) = j+i-k$ for each $i = 0, 1, \ldots, k$, so $H_{j+i+1} = H_{j+i} + H_{j+i-k}$ for each such i. Now,

$$H_{j+k+1} - cH_{j+1} = H_{j+k} + H_j - cH_{j+1} \leq cH_j + cH_{j-k} - cH_{j+1} = 0,$$

so the hypotheses of this proposition are satisfied if we replace j by $j+1$. By induction, these hypotheses are satisfied if we replace j by any larger integer; thus, as above, $H_{r+1} = H_r + H_{r-k}$ for all $r \geq j-1$. □

ALGORITHM 6.5. For $f(n) = cn + d$, where $c > 1$ and $d > 0$, compute successive terms H_i by putting $H_1 = 1$ and

$$H_{i+1} = H_i + \min_{l \leq i}\{H_l : f(H_l) \geq H_i\}.$$

Find the least integer j_0 such that $\delta(j_0) > 1$ and that each i with $\delta(j_0) \leq i \leq j_0 - 1$ satisfies $H_i \geq cH_{\delta(i)}$. Stop when an integer $j \geq j_0$ is found for which $H_j = H_{j-1} + H_{j-1-k}$ and $H_{j+t} \leq cH_{j+t-k}$ for each $t = 0, 1, \ldots, k$. Output k.

By Lemma 6.3 and Proposition 6.4, if this algorithm terminates, we have $H_{r+1} = H_r + H_{r-k}$ for all $r \geq j-1$. We implemented this algorithm and applied it to all pairs of positive integers (c, d) such that $c \leq 90$ and $d \leq 1000$; in every case the algorithm terminated rather quickly. The data is considered below.

7. Observations and Conjectures

Armed with the algorithms from the previous section, we computed the degrees for various functions $f(n) = cn + d$ for integers c, d. We computed the degrees for all pairs of integers $(c, -d)$ with $0 \leq -d < c \leq 90$, and also for all pairs of positive integers (c, d) with $c \leq 90$ and $d \leq 1000$. We now state some observations about these data, which lead us to make several conjectures.

Our first observation is that, for fixed c, positive integers d that are "close" to each other tend to produce the same degree, and similarly negative integers d that do not differ by much tend to produce the same degree. This is especially pronounced for large positive integers d: as it almost always happens that, for fixed c, all d between 50 and 1000 produce the same degree.

Next, for fixed c, the integers $d = 0, -1, \ldots, 1-c$ produce at most two different degrees, and if there are two they are consecutive integers. For fixed c and positive d, the degree tends to be smallest for $d < 20$, say, and the degree for $d = 1000$ exceeds the degree for $d = 1$ for all $c > 5$. Moreover, for $c > 5$, every $d > 30$ has larger degree than does $d = 1$. So, it seems that very small positive values of d produce a degree smaller than that produced by all larger positive d.

Finally, we observe that, for fixed c, every $d > 32$ produces a larger degree than any negative d. This suggests an interesting relationship between positive and negative values of d. Theorem 4.1 implies that, for fixed c, every d produces a degree in a certain interval whose length is something like $\log c$; our feeling is that the negative and only slightly positive values of d lead to degrees in the lower part of this interval, whereas larger positive values of d lead to degrees in the upper part of this interval. We shall not spoil the reader's fun by proving these conjectures.

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