Comparison Geometry
MSRI Publications
Volume **30**, 1997

Differential Geometric Aspects of Alexandrov Spaces

YUKIO OTSU

Abstract. We summarize the results on the differential geometric structure of Alexandrov spaces developed in [Otsu and Shioya 1994; Otsu 1995; Otsu and Tanoue a]. We discuss Riemannian and second differentiable structure and Jacobi fields on Alexandrov spaces of curvature bounded below or above.

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function, $B := \{(u,t) \in \mathbb{R}^n \times \mathbb{R} : t > f(u)\}$, and $\Gamma = \Gamma_f := \{(u, f(u)) \in \mathbb{R}^n \times \mathbb{R}\} = \partial B$. Then B is an open convex set. A hyperplane L in $\mathbb{R}^{n+1}$ is a *support* at $x \in \Gamma$ if $x \in L$ and $L \cap B = \varnothing$. We say that $x \in \Gamma$ is a *singular point* if supports at x are not unique. Let S_Γ be the set of singular points in Γ, and $S_f := \{u \in \mathbb{R}^n : (u, f(u)) \in S_\Gamma\}$. If $u \notin S_f$, then f is differentiable and the differential is continuous at u.

Theorem 1.1 [Reidemeister 1921]. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then f is a.e. differentiable: more precisely, f is C^1 on $\mathbb{R}^n \setminus S_f \subset \mathbb{R}^n$, and the Hausdorff dimension $\dim_H S_f = \dim_H S_\Gamma$ is at most $n-1$.*

If $n = 1$, the map $\mathbb{R} \to \mathbb{R}$ given by $u \mapsto df_u$ is monotone by convexity and, therefore, df_u is a.e. differentiable, that is, f is a.e. twice differentiable. In general, we have [Busemann and Feller 1935; Alexandrov 1939]:

Theorem 1.2 (Alexandrov's theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then f is a.e. twice differentiable in the sense of Stolz, that is, for a.a. $x \in \mathbb{R}^n$ there exist $A \in \mathbb{R}^n$ and an $n \times n$ symmetric matrix $B \in \mathrm{Sym}(n)$ such that*

$$f(u+h) = f(u) + Ah + \tfrac{1}{2}\, {}^t hBh + o\big(|h|^2\big)$$

1991 *Mathematics Subject Classification.* 53C20, 53C23, 53C45, 26B25.

Key words and phrases. Alexandrov space, differentiable structure, comparison geometry.

for $h \in \mathbb{R}^n$.

Consider the inner geometry of Γ. The length of a path $c : [a, b] \to \Gamma$ is

$$|c| := \sup_{a=a_0<\ldots<a_l=b} \sum_{i=1}^{l} \big|c(a_{i-1})c(a_i)\big|,$$

where $|\ |$ on the right denotes the Euclidean metric. The intrinsic distance d on Γ is defined by

$$d(p, q) = |pq| = \inf_c |c|,$$

where c is a path from p to q. Then Γ is a geodesic space, that is, for any p, $q \in \Gamma$ there is a path from p to q whose length is equal to $|pq|$; this is called a minimal segment or minimal geodesic (from p to q), and denoted by pq. Any convex function can be written as a limit of a sequence of C^∞ convex functions $\{f_i\}_{i=1}^{\infty}$; equivalently,

$$d_{pH}((\Gamma, p), (\Gamma_i, p_i)) \to 0 \quad \text{as } i \to \infty, \tag{1.1}$$

where d_{pH} denotes the pointed Hausdorff distance, $\Gamma_i = \Gamma_{f_i}$, $p = (0, f(0))$, and $p_i = (0, f_i(0))$. Note that Γ_i is a Riemannian manifold with sectional curvature ≥ 0.

Let X be a geodesic space. For p, q, $r \in X$ the *triangle* $\triangle pqr$ is a triad of segments pq, qr, rp. A *comparison triangle* for $\triangle pqr$ is a triangle $\tilde{\triangle} pqr = \triangle \tilde{p}\tilde{q}\tilde{r}$ in $\mathbb{R}^2$ with $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{r}\tilde{p}| = |rp|$, $|\tilde{r}\tilde{q}| = |qr|$. If X is a Riemannian manifold with sectional curvature ≥ 0, it has the following property:

Property 1.3 (Alexandrov convexity). *Given a triangle* $\triangle pqr$ *in* X, *there is a comparison triangle* $\tilde{\triangle} pqr$ *such that*, *for* $s \in qr$, *we have*

$$|ps| \geq |\tilde{p}\tilde{s}|,$$

where $\tilde{s} \in \tilde{q}\tilde{r}$ *with* $|\tilde{s}\tilde{q}| = |sq|$.

This property is equivalent to Toponogov's comparison theorem. Since (1.1) implies that any segment on Γ is approximated by segments on Γ_i, Alexandrov convexity is also valid on Γ.

For $k \leq 0$ a metric space X is an *Alexandrov space of curvature* $\geq k$ if X is a locally compact, complete length space of dimension $\dim_H X < \infty$, satisfying the Alexandrov convexity property with $\mathbb{R}^2$ replaced by $H^2(k)$, the simply connected space form of curvature k. Then the dimension of X is an integer, say n, and the n-dimensional Hausdorff measure V_H^n satisfies

$$0 < V_H^n(B(p, r)) < \infty$$

for any $B(p, r) := \{x \in X : |px| < r\}$.

EXAMPLES. (1) A complete Riemannian manifold of sectional curvature $\geq k$ is an Alexandrov space of curvature $\geq k$.

(2) It follows from the preceding discussion that the graph Γ of a convex function is an Alexandrov space of curvature ≥ 0 and has a natural a.e. twice differentiable structure.

(3) As the above argument illustrates, the Hausdorff limit of a sequence of Riemannian manifolds $\{M_i\}$ of curvature $\geq k$ is an Alexandrov space of curvature $\geq k$. Gromov's convergence theorem states that if, for each M_i, the absolute value of the sectional curvature is bounded above by a constant and the injectivity radius is bounded below by a positive constant, then X is a $C^{1,\alpha}$ Riemannian manifold for $0 < \alpha < 1$, that is, X has a $C^{2,\alpha}$ differentiable structure and a $C^{1,\alpha}$ Riemannian metric.

(4) Let X be an n-dimensional Alexandrov space and let $p \in X$. Then there exists a pointed Hausdorff limit of (iX, p), as $i \to \infty$; it is called a *tangent cone at* p, and is denoted by K_p. (In Section 2 we will give another definition of a tangent cone.) The point p is called *singular* if K_p is not isometric to $\mathbb{R}^n$. The tangent cone K_p is an Alexandrov space of curvature ≥ 0.

In some of these examples we find that there exists a second differentiable structure. In general Reidemeister's and Alexandrov's theorems are generalized as follows:

THEOREM 1.4 [Otsu and Shioya 1994; Otsu 1995]. *Let X be an n-dimensional Alexandrov space of curvature bounded below, and let S_X be the set of singular points on X.*

(i) *The complement $X \setminus S_X \subset X$ has a C^1 differentiable and Riemannian structure, and $\dim_H S_X \leq n-1$. The induced metric from the Riemannian structure coincides with the original metric on X.*
(ii) *There is a set $X_0 \subset X \setminus S_X$ of full measure with respect to V_H^n such that $X_0 \subset X \setminus S_X$ has an approximately second differentiable structure in the sense of Stolz.*

In this paper we do not give precise definitions for the various structures mentioned in this theorem. Instead, we give in Section 2 a rough sketch of the proof of the theorem, and show how to develop differential geometry on Alexandrov spaces of curvature bounded below. In Section 3 we discuss differential geometry on Alexandrov space of curvature bounded above.

2. Elements of Differential Geometry on Alexandrov Spaces of Curvature Bounded Below

The exponential map. For simplicity we assume that X is an n-dimensional Alexandrov space of curvature ≥ 0. Note that no segment on X branches, by

Alexandrov convexity. For $p, q, r \in X$ and segments $pq = \gamma : [0, a] \to X$ and $pr = \sigma : [0, b] \to X$, put

$$\omega(t, s) = \angle\tilde{\gamma}(t)\tilde{p}\tilde{\sigma}(s)$$

for $0 < t \le a$ and $0 < s \le b$. Then, by Alexandrov convexity, $(t, s) \mapsto \omega(t, s)$ is a monotone nonincreasing function; thus the angle between γ and σ,

$$\angle qpr := \lim_{(t,s)\searrow(0,0)} \omega(t, s),$$

is well-defined. It follows easily that X has the following property:

Property 2.1 (Toponogov convexity). *For any triangle $\triangle pqr$ there is a comparison triangle $\triangle\tilde{p}\tilde{q}\tilde{r}$ such that*

$$\angle rpq \ge \angle\tilde{r}\tilde{p}\tilde{q}, \quad \angle qrp \ge \angle\tilde{q}\tilde{r}\tilde{p}, \quad \angle pqr \ge \angle\tilde{p}\tilde{q}\tilde{r}.$$

Set $\tilde{W}_p = \{pq : q \in X\}$ and $\tilde{\Sigma}_p = (\tilde{W}_p \setminus o_p)/\sim$, where o_p is the trivial segment pp and $pq \sim pr$ implies $pq \subset pr$ or $pr \subset pq$. Denote by v_{pq} the equivalence class of pq. The *space of directions* Σ_p is the completion of $(\tilde{\Sigma}_p, \angle)$, and is a compact Alexandrov space of curvature ≥ 1 [Burago et al. 1992]. The *tangent cone* K_p is obtained from $[0, \infty) \times \Sigma_p$ by identifying together all elements of the form $(0, u_0)$, and its elements are denoted by tu_0 for $t \ge 0$ and $u_0 \in \Sigma_p$, or simply by u. We introduce a distance on K_p by setting

$$|tu_0\, sv_0| := \sqrt{t^2 + s^2 - 2st \cos\angle u_0 v_0}$$

for $tu_0, sv_0 \in K_p$. Then $(K_p, |\ |)$ is an Alexandrov space of curvature ≥ 0. By considering $\tilde{W}_p \subset K_p$, we define the exponential map $\mathrm{Exp}_p : \tilde{W}_p \to X$ by

$$\mathrm{Exp}_p |pq| v_{pq} = q.$$

Toponogov convexity implies

$$|\mathrm{Exp}_p u\, \mathrm{Exp}_p v| \le |uv|. \tag{2.1}$$

For $\delta > 0$, let W_p^δ be the set of $x \in X$ for which there exists $y \in X$ such that $x \in py$ and $|py| = (1 + \delta)|px|$. In this case y is unique, and we define a map $E_p^\delta : W_p^\delta \to X$ by setting

$$E_p^\delta(x) = y.$$

By Alexandrov convexity we have

$$|E_p^\delta(x) E_p^\delta(y)| \le (1 + \delta)|xy|, \tag{2.2}$$

and clearly

$$E_p^\delta \circ \mathrm{Exp}_p u = \mathrm{Exp}_p (1 + \delta)u. \tag{2.3}$$

Define the *cut locus* of p by $\mathrm{Cut}_p = X \setminus \bigcup_{\delta>0} W_p^\delta$. If $x \in \mathrm{Cut}_p$, then x is not an interior point of any segment from p, so this definition coincides with that of Riemannian manifold.

Proposition 2.2. *We have $V_H^n(\mathrm{Cut}_p) = 0$.*

PROOF. Because the map $E_p^\delta : W_p^\delta \cap B(p,R) \to B(p,(1+\delta)R)$ is surjective for $R > 0$, we have

$$V_H^n(B(p,(1+\delta)R)) \le (1+\delta)^n V_H^n(W_p^\delta \cap B(p,R))$$

by (2.2). As $\delta \searrow 0$ we have

$$V_H^n(B(p,R)) \le V_H^n((X \setminus \mathrm{Cut}_p) \cap B(p,R)). \qquad \square$$

Then, since X is separable, $V_H^n(S_X) = 0$ by Toponogov's splitting theorem. A more careful argument will give us $\dim_H(S_X) \le n-1$.

The first variation formula. Suppose given a Riemannian manifold M, a point $p \in M$, and a segment $\sigma : [0,a] \to M$. Applying the first variation formula for $s \mapsto |p\sigma(s)|$, we have

$$\left.\frac{d}{ds}\right|_{s=0} |p\sigma(s)| = -\cos \min_{p\sigma(0)} \angle p\sigma(0)\sigma(a).$$

In the case of a point p in an Alexandrov space X of curvature ≥ 0, we also have

$$d_p(y) = d_p(x) - |xy| \cos \min_{px} \angle pxy + o(|xy|) \tag{2.4}$$

by the Lipschitz continuity of $x \mapsto d_p(x) = |px|$ and the compactness of Σ_x.

For $p_1, \dots, p_n \in X$ we define a map $\psi : X \to \mathbb{R}^n$ by

$$\psi(x) = (|p_1 x|, \dots, |p_n x|)$$

and $g_\psi : X \setminus \bigcup_{i=1}^n \mathrm{Cut}_{p_i} \to \mathrm{Sym}(n)$ by

$$g_\psi(x) = (\cos \angle p_i x p_j).$$

If $x_0 \in X \setminus S_X$, we can choose points $p_1, \dots, p_n \in X$ so that $g_\psi(x_0) > 0$. Because the angle is continuous at a nonsingular point, as the differential is continuous at regular points of Γ, there is a neighborhood U_ψ of x_0 such that $g_\psi(x) > 0$ on U_ψ and $\psi : U_\psi \to \mathbb{R}^n$ is a homeomorphism onto an open set in $\mathbb{R}^n$. Since

$$\psi(y) = \psi(x) + |xy|(-\cos \angle p_i xy) + o(|xy|)$$

by (2.4), the set $\{(\psi, U_\psi, g_\psi)\}$ gives us an a.e. C^1 structure and Riemannian structure on $X \setminus S_X \subset X$.

Jacobi fields. For simplicity we choose $p \in X \setminus S_X$, that is, $K_p = \mathbb{R}^n$. Since $\mathrm{Exp}_p : \tilde{W}_p \to X$ is a Lipschitz map by (2.1), using the differentiable and Riemannian structure on X we conclude that Exp_p is differentiable a.e. by extending Rademacher's theorem, which states that a Lipschitz map between Euclidean spaces is differentiable a.e. Let $x \in X \setminus S_X$ be such that Exp_p is a.e. differentiable on $[0, |px|]v_{px}$. For a segment $\sigma : [0,a] \to X$ starting at x we want to construct a Jacobi field $J(t)$ from $\alpha(s,t) = \gamma_s(t)$, where γ_s is a segment from p to $\sigma(s)$ whose parameter is scaled for $[0, |px|]$. Let $\tilde{\alpha}$ be a lift of α by Exp_p

and let $w \in K_{|px|v_{px}} K_p = \mathbb{R}^n$ be such that $d\,\mathrm{Exp}_p|_{|px|v_{px}} w = \dot{\sigma}(0)$. Then $\tilde{\alpha}$ is differentiable at $s = 0$ and

$$\frac{d}{ds}\bigg|_{s=0} \tilde{\alpha}(s,t) = tw,$$

which we denote by $\tilde{J}_w(t)$. For a.a. $t \in (0, |px|]$ there exists

$$\frac{d}{ds}\bigg|_{s=0} \alpha(s,t) = d\,\mathrm{Exp}_p|_{tv_{px}} \tilde{J}_w(t).$$

This is the desired Jacobi field $J_w(t)$, that is,

$$J_w(t) = d\,\mathrm{Exp}_p|_{tv_{px}} \tilde{J}_w(t). \tag{2.5}$$

Then $w \mapsto J_w(t)$ is a well-defined linear map on $K_{|px|v_{px}} K_p = \mathbb{R}^n$.

Second differential of the distance function. As mentioned on the preceding page, the first differentiable structure is deduced from the first variation formula. Similarly, the second differentiable structure is determined by the second variation of the distance function. First we examine the second variation of the distance function on a Riemannian manifold M. For $p \in M$ and a segment $\sigma : [0,a] \to M$ with $\sigma(0) \notin \mathrm{Cut}_p$, we have

$$\frac{d^2}{ds^2}\bigg|_{s=0} |p\sigma(s)| = \Big\langle J, \frac{d}{ds}\nabla d_p \Big\rangle = \big\langle J, \nabla_{t=|p\sigma(0)|} J_w^{\perp}(t) \big\rangle, \tag{2.6}$$

where ∇d_p denotes the gradient vector of d_p and $J^{\perp}$ the orthogonal component of J with respect to ∇d_p. Therefore

$$|py| = |px| + |xy| \langle v_{xy}, \nabla d_p \rangle + \tfrac{1}{2}|xy|^2 \big\langle J_{v_{xy}}^{\perp}(t),\, \nabla J_{v_{xy}}^{\perp}(t) \big\rangle + o(|xy|^2). \tag{2.7}$$

Here we present facts that assure us that (2.7) is valid. Let $p \in X$ and let $\sigma : [0,a] \to X$ be a segment.

PROPOSITION 2.3. *The function $s \mapsto \angle p\sigma(s)\sigma(a)$ is of bounded variation; in particular, it is a.e. differentiable. For any s_0 where it is differentiable, we have*

$$\begin{aligned} |p\sigma(s)| = |p\sigma(s_0)| &- (s - s_0)\cos\angle p\sigma(s_0)\sigma(a) \\ &+ \frac{1}{2}(s-s_0)^2 \sin\angle p\sigma(s)\sigma(a)\ \frac{d}{ds}\bigg|_{s=s_0} \angle p\sigma(s)\sigma(a) + o(|s-s_0|^2) \end{aligned}$$

and

$$\frac{d}{ds}\bigg|_{s=s_0} \angle p\sigma(s)\sigma(a) \le \frac{1}{|p\sigma(s_0)|}\sin\angle p\sigma(s)\sigma(a).$$

PROOF. By Toponogov convexity,

$$\begin{aligned} \angle p\sigma(s')\sigma(a) = \pi - \angle p\sigma(s')\sigma(s) &\le \pi - \angle \tilde{p}\widetilde{\sigma(s')}\widetilde{\sigma(s)} = \angle \tilde{p}\widetilde{\sigma(s)}\widetilde{\sigma(a)} + \angle \widetilde{\sigma(s)}\tilde{p}\widetilde{\sigma(a)} \\ &< \angle p\sigma(s)\sigma(a) + (s'-s)\frac{1}{|p\sigma(s)|}\sin\angle p\sigma(s)\sigma(a) + O(|s-s_0|^2) \end{aligned}$$

for $s < s'$. (See Figure 1.) □

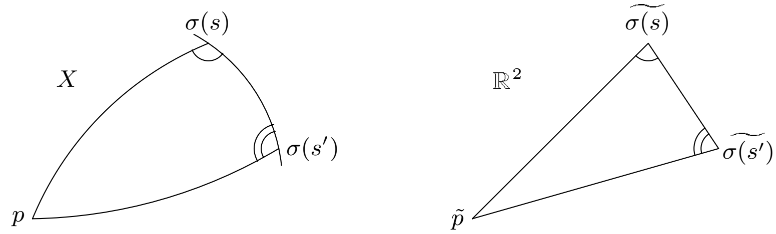


Figure 1. Comparison triangles in the proof of Proposition 2.3.

It is difficult to deduce an expansion like (2.7) from Proposition 2.3, since the above argument is restricted to segments; the estimation is not uniform on the direction; it is not clear that the second differential is a quadratic form, etc. We show here that the last term of (2.6) exists a.e. and that it is a quadratic form.

PROPOSITION 2.4. *The function* $t \mapsto |J(t)|$ *has bounded variation. In particular, for a.a.* t *there is a first differential for* $|J(t)|$, *satisfying*

$$\frac{d}{dt}|J(t)| \leq \frac{1}{t}|J(t)|,$$

and the map

$$w \mapsto \frac{d}{dt}|J_w(t)|^2$$

on $K_{|px|v_{px}}K_p$ *is a quadratic form.*

PROOF. As in Section 2.3, we consider a family of segments $\gamma_s(t)$ as a variation. By (2.2), we have

$$|\gamma_0((1+\delta)t)\gamma_s((1+\delta)t)| \leq (1+\delta)|\gamma_0(t)\gamma_s(t)|.$$

By taking $s \searrow 0$ we have $|J((1+\delta)t)| \leq (1+\delta)|J(t)|$ for a.a. t and δ, that is, $t \mapsto |J(t)|$ has bounded variation. Then

$$\frac{1}{\delta t}(|J((1+\delta)t)| - |J(t)|) \leq \frac{|J(t)|}{t}. \qquad \square$$

Although it is quite interesting to show the second differential of the distance function coincides with $d|J(t)^{\perp}|/dt$, we omit this proof because it is complicated. Then we have twice differentiability of d_p a.e., and a second differentiable structure on X a.e.

3. Elements of Differential Geometry on Alexandrov Spaces of Curvature Bounded Above

Alexandrov spaces of curvature bounded above. We say that X is a *metric space of curvature* $\leq K$ if X is a complete geodesic space with the following property:

PROPERTY 3.1 (ALEXANDROV CONVEXITY). *For any point $p \in X$ there is a convex neighborhood U of p such that, for any triangle $\triangle xyz$ in U, there is a comparison triangle $\tilde{\triangle} xyz$ such that if $s \in yz$, then*

$$|xs| \leq |\tilde{x}\tilde{s}|$$

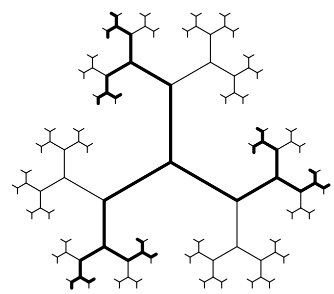
for $\tilde{s} \in \tilde{y}\tilde{z}$ with $|\tilde{s}\tilde{y}| = |sy|$. Here the comparison triangle $\triangle\tilde{x}\tilde{y}\tilde{z}$ is taken in the space form $H^2(K)$; if $K > 0$, we assume that $|xy| + |yz| + |zx| \leq 2\pi/\sqrt{K}$.

Notice that the comparison inequality is valid only on U; the situation is completely different from the case of curvature bounded below. If $x \in U$, then x is not a cut point of $p \in U$.

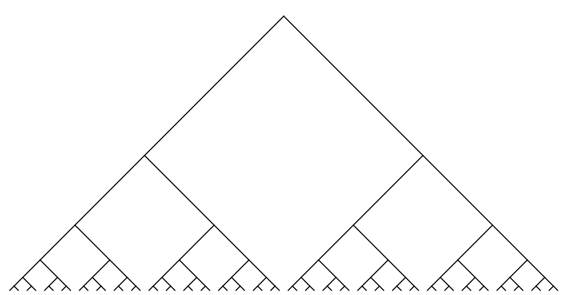
EXAMPLES. (1) A complete Riemannian manifold of sectional curvature $\leq K$.

(2) A finite graph.

(3) Let X_i be a binary tree where each edge has length 2^{-i}. Then X_i can be isometrically embedded in X_{i+1}; the figure on the right shows in thick lines a finite approximation to X_i, embedded in a finite approximation to X_{i+1}. From the sequence $X_1 \subset X_2 \subset \ldots$ we can construct an inductive limit X_∞, which is also a metric space of curvature ≤ 0. This is not locally compact.

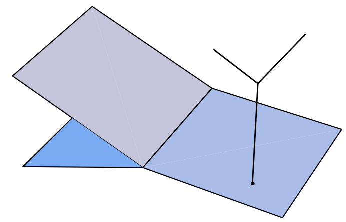


(4) Next we consider another binary tree X such that the length of each edge at height i is 2^{-i}, as in the figure below. Then X is locally compact but not geodesically complete. Notice that the Hausdorff dimension of its boundary is 1.



(5) A two-dimensional cone at whose vertex the total angle is greater than 2π.

(6) A simplicial complex which branches, as in the figure on the right.



(7) For (s,t) and $(s',t') \in \mathbb{R}^2$, define the distance between them as $|s-s'|$ if $t=t'$, $s+s'+|t-t'|$ if $s,s' \geq 0$ and $t \neq t'$, $s+\sqrt{s'^2+|t-t'|^2}$ if $s \geq 0$ and $s' \leq 0$, and as the Euclidean metric if $s,s' \leq 0$. This makes $\mathbb{R}^2$ into a metric space of curvature ≤ 0 that is not locally compact.

Angles and related concepts are defined in a similar way as for Alexandrov spaces of curvature bounded below. In the absence of additional restrictions, it is difficult to deduce for metric spaces of curvature bounded above a topological structure and a result similar to Theorem 1.4, as the examples illustrate. We therefore make the following definition: A geodesic space X is an *Alexandrov space of curvature bounded above* if X is a locally compact, geodesically complete metric space of curvature bounded above.

THEOREM 3.2 [Otsu and Tanoue a]. *Let X be an Alexandrov space of curvature bounded above. Then, for any $p \in X$ and $r > 0$, there is an integer n such that $0 < V_H^n(B(p,r)) < \infty$. Let S_X^n be the set of $x \in B(p,r)$ such that K_x is not isometric to $\mathbb{R}^n$.*

(i) *There exists a C^1 differentiable and a C^0 Riemannian structure for $B(p,r) \setminus S_X^n \subset B(p,r)$, and S_X^n is a set of V_H^n null measure.*
(ii) *There exists an a.e. second differentiable structure in the sense of Stolz on $B(p,r)$.*

We now give a description of differential geometric properties in the absence of the above restriction.

The Jacobi norm. As we know from the examples given earlier, we cannot define an exponential map. Thus we define the Jacobi norm from variation. For simplicity we assume that X is a metric space of curvature ≤ 0 and $U = X$. Let $\sigma : [0,a] \to X$ be a segment and let $\alpha(s,t) = \gamma_s(t)$ be a family of normal segments from p to $\sigma(s)$. It follows from Alexandrov convexity that

$$(1+\delta)\,|\alpha(0,t)\,\alpha(s,t)| \leq |\alpha(0,(1+\delta)t)\,\alpha(s,(1+\delta)t)|.$$

Hence for a.a. t there is

$$|J|(t) = |J|_\alpha(t) := \limsup_{h\to 0} \frac{1}{h}|\alpha(0,t)\alpha(h,t)|.$$

As in Proposition 2.4, for a.a. t there is a first differential of $|J|$ at t, and

$$\frac{d}{dt}|J|(t) \geq |J|(t) \times \frac{1}{t}.$$

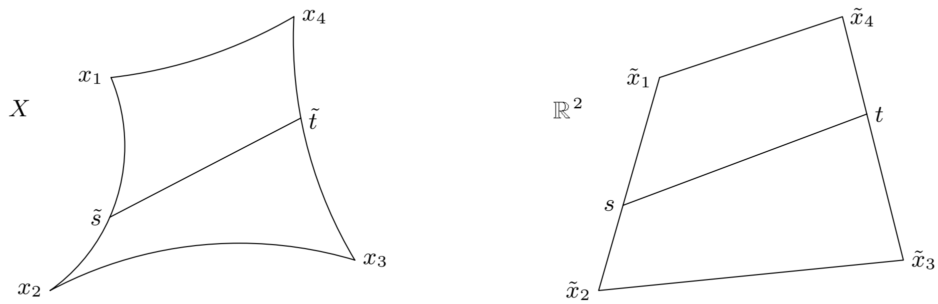


Figure 2. For a space with the Wald convexity property, $|st| \le |\tilde{s}\tilde{t}|$.

Here we examine again the Riemannian case: Let $J(t)$ be a Jacobi field on M along a segment γ. Then

$$\begin{aligned}\frac{d^2}{dt^2}|J(t)| &= \frac{d}{dt}\frac{1}{|J|}\langle J(t), \nabla_t J(t)\rangle \\ &= \frac{1}{|J|}\left(|\nabla_t J(t)|^2 - \frac{1}{|J|^2}\langle J(t), \nabla_t J(t)\rangle^2\right) + \left\langle \frac{1}{|J|}J(t), R(\dot\gamma(t), J(t))\dot\gamma(t)\right\rangle \\ &= \frac{1}{|J|}\left(|\nabla_t J(t)|^2 - \left(\frac{d}{dt}|J(t)|\right)^2\right) - K_{\langle\dot\gamma(t), J(t)\rangle}|\dot\gamma(t)\wedge J(t)|, \qquad (3.1)\end{aligned}$$

where $K_{\langle\dot\gamma(t), J(t)\rangle}$ denotes the sectional curvature of the section spanned by $\dot\gamma(t)$ and $J(t)$, and $\wedge$ denotes the exterior product of vectors. If the sectional curvature of M is nonpositive, we have

$$\frac{d^2}{dt^2}|J(t)| \ge \frac{1}{|J|}\left(|\nabla_t J(t)|^2 - \left\langle \frac{1}{|J|}J(t), \nabla_t J(t)\right\rangle^2\right) \ge 0. \qquad (3.2)$$

This argument does not hold for the case of curvature bounded below, and this is one reason why the treatment of Alexandrov spaces of curvature bounded below is difficult.

The Jacobi equation. The geometric expression of (3.2) is the following (see Figure 2):

Theorem 3.3 (Wald convexity [Reshetnyak 1968]). *Let X be a metric space of curvature ≤ 0. For any $x_1, \dots, x_4 \in X$ there exist $\tilde{x}_1, \dots, \tilde{x}_4 \in \mathbb{R}^2$ such that*

$$|x_1x_2| = |\tilde{x}_1\tilde{x}_2|, \quad |x_2x_3| = |\tilde{x}_2\tilde{x}_3|, \quad |x_3x_4| = |\tilde{x}_3\tilde{x}_4|, \quad |x_4x_1| = |\tilde{x}_4\tilde{x}_1|,$$

and that for $s \in x_ix_j$ and $t \in x_{i'}x_{j'}$ we have

$$|st| \le |\tilde{s}\tilde{t}|,$$

where $\tilde{s} \in \tilde{x}_i\tilde{x}_j$ and $\tilde{t} \in \tilde{x}_{i'}\tilde{x}_{j'}$ satisfy $|sx_i| = |\tilde{s}\tilde{x}_i|$ and $|tx_{i'}| = |\tilde{t}\tilde{x}_{i'}|$.

Applying this to $\alpha(0,t)$, $\alpha(0,t')$, $\alpha(h,t')$, $\alpha(h,t)$, we conclude that $t \mapsto |J|(t)$ is a convex function; in particular, it is continuous on $[0,|px|)$ and for a.a. t there is a second differential of $|J|$ at t, satisfying

$$\frac{d^2}{dt^2}|J|(t) \geq 0.$$

PROPOSITION 3.4. *Set* $\varphi = \angle\alpha(h,t)\alpha(0,t)\sigma(0)$, $\psi = \angle\alpha(0,t)\alpha(h,t)\sigma(h)$, $\varphi' = \angle\alpha(0,t{+}k)\alpha(h,t{+}k)p$, *and* $\psi' = \angle\alpha(h,t{+}k)\alpha(0,t{+}k)p$ (see Figure 3). *Then* $|J|$ *has a first and second differentials a.e., and*

$$\frac{d}{dt}|J|(t) = \lim_{h\to 0}\frac{-1}{h}(\cos\varphi + \cos\psi),$$
$$\frac{d^2}{dt^2}|J|(t) = \lim_{k\to 0}\lim_{h\to 0}\frac{1}{hk}(\cos\varphi + \cos\psi + \cos\varphi' + \cos\psi').$$

In the case of a Riemannian manifold, the covariant derivative of J is written as

$$\nabla J(t) = \lim_{h,k\to 0}\frac{1}{hk}(c-b-a),$$

where

$$a = \big|\alpha(h,t)\alpha(0,t)\big| v_{\alpha(0,t)\alpha(h,t)},$$
$$b = \big|\alpha(h,t{+}k)\alpha(0,t)\big| v_{\alpha(0,t)\alpha(h,t+k)},$$
$$c = \big|\alpha(0,t{+}k)\alpha(0,t)\big| v_{\alpha(0,t)\alpha(h,t+k)};$$

its existence is not clear in our case. If we set

$$|\nabla J|(t) = \lim_{k\to 0}\lim_{h\to 0}\frac{1}{hk}|c-b-a|,$$

then we have

$$\lim_{k\to 0}\lim_{h\to 0}\frac{1}{hk}(\theta+\eta-\varphi) = \frac{1}{2|J|}|\nabla J|^2,$$

where $\theta = \angle\alpha(h,t{+}k)\alpha(0,t)\alpha(0,t{+}k)$ and $\eta = \angle\alpha(h,t{+}k)\alpha(0,t)\alpha(h,t)$.

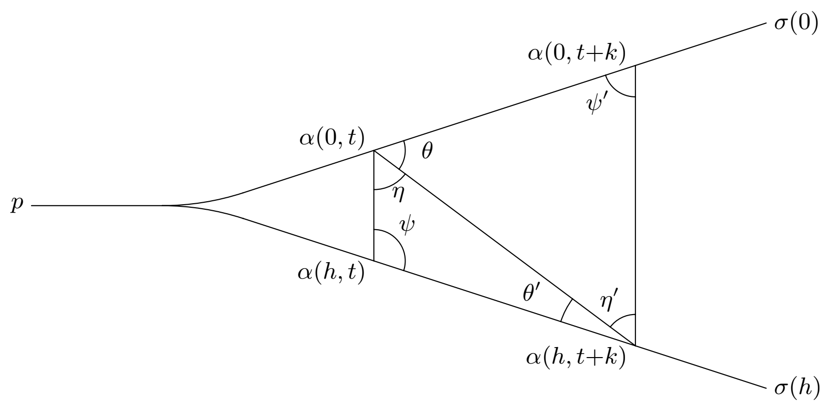


Figure 3. Angles in Proposition 3.4. In addition, $\varphi = \angle\alpha(h,t)\alpha(0,t)\sigma(0)$ and $\varphi' = \angle\alpha(0,t{+}k)\alpha(h,t{+}k)p$.

Next we consider the sectional curvature of M. If $\dim M = 2$, Gauss–Bonnet's theorem implies

$$\angle pqr + \angle qrp + \angle qrp - \pi = \int_{\triangle pqr} G\, dA,$$

where G denotes Gaussian curvature and dA the area element. Thus

$$G_x = \lim_{\triangle pqr \to x} \frac{1}{\text{area of } \triangle \tilde{p}\tilde{q}\tilde{r}} (\angle pqr + \angle qrp + \angle rpq - \pi). \tag{3.3}$$

In higher dimensions we have a similar result.

In the case of metric spaces of curvature bounded above we define the *connection norm* of $|J|$ as

$$|\nabla_t J|(t) = \left(|J| \limsup_{k\to 0} \limsup_{h\to 0} \frac{1}{hk}(\theta + \eta - \varphi + \theta' + \eta' - \varphi') \right)^{\frac{1}{2}},$$

and the *sectional curvature* by

$$K_\alpha(t) = \limsup_{k\to 0} \limsup_{h\to 0} \frac{1}{\tilde{h}k}(\psi + \eta + \theta' - \pi + \psi' + \eta' + \theta - \pi),$$

where $\theta' = \angle\alpha(0,t)\alpha(h,t{+}k)\alpha(h,t)$, $\eta' = \angle\alpha(0,t)\alpha(h,t{+}k)\alpha(0,t{+}k)$, and $\tilde{h} = |\alpha(0,t)\alpha(h,t)|$.

Then:

Theorem 3.5 [Otsu and Tanoue a]. *The second differential of the norm of Jacobi field*, *the sectional curvature*, *and* $|\nabla_t J|(t)$ *are well defined on* α *a.e. If they are well defined at* $(0,t)$, *then*

$$\frac{d^2}{dt^2}|J|(t) = \frac{1}{|J|}|\nabla_t J|(t)^2 - K_\alpha(t) \times |J|(t).$$

Consider a C^∞ hypersurface determined by $f : \mathbb{R}^n \to \mathbb{R}$. The first fundamental form is

$$g_{ij} = \delta_{ij} + \partial_i f \partial_j f,$$

and the curvature tensor is written by its second differentials, that is, the third differentials of f. However, the Theorema Egregium of Gauss states the curvature is determined by the eigenvalues of the second fundamental form, which is determined by the second differentials of f. Thus the third differentials kill each other and the sectional curvature is determined by the second differential structure of the graph. This observation explains why we can treat the above quantities without invoking higher differentiable structures.

Remarks. (1) The case of two-dimensional Alexandrov spaces of curvature bounded below is studied in [Machigashira 1995], where in particular the sectional curvature and a generalization of Gauss–Bonnet theorem are given. Note that here parallel translation along a segment is trivially determined.

We also mention Petrunin's work on parallel translation in Alexandrov spaces of curvature bounded below [Petrunin a; Berestvskii and Nikolaev 1993].

(2) Recently there have been several studies on harmonic maps on simply connected Alexandrov spaces of curvature ≤ 0 [Gromov and Schoen 1992; Jost a; Korevaar and Schoen 1993]. These results would seem to have close connections with ours.

References

[Alexandrov 1939] A. D. Alexandrov, "Almost everywhere existence of second differentials of convex functions and convex surfaces connected with it", *Leningrad State Univ. Ann. (Uchenye Zapiski) Math. Ser.* **6** (1939), 3–35.

[Alexandrov 1951] A. D. Alexandrov, "A theorem on triangles in a metric space and some of its applications", *Trudy Mat. Inst. Steklov* **38** (1951), 4–23.

[Alexander et al. 1993] S. Alexander, I. D. Berg, and R. L. Bishop, "Geometric curvature bounds in Riemannian manifolds with boundary", *Trans. Amer. Math. Soc.* **339** (1993), 703–716.

[Berestvskii and Nikolaev 1993] V. N. Berestvskii and I. G. Nikolaev, "Multidimensional generalized Riemannian spaces", pp. 165–243 in *Geometry IV: nonregular Riemannian geometry*, edited by Y. G. Reshetnyak, Encycl. Math. Sciences **70**, Springer, Berlin, 1993.

[Burago et al. 1992] Y. Burago, M. Gromov, and G. Perelman, "A. D. Aleksandrov spaces with curvatures bounded below", *Uspehi Mat. Nauk* **47**:2 (1992), 3–51, 222; translation in *Russian Math. Surveys* **47**:2 (1992), 1–58.

[Busemann 1958] H. Busemann, *Convex Surfaces*, Tracts in Pure and Appl. Math. **6**, Interscience, New York, 1958.

[Busemann and Feller 1935] H. Busemann and W. Feller, "Krümmungseigenschaften Konvexer Flächen", *Acta Math.* **66** (1935), 1–47.

[Evans and Gariepy 1992] L. Evans and R. Gariepy, *Measure theory and fine properties of functions*, CRC Press, Boca Raton (FL), 1992.

[Gromov 1981] M. Gromov, *Structures métriques pour les variétés riemanniennes*, edited by J. Lafontaine and P. Pansu, CEDIC, Paris, 1981.

[Gromov and Schoen 1992] M. Gromov and R. Schoen, "Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one", *Pub. Math. IHES* **76** (1992), 165–246.

[Jost a] J. Jost, "Equilibrium Maps between metric spaces", preprint.

[Korevaar and Schoen 1993] N. Korevaar and R. Schoen, "Sobolev spaces and harmonic maps for metric space targets", *Comm. Anal. Geom.* **1** (1993), 561–659.

[Machigashira 1995] Y. Machigashira, "The Gaussian curvature of Alexandrov surfaces", preprint, Saga University, 1995.

[Otsu 1995] Y. Otsu, "Almost everywhere existence of second differentiable structure of Alexandrov spaces", preprint, University of Tokyo, 1995.

[Otsu and Shioya 1994] Y. Otsu and T. Shioya, "The Riemannian structure of Alexandrov spaces", *J. Diff. Geom.* **39** (1994), 629–658.

[Otsu and Tanoue a] Y. Otsu and H. Tanoue, "The Riemannian structure of Alexandrov spaces with curvature bounded above", in preparation.

[Petrunin a] A. Petrunin, "Parallel transportation and second variation", preprint.

[Rademacher 1919] H. Rademacher, "Über partielle und totale Differenzierbarkeit von Funktionen mehrerer Variabeln und über die Transformation der Doppelintegrale", *Math. Ann.* **79** (1919), 340–359.

[Reidemeister 1921] K. Reidemeister, "Über die singulären Randpunkte eines konvexen Körpers", *Math. Ann.* **83** (1921), 116–118.

[Reshetnyak 1968] Y. G. Reshetnyak, "Inextensible mapping in a space of curvature no greater than K", *Sib. Math. Zhur.* **9** (1968), 918–927.

[Shiohama 1993] K. Shiohama, *An introduction to the geometry of Alexandrov spaces*, Lecture Notes Series **8**, Seoul National University, 1993.

Yukio Otsu
University of Tokyo
Department of Mathematical Sciences
Hongo, Tokyo 113
Japan