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Real Hypersurfaces in Complex Space Forms

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Abstract. The study of real hypersurfaces in complex space forms has been an active field of study over the past decade. This article attempts to give the necessary background material to access this field, as well as a detailed construction of the important examples of hypersurfaces in complex projective and complex hyperbolic space. Following this we give a survey of the major classification results, including such topics as restrictions on the shape operator, the η-parallel condition, and restrictions on the Ricci tensor. We conclude with a brief discussion of some additional areas of study and some open problems. A comprehensive bibliography is included.

Introduction

The study of real hypersurfaces in complex projective space $\mathbb{C}P^n$ and complex hyperbolic space $\mathbb{C}H^n$ has been an active field over the past decade. Although these ambient spaces might be regarded as the simplest after the spaces of constant curvature, they impose significant restrictions on the geometry of their hypersurfaces. For instance, they do not admit umbilic hypersurfaces and their geodesic spheres do not have constant curvature. They also do not admit Einstein hypersurfaces. M. Okumura [1978] remarked that there was a poverty of vocabulary for describing the differential geometric properties of the hypersurfaces that can arise. That situation has since been improved.

One can regard $\mathbb{C}P^n$ as a projection from S^{2n+1} with fibre S^1. H. B. Lawson [1970] was the first to exploit this idea to study a hypersurface in $\mathbb{C}P^n$ by lifting it to an S^1-invariant hypersurface of the sphere. He identified certain hypersurfaces called *equators* of $\mathbb{C}P^n$ which are minimal and lift to Clifford minimal hypersurfaces of the sphere. Subsequently, other investigators explored properties that lifted to familiar properties of hypersurfaces in S^{2n+1}.

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R. Takagi's classification [1973] of the homogeneous real hypersurfaces of $\mathbb{C}P^n$ was important in its own right, but it also identified a whole list of hypersurfaces, gave them names (type A1, type B, etc.), and focused attention on them. Other geometers began to study them and to derive new characterizations of various subsets of the list.

Another important notion that developed was the relevance of the *structure vector* of a hypersurface. It is defined by $W = -J\xi$ where J is the complex structure and ξ is the unit normal field. In early investigations, it was found that computations were more tractable when W was a principal vector. Further, it was observed that W is principal for all homogeneous hypersurfaces in $\mathbb{C}P^n$. Later, geometric characterizations of this property were found and hypersurfaces that satisfy it are now called *Hopf hypersurfaces*.

It has also developed that certain interesting classes of hypersurfaces can be characterized by simply stated conditions on the so-called holomorphic distribution $W^\perp$. For example, the notions of η-umbilical and pseudo-Einstein have arisen. These are the appropriate analogues of umbilical and Einstein, respectively, and essentially say that the indicated property holds on $W^\perp$.

The homogeneous hypersurfaces of $\mathbb{C}P^n$ all have constant principal curvatures, and Hopf hypersurfaces with constant principal curvatures have been determined, both for $\mathbb{C}P^n$ and for $\mathbb{C}H^n$. In real space forms, constant principal curvature hypersurfaces are isoparametric and have many nice properties related to parallel families and focal sets. The various equivalent definitions of *isoparametric* that can be used in real space forms lead to distinct classes of hypersurfaces in $\mathbb{C}P^n$. Several of these alternatives have been investigated but the constant principal curvature hypersurfaces have been studied most intensively.

The study of hypersurfaces in $\mathbb{C}H^n$ has followed developments in $\mathbb{C}P^n$, often with similar results, but sometimes with differences. For example, a Hopf hypersurface with constant principal curvatures in $\mathbb{C}P^n$ must have 2, 3, or 5 distinct principal curvatures. For $\mathbb{C}H^n$, 2 and 3 are the only possibilities. On the other hand, $\mathbb{C}H^n$ admits a wider variety of hypersurfaces with a specific number (say 2) of principal curvatures.

In this article, we will present the fundamental definitions and results necessary for reaching the frontiers of research in the field. We will state the known classification results and provide proofs of many of them. For those proofs that we cannot include because of time and space limitations, we provide appropriate pointers to the literature.

In Section 1 we construct the standard models of spaces of constant holomorphic sectional curvature, and give the essential background for studying real hypersurfaces. In Section 2 we discuss the notion of Hopf hypersurfaces, and show that the shape operator satisfies rather stringent conditions for these hypersurfaces. In Section 3 we list the standard examples of real hypersurfaces that occur in spaces of constant holomorphic sectional curvature. The classification results are discussed in Sections 4–7. Finally, in Sections 8 and 9, we discuss areas for further study.

We conclude this section by mentioning a few notational conventions. In addition to the usual end-of-proof symbol □, we use ◁ to conclude the statement of a theorem whose proof is to be omitted.

For Hopf hypersurfaces, the shape operator A preserves the holomorphic distribution $W^\perp$. Rather than say that λ is a principal curvature whose corresponding principal vectors lie in $W^\perp$, we often say that λ is a principal curvature "on $W^\perp$". This will allow us to avoid repeating a wordy and awkward phrase.

When X and Y are vectors, $X \wedge Y$ will denote the linear transformation satisfying

$$(X \wedge Y)Z = \langle Y, Z\rangle X - \langle X, Z\rangle Y$$

where $\langle\ ,\ \rangle$ is the inner product. Finally, since covariant differentiation acts as a derivation on the algebra of tensor fields, and commutes with contractions, the curvature operator $R(X, Y)$ can operate in the same way. For any tensor field T, the tensor field defined by

$$R(X, Y) \cdot T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T$$

is abbreviated $R \cdot T$. For instance, if T is a tensor field of type $(1, 1)$,

$$(R \cdot T)(X, Y, Z) = (R(X, Y) \cdot T)Z = R(X, Y)(TZ) - T(R(X, Y)Z).$$

1. Preliminaries

In this section we construct the standard models of spaces of constant holomorphic sectional curvature. We first construct $\mathbb{C}P^n$ and then $\mathbb{C}H^n$, and then we take a unified approach to the discussion of spaces of constant holomorphic sectional curvature. We then discuss the geometry of hypersurfaces and their lifts.

Complex Projective Space. We first introduce the complex projective space $\mathbb{C}P^n$ and the basic equations for studying its hypersurfaces. For $z = (z_0, \ldots, z_n)$, $w = (w_0, \ldots, w_n)$ in $\mathbb{C}^{n+1}$, write

$$F(z, w) = \sum_{k=0}^{n} z_k \bar{w}_k$$

and let $\langle z, w\rangle = \operatorname{Re} F(z, w)$, the real part of $F(z, w)$. The $(2n + 1)$-sphere $S^{2n+1}(r)$ of radius r is defined by

$$S^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = r^2\}.$$

We may consider $\mathbb{C}^{n+1}$ as $\mathbb{R}^{2n+2}$ and define $u, v \in \mathbb{R}^{2n+2}$ by

$$z_k = u_{2k} + u_{2k+1} i, \qquad w_k = v_{2k} + v_{2k+1} i.$$

Then

$$\langle z,w\rangle = \langle u,v\rangle = \sum_{k=0}^{2n+1} u_k v_k.$$

We will use $\langle z,w\rangle$ and $\langle u,v\rangle$ interchangeably. When desired, we can work exclusively in real terms by introducing the operator J for multiplication by the complex number i. Note that for $z \in S^{2n+1}(r)$,

$$T_z S^{2n+1}(r) = \{w \in \mathbb{C}^{n+1} : \langle z,w\rangle = 0\}.$$

Restricting $\langle\ ,\ \rangle$ to $S^{2n+1}(r)$ gives a Riemannian metric whose Levi-Civita connection $\widetilde{\nabla}$ satisfies

$$D_X Y = \widetilde{\nabla}_X Y - \langle X,Y\rangle \frac{z}{r^2}$$

for X, Y tangent to $S^{2n+1}(r)$ at z, where D is the Levi-Civita connection of $\mathbb{R}^{2n+2}$. The usual calculations of the Gauss equation yield that the curvature tensor $\tilde{R}$ of $\widetilde{\nabla}$ satisfies

$$\tilde{R}(X,Y) = \frac{1}{r^2}\ X \wedge Y. \tag{1.1}$$

Let $V = Jz$ and write down the orthogonal decomposition into so-called vertical and horizontal components,

$$T_z S^{2n+1}(r) = \operatorname{span}\{V\} \oplus V^{\perp}.$$

Let π be the canonical projection of $S^{2n+1}(r)$ to complex projective space $\mathbb{C}P^n$,

$$\pi : S^{2n+1}(r) \to \mathbb{C}P^n.$$

Complex Hyperbolic Space. Next, we introduce the complex hyperbolic space $\mathbb{C}H^n$. The construction is parallel to that of $\mathbb{C}P^n$ with some important differences. For z, w in $\mathbb{C}^{n+1}$, write

$$F(z,w) = -z_0\bar{w}_0 + \sum_{k=1}^{n} z_k \bar{w}_k$$

and let $\langle z,w\rangle = \operatorname{Re} F(z,w)$. The anti–de Sitter space of radius r in $\mathbb{C}^{n+1}$ is defined by

$$H_1^{2n+1}(r) = \{z \in \mathbb{C}^{n+1} : \langle z,z\rangle = -r^2\}.$$

We denote $H_1^{2n+1}(r)$ by $\mathbb{H}$ for short. We use the same identification of $\mathbb{C}^{n+1}$ with $\mathbb{R}^{2n+2}$ so that

$$\langle z,w\rangle = \langle u,v\rangle = -u_0v_0 - u_1v_1 + \sum_{k=2}^{2n+1} u_k v_k.$$

For $z \in \mathbb{H}$,

$$T_z\mathbb{H} = \{w \in \mathbb{C}^{n+1} : \langle z,w\rangle = 0\}.$$

Restricting $\langle\ ,\ \rangle$ to $\mathbb{H}$ gives a Lorentz metric whose Levi-Civita connection $\widetilde{\nabla}$ satisfies

$$D_X Y = \widetilde{\nabla}_X Y + \langle X,Y\rangle \frac{z}{r^2}$$

for X, Y tangent to $\mathbb{H}$ at z. The Gauss equation takes the form

$$\tilde{R}(X,Y) = -\frac{1}{r^2}\ X \wedge Y. \tag{1.2}$$

Again take $V = Jz$ and we get the analogous orthogonal decomposition

$$T_z\mathbb{H} = \operatorname{span}\{V\} \oplus V^\perp.$$

Denote by $\mathbb{C}H^n$ the image of $\mathbb{H}$ by the canonical projection π to complex projective space,

$$\pi : \mathbb{H} \to \mathbb{C}H^n \subset \mathbb{C}P^n.$$

Thus, topologically, $\mathbb{C}H^n$ is an open subset of $\mathbb{C}P^n$. However, as Riemannian manifolds, they have quite different structures.

Complex Space Forms From here we make a uniform exposition covering both $\mathbb{C}P^n$ and $\mathbb{C}H^n$. When convenient, we make use of the letter ε to distinguish the two cases. It denotes the sign of the constant holomorphic sectional curvature $4c = 4\varepsilon/r^2$. For example, (1.1) and (1.2) could be written as

$$\tilde{R}(X,Y) = \frac{\varepsilon}{r^2}\ X \wedge Y.$$

We also use $\tilde{M}$ to stand for either $\mathbb{C}P^n$ or $\mathbb{C}H^n$ and $\tilde{M}'$ for $S^{2n+1}(r)$ or $\mathbb{H}$.

Note that $\pi_* V = 0$ but that π_* is an isomorphism on $V^\perp$. Let z be any point of $\tilde{M}'$. For $X \in T_{\pi z}\tilde{M}$, let X^L be the vector in $V_z^\perp$ that projects to X. X^L is called the *horizontal lift* of X to z. Define a Riemannian metric on $\tilde{M}$ by $\langle X, Y\rangle = \langle X^L, Y^L\rangle$. It is well-defined since the metric on $\tilde{M}'$ is invariant by the fibre S^1. Since $V^\perp$ is J-invariant, $\tilde{M}$ can be assigned a complex structure (also denoted by J) by $JX = \pi_*(JX^L)$. It is easy to check that $\langle\ ,\ \rangle$ is Hermitian on $\tilde{M}$ and that its Levi-Civita connection $\tilde{\nabla}$ satisfies

$$\tilde{\nabla}_X Y = \pi_*(\tilde{\tilde{\nabla}}_{X^L} Y^L). \tag{1.3}$$

We also note that on $\tilde{M}'$

$$\tilde{\tilde{\nabla}}_{X^L} V = \tilde{\tilde{\nabla}}_V X^L = JX^L = (JX)^L \tag{1.4}$$

while

$$\tilde{\tilde{\nabla}}_V V = 0.$$

See [O'Neill 1966; Gray 1967] for background on Riemannian submersions.

THEOREM 1.1. *The curvature tensor $\tilde{R}$ of $\tilde{M}$ satisfies*

$$\tilde{R}(X,Y)Z = \frac{\varepsilon}{r^2}(X \wedge Y + JX \wedge JY + 2\langle X, JY\rangle J)Z.$$

In particular, the sectional curvature of a holomorphic plane spanned by X and JX is $4\varepsilon/r^2$ so that $\tilde{M}$ is a space of constant holomorphic sectional curvature. ◁

The Riemannian metrics we have just constructed are known as the *Fubini–Study metric* on $\mathbb{C}P^n$ and the *Bergman metric* on $\mathbb{C}H^n$ respectively. See [Kobayashi and Nomizu 1969, Chapter IX] for additional information on these metrics.

Hypersurfaces in Complex Space Forms. Let $\tilde{M}(c)$ be a space of constant holomorphic sectional curvature $4c$ with real dimension $2n$ and Levi-Civita connection $\widetilde{\nabla}$. For an immersed manifold $f : M^{2n-1} \to \tilde{M}$, the Levi-Civita connection ∇ of the induced metric and the shape operator A of the immersion are characterized respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle AX, Y\rangle \xi$$

and

$$\widetilde{\nabla}_X \xi = -AX$$

where ξ is a local choice of unit normal. We omit mention of the immersion f for brevity of notation. Let

$$J : T\tilde{M} \to T\tilde{M}$$

be the complex structure with properties $J^2 = -I$, $\widetilde{\nabla} J = 0$, and $\langle JX, JY\rangle = \langle X, Y\rangle$. Define the *structure vector*

$$W = -J\xi.$$

Clearly $W \in TM$, and $|W| = 1$. Write $a = \langle AW, W\rangle$. We reserve the symbols W and a for these purposes throughout.

Define a skew-symmetric $(1,1)$-tensor φ from the tangential projection of J by

$$JX = \varphi X + \langle X, W\rangle \xi. \tag{1.5}$$

Then, since $-X = J^2X = J(\varphi X + \langle X, W\rangle\xi) = \varphi^2 X + \langle \varphi X, W\rangle \xi + \langle X, W\rangle J\xi$, we see that

$$\varphi^2 X = -X + \langle X, W\rangle W. \tag{1.6}$$

It is easy to check that

$$\langle \varphi X, \varphi Y\rangle = \langle X, Y\rangle - \langle X, W\rangle\langle Y, W\rangle. \tag{1.7}$$

Putting $X = W$ in (1.5) gives $\varphi W = 0$. Noting that $\varphi^2 = -I$ on $W^\perp = \{X \in TM : \langle X, W\rangle = 0\}$ we see that φ has rank $2n-2$ and that

$$\ker\, \varphi = \operatorname{span}\{W\}.$$

Such a φ determines an almost contact metric structure [Blair 1976, pp. 19–21] and $W^\perp$ is called the *holomorphic distribution*.

In the usual way, we derive the Gauss and Codazzi equations:

$$R(X, Y) = AX \wedge AY + c\big(X \wedge Y + \varphi X \wedge \varphi Y + 2\langle X, \varphi Y\rangle \varphi\big), \tag{1.8}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\big(\langle X, W\rangle \varphi Y - \langle Y, W\rangle \varphi X + 2\langle X, \varphi Y\rangle W\big). \tag{1.9}$$

COROLLARY 1.2. *We have* $\big\langle (\nabla_X A)Y - (\nabla_Y A)X,\, W \big\rangle = 2c\langle X, \varphi Y\rangle$ *and*

$$\big\langle (\nabla_X A)W,\, W \big\rangle = \big\langle (\nabla_W A)X,\, W \big\rangle = \langle (\nabla_W A)W,\, X\rangle.$$

PROOF. The first equation follows by taking the inner product of the Codazzi equation with W, and the second follows by letting $Y = W$. □

From equation (1.8) we get the Ricci tensor S of type $(1,1)$ defined by

$$\langle SX, Y\rangle = \operatorname{trace}\{Z \to R(Z,X)Y\} \tag{1.10}$$

as

$$SX = (2n+1)cX - 3c\langle X, W\rangle W + (\operatorname{trace} A)AX - A^2X. \tag{1.11}$$

The scalar curvature is

$$s = \operatorname{trace} S = 4(n^2-1)c + (\operatorname{trace} A)^2 - \operatorname{trace} A^2.$$

The mean curvature is $\mathfrak{m} = \operatorname{trace} A$ and we reserve the symbol $\mathfrak{m}$ for this purpose throughout.

PROPOSITION 1.3. *We have* $\nabla_X W = \varphi AX$ *and*

$$(\nabla_X \varphi)Y = \langle Y, W\rangle AX - \langle AX, Y\rangle W.$$

PROOF. For the first equality,

$$\begin{aligned}\nabla_X W &= -\nabla_X(J\xi) = -\widetilde{\nabla}_X(J\xi) + \langle AX, J\xi\rangle\xi \\ &= -J\widetilde{\nabla}_X\xi + \langle AX, J\xi\rangle\xi = JAX - \langle JAX, \xi\rangle\xi = \varphi AX.\end{aligned}$$

For the second,

$$\begin{aligned}(\nabla_X\varphi)Y &= \nabla_X(\varphi Y) - \varphi\nabla_X Y = \nabla_X(JY - \langle Y, W\rangle\xi) - \varphi\nabla_X Y \\ &= \widetilde{\nabla}_X(JY - \langle Y, W\rangle\xi) - \langle A\varphi Y, X\rangle\xi - \varphi\nabla_X Y \\ &= J(\nabla_X Y + \langle AX, Y\rangle\xi) - X\langle Y, W\rangle\xi + \langle Y, W\rangle AX - \langle A\varphi Y, X\rangle\xi - \varphi\nabla_X Y \\ &= \varphi\nabla_X Y + \langle \nabla_X Y, W\rangle\xi - \langle AX, Y\rangle W - \langle \nabla_X Y, W\rangle\xi \\ &\qquad - \langle Y, \varphi AX\rangle\xi + \langle Y, W\rangle AX - \langle A\varphi Y, X\rangle\xi - \varphi\nabla_X Y \\ &= \langle Y, W\rangle AX - \langle AX, Y\rangle W.\end{aligned}$$

□

PROPOSITION 1.4. *If* $c \neq 0$ *then* ∇W *cannot be identically zero. Equivalently,* φA *cannot be identically zero.*

PROOF. By Proposition 1.3, $\nabla_X W = 0$ if and only if $\varphi AX = 0$. Suppose that this condition holds for all X. Then $AX = \langle AX, W\rangle W$. Thus, for all X and Y,

$$(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y \in \operatorname{span}\{W\}.$$

Applying the Codazzi equation, we have

$$c(\langle X, W\rangle \varphi Y - \langle Y, W\rangle \varphi X) \in \operatorname{span}\{W\}.$$

In particular, put $Y = W$ to get that $-c\varphi X$ lies in the span of W for all X. This is clearly impossible since $c \neq 0$. □

THEOREM 1.5. *Let M^{2n-1}, where $n \geq 2$, be a hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the shape operator A cannot be parallel. Also, no identity of the form $A = \lambda I$ can hold, even with λ nonconstant. In particular, umbilic hypersurfaces cannot occur.*

PROOF. Let us first assume that $A = \lambda I$. The Codazzi equation (1.9) becomes

$$(X\lambda)Y - (Y\lambda)X = c(\langle X, W\rangle \varphi Y - \langle Y, W\rangle \varphi X + 2\langle X, \varphi Y\rangle W).$$

If we put $Y = W$ in this equation, it simplifies to

$$(X\lambda)W - (W\lambda)X = -c\varphi X.$$

For $X \neq 0$ orthogonal to W, the set $\{X, \varphi X, W\}$ is linearly independent, and so $c = 0$ which contradicts the hypothesis. Now suppose that $\nabla A = 0$. Take $X \neq 0$ orthogonal to W and $Y = W$ in the Codazzi equation, to get $-c\varphi X = 0$, another contradiction. □

The nonexistence of umbilic hypersurfaces was proved by Tashiro and Tachibana [1963].

Lifts of Hypersurfaces in $\tilde{M}$ to $\tilde{M}'$. Once again, we let $\tilde{M}$ represent $\mathbb{C}P^n$ or $\mathbb{C}H^n$ and $\tilde{M}'$ represent $S^{2n+1}(r)$ or $\mathbb{H}$ respectively, with the canonical projection

$$\pi : \tilde{M}' \to \tilde{M}.$$

Now consider a hypersurface M in $\tilde{M}$. Then $M' = \pi^{-1}M$ is an S^1-invariant hypersurface in $\tilde{M}'$ (Lorentzian in the case $\tilde{M}' = \mathbb{H}$). If ξ is a unit normal for M, then $\xi' = \xi^L$ is a unit normal for M'. The induced connection ∇' and the shape operator A' for M' satisfy

$$\tilde{\tilde{\nabla}}_X Y = \nabla'_X Y + \langle A'X, Y\rangle \xi', \qquad \tilde{\tilde{\nabla}}_X \xi' = -A'X,$$

and the more familiar form of the Codazzi equation

$$(\nabla'_X A')Y = (\nabla'_Y A')X.$$

There is also a Gauss equation, but we will not have occasion to use it. We also have $W^L = U = -J\xi'$ where $W = -J\xi$ is the structure vector introduced earlier.

LEMMA 1.6. *For X and Y tangent to $\tilde{M}$,*

$$(\widetilde{\nabla}_X Y)^L = \widetilde{\nabla}_{X^L} Y^L + \varepsilon \langle JX^L, Y^L \rangle \frac{1}{r^2} V. \tag{1.12}$$

PROOF. By (1.3), π_* applied to each side of (1.12) yields the same result. Thus, it only remains to check that the right side is horizontal. However,

$$\begin{aligned}\langle \widetilde{\nabla}_{X^L} Y^L, V \rangle + \varepsilon \langle JX^L, Y^L \rangle \frac{1}{r^2} \langle V, V \rangle &= -\langle Y^L, \widetilde{\nabla}_{X^L} V \rangle + \langle JX^L, Y^L \rangle \varepsilon^2 \\ &= -\langle Y^L, JX^L \rangle + \langle JX^L, Y^L \rangle = 0. \quad \square\end{aligned}$$

LEMMA 1.7. (i) $\widetilde{\nabla}_V \xi' = J\xi' = -U$, *so* $A'V = U$.
(ii) *For X tangent to M*, $(AX)^L = A'X^L - \langle X^L, U \rangle \varepsilon r^{-2} V$.
(iii) *In particular*, $(AW)^L = A'U - \varepsilon r^{-2} V$.
(iv) *If* $AW = aW$, *then* $A'U = aU + \varepsilon r^{-2} V$.

PROOF. To verify the first assertion, note that $\widetilde{\nabla}_V \xi' = \widetilde{\nabla}_V \xi^L = J\xi^L = -U$. For the second, we compute

$$-A'X^L = \widetilde{\nabla}_{X^L} \xi^L = (\widetilde{\nabla}_X \xi)^L - \varepsilon \langle JX^L, \xi^L \rangle r^{-2} V,$$

and hence

$$A'X^L = (AX)^L + \varepsilon \langle X^L, U \rangle r^{-2} V.$$

Assertions (iii) and (iv) are special cases of (ii). $\square$

We now look at the relationship between the covariant derivatives of the respective shape operators of M and M'.

THEOREM 1.8. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the shape operator A' of $M' = \pi^{-1} M$ satisfies*

$$\pi_*((\nabla'_{X^L} A')Y^L) = (\nabla_X A)Y + c\big(\langle \varphi X, Y \rangle W + \langle Y, W \rangle \varphi X\big)$$

for all X, Y tangent to M.

PROOF. We begin with the fundamental identity,

$$(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y),$$

and take the horizontal lift of each side. In the following equation, all equalities are to be understood mod V. We freely use (1.3), (1.4) and the results of

Lemmas 1.6 and 1.7.

$$\begin{aligned}
\big((\nabla_X A)Y\big)^L &= \big(\widetilde{\nabla}_X(AY) - \langle AX, AY\rangle\xi\big)^L - A'(\nabla_X Y)^L \\
&= \widetilde{\nabla}_{X^L}(AY)^L - \langle (AX)^L, (AY)^L\rangle\xi' - A'\big((\widetilde{\nabla}_X Y)^L - \langle (AX)^L, Y^L\rangle\xi'\big) \\
&= \widetilde{\nabla}_{X^L}(A'Y^L - \varepsilon r^{-2}\langle Y^L, U\rangle V) \\
&\quad - \big(\langle A'X^L, A'Y^L\rangle + r^{-4}\langle X^L, U\rangle\langle Y^L, U\rangle\langle V, V\rangle\big)\xi' \\
&\quad + 2\varepsilon r^{-2}\langle X^L, U\rangle\langle Y^L, U\rangle\xi' \\
&\quad - A'\big(\widetilde{\nabla}_{X^L}Y^L + \varepsilon r^{-2}\langle JX^L, Y^L\rangle V - \langle (AX)^L, Y^L\rangle\xi'\big) \\
&= \nabla'_{X^L}(A'Y^L) + \langle A'X^L, A'Y^L\rangle\xi' - \varepsilon r^{-2}\langle Y^L, U\rangle JX^L \\
&\quad - \langle A'X^L, A'Y^L\rangle\xi' + \varepsilon r^{-2}\langle X^L, U\rangle\langle Y^L, U\rangle\xi' \\
&\quad - A'\big(\nabla'_{X^L}Y^L + \langle A'X^L, Y^L\rangle\xi' + \varepsilon r^{-2}\langle JX^L, Y^L\rangle V - \langle (AX)^L, Y^L\rangle\xi'\big) \\
&= (\nabla'_{X^L}A')Y^L - \varepsilon r^{-2}\langle Y^L, U\rangle(JX)^L \\
&\quad + \varepsilon r^{-2}\langle X^L, U\rangle\langle Y^L, U\rangle\xi' - \varepsilon r^{-2}\langle JX, Y\rangle U \\
&= (\nabla'_{X^L}A')Y^L - \varepsilon r^{-2}\langle Y, W\rangle(\varphi X)^L - \varepsilon r^{-2}\langle Y, W\rangle\langle X, W\rangle\xi^L \\
&\quad + \varepsilon r^{-2}\langle X, W\rangle\langle Y, W\rangle\xi^L - \varepsilon r^{-2}\langle \varphi X, Y\rangle W^L,
\end{aligned}$$

from which the result follows. □

Note that the Codazzi equation (1.9) for M in $\tilde{M}$ is a consequence of Theorem 1.8 together with the Codazzi equation for M' in $\tilde{M}'$. We also look at the vertical component of the covariant derivative of A' and observe the following nice relationship. The proof is a straightforward application of the same methods used in Theorem 1.8.

Proposition 1.9. *Under the hypothesis of Theorem* 1.8,

$$\big\langle (\nabla'_{X^L}A')Y^L,\, V\big\rangle = \langle (\varphi A - A\varphi)X,\, Y\rangle.$$

Therefore $(\nabla'_{X^L}A')Y^L$ *is horizontal for all* X *and* Y *if and only if* φ *and* A *commute.* ◁

A $(1,1)$ tensor A is said to be a *Codazzi tensor* with respect to a semi-Riemannian metric $\langle\ ,\ \rangle$ if, for all tangent vectors X and Y,

$$\langle AX, Y\rangle = \langle X, AY\rangle \qquad \text{and} \qquad (\nabla_X A)Y = (\nabla_Y A)X,$$

where ∇ is the Levi-Civita connection.

Lemma 1.10. *Let A be a Codazzi tensor. Assume that there are constants α and β such that $A^2 = \alpha A + \beta I$. If $\alpha^2 + 4\beta \neq 0$, then $\nabla A = 0$. Furthermore, if $\alpha^2 + 4\beta < 0$, the tangent space splits into spacelike and timelike subspaces of equal dimensions.*

PROOF. Differentiating the quadratic condition yields

$$(\nabla_X A)A + A(\nabla_X A) = \alpha(\nabla_X A) \tag{1.13}$$

so that

$$A(\nabla_X A)A = (\alpha A - A^2)\nabla_X A = -\beta\nabla_X A. \tag{1.14}$$

On the other hand, we can write (1.13) in the form

$$\langle(\nabla_Z A)Y, AX\rangle + \langle(\nabla_Z A)X, AY\rangle = \alpha\langle(\nabla_Z A)X, Y\rangle$$

where we have used the symmetry of A and $\nabla_Z A$. Applying the Codazzi equation and then replacing Z by AZ yields

$$\langle(\nabla_Y A)AZ, AX\rangle + \langle(\nabla_X A)AZ, AY\rangle = \alpha\langle(\nabla_X A)AZ, Y\rangle.$$

Again using symmetry and the Codazzi equation, along with (1.14), we have

$$\begin{aligned} -2\beta\big\langle(\nabla_X A)Y,\, Z\big\rangle &= \alpha\big\langle A(\nabla_X A)Y,\, Z\big\rangle \\ &= \alpha\big\langle(\nabla_X A)AZ,\, Y\big\rangle. \end{aligned}$$

Since the left side is symmetric in Y and Z, we have

$$\alpha(\nabla_X A)A = \alpha A(\nabla_X A) = -2\beta\nabla_X A$$

so that, in view of (1.13),

$$(\alpha^2 + 4\beta)\nabla_X A = 0.$$

If $\alpha^2 + 4\beta < 0$, there is a constant γ such that $P = \gamma(A - \frac{1}{2}\alpha I)$ is a symmetric transformation satisfying $P^2 = -I$. In fact, $\gamma = \big(-(\beta + \frac{1}{4}\alpha^2)\big)^{-1/2}$. The result follows immediately. □

Using Theorem 1.8, we can strengthen the result that the shape operator cannot be parallel. In fact, its covariant derivative cannot vanish even at one point. Specifically:

THEOREM 1.11. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the shape operator A satisfies $|\nabla A|^2 \geq 4c^2(n-1)$.*

PROOF. Let

$$T(X,Y) = (\nabla_X A)Y + c(\langle\varphi X, Y\rangle W + \langle Y, W\rangle\varphi X),$$

the right side of the equation in Theorem 1.8. Then

$$0 \leq |T|^2 = |\nabla A|^2 + 2ck_1 + c^2k_2,$$

where k_1 is the sum of

$$\langle(\nabla_X A)Y, \langle\varphi X, Y\rangle W + \langle Y, W\rangle\varphi X\rangle$$

as X and Y range over an orthonormal basis, while k_2 is the sum of

$$\langle\varphi X, Y\rangle^2 + \langle Y, W\rangle^2|\varphi X|^2$$

over the same range of X and Y. Summing over Y and using the fact that $\nabla_X A$ is symmetric, we see that k_1 is equal to the sum over X of

$$\langle \varphi X, (\nabla_X A)W \rangle + \langle W, (\nabla_X A)\varphi X \rangle = 2\langle \varphi X, (\nabla_X A)W \rangle.$$

By the Codazzi equation,

$$\langle \varphi X, (\nabla_X A)W \rangle = \langle \varphi X, (\nabla_W A)X \rangle - c\langle \varphi X, \varphi X \rangle.$$

Now note that $(\nabla_W A)\varphi$ has zero trace while the trace of φ^2 is $-2(n-1)$. Thus we can calculate that $k_1 = -4c(n-1)$, while $k_2 = 4(n-1)$. Therefore

$$|\nabla A|^2 - 8c^2(n-1) + 4c^2(n-1) \geq 0,$$

which gives the desired result. □

As a byproduct of this proof, we also see that equality holds if and only if $T = 0$. This means that

$$(\nabla_X A)Y = -c(\langle \varphi X, Y \rangle W + \langle Y, W \rangle \varphi X).$$

We will discuss this further at the beginning of Section 4. See Corollary 4.4.

2. Hopf Hypersurfaces: When W Is Principal

If W is a principal vector, M is called a *Hopf hypersurface*. Hopf hypersurfaces have several nice characterizations. The notion makes sense in any Kähler ambient space, and corresponds to the property that the integral curves of W are geodesics. Tubes over complex submanifolds are known to be Hopf.

A fundamental fact about Hopf hypersurfaces is that the principal curvature a corresponding to W is constant for complex space forms of nonzero curvature. For $c > 0$, the proof is fairly direct, but for $c < 0$, it is rather lengthy, and involves formidable computation. Nevertheless, it is of significance for the geometry of real hypersurfaces in complex space forms, and so we will include it here. When $c = 0$, a need not be constant, but nonconstancy puts rather strong restrictions on A.

THEOREM 2.1. *Let* M^{2n-1}, *where* $n \geq 2$, *be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature* $4c$, *and let* a *be the principal curvature corresponding to* W.

(i) *If* $c \neq 0$, *then* a *must be constant*.
(ii) *If* $c = 0$ *and* $\operatorname{grad} a \neq 0$ *at some point*, *then* $A|_{W^\perp} = 0$ *in a neighborhood of this point*. *Consequently*, *the number of distinct principal curvatures is* 1 *or* 2 *in this neighborhood*. ◁

We first remark that $a = \langle AW, W \rangle$ is a smooth function whether or not W is principal. The first few lemmas establish a relationship between the shape operator A and the structure tensor φ. A consequence of these preliminary results is the proof of the theorem in the case when $c \geq 0$. The remainder of the section is necessary to establish the result when $c < 0$.

Throughout this section, M^{2n-1}, where $n \geq 2$, will be a real hypersurface in a space of constant holomorphic sectional curvature $4c$, and X, Y, and Z will be vectors tangent M.

LEMMA 2.2. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. Then*

$$A\varphi A - \frac{a}{2}(A\varphi + \varphi A) - c\varphi = 0 \tag{2.1}$$

and

$$\operatorname{grad} a = (Wa)W. \tag{2.2}$$

PROOF. Expand $(\nabla_X A)W$, by making use of Proposition 1.3 to calculate that

$$(\nabla_X A)W = \nabla_X(AW) - A\nabla_X W = (Xa)W + (aI - A)\varphi AX. \tag{2.3}$$

Then, by Corollary 1.2,

$$Xa = \langle(\nabla_X A)W, W\rangle = \langle(\nabla_W A)X, W\rangle, \tag{2.4}$$

and $\langle(\nabla_W A)W, X\rangle = (Wa)\langle W, X\rangle$. Since $\langle \operatorname{grad} a, X\rangle = Xa$, we have $\operatorname{grad} a = (Wa)W$. Now, using (2.3) and (2.4),

$$\begin{aligned}\langle(\nabla_X A)Y, W\rangle &= \langle(\nabla_X A)W, Y\rangle \\ &= (Wa)\langle W, X\rangle\langle W, Y\rangle + \langle(aI - A)\varphi AX, Y\rangle.\end{aligned} \tag{2.5}$$

Interchanging X and Y in (2.5) and subtracting, we calculate

$$\langle(\nabla_X A)Y, W\rangle - \langle(\nabla_Y A)X, W\rangle = \langle(aI - A)\varphi AX, Y\rangle - \langle(aI - A)\varphi AY, X\rangle.$$

Comparing this with Corollary 1.2 we see that

$$\begin{aligned}2c\langle X, \varphi Y\rangle &= \langle(aI - A)\varphi AX, Y\rangle - \langle(aI - A)\varphi AY, X\rangle \\ &= -\langle X, A\varphi(aI - A)Y\rangle - \langle X, (aI - A)\varphi AY\rangle.\end{aligned}$$

Since this is true for all tangent X and Y, we get

$$2c\varphi Y = -a(A\varphi Y + \varphi AY) + 2A\varphi AY,$$

and so

$$A\varphi A - \frac{a}{2}(A\varphi + \varphi A) - c\varphi = 0. \qquad \square$$

Here is an immediate consequence of this lemma.

COROLLARY 2.3. (i) *If $X \in W^{\perp}$ and $AX = \lambda X$, then*

$$(\lambda - \frac{a}{2})A\varphi X = (\frac{\lambda a}{2} + c)\varphi X.$$

(ii) *If a nonzero $X \in W^{\perp}$ satisfies $AX = \lambda X$ and $A\varphi X = \mu\varphi X$, then*

$$\lambda\mu = \frac{\lambda+\mu}{2}a + c.$$

(iii) *If T_λ is φ-invariant, then $\lambda^2 = a\lambda + c$. (The notation T_λ is used for the set of principal vectors for a principal curvature λ).* ◃

LEMMA 2.4. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. Then $(Wa)(\varphi A + A\varphi) = 0$.*

PROOF. Let $\beta = Wa$, so that $\operatorname{grad} a = \beta W$. Then

$$\begin{aligned}&\langle \nabla_X(\operatorname{grad} a), Y\rangle - \langle \nabla_Y(\operatorname{grad} a), X\rangle \\ &\quad = X\langle \operatorname{grad} a, Y\rangle - \langle \operatorname{grad} a, \nabla_X Y\rangle - Y\langle \operatorname{grad} a, X\rangle + \langle \operatorname{grad} a, \nabla_Y X\rangle \\ &\quad = XYa - YXa - \operatorname{grad} a\langle \nabla_X Y - \nabla_Y X\rangle = ([X,Y] - (\nabla_X Y - \nabla_Y X))a = 0.\end{aligned}$$

(This is, of course, true for any function a and expresses the symmetry of the Hessian.) So, since $\beta W = \operatorname{grad} a$,

$$\begin{aligned}0 &= \langle \nabla_X(\beta W), Y\rangle - \langle \nabla_Y(\beta W), X\rangle \\ &= X\beta\langle W, Y\rangle + \beta\langle \varphi AX, Y\rangle - Y\beta\langle W, X\rangle - \beta\langle \varphi AY, X\rangle \\ &= (X\beta)\langle W, Y\rangle - (Y\beta)\langle W, X\rangle + \beta\langle (\varphi A + A\varphi)X, Y\rangle. \end{aligned} \tag{2.6}$$

If we set $Y = W$ in this equation, we get

$$0 = X\beta - (W\beta)\langle W, X\rangle + \beta\langle A\varphi X, W\rangle,$$

where the last term is zero since $AW = aW$ and $\varphi W = 0$. Thus

$$X\beta = (W\beta)\langle W, X\rangle.$$

Using this to simplify equation (2.6) and noticing that this is true for all X proves the lemma. □

COROLLARY 2.5. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. If $\varphi A = A\varphi$, then $AW = aW$. If, in addition, $c \neq 0$, then a must be constant.*

PROOF. Start with $a = \langle AW, W\rangle$. Because $\varphi AW = A\varphi W = 0$, AW lies in the span of W; that is, $AW = \langle AW, W\rangle W = aW$. Now suppose that $\beta = Wa \neq 0$ at some point. From Lemma 2.4, we have $A\varphi = -\varphi A = -A\varphi$ so that $A\varphi = 0$, and hence $c\varphi = 0$ by (2.1). If $c \neq 0$, we have a contradiction. If $c = 0$, there is no contradiction, but A must vanish on $W^{\perp}$ in a neighborhood of the point in question. □

A cylinder of the form $\Gamma \times \mathbb{C}^n$, where Γ is any plane curve other than a circle or line, shows that a need not be constant when $c = 0$.

Corollary 2.6. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. Suppose $AW = aW$. If $c > 0$, then a is constant. If $c = 0$, then either a is constant or $g \leq 2$, where g is the number of distinct eigenvalues of A.*

Proof. By Lemma 2.4, if $\beta \neq 0$ at some point, then $\varphi A + A\varphi = 0$ in some neighborhood of this point, and (2.1) reduces to

$$\varphi A^2 + c\varphi = 0.$$

Now, for any eigenvector of A, say X, orthogonal to W, we have

$$0 = \varphi(A^2 + cI)X = \varphi(\lambda^2 + c)X,$$

where λ is the eigenvalue of A for X. Hence $\lambda^2 + c = 0$. For $c > 0$, this is a contradiction. If $c = 0$, then $\lambda = 0$, and there can be no eigenvalues other than 0 and a. □

To complete the proof of Theorem 2.1, the main result of this section, we must verify the theorem for the case when $c < 0$ The purpose of this next series of lemmas is to compute the explicit form of $\nabla_X A$. This is accomplished in Lemma 2.9. We can then compute $(R(X,Y)\cdot A)Z$ and use this to prove that $A\varphi + \varphi A$ cannot be zero and hence that $Wa = 0$. Our proof follows [Ki and Suh 1990].

Lemma 2.7. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. If $A\varphi + \varphi A = 0$, then*

$$\begin{aligned}(\nabla_X A)AY + A(\nabla_X A)Y &= 2a\beta\langle X,W\rangle\langle Y,W\rangle W\\ &\quad + (a^2+c)(\langle\varphi AX,Y\rangle W + \langle Y,W\rangle\varphi AX), \quad (2.7)\\ (\nabla_X A)AY - (\nabla_Y A)AX &= 2ca\langle\varphi X,Y\rangle W + a^2(\langle Y,W\rangle\varphi AX - \langle X,W\rangle\varphi AY). \quad (2.8)\end{aligned}$$

Proof. First note that $A\varphi = -\varphi A$ implies that $\varphi AW = 0$ and hence that W is principal, so we can write $AW = aW$. Also, (2.1) simplifies to $\varphi(A^2 + cI) = 0$, and from this we compute

$$\begin{aligned}0 &= (\nabla_X(\varphi(A^2+cI)))Y\\ &= (\nabla_X\varphi)(A^2+cI)Y + \varphi(\nabla_X A)AY + \varphi A(\nabla_X A)Y. \quad (2.9)\end{aligned}$$

Proposition 1.3 allows us to rewrite the first term, and equation (2.9) becomes

$$0 = (a^2+c)\langle Y,W\rangle AX - \langle(A^3+cA)X,Y\rangle W + \varphi(\nabla_X A)AY + \varphi A(\nabla_X A)Y.$$

Applying φ to this equality gives

$$\varphi((a^2+c)\langle Y,W\rangle AX) + \varphi^2((\nabla_X A)AY) + \varphi^2(A(\nabla_X A)Y) = 0. \quad (2.10)$$

Using (1.6), the second term of (2.10) is

$$\varphi^2(\nabla_X A)AY = -(\nabla_X A)AY + \langle(\nabla_X A)AY,W\rangle W.$$

From Proposition 1.3 and the fact that $\varphi A^2 = -c\varphi$, we calculate directly that

$$\begin{aligned}\langle(\nabla_X A)AY, W\rangle &= \langle AY, (Xa)W + (aI - A)\varphi AX\rangle \\ &= a\beta\langle X, W\rangle\langle Y, W\rangle + a\langle A\varphi AX, Y\rangle - \langle A^2\varphi AX, Y\rangle \\ &= a\beta\langle X, W\rangle\langle Y, W\rangle + ca\langle\varphi X, Y\rangle + c\langle\varphi AX, Y\rangle.\end{aligned}$$

Again using (1.6), we can rewrite the third term of (2.10) as

$$\varphi^2 A(\nabla_X A)Y = -A(\nabla_X A)Y + \langle A(\nabla_X A)Y, W\rangle W,$$

and we can calculate directly that

$$\langle A(\nabla_X A)Y, W\rangle = a\langle(\nabla_X A)Y, W\rangle a\beta\langle X, W\rangle\langle Y, W\rangle + a^2\langle\varphi AX, Y\rangle - ac\langle\varphi X, Y\rangle.$$

All of the information allows us to rewrite (2.10) as

$$\begin{aligned}&(\nabla_X A)AY + A(\nabla_X A)Y \\ &\qquad = (a^2 + c)\langle Y, W\rangle\varphi AX + 2a\beta\langle X, W\rangle\langle Y, W\rangle W + (a^2 + c)\langle\varphi AX, Y\rangle W,\end{aligned}$$

which is equation (2.7). Then, interchanging X and Y in (2.7) and subtracting gives

$$\begin{aligned}&(\nabla_X A)AY - (\nabla_Y A)AX + c(\langle X, W\rangle A\varphi Y - \langle Y, W\rangle A\varphi X + 2a\langle X, \varphi Y\rangle W) \\ &\qquad = (a^2 + c)\big(\langle Y, W\rangle\varphi AX - \langle X, W\rangle\varphi AY + \langle\varphi AX, Y\rangle W - \langle\varphi AY, X\rangle W\big).\end{aligned}$$

Equation (2.8) results from this. ☐

Lemma 2.8. *Let M be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. If $\varphi A + A\varphi = 0$, then*

$$\begin{aligned}(\nabla_X A)AY - A(\nabla_X A)Y &= (a^2 - c)(\langle Y, W\rangle\varphi AX - \langle\varphi AX, Y\rangle W) \\ &\qquad - 2ac(\langle W, Y\rangle\varphi X + \langle W, X\rangle\varphi Y + \langle\varphi Y, X\rangle W).\end{aligned}$$

Proof. Taking the inner product of Z with $(\nabla_X A)AY$ and using the Codazzi equation results in

$$\begin{aligned}\langle(\nabla_X A)AY, Z\rangle &= \langle AY, (\nabla_X A)Z\rangle \\ &= \langle AY, (\nabla_Z A)X\rangle + c\langle AY, \langle X, W\rangle\varphi Z - \langle Z, W\rangle\varphi X + 2\langle X, \varphi Z\rangle W\rangle \\ &= \langle AY, (\nabla_Z A)X\rangle \\ &\quad + c\big(\langle X, W\rangle\langle A\varphi Y, Z\rangle - \langle Z, W\rangle\langle A\varphi X, Y\rangle + 2a\langle W, Y\rangle\langle X, \varphi Z\rangle\big).\end{aligned}$$

Reversing X and Y and subtracting the two equations, we get

$$\begin{aligned}&\langle(\nabla_X A)AY, Z\rangle - \langle(\nabla_Y A)AX, Z\rangle = \langle AY, (\nabla_Z A)X\rangle - \langle AX, (\nabla_Z A)Y\rangle \\ &\quad + c\big(\langle X, W\rangle\langle A\varphi Y, Z\rangle - \langle Y, W\rangle\langle A\varphi X, Z\rangle\big) + 2ac\big(\langle W, Y\rangle\langle X, \varphi Z\rangle - \langle W, X\rangle\langle Y, \varphi Z\rangle\big).\end{aligned}$$

Then the coefficient of X on the right hand side is

$$\begin{aligned}&(\nabla_Z A)AY - A(\nabla_Z A)Y + c(\langle A\varphi Y, Z\rangle W - \langle Y, W\rangle A\varphi Z) \\ &\qquad + 2ac(\langle W, Y\rangle\varphi Z - \langle Y, \varphi Z\rangle W). \qquad (2.11)\end{aligned}$$

Now consider equation (2.8). If we take the inner product of this equation with Z, the coefficient of X on the right hand side is

$$-2ca\langle W, Z\rangle\varphi Y + a^2(\langle Y, W\rangle\varphi AZ - \langle\varphi AY, Z\rangle W).$$

Since this represent the same quantity as (2.11), the two expressions can be equated, and we get

$$\begin{aligned}(\nabla_Z A)AY - A(\nabla_Z A)Y &= (a^2 - c)\langle Y, W\rangle\varphi AZ \\ &\quad - 2ac\big(\langle W, Y\rangle\varphi Z + \langle W, Z\rangle\varphi Y + \langle\varphi Y, Z\rangle W\big) - (a^2 - c)\langle\varphi AY, Z\rangle W.\end{aligned}$$

The statement of the lemma can be obtained by replacing Z with X. □

LEMMA 2.9. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If $\varphi A + A\varphi = 0$, then*

$$\begin{aligned}(\nabla_X A)Y = \beta\langle X, W\rangle\langle Y, W\rangle W + a\big(\langle X, W\rangle\varphi AY + \langle Y, W\rangle\varphi AX + \langle\varphi AX, Y\rangle W\big) \\ + c\big(\langle\varphi Y, X\rangle W - \langle W, Y\rangle\varphi X\big).\end{aligned}$$

PROOF. Adding the result of Lemma 2.8 to the first equation from Lemma 2.9 we get

$$\begin{aligned}(\nabla_X A)AY = a\beta\langle X, W\rangle\langle Y, W\rangle W + c\langle\varphi AX, Y\rangle W + a^2\langle Y, W\rangle\varphi AX \\ - ac\big(\langle Y, W\rangle\varphi X + \langle X, W\rangle\varphi Y + \langle\varphi Y, X\rangle W\big).\end{aligned} \tag{2.12}$$

Notice that $(A^2 + cI)Y = (a^2 + c)\langle Y, W\rangle W$, since $\varphi(A^2 + cI) = 0$. Also recall that $A\varphi A = c\varphi$. Replacing Y by AY in (2.12) we get

$$\begin{aligned}(\nabla_X A)A^2Y = a^2\beta\langle X, W\rangle\langle Y, W\rangle W + c^2\langle\varphi X, Y\rangle W \\ + a^3\langle Y, W\rangle\varphi AX - a^2c\langle Y, W\rangle\varphi X - ac\langle X, W\rangle\varphi AY - ac\langle\varphi AY, X\rangle W.\end{aligned} \tag{2.13}$$

On the other hand, by computing it directly we see that

$$\begin{aligned}(\nabla_X A)A^2Y &= -c(\nabla_X A)Y + (a^2 + c)\langle Y, W\rangle(\nabla_X A)W \\ &= -c(\nabla_X A)Y + (a^2 + c)\beta\langle Y, W\rangle\langle X, W\rangle W \\ &\quad + (a^2 + c)\langle Y, W\rangle a\varphi AX - (a^2 + c)c\langle Y, W\rangle\varphi X \\ &= -c(\nabla_X A)Y + a^2\beta\langle X, W\rangle\langle Y, W\rangle W + \beta c\langle X, W\rangle\langle Y, W\rangle W \\ &\quad + a^3\langle Y, W\rangle\varphi AX + ac\langle Y, W\rangle\varphi AX - a^2c\langle Y, W\rangle\varphi X - c^2\langle Y, W\rangle\varphi X.\end{aligned}$$

Now after equating this with (2.13), we can make several cancellations to get

$$\begin{aligned}c(\nabla_X A)Y = c\beta\langle X, W\rangle\langle Y, W\rangle W + ac\langle\varphi AY, X\rangle W \\ + ac(\langle Y, W\rangle\varphi AX + \langle X, W\rangle\varphi AY) - c^2\langle Y, W\rangle\varphi X - c^2\langle\varphi X, Y\rangle W.\end{aligned}$$

Finally, using the assumption that $4c \neq 0$, we obtain the desired conclusion. □

In this final pair of lemmas we show that if $A\varphi + \varphi A = 0$, then $c = 0$. This is accomplished by computing

$$\sum (R(e_i, \varphi e_i) \cdot A)Z,$$

where $\{e_i\}$ is an orthonormal basis for $W^\perp$, in two different ways. The first way, described in Lemma 2.10, is to use Lemma 2.9, and compute the sum directly. The second way, described in Lemma 2.11, uses the Gauss equation.

Lemma 2.10. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose $A\varphi + \varphi A = 0$ and $X \in W^\perp$. Then direct calculation of $(R(X, \varphi X) \cdot A)Z$ using Lemma 2.9 yields*

$$(R(X, \varphi X) \cdot A)Z = c(\langle X, Y\rangle \varphi AX - \langle X, \varphi Z\rangle AX + \langle X, AZ\rangle \varphi X + \langle X, \varphi AZ\rangle X).$$

Consequently, if $\{e_i\}$ is an orthonormal basis for $W^\perp$, we have

$$\sum (R(e_i, \varphi e_i) \cdot A)Z = 4c\varphi AZ$$

for all tangent vectors Z.

Proof. We will first calculate $\nabla_X((\nabla_{\varphi X} A)Z)$, $(\nabla_{\varphi X} A)(\nabla_X Z)$ and $(\nabla_{\nabla_X \varphi X} A)Z$ separately, and then put the pieces together to get the desired results. We will make the first calculation in detail, and simply state the results for the others.

Direct calculation, using previous results, in particular Lemma 2.9, and repeated use of (1.7) and Proposition 1.3 allows us to conclude that

$$\begin{aligned}
&\nabla_X\big((\nabla_{\varphi X} A)Z\big)\\
&\quad = \nabla_X\big(a(\langle Z, W\rangle \varphi A\varphi X + \langle \varphi A\varphi X, Z\rangle W) + c(\langle \varphi Z, \varphi X\rangle W - \langle W, Z\rangle \varphi^2 X)\big)\\
&\quad = a\big(\langle \nabla_X Z, W\rangle AX + \langle Z, \nabla_X W\rangle AX + \langle Z, W\rangle \nabla_X(AX)\\
&\qquad\qquad + \langle \nabla_X(AX), Z\rangle W + \langle AX, \nabla_X Z\rangle W + \langle AX, Z\rangle \nabla_X W\big)\\
&\qquad + c\big(\langle \nabla_X X, Z\rangle W + \langle X, \nabla_X Z\rangle W + \langle X, Z\rangle \nabla_X W\\
&\qquad\qquad + \langle \nabla_X W, Z\rangle X + \langle W, \nabla_X Z\rangle X + \langle W, Z\rangle \nabla_X X\big), \qquad (2.14)
\end{aligned}$$

where we have used (2.2) to dispose of Xa. Then, using the fact that

$$\nabla_X(AX) = A(\nabla_X X) + (\nabla_X A)X = A(\nabla_X X) + a\langle \varphi AX, X\rangle W + c\langle \varphi X, X\rangle W,$$

and noting that $\langle \varphi X, X\rangle = 0$, we rewrite (2.14) as

$$\begin{aligned}
&\nabla_X\big((\nabla_{\varphi X} A)Z\big)\\
&\quad = a\big(\langle \nabla_X Z, W\rangle AX + \langle Z, \varphi AX\rangle AX + \langle Z, W\rangle A(\nabla_X X) + a\langle \varphi AX, X\rangle \langle Z, W\rangle W\\
&\qquad + \langle A(\nabla_X X), Z\rangle W + a\langle \varphi AX, X\rangle \langle W, Z\rangle W + \langle AX, \nabla_X Z\rangle W + \langle AX, Z\rangle \varphi AX\big)\\
&\qquad + c\big(\langle \nabla_X X, Z\rangle W + \langle X, \nabla_X Z\rangle W + \langle X, Z\rangle \varphi AX\\
&\qquad\qquad + \langle \varphi AX, Z\rangle X + \langle W, \nabla_X Z\rangle X + \langle W, Z\rangle \nabla_X X\big). \qquad (2.15)
\end{aligned}$$

Similar calculations can be performed to compute

$$(\nabla_{\varphi X}A)(\nabla_X Z) = a\big(\langle \nabla_X Z, W\rangle AX + \langle AX, \nabla_X Z\rangle W\big) + c\big(\langle \nabla_X Z, X\rangle W + \langle W, \nabla_X Z\rangle X\big), \quad (2.16)$$

and

$$\begin{aligned}(\nabla_{\nabla_X \varphi X}A)Z = &-\beta\langle AX, X\rangle\langle Z, W\rangle W \\ &+ a\big(\langle Z, W\rangle A\nabla_X X - \langle AX, X\rangle \varphi AZ - \langle Z, W\rangle\langle \nabla_X X, W\rangle aW \\ &\quad + \langle A\nabla_X X, Z\rangle W - a\langle \nabla_X X, W\rangle\langle W, Z\rangle W\big) \\ &+ c\big(\langle Z, \nabla_X X\rangle W - \langle Z, W\rangle\langle \nabla_X X, W\rangle W \\ &\quad + \langle W, Z\rangle \nabla_X X - c\langle W, Z\rangle\langle \nabla_X X, W\rangle W\big). \quad (2.17)\end{aligned}$$

Now define

$$\begin{aligned}N(X, Z) &= (\nabla_X \nabla_{\varphi X}A - \nabla_{\nabla_X \varphi X}A)Z \\ &= \nabla_X\big((\nabla_{\varphi X}A)Z\big) - (\nabla_{\varphi X}A)(\nabla_X Z) - (\nabla_{\nabla_X \varphi X}A)Z.\end{aligned}$$

Substituting the values from equations (2.15), (2.16), and (2.17), we get

$$\begin{aligned}N(X, Z) = &\ \beta\langle AX, X\rangle\langle Z, W\rangle W \\ &+ a\big(\langle Z, \varphi AX\rangle AX + \langle AX, Z\rangle \varphi AX + \langle AX, X\rangle \varphi AZ\big) \\ &+ c\big(\langle X, Z\rangle \varphi AX + \langle \varphi AX, Z\rangle X - 2\langle Z, W\rangle\langle \varphi AX, X\rangle W\big). \quad (2.18)\end{aligned}$$

Also, notice that, since we are assuming that X is orthogonal to W, we have

$$N(\varphi X, Z) = (\nabla_{\varphi X}\nabla_{\varphi^2 X}A - \nabla_{\nabla_{\varphi X}\varphi^2 X}A)Z - (\nabla_X \nabla_{\varphi X}A - \nabla_{\nabla_X \varphi X})Z,$$

from which we conclude that

$$(R(X, \varphi X)\cdot A)Z = N(X, Z) + N(\varphi X, Z).$$

Now, by direct calculation using (2.18), we see that

$$\begin{aligned}N(\varphi X, Z) = &-\beta\langle AX, X\rangle\langle Z, W\rangle W \\ &+ a\big(\langle Z, AX\rangle A\varphi X + \langle A\varphi X, Z\rangle AX - \langle AX, X\rangle \varphi AZ\big) \\ &+ c\big(\langle \varphi X, Z\rangle AX + \langle AX, Z\rangle \varphi X - 2\langle Z, W\rangle\langle X, A\varphi X\rangle W\big).\end{aligned}$$

Hence

$$(R(X, \varphi X)\cdot A)Z = c\big(\langle X, Z\rangle \varphi AX + \langle X, \varphi AZ\rangle X - \langle X, \varphi Z\rangle AX + \langle X, AZ\rangle \varphi X\big).$$

Now let $\{e_i\}$ be an orthonormal basis of $W^\perp$. Then

$$\sum (R(e_i, \varphi e_i)\cdot A)Z = c\Big(\varphi AZ + \varphi AZ - A\varphi Z + \varphi AZ\Big) = 4c\varphi AZ. \qquad \square$$

LEMMA 2.11. *Under the conditions of Lemma* 2.10, *the Gauss equation yields*

$$\sum (R(e_i, \varphi e_i)\cdot A)Z = -4c(2n+1)\varphi AZ.$$

Proof. First look at the calculations for any $(1,1)$ tensor T, and evaluate

$$(TX \wedge T\varphi X)AZ - A(TX \wedge T\varphi X)Z.$$

Two of the terms needed in calculating the desired equality are of this form for various tensors T. Expand this to get

$$\begin{aligned}&(TX \wedge T\varphi X)AZ - A(TX \wedge T\varphi X)Z \\ &\qquad = \langle T\varphi X, AZ\rangle TX - \langle TX, AZ\rangle T\varphi X - \langle T\varphi X, Z\rangle ATX + \langle TX, Z\rangle AT\varphi X.\end{aligned}$$

Summing this over $X = e_i$, the right side becomes

$$-T(\varphi T^* AZ) - T\varphi T^* AZ + AT\varphi T^* Z + AT\varphi T^* Z = -2T\varphi T^* AZ + 2AT\varphi T^* Z, \tag{2.19}$$

where T^* is the transpose of T. Now we can look at this for specific choices of T. In the case when $T = I$, (2.19) becomes

$$2(A\varphi - \varphi A)Z = -4\varphi AZ.$$

When $T = A$, (2.19) is

$$-2A\varphi A^2 Z + 2A^2\varphi AZ = -2(A\varphi A)AZ + 2A(A\varphi A)Z = -4c\varphi AZ.$$

Here we have substituted for $A\varphi A$ using (2.1). Now

$$\begin{aligned}R(e_i, \varphi e_i) &= Ae_i \wedge A\varphi e_i + c(e_i \wedge \varphi e_i + \varphi e_i \wedge \varphi^2 e_i + 2\langle e_i, \varphi^2 e_i\rangle \varphi) \\ &= Ae_i \wedge A\varphi e_i + 2c(e_i \wedge \varphi e_i) - 2c\varphi.\end{aligned}$$

Summation of the last term in $(R(e_i, \varphi e_i) \cdot A)Z$ gives $-4c(2n-2)\varphi AZ$. Using this and (2.19) with $T = I$ and $T = A$, we get

$$\sum (R(e_i, \varphi e_i) \cdot A)Z = -4c(2n+1)\varphi AZ,$$

which is the claim of the lemma. □

Corollary 2.12. *Let M^{2n-1}, $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $A\varphi + \varphi A$ cannot be identically zero.*

Proof. Suppose $A\varphi + \varphi A = 0$. From Lemmas 2.10 and 2.11 we see that for all tangent vectors Z,

$$8c(n+1)\varphi AZ = 0.$$

Therefore, $\varphi A = 0$. This is impossible by Proposition 1.4. □

Now all the ingredients for proving that a is constant have been assembled.

Proof of Theorem 2.1. In view of Corollary 2.12 and Lemma 2.4, we must have $Wa = 0$. This implies that $\operatorname{grad} a = 0$ by Lemma 2.2. Thus a is constant. □

Lemmas 2.7 and 2.8 are valid for $c = 0$, and, in fact, $\varphi A + A\varphi = 0$ leads to no contradiction in this case. However, the only new information that can be derived from these lemmas is that $A(\nabla_X A)Y = 0$ when X and Y are in $W^\perp$. Even this can be deduced more easily by differentiating the identity $AY = 0$.

We finish this section with two applications that make use of the preceding material. The first one will be useful for later classifications.

LEMMA 2.13. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $W\mathfrak{m} = 0$.*

PROOF. By the Codazzi equation (1.9), we have $(\nabla_W A)X = (\nabla_X A)W + c\varphi X$. Thus

$$\begin{aligned} W\mathfrak{m} = W(\operatorname{trace} A) &= \operatorname{trace} \nabla_W A = \operatorname{trace}\{X \mapsto a\nabla_X W - A\nabla_X W + c\varphi X\} \\ &= \operatorname{trace}(a\varphi A - A\varphi A + c\varphi) = 0. \end{aligned}$$ □

The second application shows that there is a lower bound on the rank of A, even pointwise.

PROPOSITION 2.14. *Let M^{2n-1}, where $n \geq 2$, be a hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the rank of the shape operator A is ≥ 2 at some point.*

PROOF. Suppose that the rank of A is ≤ 1 everywhere. Since M cannot be umbilic, by Theorem 1.5, there is an open connected set $\mathcal{U}$ where the rank is 1. We restrict our attention to $\mathcal{U}$. Let λ be the nonzero principal curvature with (one-dimensional) principal subspace T_λ. If X and Y are vector fields in the principal distribution $T_0 = \ker A = T_\lambda^\perp$, then Codazzi's equation gives

$$\begin{aligned} c(\langle X, W\rangle\varphi Y - \langle Y, W\rangle\varphi X + 2\langle X, \varphi Y\rangle W) &= (\nabla_X A)Y - (\nabla_Y A)X \\ &= -A(\nabla_X Y - \nabla_Y X). \end{aligned} \tag{2.20}$$

Clearly the right (and hence the left) side of this equation lies in $T_0^\perp = T_\lambda$. In particular, taking the inner product with W yields

$$3\langle X, W\rangle\langle\varphi Y, X\rangle = 0. \tag{2.21}$$

If $W \notin T_\lambda$, we can choose $X \in T_0$ which is not orthogonal to W. For this X, we have $\langle\varphi Y, X\rangle = 0$ for all $Y \in T_0$ by (2.21). Thus $\varphi X \in T_\lambda$. At any point where $\varphi X = 0$, we have $X = \langle X, W\rangle W$ by (1.6) so that $W \in \operatorname{span}\{X\} \subset T_0$. If $\varphi X \neq 0$, on the other hand, we have

$$0 = \langle\varphi W, X\rangle = -\langle W, \varphi X\rangle$$

so that $W \in T_\lambda^\perp = T_0$. In either case, we can conclude that $AW = 0$.

The net result is that either $AW \equiv \lambda W$ or that $AW \equiv 0$ on $\mathcal{U}$. Thus $\mathcal{U}$ is a Hopf hypersurface and we can apply Corollary 2.3(i). In the first case, we get

$$-\frac{\lambda}{2}A\varphi X = c\varphi X$$

for any $X \in T_0 = W^\perp$, which contradicts the fact that the rank is 1. If $AW = 0$, on the other hand, we have $\varphi X \in T_\lambda$ for any X orthogonal to W in T_0. This comes from setting $Y = W$ in (2.20). Corollary 2.3 then yields $c\varphi X = 0$, another contradiction. □

For other theorems involving lower bounds on the rank, see [Suh 1991].

3. Hypersurfaces in Complex Space Forms

In this section we will describe the standard examples of real hypersurfaces in spaces of constant holomorphic sectional curvature. We start with real hypersurfaces in complex hyperbolic space, and discuss them in detail.

3A. Examples in Complex Hyperbolic Space. We begin with a lemma that will be useful for discussing the first class of examples, the horospheres. The proof is a straightforward calculation.

LEMMA 3.1. *In $\mathbb{C}^{n+1}$, let $p = (1, 1, 0, \ldots, 0)$ and let $\eta = (z_0 - z_1)p$. Then treating z as the position vector field, we have, for any vector field Z in $\mathbb{C}^{n+1}$,*

(i) $\eta = (z_0 - z_1)p = -F(z,p)p$.
(ii) $\langle \eta, z\rangle = -|z_0 - z_1|^2 = -|F(z,p)|^2$.
(iii) $\langle \eta, Z\rangle = -\operatorname{Re}(F(z,p)\overline{F(Z,p)}) = \langle D_Z\eta, z\rangle$.
(iv) $D_Z\eta = -F(Z,p)p$.
(v) $Z\langle \eta, z\rangle = 2\langle \eta, Z\rangle$. ◁

We will use the notation introduced in Section 1. In particular, r is a positive number and the holomorphic curvature of $\mathbb{C}H^n$ is $4c = -4/r^2$. All of the examples, both in $\mathbb{C}H^n$ and in $\mathbb{C}P^n$, are tubes of some sort, and we will use these descriptive names for the purposes of identification. However, we will not attempt to justify these names, but refer the reader to [Cecil and Ryan 1982; Kimura 1986a; Berndt 1989a; 1990] for a full discussion.

The horospheres: Type A0. The first class of examples to be considered were called "self tubes" by Montiel [1985] and are now called horospheres. They form a one-parameter family, parametrized by $t > 0$. Let

$$M' = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = -r^2,\ |z_0 - z_1|^2 = t\}.$$

Then the corresponding horosphere is $M = \pi M'$. We now investigate the geometry of M and its relation to that of M'.

LEMMA 3.2. *M' is a hypersurface in $\mathbb{H}$. For $z \in M'$,*

$$T_zM' = \{Z \in \mathbb{C}^{n+1} : \langle Z, z\rangle = 0,\ \langle Z, \eta\rangle = 0\}.$$

PROOF. First we show that z and η are linearly independent. Both are nonzero, since $\langle \eta, z\rangle = -t \neq 0$. However, $\langle \eta, \eta\rangle = \operatorname{Re}(F(z,p)\overline{F(z,p)}F(p,p)) = 0$ so z cannot be a multiple of η.

The map $\Phi : \mathbb{C}^{n+1} \to \mathbb{R}^2$, given by $\Phi(z) = (\langle z, z\rangle, \langle \eta, z\rangle)$ implicitly defines M'. That is

$$M' = \Phi^{-1}(-r^2, -t).$$

Its rank is 2 since for $Z \in \mathbb{C}^{n+1}$, we have $\Phi_* Z = 0$ if and only if $Z\langle z, z\rangle = 0 = Z\langle \eta, z\rangle$, if and only if $\langle z, Z\rangle = 0 = \langle \eta, Z\rangle$. Thus $\ker \Phi_* = (\operatorname{span}\{z, \eta\})^\perp$, which has dimension $2n$.

Let $\xi' = (r/t)\eta - (1/r)z$. Then ξ' is orthogonal to $T_z M'$ (since η and z are) and coefficients have been chosen so that $\langle \xi', z\rangle = 0$ and $\langle \xi', \xi'\rangle = 1$. Then $\{z/r, \xi'\}$ can be completed to an orthonormal basis for C^{n+1} by adding one timelike and $2n-1$ spacelike vectors, so M' is a Lorentz hypersurface of $\mathbb{H}$. □

The proof of the next lemma can be verified directly from the definitions, and the background developed in Section 1.

Lemma 3.3. *Let $U = -J\xi'$ and $V = Jz$. Then $\{r^{-1}z, \xi', r^{-1}V, U = -J\xi'\}$ is an orthonormal set of vectors with $\{U, V\}$ in $T_z M'$ and $\{z, \xi'\}$ is orthogonal to $T_z M'$. The shape operator A' satisfies $A'V = U$, $A'U = -r^{-2}V + 2r^{-1}U$. For $Z \in T_z M' \cap \{U, V\}^\perp$, $A'Z = r^{-1}Z$.* ◁

We now look at the geometry of $M = \pi M'$ which is a hypersurface in $\mathbb{C}H^n$. First note that ξ' is the horizontal lift of the unit normal ξ of M. For $W = -J\xi$, we have

$$\widetilde{\nabla}_W \xi = \pi_*(\widetilde{\nabla}_{W^L} \xi^L) = \pi_*\Big(\frac{1}{r^2}V - \frac{2}{r}U\Big) = -\frac{2}{r}W.$$

For X in $W^\perp$,

$$\widetilde{\nabla}_X \xi = \pi_*(\widetilde{\nabla}_{X^L} \xi^L) = -\frac{1}{r}X,$$

since $X^L \in \{U, V\}^\perp$. Thus we have established the following.

Theorem 3.4. *The horospheres* (*Type A*0) *are hypersurfaces of complex hyperbolic space that have two distinct principal curvatures*: $\lambda = 1/r$ *of multiplicity* $2n-2$, *and* $a = 2/r$ *of multiplicity* 1. ◁

Tubes around complex hyperbolic spaces: Types A1, A2. Such tubes also form a one-parameter family, parametrized initially by $b > 0$ and later by u. Begin by writing $\mathbb{C}^{n+1} = \mathbb{C}^{p+1} \times \mathbb{C}^{q+1}$ where $p, q \geq 0$ and $p + q = n - 1 > 0$. Let

$$M' = \{z = (z_1, z_2) \in \mathbb{C}^{n+1} : F_1(z_1, z_1) = -(b^2 + r^2),\ F_2(z_2, z_2) = b^2\}$$

where F_1 and F_2 are the restrictions of F to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$ respectively. Then M' is the Cartesian product of an anti–de Sitter space and a sphere whose radii have been chosen so that M' lies in $\mathbb{H}$. Specifically,

$$M'^{2n} = H_1^{2p+1}((b^2 + r^2)^{1/2}) \times S^{2q+1}(b).$$

Write $b = r \sinh u$, $(b^2 + r^2)^{1/2} = r \cosh u$, $\lambda_1 = r^{-1} \tanh u$, $\lambda_2 = r^{-1} \coth u$, and $c = -r^{-2}$, so that $\lambda_1 \lambda_2 + c = 0$. These turn out to be principal curvatures of M'.

Let $\xi' = -(\lambda_1 z_1 + \lambda_2 z_2)$. Note that we identify z_1 and $(z_1, 0)$ for convenience of notation. Similarly for z_2. It is easy to check that ξ' is a unit normal vector for M'.

Let $V = Jz$. Note that V is tangent to M' since $\langle V, z\rangle = \langle Jz, z\rangle = 0$ and $\langle V, \xi'\rangle = -\langle Jz_1 + Jz_2, \lambda_1 z_1 + \lambda_2 z_2\rangle = -(\lambda_1\langle Jz_1, z_1\rangle + \lambda_2\langle Jz_2, z_2\rangle) = 0$. Both λ_1 and λ_2 satisfy the equation

$$\lambda^2 - \alpha\lambda + \frac{1}{r^2} = 0 \quad \text{with } \alpha = \frac{2}{r}\coth 2u.$$

A routine calculation yields the following two lemmas.

LEMMA 3.5. *For X tangent to H_1^{2p+1}, $A'X = \lambda_1 X$. For X tangent to S^{2q+1}, $A'X = \lambda_2 X$. If $U = -J\xi'$, then $A'V = U$ and $A'U = \alpha U - Vr^{-2}$.* ◁

LEMMA 3.6. *$\{z/r, \xi', V/r, U\}$ is an orthonormal set with ξ' and U spacelike. The other two vectors are timelike.* ◁

The hypersurface $\pi M'$ is denoted by $M_{2p+1,2q+1}$. At a typical point $z \in M'$, the horizontal subspace of $T_z M'$ is the orthogonal direct sum

$$(\operatorname{span}\{U\}) \oplus T_1 \oplus T_2,$$

where

$$T_1 = \{Z \in T_z H_1^{2p+1} : \langle Z, U\rangle = 0, \langle Z, V\rangle = 0\},$$
$$T_2 = \{Z \in T_z S^{2q+1} : \langle Z, U\rangle = 0, \langle Z, V\rangle = 0\}.$$

Note that T_1 and T_2 are J-invariant. Thus

$$T_{\pi z} M_{2p+1,2q+1} = (\operatorname{span}\{W\}) \oplus \pi_* T_1 \oplus \pi_* T_2.$$

We calculate the shape operator A using Lemma 1.7.

$$AW = \pi_*(A'U) - \frac{\varepsilon}{r^2}\pi_* V = \pi_*\Big(A'U + \frac{1}{r^2}V\Big)$$
$$= \pi_*\Big(\alpha U - \frac{1}{r^2}V + \frac{1}{r^2}V\Big) = \pi_*(\alpha U) = \alpha W.$$

For $X \in \pi_* T_1$,

$$AX = \pi_*\Big(A'X^L - \langle X^L, U\rangle\frac{\varepsilon}{r^2}V\Big) = \pi_*\lambda_1 X^L = \lambda_1 X.$$

Similarly, $AX = \lambda_2 X$ for $X \in \pi_* T_2$. When $p = 0$, M is a geodesic hypersphere with principal curvatures λ_2 (of multiplicity $2n-2$) and α of multiplicity 1. The radius of the sphere is ru. When $q = 0$, M is a tube of radius ru over a complex hyperbolic hyperplane. There are only two principal curvatures λ_1 and α. These are the Type A1 hypersurfaces. The rest are Type A2 and have three distinct principal curvatures. Again, they are tubes of radius ru about complex hyperbolic spaces of codimension greater than 1. Summarizing this information, we have the following results.

THEOREM 3.7. *The tubes around complex hyperbolic hyperplanes* (*Type A*1) *in complex hyperbolic space have two distinct principal curvatures*: $\lambda_1 = (1/r)\tanh u$ *of multiplicity* $2n-2$ *and* $a = (2/r)\coth 2u$ *of multiplicity* 1. ◁

THEOREM 3.8. *The geodesic spheres* (*Type A*1) *in complex hyperbolic space have two distinct principal curvatures*: $\lambda_2 = (1/r)\coth u$ *of multiplicity* $2n-2$ *and* $a = (2/r)\coth 2u$ *of multiplicity* 1. ◁

THEOREM 3.9. *The Type A*2 *hypersurfaces in complex hyperbolic space have three distinct principal curvatures*: $\lambda_1 = (1/r)\tanh u$ *of multiplicity* $2p$, $\lambda_2 = (1/r)\coth u$ *of multiplicity* $2q$, *and* $a = (2/r)\coth 2u$ *of multiplicity* 1, *where* $p > 0$, $q > 0$, *and* $p + q = n - 1$. ◁

Tubes around real hyperbolic space: Type B. Again, these examples form a one-parameter family. We begin with $t > r^4$ but later find a more convenient parameter u. Let

$$M' = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle = -r^2,\ |F(z, \bar{z})|^2 = t\}.$$

Write $Q(z) = F(z, \bar{z})$ and $\eta = Q(z)\bar{z}$.

LEMMA 3.10. *M' is a hypersurface in $\mathbb{H}$. For $z \in M'$,*

$$T_zM' = \{Z \in \mathbb{C}^{n+1} : \langle Z, z\rangle = 0,\ \langle Z, \eta\rangle = 0\}.$$

PROOF. First we note that z and η are nonzero. In fact, $\langle \eta, \eta\rangle = |Q(z)|^2\langle \bar{z}, \bar{z}\rangle = -tr^2$. Further, they are linearly independent since if $\eta = \rho z$ then $-\rho r^2 = \langle \eta, z\rangle = \operatorname{Re} F(F(z, \bar{z})\bar{z}, z) = |F(z, \bar{z})|^2 = t$. On the other hand, by considering $\langle \eta, \eta\rangle$, we have $\rho^2 = t$. Thus $tr^4 = \rho^2 r^4 = (-\rho r^2)^2 = t^2$. In other words, $t(r^4 - t) = 0$, which is a contradiction.

The map $\Phi : \mathbb{C}^{n+1} \to R^2$, given by $\Phi(z) = (\langle z, z\rangle, |Q(z)|^2)$ implicitly defines M'. That is,

$$M' = \Phi^{-1}(-r^2, t).$$

Its rank is 2 since, for $Z \in \mathbb{C}^{n+1}$, we have $\Phi_* Z = 0$ if and only if $Z\langle z, z\rangle = 0 = Z\langle \eta, z\rangle$, if and only if $\langle z, Z\rangle = 0 = \langle \eta, Z\rangle$. To see this, note that

$$\begin{aligned} Z(F(z, \bar{z})F(\bar{z}, z)) &= (F(Z, \bar{z}) + F(z, \bar{Z}))F(\bar{z}, z) + F(z, \bar{z})(F(\bar{Z}, z) + F(\bar{z}, Z)) \\ &= 2\operatorname{Re}(F(Z, \bar{z})F(\bar{z}, z)) + 2(\operatorname{Re} F(z, \bar{Z})F(\bar{z}, z)) \\ &= 2\operatorname{Re} F(Z, Q(z)\bar{z}) + 2\operatorname{Re} F(\overline{Q(z)}z, \bar{Z}) \\ &= 4\langle Z, Q(z)\bar{z}\rangle = 4\langle Z, \eta\rangle. \end{aligned}$$ □

From the lemma we see that η is normal to M' but not tangent to $\mathbb{H}$. However, it is easy to verify that

$$\xi_0 = r^2\eta + tz$$

satisfies $\langle \xi_0, z\rangle = 0$. Furthermore,

$$F(\xi_0, \xi_0) = r^4\langle \eta, \eta\rangle + t^2F(z, z) + 2r^2\operatorname{Re} F(\eta, tz) = -r^6t - t^2r^2 + 2t^2r^2 = r^2t(t - r^4),$$

so that we can define a unit normal ξ' by

$$\xi_0 = -rt^{1/2}(t-r^4)^{1/2}\xi'. \tag{3.1}$$

We now describe the shape operator A' of M'. First let $V = Jz$ as in earlier examples. Note that V is tangent to M' since $F(Q(z)\bar{z}, V) = -iF(Q(z)\bar{z}, z) = -iQ(z)\overline{Q(z)}$ is purely imaginary. Also the vector field $U_0 = -J\xi_0 = -tV - r^2 J\eta$ is tangent to M' and orthogonal to V. In computing A', we first work with ξ_0 to avoid the normalization factor involved in (3.1).

Lemma 3.11. (i) *For Z tangent to M',* $D_Z\xi_0 = 2r^2F(Z,\bar{z})\bar{z} + r^2Q(z)\bar{Z} + tZ$.
(ii) *In particular,* $D_V\xi_0 = -U_0$.
(iii) $D_{U_0}\xi_0 = (t-r^4)(2U_0 + tV)$.
(iv) *There are two $n-1$ dimensional subspaces $\mathcal{V}^+$ and $\mathcal{V}^-$, orthogonal both to $\{U, V\}$ and to each other, such that $D_X\xi_0 = (t + r^2t^{1/2})X$ for $X \in \mathcal{V}^+$, and $D_X\xi_0 = (t - r^2t^{1/2})X$ for $X \in \mathcal{V}^-$.*

Proof. We compute

$$\begin{aligned} D_Z\xi_0 &= D_Z(r^2F(z,\bar{z})\bar{z}) + D_Z(tz) = 2r^2F(Z,\bar{z})\bar{z} + r^2F(z,\bar{z})\bar{Z} + tZ \\ &= 2r^2F(Z,\bar{z})\bar{z} + r^2Q(z)\bar{Z} + tZ, \end{aligned}$$

which proves the general formula (i). Now, if $Z = V = iz$, we get

$$D_V\xi_0 = 2r^2F(iz,\bar{z})\bar{z} + r^2Q(z)(-i\bar{z}) + tiz = ir^2Q(z)\bar{z} + tiz = i\xi_0 = -U_0.$$

For (iii) we compute

$$D_{U_0}\xi_0 = 2r^2F(U_0,\bar{z})\bar{z} + r^2Q(z)\bar{U}_0 + tU_0.$$

It is straightforward to verify that

$$F(U_0,\bar{z}) = -i(t-r^4)Q(z)$$

and that

$$Q(z)\bar{U}_0 = it(r^2z + \eta),$$

from which it follows by substitution that

$$D_{U_0}\xi_0 = (t-r^4)(2U_0 + tV).$$

We now prove (iv). Choose α in the range $0 \le \alpha \le \pi/2$ and such that $Q(z) = e^{2i\alpha}|Q(z)|$. Let X be a totally real tangent vector (that is, $\bar{X} = X$). Then $Z = e^{i\alpha}X = \cos\alpha X + \sin\alpha JX$ satisfies

$$\begin{aligned} D_Z\xi_0 &= r^2Q(z)e^{-i\alpha}X + te^{i\alpha}X = r^2Q(z)e^{-2i\alpha}Z + tZ \\ &= (r^2|Q(z)| + t)Z = (t + r^2t^{1/2})Z. \end{aligned}$$

Similarly, if $Z = e^{i\alpha}JX = e^{i(\alpha+\pi/2)}X$, then

$$D_Z\xi_0 = r^2Q(z)e^{-i\alpha}(-iX) + te^{i\alpha}(iX) = -r^2Q(z)e^{-2i\alpha}Z + tZ = (t - r^2t^{1/2})Z.$$

Now

$$\mathcal{V}^+ = \{e^{i\alpha}X : X \in T_zM',\, \bar{X} = X,\, \langle X, U_0\rangle = \langle X, V\rangle = 0\}$$

and $\mathcal{V}^- = J\mathcal{V}^+$ have the properties stated in (iv). Note that $\mathcal{V}^-$ is obtained from "totally imaginary" tangent vectors as follows:

$$\mathcal{V}^- = \{e^{i\alpha}X : X \in T_zM',\, \bar{X} = -X,\, \langle X, U_0\rangle = \langle X, V\rangle = 0\},$$

and that J interchanges the two principal spaces $\mathcal{V}^+$ and $\mathcal{V}^-$. □

THEOREM 3.12. *The Type B hypersurfaces in complex hyperbolic space have three principal curvatures, namely,* $\lambda_1 = (1/r)\coth u$ *of multiplicity* $n-1$, $\lambda_2 = (1/r)\tanh u$ *of multiplicity* $n-1$, *and* $a = (2/r)\tanh 2u$ *of multiplicity* 1. *These curvatures are distinct unless* $\coth u = \sqrt{3}$, *in which case* λ_1 *and* a *coincide to make a principal curvature of multiplicity* n.

PROOF. Taking the normalization factor from (3.1) into account, we get

$$A'X = \frac{1}{r}\frac{(t^{1/2}+r^2)}{(t-r^4)^{1/2}}X = \lambda_1 X$$

for $X \in \mathcal{V}^+$, and

$$A'X = \frac{1}{r}\frac{(t^{1/2}-r^2)}{(t-r^4)^{1/2}}X = \lambda_2 X$$

for $X \in \mathcal{V}^-$. (These equations serve to define the principal curvatures λ_1 and λ_2.) Also, we get

$$D_{U_0}\xi' = -\frac{(t-r^4)^{1/2}}{rt^{1/2}}(2U_0 + tV).$$

Applying Lemma 1.7, $\mathcal{V}^+$ and $\mathcal{V}^-$ are horizontal subspaces projecting to $T_{\pi z}M$ and the principal curvatures are preserved. Also, we get

$$AW = \frac{(t-r^4)^{1/2}}{rt^{1/2}}(2W).$$

It is easy to verify that $\lambda_1\lambda_2 = 1/r^2 = -c$, so that there is a unique positive number u satisfying $\lambda_1 = (1/r)\coth u$ and $\lambda_2 = (1/r)\tanh u$. A direct calculation shows that $a = (2/r)\tanh 2u$. □

The Type B hypersurfaces are tubes of radius ru around real hyperbolic space $\mathbb{R}H^n$.

3B. Examples in Complex Projective Space. We now discuss the standard examples of real hypersurfaces in complex projective space. Since these examples may be more widely know than those in complex hyperbolic space, and since the constructions often resemble those in complex hyperbolic space, we will leave some of the details to the reader. Here r is a positive constant and $c = 1/r^2$.

Tubes around complex projective spaces: Types A1, A2. Begin by writing $\mathbb{C}^{n+1} = \mathbb{C}^{p+1} \times \mathbb{C}^{q+1}$, where $p, q \geq 0$ and $p + q = n - 1 > 0$. Choose b so that $0 < b < r$. Let

$$M' = \{z = (z_1, z_2) \in \mathbb{C}^{n+1} : F_1(z_1, z_1) = r^2 - b^2,\ F_2(z_2, z_2) = b^2\},$$

where F_1 and F_2 are the restrictions of F to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$ respectively. Then M' is the Cartesian product of spheres whose radii have been chosen so that M' lies in $S^{2n+1}(r)$. Specifically,

$$M'^{2n} = S^{2p+1}\big((r^2 - b^2)^{1/2}\big) \times S^{2q+1}(b).$$

Write $b = r \sin u$, so $(r^2 - b^2)^{1/2} = r \cos u$. We can choose u so that $0 < u < \pi/2$. Write $\lambda_1 = -(1/r) \tan u$ and $\lambda_2 = (1/r) \cot u$. Since $c = r^{-2}$ we have $\lambda_1 \lambda_2 + c = 0$. The numbers λ_1 and λ_2 turn out to be the principal curvatures of M'.

The principal curvatures λ_1, λ_2, and α project as in the hyperbolic case. There is only one kind of Type A1 hypersurface since tubes over complex projective hyperplanes are also geodesic spheres. The principal curvatures of one are related to those of the other by replacing the parameter u by $\frac{\pi}{2} - u$. We summarize as follows.

THEOREM 3.13. *The geodesic spheres* (*Type A*1) *in complex projective space have two distinct principal curvatures*: $\lambda_2 = (1/r) \cot u$ *of multiplicity* $2n - 2$ *and* $a = (2/r) \cot 2u$ *of multiplicity* 1. ◁

THEOREM 3.14. *The Type A*2 *hypersurfaces in complex projective space have three distinct principal curvatures*: $\lambda_1 = -(1/r) \tan u$ *of multiplicity* $2p$, $\lambda_2 = (1/r) \cot u$ *of multiplicity* $2q$, *and* $a = (2/r) \cot 2u$ *of multiplicity* 1, *where* $p > 0$, $q > 0$, *and* $p + q = n - 1$. ◁

Tubes around the complex quadric: Type B. Again, such tubes form a one-parameter family. We begin with $t < r^4$ but later find a more convenient parameter u. Let

$$M' = \{z \in \mathbb{C}^{n+1} : \langle z, z \rangle = r^2,\ |F(z, \bar{z})|^2 = t\}.$$

Write $Q(z) = F(z, \bar{z})$ and $\eta = Q(z)\bar{z}$. Then we have,

THEOREM 3.15. *The Type B hypersurfaces in complex projective space have three distinct principal curvatures*: $\lambda_1 = -(1/r) \cot u$ *of multiplicity* $n - 1$, $\lambda_2 = (1/r) \tan u$ *of multiplicity* $n - 1$, *and* $a = (2/r) \tan 2u$ *of multiplicity* 1. ◁

The Type B hypersurfaces are also tubes over totally geodesic real projective spaces $\mathbb{R}P^n$. The parameter u is chosen so that the tubes have radius ru. Then the tubes over the complex quadric have radius $r(\pi/4 - u)$. This is taken into account in the statement of Theorem 6.1. For more detail—in particular, the relationship to focal sets—see [Cecil and Ryan 1982].

Examples with Five Principal Curvatures. There are three additional standard types of hypersurfaces in $\mathbb{C}P^n$ and they have five distinct principal curvatures. They are listed at the end of Section 3 (page 261), but we will not describe their construction in detail. However we will list the principal curvatures for future reference.

THEOREM 3.16. *The Type C hypersurfaces in complex projective space have five distinct principal curvatures*,

(i) $\lambda_1 = -(1/r)\cot u$ *of multiplicity* $n-3$,
(ii) $\lambda_2 = (1/r)\cot(\pi/4 - u)$ *of multiplicity* 2,
(iii) $\lambda_3 = (1/r)\cot(\pi/2 - u)$ *of multiplicity* $n-3$,
(iv) $\lambda_4 = (1/r)\cot(3\pi/4 - u)$ *of multiplicity* 2,
(v) $a = -(2/r)\cot 2u$ *of multiplicity* 1.

These hypersurfaces occur for $n \geq 5$, n *odd*. ◁

THEOREM 3.17. *The Type D hypersurfaces in complex projective space have five distinct principal curvatures*,

(i) $\lambda_1 = -(1/r)\cot u$ *of multiplicity* 4,
(ii) $\lambda_2 = (1/r)\cot(\pi/4 - u)$ *of multiplicity* 4,
(iii) $\lambda_3 = (1/r)\cot(\pi/2 - u)$ *of multiplicity* 4,
(iv) $\lambda_4 = (1/r)\cot(3\pi/4 - u)$ *of multiplicity* 4,
(v) $a = -(2/r)\cot 2u$ *of multiplicity* 1.

This hypersurface occurs only in $\mathbb{C}P^9$. ◁

THEOREM 3.18. *The Type E hypersurfaces in complex projective space have five distinct principal curvatures*,

(i) $\lambda_1 = -(1/r)\cot u$ *of multiplicity* 8,
(ii) $\lambda_2 = (1/r)\cot(\pi/4 - u)$ *of multiplicity* 6,
(iii) $\lambda_3 = (1/r)\cot(\pi/2 - u)$ *of multiplicity* 8,
(iv) $\lambda_4 = (1/r)\cot(3\pi/4 - u)$ *of multiplicity* 6,
(v) $a = -(2/r)\cot 2u$ *of multiplicity* 1.

This hypersurface occurs only in $\mathbb{C}P^{15}$. ◁

3C. Summary: Takagi's list and Montiel's list. In this section, we list, for reference purposes, standard examples of hypersurfaces in complex space forms. These examples are so prevalent in the subject that they have acquired a standard nomenclature. In $\mathbb{C}P^n$, they divide into five types, A–E, while $\mathbb{C}H^n$ has just two types. Types are further subdivided, e.g., A1, A2. The list is as follows. In complex projective space, $\mathbb{C}P^n$:

(A1) Geodesic spheres.
(A2) Tubes over totally geodesic complex projective spaces $\mathbb{C}P^k$, where $1 \leq k \leq n-2$.

(B) Tubes over complex quadrics and $\mathbb{R}P^n$.
(C) Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^m$ where $2m+1=n$ and $n \geq 5$.
(D) Tubes over the Plücker embedding of the complex Grassmann manifold $G_{2,5}$. Occur only for $n=9$.
(E) Tubes over the canonical embedding of the Hermitian symmetric space $\mathrm{SO}(10)/U(5)$. Occur only for $n=15$.

This list consists precisely of the homogeneous real hypersurfaces in $\mathbb{C}P^n$ as determined by Takagi [1973], and is often referred to as "Takagi's list". The list itself with the type names is given in [Takagi 1975a]. In addition, every Hopf hypersurface with constant principal curvatures is an open subset of one of these. Many authors contributed to this result, which was completed by M. Kimura [Kimura 1986a]. In complex hyperbolic space $\mathbb{C}H^n$ the list is as follows:

(A0) Horospheres.
(A1) Geodesic spheres and tubes over totally geodesic complex hyperbolic hyperplanes.
(A2) Tubes over totally geodesic $\mathbb{C}H^k$, where $1 \leq k \leq n-2$.
(B) Tubes over totally real hyperbolic space $\mathbb{R}H^n$.

These hypersurfaces are homogeneous, but there is yet no classification theorem for homogeneous hypersurfaces in $\mathbb{C}H^n$. However, every Hopf hypersurface with constant principal curvatures must be one of these. This classification was begun by S. Montiel [1985] (who also described the examples in detail) and completed by J. Berndt [1989a]. We refer to the list as "Montiel's list".

In subsequent sections, we will characterize certain subsets of these lists in terms of properties of the shape operator, Ricci tensor and other geometric objects.

4. Restrictions on the Shape Operator and the Number of Principal Curvatures

We recall that the principal spaces of the Type A hypersurfaces are invariant by the structure tensor φ. One of the first classification theorems in this subject is that this property is a characterization for Type A hypersurfaces.

Theorem 4.1. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $\varphi A = A\varphi$ if and only if M is an open subset of a Type A hypersurface.*

As a first step in proving Theorem 4.1, we show that this property is equivalent to parallelism of the shape operator of the lifted hypersurface M'.

Lemma 4.2. *Under the hypothesis of Theorem 4.1, $\varphi A = A\varphi$ if and only if A' is parallel.*

PROOF. If A' is parallel, then $\varphi A = A\varphi$ by Proposition 1.9. Conversely, suppose $\varphi A = A\varphi$. Then $(A^2 - aA - cI)\varphi = 0$ by Lemma 2.2 and Corollary 2.5, so $A|_{W^\perp}$ satisfies the quadratic equation

$$t^2 - at - c = 0. \tag{4.1}$$

Lifting this condition to M' using Lemma 1.7, we see that A' satisfies (4.1) on $\{U, V\}^\perp$. Again using Lemma 1.7, we have

$$(A')^2 U = A'(aU + cV) = aA'U + cA'V = aA'U + cU$$

and

$$(A')^2 V = A'U = aU + cV = aA'V + cV.$$

Applying Lemma 1.10 to A', we see that A' is parallel provided that $a^2 + 4c \neq 0$. Suppose now that $a^2 + 4c = 0$. Without loss of generality we can assume that a is positive so that $a = 2/r$. Let

$$P = A' - \tfrac{1}{2}aI = A' - \frac{1}{r}I$$

so that (4.1) means that $P^2 = 0$. Note that $\nabla' A' = \nabla' P$ and that

$$\ker P = \{U, V\}^\perp \oplus \operatorname{span}\{rU - V\} = \{rU - V\}^\perp.$$

Writing $\bar{U} = rU - V$ and $\bar{V} = rU + V$, we have $P\bar{U} = 0$ and $P\bar{V} = (2/r)\bar{U}$. Take $Z \in \ker P$ and and differentiate $\langle Z, \bar{U}\rangle = 0$ with respect to an arbitrary tangent vector X. Using the fact that $\bar{U} = -J(r\xi + z)$, we arrive at

$$\langle \nabla'_X Z, \bar{U}\rangle = -r\langle Z, JPX\rangle.$$

On the other hand,

$$PX \in (\ker P) \cap (\ker P)^\perp = \operatorname{span} \bar{U}$$

so that $JPX \in \operatorname{span}\{r\xi + z\}$ which is normal to M'. Thus

$$\langle \nabla'_X Z, \bar{U}\rangle = 0. \tag{4.2}$$

Now take any tangent vectors X and Y and let Z be in $\ker P$. Then

$$\langle (\nabla'_X P)Y, Z\rangle = \langle \nabla'_X (PY), Z\rangle = -\langle PY, \nabla'_X Z\rangle = 0$$

since $\nabla'_X Z \in \ker P$ by (4.2). Noting that any expression of the form $\langle (\nabla'_X P)Y, Z\rangle$ is symmetric in its three arguments (by the Codazzi equation), we have shown that such an expression is zero if any of its three arguments lies in $\ker P$. To complete the proof that $\nabla' P = 0$, we need only show that $\langle (\nabla'_X P)Y, Z\rangle = 0$ when $X = Y = Z = \bar{V}$. For this Z we have $\langle PZ, Z\rangle = 2r^{-2}\langle \bar{U}, \bar{V}\rangle = 4r$, so that

$$\langle (\nabla'_Z P)Z, Z\rangle + 2\langle PZ, \nabla'_Z Z\rangle = 0.$$

On the other hand, if we differentiate $\langle Z, \bar{U}\rangle$, we get

$$\langle \nabla'_Z Z, \bar{U}\rangle = -\langle Z, \nabla'_Z \bar{U}\rangle = -\langle Z, rD_Z U - D_Z V\rangle.$$

Substituting $U = -J\xi$ and $V = Jz$, we show that the second argument of the inner product is $-J(r^{-1}\bar{V} + 2r^{-1}\bar{U})$ which is a normal vector to M'. Thus $\langle PZ, \nabla'_Z Z\rangle = 0$ and hence $\langle(\nabla'_Z P)Z, Z\rangle = 0$. This completes the proof that A' is parallel. □

Sketch of proof of Theorem 4.1. Suppose that $\varphi A = A\varphi$. By Lemma 4.2, A' is parallel. It also follows from Lemma 1.10 that $a^2 + 4c \geq 0$. This is because $n \geq 2$ and the tangent space to M' cannot have a timelike subspace of dimension greater than one. If $a^2 + 4c > 0$, then (4.1) has two distinct roots. The classification is fairly straightforward in this case. If $a^2 + 4c = 0$, it is a little more difficult and leads to the horosphere. For details, see [Ryan 1971; Okumura 1975; Montiel and Romero 1986]. □

There are other characterizations of the Type A hypersurfaces. Theorem 1.8 together with Proposition 1.9 yield the following formula for ∇A.

Theorem 4.3. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then $\varphi A = A\varphi$ if and only if*

$$(\nabla_X A)Y = -c(\langle\varphi X, Y\rangle W + \langle Y, W\rangle\varphi X). \tag{4.3}$$

◁

Corollary 4.4. *Equality occurs in Theorem 1.11 if and only if M is an open subset of a Type A hypersurface.* ◁

Theorem 4.1 is due to Okumura [1975] for $\mathbb{C}P^n$ and to Montiel and Romero [1986] for $\mathbb{C}H^n$. Theorem 1.11 and Corollary 4.4 were proven by Y. Maeda [1976] for $\mathbb{C}P^n$ and by B.-Y. Chen, G. D. Ludden, and Montiel [1984] for $\mathbb{C}H^n$. Also, a generalization of Theorem 1.11 and Corollary 4.4 to Type B hypersurfaces can be found in [Ki et al. 1990a]. As we saw in Theorem 1.5, there are no real hypersurfaces for which A is parallel. A $(1,1)$ tensor field T is said to be *cyclic parallel* if the cyclic sum

$$\langle(\nabla_X T)Y, Z\rangle + \langle(\nabla_Y T)Z, X\rangle + \langle(\nabla_Z T)X, Y\rangle$$

vanishes for all X, Y, and Z. This provides yet another characterization of the Type A hypersurfaces. Relevant references are [Chen and Vanhecke 1981; Chen et al. 1984; Ki 1988; Ki and Kim 1989].

Theorem 4.5. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the shape operator A is cyclic parallel if and only if (4.3) holds.*

Proof. First assume (4.3). Then

$$\begin{aligned}
\langle(\nabla_X A)Y, Z\rangle &= -c(\langle\varphi X, Y\rangle\langle W, Z\rangle + \langle Y, W\rangle\langle\varphi X, Z\rangle),\\
\langle(\nabla_Y A)Z, X\rangle &= -c(\langle\varphi Y, Z\rangle\langle W, X\rangle + \langle Z, W\rangle\langle\varphi Y, X\rangle),\\
\langle(\nabla_Z A)X, Y\rangle &= -c(\langle\varphi Z, X\rangle\langle W, Y\rangle + \langle X, W\rangle\langle\varphi Z, Y\rangle).
\end{aligned}$$

The right sides sum to zero by skew-symmetry of φ. Thus A is cyclic parallel.

Conversely, suppose that A is cyclic parallel. Then we obtain (4.3) by applying the Codazzi equation twice, as follows:

$$\begin{aligned}
-\langle(\nabla_X A)Y, Z\rangle &= \langle(\nabla_Y A)Z, X\rangle + \langle(\nabla_Z A)X, Y\rangle \\
&= \langle Z, (\nabla_Y A)X\rangle + \langle(\nabla_Z A)X, Y\rangle \\
&= \langle(\nabla_X A)Y, Z\rangle \\
&\quad - c(\langle X, W\rangle\langle\varphi Y, Z\rangle - \langle Y, W\rangle\langle\varphi X, Z\rangle + 2\langle X, \varphi Y\rangle\langle W, Z\rangle) \\
&\quad + \langle(\nabla_Z A)X, Y\rangle, \\
-2\langle(\nabla_X A)Y, Z\rangle &= \langle(\nabla_Z A)X, Y\rangle \\
&\quad - c(\langle X, W\rangle\langle\varphi Y, Z\rangle - \langle Y, W\rangle\langle\varphi X, Z\rangle + 2\langle X, \varphi Y\rangle\langle W, Z\rangle), \\
-2\langle Y, (\nabla_X A)Z\rangle &= \langle(\nabla_X A)Z, Y\rangle \\
&\quad - c(\langle X, W\rangle\langle\varphi Z, Y\rangle - \langle Z, W\rangle\langle\varphi X, Y\rangle + 2\langle X, \varphi Z\rangle\langle W, Y\rangle) \\
&\quad - c(\langle X, W\rangle\langle\varphi Y, Z\rangle - \langle Y, W\rangle\langle\varphi X, Z\rangle + 2\langle X, \varphi Y\rangle\langle W, Z\rangle), \\
-3\langle Y, (\nabla_X A)Z\rangle &= 3c(\langle Y, W\rangle\langle\varphi X, Z\rangle + \langle Z, W\rangle\langle\varphi X, Y\rangle).
\end{aligned}$$

That is,

$$(\nabla_X A)Y = -c(\langle\varphi X, Y\rangle W + \langle Y, W\rangle\varphi X). \qquad \square$$

Theorem 4.5 is also trivially true when $c = 0$. In that case, A is cyclic parallel if and only if $\nabla A = 0$ because of the Codazzi equation. Also, when $c = 0$, (4.3) means that $\nabla A = 0$.

The "standard examples" listed in the summary at the end of Section 3 have constant principal curvatures, and there are classification theorems using this hypothesis. It is possible to get some results by merely imposing a limit on the *number* of distinct principal curvatures. As we noted in Theorem 1.5, there is no possibility of an umbilic hypersurface. The next two theorems show what can happen when the number of distinct principal curvatures is 2. For proofs, we refer to [Cecil and Ryan 1982; Montiel 1985].

THEOREM 4.6. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Suppose that the number g of distinct principal curvatures is ≤ 2 at each point. Then M is an open subset of a geodesic hypersphere.* ◁

THEOREM 4.7. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c < 0$. Suppose that the number g of distinct principal curvatures is ≤ 2 at each point. Then M is an open subset of one of the following*:

(i) *a geodesic sphere* (*Type* A1);
(ii) *a tube over a complex hyperbolic hyperplane* (*Type* A1);
(iii) *a horosphere* (*Type* A0);

(iv) *a tube of radius*

$$r\log\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)=\frac{r}{2}\log\bigl(2+\sqrt{3}\bigr)$$

over a totally real hyperbolic space (Type B). ◁

Possibility (iv) occurs when u is chosen so that $\coth u = 2\tanh 2u$. One principal subspace includes W and therefore has dimension n.

Since A cannot be parallel, we look for similar but weaker conditions that can be satisfied and can serve as characterizing properties. The condition $R\cdot A=0$ is sometimes called "semi-parallel", and has been of interest for hypersurfaces in real space forms. However, here we find it is also too strong, as is shown by the next sequence of results.

Lemma 4.8. *Let M^{2n-1}, where $n\geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c\neq 0$. Suppose that p is a point where $R\cdot A=0$. If λ and μ are distinct principal curvatures at p, with associated principal orthonormal vectors X and Y, then*

$$\lambda\mu+c\bigl(1+3\langle\varphi X,Y\rangle^2\bigr)=0.$$

Proof. Since $A(R(X,Y)Y=R(X,Y)AY=\mu R(X,Y)Y$, we have $R(X,Y)Y\in T_\mu(p)$ so that $\langle R(X,Y)Y,X\rangle=0$. On the other hand, by the Gauss equation,

$$\begin{aligned}&\langle R(X,Y)Y,X\rangle\\&\qquad=(\lambda\mu+c)+c\langle\varphi Y,Y\rangle\langle\varphi X,X\rangle-c\langle\varphi X,Y\rangle\langle\varphi Y,X\rangle+2c\langle X,\varphi Y\rangle\langle\varphi Y,X\rangle\\&\qquad=\lambda\mu+c(1+3\langle\varphi X,Y\rangle^2).\end{aligned}$$

□

Theorem 4.9. *Let M^{2n-1}, where $n\geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c\neq 0$. Then $R\cdot A$ never vanishes.*

Proof. Suppose $R\cdot A=0$ at a point p. Take $Y=W$ in Lemma 4.8. This shows that $A=\lambda I$ on $W^\perp$ where $\lambda a+c=0$. In particular, for any $X\in W^\perp$, $AX=\lambda X$ and $A\varphi X=\lambda\varphi X$. By Corollary 2.3, $0\neq\lambda^2=\lambda a+c=0$, which is a contradiction. □

Theorem 4.10. *Let M^{2n-1}, where $n\geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c>0$. Then $R\cdot A$ cannot be identically zero.*

Proof. Suppose that $R\cdot A=0$. As a result of Lemma 4.8, any two distinct principal curvatures have opposite signs. Therefore, there can be at most two of them at each point. Thus M must be an open subset of a geodesic hypersphere by Theorem 4.6. Since geodesic hyperspheres are Hopf hypersurfaces, this contradicts Theorem 4.9. □

Theorem 4.10 is due to S. Maeda [1983]. Note that it deals only with the case $c > 0$. Also, because of the hypotheses required for Theorem 4.6, the proof only applies when $n \geq 3$. However, there is a direct proof for $n = 2$, both for positive and for negative c. For the proof, we refer to [Niebergall and Ryan 1996].

THEOREM 4.11. *Let M^3 be a real hypersurface in a complex space form $\tilde{M}$ of constant holomorphic sectional curvature $4c \neq 0$. Then $R \cdot A$ cannot be identically zero.* ◁

In the next two theorems, we use the hypothesis of constant principal curvatures.

THEOREM 4.12. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Suppose that the number of principal curvatures is 3 at each point and that these principal curvatures are constant. Then M is an open subset of a hypersurface of Type $A2$ or Type B.* ◁

It is not known whether a similar theorem holds for complex hyperbolic space. In order to get further results along this line for either ambient space, we need to restrict our attention to Hopf hypersurfaces.

THEOREM 4.13. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If the principal curvatures are constants, then M is an open subset of a member of Takagi's list or Montiel's list.* ◁

Theorem 4.12 is due to Takagi for $n \geq 3$ and to Q.-M. Wang for $n = 2$. Theorem 4.13 is due to Kimura for $c > 0$ and to Berndt for $c < 0$. Relevant references are [Takagi 1975a; 1975b; Li 1988; Wang 1983; Kimura 1986a; Berndt 1989a; 1990].

Böning [1995] obtained the following result, which we may regard as a generalization of Theorems 4.5 and 4.6 to the case $g = 3$. Note that he assumes that M is Hopf with $a^2 + 4c \neq 0$. He proves that under the stated hypotheses, the principal curvatures must be constants. Then Theorem 4.13 can be applied. It is not known whether the hypothesis that $a^2 + 4c \neq 0$ is necessary.

THEOREM 4.14. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that $a^2 + 4c \neq 0$ and the number of distinct principal curvatures is equal to 3 at each point. Then M is an open subset of a member of Takagi's list or Montiel's list.* ◁

Further results involving assumptions on the principal curvatures may be found in [Chen 1996; Deshmukh and Al-Gwaiz 1992; Ki and Takagi 1992; Kon 1980; Xu 1992]. Semi-parallelism and semi-symmetry are also discussed in [Kimura and Maeda 1993; Vernon 1991]. For extensions in other directions, including the indefinite case and the case of minimal hypersurfaces, see [Berndt et al. 1995; Bejancu and Duggal 1993; Garay and Romero 1990; Gotoh 1994; Kim and Pyo

1991; Maeda 1984; Martínez and Ros 1984; Miquel 1994; Nagai 1995; Shen 1985; Vernon 1989; Vernon 1987; Udagawa 1987]. The constructions in [Fornari et al. 1993] are also of interest, although their main result is incorrect, as noted in the errata to that paper.

5. The η-Parallel Condition

It is clear that the behavior of the structure vector W is crucial whenever we work with real hypersurfaces in complex space forms. The Hopf condition takes this into account. The next set of conditions we study will also do so. Specifically, we will take some familiar condition that is too strong to be useful for classification, and weaken it by only insisting that it apply on the holomorphic distribution $W^\perp$. We begin with η-parallelism which essentially restricts the $\nabla A = 0$ condition to $W^\perp$.

A $(1,1)$ tensor T on a hypersurface in a complex space form is said to be *η-parallel* if $\langle (\nabla_X T)Y, Z\rangle = 0$ for all X, Y, and $Z \in W^\perp$. Further, T is said to be *cyclic* η-parallel if the cyclic sum of this same expression vanishes. That is,

$$\langle (\nabla_X T)Y, Z\rangle + \langle (\nabla_Y T)Z, X\rangle + \langle (\nabla_Z T)X, Y\rangle = 0$$

for all X, Y, and $Z \in W^\perp$.

Lemma 5.1. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Let X be a (smooth) principal vector field in $W^\perp$ with associated principal curvature λ. Then $W\lambda = 0$. Further, if A is η-parallel, λ must be constant.*

Proof. The Codazzi equation gives

$$(\nabla_X A)W - (\nabla_W A)X = -c\varphi X,$$

which, using the fact that a is constant (Theorem 2.1), can be rewritten as

$$(aI - A)\varphi AX - (W\lambda)X - (\lambda I - A)\nabla_W X = -c\varphi X.$$

Taking the inner product with X yields $W\lambda = 0$.

Suppose now that A is η-parallel. Then if $Y \in W^\perp$, we have

$$0 = \langle (\nabla_Y A)X, X\rangle = (Y\lambda)\langle X, X\rangle + \langle (\lambda I - A)\nabla_Y X, X\rangle = (Y\lambda)\langle X, X\rangle,$$

so $Y\lambda = 0$. This proves that λ is constant. □

Lemma 5.2. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If M has η-parallel shape operator, then its principal curvatures are constant.*

Proof. Suppose M has η-parallel shape operator. Let p be a point where the maximum number of principal curvatures are distinct. The $2n-1$ principal curvature functions, numbered in nonincreasing order, are continuous. The set $\mathcal{U}$

of points where they assume the values taken at p is clearly closed by continuity. On the other hand, p must have a neighborhood where the distinct principal curvatures have constant multiplicities. Lemma 5.1 shows that these principal curvatures are constant in such a neighborhood. This shows that the set $\mathcal{U}$ is also open. We conclude that the principal curvatures are constant on M. (We have implicitly assumed that M is orientable. If not, apply the same argument to the twofold covering to reach the same conclusion.) □

We make the observation that the expression $\langle(\nabla_X A)Y, Z\rangle$ restricted to $W^\perp$ is symmetric in its three arguments. This is immediate from the Codazzi equation and the symmetry of A.

THEOREM 5.3. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then M has η-parallel shape operator if and only if it is an open subset of a Type A or Type B hypersurface from Takagi's list or Montiel's list.*

PROOF. In light of Theorem 4.13, there are two things to prove. First that every Type A or Type B hypersurface has η-parallel shape operator, and second, that hypersurfaces of type C, D, or E do not. The first part is easy. Since M is Hopf, $W^\perp$ is spanned by principal vector fields locally. Suppose M is a Type A or Type B hypersurface. Since the number of distinct principal curvatures on $W^\perp$ is 1 or 2, the arguments in any expression of the form $\langle(\nabla_X A)Y, Z\rangle$ with X, Y, and Z principal vectors in $W^\perp$, can be permuted so that Y and Z belong to the same principal distribution, say that of λ. Then

$$\langle(\nabla_X A)Y, Z\rangle = \langle(\lambda I - A)\nabla_X Y, Z\rangle = \langle\nabla_X Y, (\lambda I - A)Z\rangle = 0,$$

where we have used the fact that λ is constant. We write the second half of the proof as a separate lemma. □

LEMMA 5.4. *Hypersurfaces of types C, D, and E in complex projective space are not η-parallel.*

PROOF. Denote the four distinct principal curvatures (other than a) by λ, μ, ρ, and σ, where T_λ and T_μ are φ-invariant, while φ interchanges T_ρ and T_σ. Assume that A is η-parallel. Choose $X \in T_\lambda$, $Z \in T_\rho$ (nonzero vectors) and compute $\langle R(X, \varphi X)Z, \varphi Z\rangle$ in two ways, once directly and once using the Gauss equation. This will lead to a contradiction. We first observe that since A is η-parallel, ∇_X takes any principal distribution $T_r \subseteq W^\perp$ into the span of T_r and W provided that $X \in W^\perp$. In the curvature calculation,

$$\langle\nabla_{\varphi X} Z, W\rangle = -\langle Z, \nabla_{\varphi X} W\rangle = -\langle Z, \varphi A\varphi X\rangle = \lambda\langle Z, X\rangle = 0.$$

Therefore, $\nabla_{\varphi X} Z \in T_\rho$ and $\nabla_X \nabla_{\varphi X} Z$ contributes nothing to the curvature since it has no T_σ component. Similarly $\nabla_{\varphi X}\nabla_X Z$ makes no contribution. Finally, the

$W^\perp$ component of $[X, \varphi X]$ makes no contribution since differentiation in a $W^\perp$ direction takes Z to T_ρ. However,

$$\begin{aligned}\langle [X,\varphi X], W\rangle &= \langle \nabla_X(\varphi X), W\rangle - \langle \nabla_{\varphi X} X, W\rangle \\ &= \langle \varphi X, \varphi A X\rangle + \langle X, \varphi A \varphi X\rangle = -2\lambda\langle X, X\rangle.\end{aligned}$$

So far, we have

$$\langle R(X,\varphi X)Z, \varphi Z\rangle = 2\lambda\langle \nabla_W Z, \varphi Z\rangle\langle X, X\rangle.$$

We now simplify the right side further, as follows:

$$\langle (\nabla_W A)Z, \varphi Z\rangle = \langle (\rho I - A)\nabla_W Z, \varphi Z\rangle = (\rho - \sigma)\langle \nabla_W Z, \varphi Z\rangle.$$

Also,

$$\langle (\nabla_Z A)W, \varphi Z\rangle = \langle (aI - A)\nabla_Z W, \varphi Z\rangle = (a - \sigma)\langle \varphi A Z, \varphi Z\rangle = (a-\sigma)\rho\langle Z, Z\rangle.$$

Now apply the Codazzi equation to get

$$(\rho - \sigma)\langle \nabla_W Z, \varphi Z\rangle = ((a-\sigma)\rho - c)\langle Z, Z\rangle = \frac{a}{2}(\rho - \sigma)\langle Z, Z\rangle,$$

the last equality using the relationship between ρ and σ implicit in Corollary 2.3. Specifically, it is

$$\rho\sigma = \frac{\rho+\sigma}{2}a + c.$$

Thus we get

$$\langle R(X,\varphi X)Z, \varphi Z\rangle = \lambda a\langle X, X\rangle\langle Z, Z\rangle.$$

On the other hand, using the Gauss equation, we obtain

$$\langle R(X,\varphi X)Z, \varphi Z\rangle = -2c\langle X, X\rangle\langle Z, Z\rangle.$$

Since $\lambda a + 2c \neq 0$, as can be seen from Corollary 2.3, we have the desired contradiction. □

Theorem 5.3 was proved by Kimura and S. Maeda [1989] for complex projective space. They also produced a class of examples which showed that the Hopf hypothesis is necessary (ruled real hypersurfaces). Suh [1990] extended it to complex hyperbolic space. In addition, the following characterization [Ki and Suh 1994] uses a condition that is stronger than η-parallelism. However, it does not assume that the hypersurface is Hopf, and establishing this is the heart of the proof.

THEOREM 5.5. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Assume that*

$$(\nabla_X A)Y = -c\langle \varphi X, Y\rangle W$$

and

$$\langle (A\varphi - \varphi A)X,\ Y\rangle = 0$$

for all X and Y in $W^{\perp}$. Then M is an open subset of a Type A hypersurface from Takagi's list or Montiel's list. ◁

For further results along these lines, see [Hamada 1995; Suh 1995].

6. Conditions on the Ricci Tensor

We recall from Section 1 that the $(1, 1)$ Ricci tensor is denoted by S. A Riemannian manifold for which S is a constant multiple of the identity is called an *Einstein space*. A weaker condition is the *Ricci-parallel* condition which says that $\nabla S = 0$. As we shall see, both are too strong to be satisfied by a real hypersurface. A real hypersurface in a complex space form is said to be *pseudo-Einstein* if there are constants ρ and σ such that

$$SX = \rho X + \sigma\langle X, W\rangle W \tag{6.1}$$

for all tangent vectors X. (The terms *quasi-Einstein* and *η-Einstein* have also been used for this notion.) The following theorem classifies pseudo-Einstein hypersurfaces in $\mathbb{C}P^n$, and in fact proves a stronger result, namely that if a condition of the form (6.1) is satisfied, the coefficients are automatically constants. The proof can be found in [Cecil and Ryan 1982].

THEOREM 6.1. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Suppose that there are smooth functions ρ and σ such that $SX = \rho X + \sigma\langle X, W\rangle W$ all tangent vectors X. Then ρ and σ must be constant and M is an open subset of one of*

(i) *a geodesic sphere* (*as in Theorem* 3.13),
(ii) *a tube of radius ur over a complex projective subspace $\mathbb{C}P^p$, with $1 \leq p \leq n-2$, $0 < u < \pi/2$, and $\cot^2 u = p/q$* (*notation as in Theorem* 3.14 *with $\lambda_1^2 = qc/p$ and $\lambda_2^2 = pc/q$*), *or*
(iii) *a tube of radius ur over a complex quadric Q^{n-1} where $0 < u < \pi/4$ and $\cot^2 2u = n-2$* (*as in Theorem* 3.15). ◁

Theorem 6.1 was proved by M. Kon [1979] under the assumption that ρ and σ are constant. For complex hyperbolic space, the analogous theorem was proved by Montiel [1985].

THEOREM 6.2. *Let M^{2n-1}, where $n \geq 3$, be a real pseudo-Einstein hypersurface in a complex space form of constant holomorphic sectional curvature $4c < 0$. Then M is an open subset of one of*

(i) *a geodesic sphere*,
(ii) *a tube over a complex hyperbolic hyperplane*, *or*
(iii) *a horosphere.* ◁

It is not trivial to prove directly that every pseudo-Einstein hypersurface is Hopf, even though we can observe it from Theorems 6.1 and 6.2, at least for $n \geq 3$.

When $\sigma + 3c \neq 0$, there is a straightforward proof, valid for $n = 2$ as well. However, if $\sigma + 3c = 0$, one essentially has to complete the classification. Since we have not included proofs of Theorems 6.1 and 6.2, it is worthwhile to present a few of the basic equations. Using (6.1) and (1.11), we observe that any pseudo-Einstein hypersurface satisfies

$$(A^2 - \mathfrak{m}A)X = -(\sigma + 3c)\langle X, W\rangle W + ((2n+1)c - \rho)X.$$

Further, if the hypersurface is known to be Hopf, then

$$a^2 - \mathfrak{m}a = -(\sigma + \rho - 2(n-1)c)$$

while on $W^\perp$, any principal curvature λ must satisfy

$$\lambda^2 - \mathfrak{m}\lambda - ((2n+1)c - \rho) = 0.$$

In fact, a Hopf hypersurface will be pseudo-Einstein if and only if any two principal curvatures λ_1 and λ_2 on $W^\perp$ satisfy

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \mathfrak{m}) = 0. \tag{6.2}$$

The reason that hypersurfaces of types A2 and B in $\mathbb{C}H^n$ cannot be pseudo-Einstein is that the necessary condition $\lambda_1 + \lambda_2 = \mathfrak{m}$ cannot hold when all the principal curvatures have the same sign, which is the case for $\mathbb{C}H^n$. In $\mathbb{C}P^n$, on the other hand, the signs of λ_1 and λ_2 differ, so in each family there is a one choice of radius for which the hypersurface will be pseudo-Einstein.

In the rest of this section we study several conditions that represent ways of weakening the Ricci-parallel condition $\nabla S = 0$. Surprisingly, many of these turn out to be equivalent to, or to imply, the pseudo-Einstein condition.

When a hypersurface is pseudo-Einstein, it is easy to check that ∇S satisfies the identity

$$(\nabla_X S)Y = \sigma(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX), \tag{6.3}$$

where the constant σ is as in (6.1). We shall now investigate how a condition of the form of (6.3) restricts a hypersurface. A routine calculation yields the following information on ∇S.

Proposition 6.3. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If there is a function κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX), \tag{6.4}$$

then

$$|\nabla S|^2 = 2\kappa^2(\operatorname{trace} A^2 - |AW|^2). \qquad \triangleleft$$

We now look at the standard examples in light of the condition (6.4).

Theorem 6.4. *Let M^{2n-1}, where where $n \geq 2$, be a member of Takagi's list or Montiel's list satisfying the hypothesis of Proposition* 6.3. *Then M is pseudo-Einstein. Specifically it is a Type $A0$ or $A1$ hypersurface or one of the Type $A2$ or B hypersurfaces occurring in case* (ii) *or* (iii) *of Theorem* 6.1. *The latter two occur only when $c > 0$ and $n \geq 3$.*

Proof. We consider first a Type A hypersurface and derive an expression for ∇S. From Theorem 4.1 and formulas (1.10) and (4.3), we calculate

$$(\nabla_X S)Y = -2c(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX) + c(a - \mathfrak{m})(\langle \varphi X, Y\rangle W + \langle Y, W\rangle \varphi X). \tag{6.5}$$

This implies

$$(\nabla_W S)Y = 0,$$

and hence both sides of (6.4) vanish when $X = W$. Now suppose that $X \in W^\perp$. If the hypersurface is of type A0 or A1, then $\mathfrak{m} - a = (2n-2)\lambda$ where λ is the principal curvature for $W^\perp$ so that (6.4) holds with $\kappa = -2nc$. However, a Type A2 hypersurface will have linearly independent values of X corresponding to distinct principal curvatures. The only way that (6.4) can be satisfied is if $\mathfrak{m} - a = 0$. This occurs for just one choice of radius for each value of p (see Theorem 3.14), namely the one that makes the hypersurface pseudo-Einstein. For a Type B hypersurface, we note that the principal curvatures satisfy $\lambda_1\lambda_2 = -c$ so that neither is zero. If (6.4) holds, then (as we shall see later) so does (6.7). Hence $\mathfrak{m}\lambda_1 - \lambda_1^2 = \mathfrak{m}\lambda_2 - \lambda_2^2$ which is precisely the condition for M to be pseudo-Einstein. It will become clear in the proofs of the next few theorems that hypersurfaces of types C, D, or E cannot satisfy an equation of the form (6.4). □

Using Proposition 6.3 and the information on the standard examples in Section 3, we can compute the following information.

Corollary 6.5. *For the hypersurfaces of Theorem* 6.4, $|\nabla S|^2$ *is equal to*

(i) $16n^2(n-1)|c|^3$ *for Type A0*,

(ii) $16n^2(n-1)|c|^3 \tanh^2 u$ *for Type A1 with* $c < 0$,

(iii) $16n^2(n-1)|c|^3 \cot^2 u$ *for Type A1 with* $c > 0$,

(iv) $16(n-1)|c|^3$ *for Type A2*,

(v) $16n(n-1)(2n-1)^2|c|^3/(n-2)$ *for Type B*. ◁

In particular, the value of κ occurring in any equation of the form (6.4) is nonzero when M is one of the standard examples. Thus:

Corollary 6.6. *For the hypersurfaces in Takagi's and Montiel's lists, the Ricci tensor is never parallel, that is, ∇S never vanishes. In particular, none of these hypersurfaces are Einstein spaces.* ◁

This also proves, for $n \geq 3$, the remark made in the introduction that there are no Einstein hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. This statement is also true for $n = 2$. A proof can be found in [Niebergall and Ryan 1996]. As a consequence of the calculations performed in proving Theorem 6.4, we can make the following further observation.

THEOREM 6.7. *The Type A*0 *and Type A*1 *hypersurfaces in* $\mathbb{C}P^n$ *and* $\mathbb{C}H^n$, *where* $n \geq 2$, *satisfy*

$$(\nabla_X S)Y = -2nc\lambda(\langle \varphi X, Y\rangle W + \langle Y, W\rangle \varphi X)$$

where λ *is the principal curvature for the principal space* $W^\perp$. $\triangleleft$

We now look at the converse of formula (6.3). As a first step we show that (6.4) implies the Hopf condition if $\kappa \neq 0$.

LEMMA 6.8. *Let* M^{2n-1}, *where* $n \geq 2$, *be a real hypersurface in a complex space form of constant holomorphic sectional curvature* $4c \neq 0$. *If there is a nonvanishing function* κ *satisfying* (6.4), *then* M *is a Hopf hypersurface*, *and* $\mathfrak{m}^2 - \operatorname{trace} A^2$ *is constant*.

PROOF. We first differentiate (1.11) to find the general expression for ∇S in terms of ∇A to get

$$(\nabla_X S)Y = -3c(\langle Y, \varphi AX\rangle W + \langle Y, W\rangle \varphi AX) + (X\mathfrak{m})AY + \mathfrak{m}(\nabla_X A)Y - (\nabla_X(A^2))Y. \tag{6.6}$$

Taking the trace of (6.6) and of (6.4) with respect to Y, we get

$$\begin{aligned}\mathfrak{m}(X\mathfrak{m}) + \mathfrak{m}\operatorname{trace}(\nabla_X A) - \operatorname{trace}(\nabla_X A^2) &= 2\mathfrak{m}(X\mathfrak{m}) - X(\operatorname{trace} A^2)\\ &= X(\mathfrak{m}^2 - \operatorname{trace} A^2),\end{aligned}$$

and we conclude that $\mathfrak{m}^2 - \operatorname{trace} A^2$ is constant.

On the other hand, using the fact that $\nabla_X A^2 = (\nabla_X A)A + A\nabla_X A$ in (6.6), substituting for $(\nabla_X A)Y$ and $(\nabla_X A)AY$ from the Codazzi equation, and taking the trace with respect to X gives

$$\begin{aligned}&-3c\langle Y, \varphi AW\rangle + \langle \operatorname{grad}\mathfrak{m}, AY\rangle + \mathfrak{m}(\operatorname{trace}(\nabla_Y A))\\ &\qquad - \operatorname{trace}(\nabla_{AY} A) - (\operatorname{trace} A\nabla_Y A + c\langle A\varphi Y, W\rangle + 2c\langle AW, \varphi Y\rangle)\\ &\qquad\qquad = \kappa\langle \varphi AW, Y\rangle.\end{aligned}$$

This equation simplifies to

$$\kappa\langle \varphi AW, Y\rangle = \tfrac{1}{2}\langle \operatorname{grad}(\mathfrak{m}^2 - \operatorname{trace} A^2), Y\rangle.$$

Since the right side vanishes, we must have $\varphi AW = 0$ so that M is a Hopf hypersurface. $\square$

LEMMA 6.9. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that there is a function κ satisfying* (6.4). *Then either a is zero or $\mathfrak{m}$ is constant. Furthermore,*

$$(\kappa + 3c)\varphi AX = ((\mathfrak{m}a - a^2)I - \mathfrak{m}A + A^2)\varphi AX \tag{6.7}$$

for all $X \in W^\perp$.

PROOF. Combine (6.4) and (6.6) and take the inner product with W. The resulting equation gives $X(a\mathfrak{m}) = 0$ and (6.7). □

LEMMA 6.10. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that there is a function κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi AX, Y \rangle W + \langle Y, W \rangle \varphi AX).$$

If $a/2$ occurs as a principal curvature on $W^\perp$ at every point of M, then $c < 0$ and M is an open subset of a horosphere.

PROOF. To make the argument more general, we first assume only that there is a point where $a/2$ occurs as a principal curvature on $W^\perp$ and work at this point. Then $a^2 + 4c = 0$ by Corollary 2.3 and hence a is not zero. If, in addition, 0 is a principal curvature, $X \in T_0$ implies that $A\varphi X = \frac{1}{2}a\varphi X$, again by Corollary 2.3. Substituting φX for X in (6.7), we get

$$\kappa + 3c = \mathfrak{m}a - a^2.$$

This reduces (6.7) to

$$(\mathfrak{m}A - A^2)\varphi AX = 0, \tag{6.8}$$

which holds for all $X \in W^\perp$.

Now let $\mathcal{V} = T_0 \oplus \varphi T_0$ and note that $A\mathcal{V} \subseteq \mathcal{V}$. Then $\tilde{\mathcal{V}} = (\mathcal{V} \oplus \operatorname{span}\{W\})^\perp$ is A-invariant and φ-invariant. From (6.8) we see that the only possible principal curvature on $\tilde{\mathcal{V}}$ has the value $\mathfrak{m} = a/2$. Whether $\tilde{\mathcal{V}}$ is the zero subspace or not, we are led to the absurd conclusion that $\mathfrak{m} = \operatorname{trace} A = a + (k-1)\mathfrak{m} = (k+1)\mathfrak{m}$ where k is the rank of A. The result is that if $\frac{a}{2}$ occurs as a principal curvature (with principal vector in $W^\perp$) at some point, then 0 is not a principal curvature at that point.

We now continue to work at a point where $a/2$ is a principal curvature on $W^\perp$. Since 0 is not a principal curvature,

$$(\kappa + 3c)I = (\mathfrak{m}a - a^2)I - \mathfrak{m}A + A^2.$$

holds on all of $W^\perp$. The fact that $a/2$ is a principal curvature allows us to compute that $\kappa = \mathfrak{m}a/2$. Further, the equation two lines above shows that any principal curvature not equal to $a/2$ must be equal to $\mathfrak{m} - a/2$.

Now assume the full hypothesis of the lemma. Recall that $a \neq 0$ and so $\mathfrak{m}$ is constant by Lemma 6.9. If three principal curvatures are distinct at any point, then there is a neighborhood where they are all distinct, hence constant by the arguments of the preceding paragraph. Since none of the standard examples with three distinct principal curvature satisfy $a^2 + 4c = 0$, we have a contradiction in view of Theorem 4.13. We therefore must have exactly two principal curvatures at each point, a and $a/2$. Again by Theorem 4.13, M must be an open subset of a horosphere. □

Lemma 6.11. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that there is a constant κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX).$$

If $a/2$ does not occur as a principal curvature on $W^\perp$, M is an open subset of a Type A or Type B hypersurface provided $a \neq 0$. If $n \geq 3$, the same conclusion holds without the assumption that $a \neq 0$.

Proof. Choose a point p where the maximum number of principal curvatures are distinct and work in a neighborhood of that point. Suppose first that 0 occurs as a principal curvature at p so that $AX = 0$ for a nonzero $X \in W^\perp$. Again invoking Corollary 2.3, we have that $a \neq 0$ and $-2c/a$ is also a principal curvature with principal vector φX. Using the fact that $\frac{1}{2}$ is not a principal curvature, we can verify that 0 and $-2c/a$ have equal multiplicities. Formula (6.8) still holds and arguing as in Lemma 6.10, we find that $\kappa + 3c - \mathfrak{m}a + a^2 = 0$ and hence that any further principal curvature λ must satisfy $\lambda = \mathfrak{m}$. In addition, T_λ must be φ-invariant and so $\mathfrak{m}^2 = a\mathfrak{m} + c$ by Corollary 2.3. If all four principal curvatures we have identified are distinct, then there are also four distinct principal curvatures at nearby points. The zero principal curvature must remain zero since (6.8) does not allow for more than two distinct principal curvatures on $W^\perp$. As a consequence, M has four distinct constant principal curvatures in a neighborhood of p. This is impossible by Theorem 4.13. There is still the possibility that $a = \mathfrak{m}$ or that $a = -2c/a$ so that only three principal curvatures are distinct. In the first case, $\mathfrak{m} = a$ must be a principal curvature of the same multiplicity nearby. If the zero principal curvature were to become nonzero nearby, then (6.8) would imply that this nonzero value must be equal to $\mathfrak{m}$, which will not be true for sufficiently nearby points. If $a = -2c/a$, the same argument shows that the zero principal curvature remains zero nearby. In either case, M has three distinct principal curvatures, all constant, in a neighborhood of p. Observing that none of the standard examples has this particular configuration of principal curvatures, we conclude that 0 cannot occur as a principal curvature on $W^\perp$ at p.

Now that we know that 0 is excluded, there are two possibilities arising from (6.7). The first is that $a \neq 0$ in which case $\mathfrak{m}$ is constant and any principal curvature on $W^\perp$ satisfies a quadratic equation with constant coefficients in a neighborhood of p. Thus, M is locally a Type A or Type B hypersurface. If $a = 0$, then $\mathfrak{m}$ is not automatically constant. However, the quadratic equation is still satisfied. If only two principal curvatures are distinct at p, then the same holds in a neighborhood and locally M is one of the standard examples as in Theorems 4.6 and 4.7 provided that $n \geq 3$. On the other hand, if three principal curvatures are distinct, say 0, λ, and μ, then λ and μ satisfy the quadratic equation

$$\kappa + 3c + \mathfrak{m}t - t^2 = 0.$$

Noting that $a = 0$, we observe that $\lambda\mu = c$ by Corollary 2.3. Thus $\kappa = -4c$. If the two eigenspaces are φ-invariant, then each principal curvature is constant, again by Corollary 2.3. If not, then $\lambda + \mu = \mathfrak{m}$ and we get $(n-2)\mathfrak{m} = 0$. Thus $\mathfrak{m} = 0$ and the principal curvatures are constants provided that $n \geq 3$. □

We can now state the result that we have been working towards. It determines the hypersurfaces that can satisfy a condition of the form of (6.4).

THEOREM 6.12. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If there is a nonzero constant κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX),$$

then M is an open subset of a pseudo-Einstein hypersurface as listed in Theorems 6.1 and 6.2.

PROOF. From Lemma 6.8, M is a Hopf hypersurface. Suppose there is a point p where $a/2$ is not a principal curvature on $W^\perp$. Then Lemma 6.11 shows that the set where all principal curvatures have the same values as they have at p is open. Since this set is also closed, M is an open subset of a Type A or Type B hypersurface. However, as we have seen in Theorem 6.4, a Type A or Type B hypersurface satisfying the given condition must be pseudo-Einstein. On the other hand, if no such p exists, the desired conclusion follows from Lemma 6.10. □

Theorem 6.12 was proved by Kimura [1986b] for the case of $\mathbb{C}P^n$. Although the assumption that $n \geq 3$ is implicit in his proof (which relies on [Cecil and Ryan 1982] to handle the possibility that $a = 0$), it is omitted from the stated hypothesis.

The above argument also holds for $n = 2$, provided that we assume $a \neq 0$. We state the result separately as follows.

THEOREM 6.13. *Let M^3 be a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$. Suppose that $a = \langle AW, W\rangle \neq 0$. If there is a nonzero constant κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi AX, Y\rangle W + \langle Y, W\rangle \varphi AX),$$

M is an open subset of a hypersurface of type A0 or type A1. ◁

The Type A0 and A1 hypersurfaces are also characterized by a refinement of this condition as follows [Kimura and Maeda 1992; Taniguchi 1994; Choe 1995].

THEOREM 6.14. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then there is a nonzero constant κ such that*

$$(\nabla_X S)Y = \kappa(\langle \varphi X, Y\rangle W + \langle Y, W\rangle \varphi X)$$

if and only if M is an open subset of a hypersurface of type A0 or type A1. ◁

If $n \geq 3$, one need not specify that κ is constant in Theorem 6.14. If κ is assumed to be a function, it turns out to be constant. However, $\kappa \neq 0$ is essential, although this hypothesis is missing from [Choe 1995, Theorem 3.2].

We remarked in Corollary 6.6 that none of the standard examples have parallel Ricci tensor. It follows from Lemmas 6.9 through 6.11 that Hopf hypersurfaces cannot have parallel Ricci tensor, at least when $n \geq 3$. In fact, the following stronger result is known [Ki 1989]. We will not give the proof here, since it will be a consequence of Theorem 6.29.

THEOREM 6.15. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the Ricci tensor of M cannot be parallel everywhere.* ◁

Ki and Suh [1992] proved yet another characterization of the Type A hypersurfaces. Recalling (6.5), which is an expression for ∇S, the following could be regarded a kind of converse.

THEOREM 6.16. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Suppose that the mean curvature $\mathfrak{m} = \operatorname{trace} A$ and $a = \langle AW, W\rangle$ are constants. Then*

$$\begin{aligned}(\nabla_X S)Y = -c\mathfrak{m}(\langle Y, \varphi X\rangle W + \langle Y, W\rangle \varphi X)\\ + c(\langle Y, \varphi X\rangle AW + \langle AW, Y\rangle \varphi X) - 2c(\langle Y, W\rangle \varphi AX + \langle \varphi AX, Y\rangle W)\end{aligned}$$

if and only if M is a Type A hypersurface from Takagi's list. ◁

Kimura and Maeda [1991] investigated the consequences of assuming only that S is parallel in the direction of the structure vector W. Although we will go into some detail concerning a similar condition on the shape operator in the next section, we will merely state their theorem concerning for $\nabla_W S$. See [Maeda 1993] for a related result.

THEOREM 6.17. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Assume that the mean curvature is constant. If $\nabla_W S = 0$, then M is an open subset of a hypersurface from Takagi's list. There is a restriction on the radii that can occur.* ◁

For further results along these lines, see [Cho et al. 1991; Kim 1988b; Maeda 1994].

We now look at a condition analogous to the one characterizing the Type A hypersurfaces in Theorem 4.1. However, we ask only that φ commute with S, not with A. This condition is significantly weaker as it allows at least some of each type of homogeneous hypersurface as well as as certain nonhomogeneous ones. Relevant references are [Aiyama et al. 1990; Ki and Suh 1990; Kimura 1987b]. The original statements have been modified to take certain corrections into account.

THEOREM 6.18. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. If $\varphi S = S\varphi$, then M is an open subset of a hypersurface from Takagi's list or a certain nonhomogeneous hypersurface. Although all types A–E occur, there is a restriction on the radii of the tubes.* ◁

THEOREM 6.19. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c < 0$. If $\varphi S = S\varphi$, then M is an open subset of a hypersurface of type A from Montiel's list.* ◁

A Riemannian manifold is said to have *harmonic curvature* if its Ricci tensor S is a Codazzi tensor, i.e., $(\nabla_X S)Y = (\nabla_Y S)X$. Concerning this condition, we can state the following [Ki 1989; Kwon and Nakagawa 1989a; Kim 1988a].

THEOREM 6.20. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then M cannot have harmonic curvature, that is, $(\nabla_X S)Y - (\nabla_Y S)X$ cannot vanish identically.* ◁

Other versions of the harmonicity condition are pursued in [Ki and Nakagawa 1991; Ki et al. 1989; Ki et al. 1990b].

We now proceed to look at further conditions on the ∇S. First of all, it is possible for S to be cyclic-parallel, but the condition is still a rather strong one [Kwon and Nakagawa 1988].

THEOREM 6.21. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. If the Ricci tensor S is cyclic parallel, then M is an open subset of a hypersurface from Takagi's list. There is a restriction on the radii that can occur.* ◁

As long as we assume the Hopf condition, η-parallelism of S turns out to be strong enough to characterize Type A and Type B hypersurfaces. Suh [1990] proved the following theorem.

THEOREM 6.22. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then M has η-parallel Ricci tensor if and only if it is an open subset of a Type A or Type B hypersurface from Takagi's list or Montiel's list.* ◁

For $n \geq 3$, the same conclusion was obtained by J.-H. Kwon and H. Nakagawa [1989b] under a weaker assumption. We will present a full account of their result, organized as a sequence of lemmas preceded by a statement of the theorem and the core of the proof.

THEOREM 6.23. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the Ricci tensor of M is cyclic η-parallel if and only if M is an open subset of a Type A or Type B hypersurface from Takagi's list or Montiel's list.*

PROOF. Suppose that S is cyclic η-parallel. Let p be a point where the maximum number of principal curvatures are distinct. Let $\mathcal{V}$ be a neighborhood of p where we can find an orthonormal basis of principal vectors. Then Lemma 6.27 (to be proved below) shows that $\mathfrak{m}$ is constant. For any two distinct principal curvatures λ and μ, we have from Lemma 6.26 below that

$$(2\lambda - \mathfrak{m})X\lambda = (2\lambda + 4\mu - 3\mathfrak{m})X\mu = 0,$$

where X is the principal vector corresponding to λ. Let $\mathcal{U} = \{x \in \mathcal{V} : 2\lambda \neq \mathfrak{m}\}$. Then $X\lambda = 0$ on $\bar{\mathcal{U}}$. For any point in the complement of $\bar{\mathcal{U}}$, if $X\lambda \neq 0$, then $2\lambda = m$ in a neighborhood of this point, which contradicts the fact that $X\mathfrak{m} = 0$. We conclude that $X\lambda = 0$ on all of $\mathcal{V}$. Similarly, we see that $X\mu = 0$ on the closure of the set where $2\lambda + 4\mu - 3\mathfrak{m} \neq 0$. On the complement, however, we have $4\mu = 3\mathfrak{m} - 2\lambda$. By the same argument as before, we see that $X\mu = 0$ there as well. We have shown that every principal curvature function is constant along every direction in $W^{\perp}$. By Lemma 5.1, it is also constant in the W direction. We conclude that $\mathcal{V}$ has constant principal curvatures. By Theorem 4.13, it is an open subset of a member of Takagi's list or Montiel's list.

It remains to check which of the standard examples actually have cyclic η-parallel Ricci tensor. This is covered in the next theorem, Theorem 6.24. To complete the proof, observe that the set of points where the principal curvatures match their values at p is open since any such point will be a point where the maximum number of principal curvatures are distinct. Also, it will have a neighborhood where the principal curvatures are constant. On the other hand, such a set is closed by continuity. □

THEOREM 6.24. *Let M^{2n-1}, where $n \geq 2$, be a member of Takagi's list or Montiel's list. Then the Ricci tensor is cyclic η-parallel if and only if the shape operator is η-parallel; that is if M is of type A or type B.*

PROOF. Lemma 6.25 below shows that in the case of the standard examples, the cyclic sum of the expression $\langle(\nabla_X S)Y, Z\rangle$ over principal vectors X, Y, and Z in $W^{\perp}$ is equal to

$$(2(\lambda + \mu + \sigma) - 3\mathfrak{m})\langle(\nabla_X A)Y, Z\rangle,$$

where λ, μ, and σ are the respective principal curvatures. Thus a hypersurface with η-parallel shape operator will have an η-parallel Ricci tensor. In view of Theorem 5.3, it remains to show that hypersurfaces of types C, D, and E in $\mathbb{C}P^n$ do not have η-parallel Ricci tensor.

Let M be a hypersurface of type C, D, or E. Assume that the Ricci tensor is cyclic η-parallel. Pick any three of the four principal curvatures whose principal spaces lie in $W^\perp$. Call them λ, μ, and σ. See Theorems 3.16 to 3.18 for the values. For a particular type (say type C), note that the expression $2(\lambda+\mu+\sigma) - 3\mathfrak{m}$ varies continuously with u so that among all the hypersurfaces in the one-parameter family, only a finite number will give a value of zero for this expression. Thus, except for a finite number of values of u, the corresponding expression involving ∇A will vanish. This holds for any combination of three distinct principal curvatures and corresponding principal vectors. By continuity, it holds for all values of u as well. On the other hand, the argument given in the proof of Theorem 5.3 takes care of the case of vectors which do not belong to distinct principal spaces. The net result is that $\langle(\nabla_X A)Y, Z\rangle = 0$ on $W^\perp$. Since this contradicts Theorem 5.3, we must conclude that no hypersurface of type C, D, or E can have cyclic η-parallel Ricci tensor. $\square$

LEMMA 6.25. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Then the cyclic sum of the expression $\langle(\nabla_X S)Y, Z\rangle$ over principal vectors X, Y, and Z in $W^\perp$ is equal to*

$$(2(\lambda+\mu+\sigma)-3\mathfrak{m})(\langle(\nabla_X A)Y, Z\rangle-(\lambda(Z\mathfrak{m})\langle X, Y\rangle+\mu(X\mathfrak{m})\langle Y, Z\rangle+\sigma(Y\mathfrak{m})\langle Z, X\rangle)),$$

where λ, μ, and σ are the respective principal curvatures. $\triangleleft$

LEMMA 6.26. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that the Ricci tensor S is cyclic η-parallel. Suppose that X and Y are smooth orthonormal principal vector fields in $W^\perp$ with corresponding principal curvatures λ and μ. Then*

$$(2\lambda - \mathfrak{m})X\lambda = \lambda X\mathfrak{m},$$
$$(2\lambda + 4\mu - 3\mathfrak{m})X\mu = \mu X\mathfrak{m}. \quad \triangleleft$$

LEMMA 6.27. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. Suppose that the Ricci tensor S is cyclic η-parallel. Suppose further that the principal curvatures have constant multiplicities. Then the mean curvature $\mathfrak{m}$ is constant.*

PROOF. First note that $W\mathfrak{m} = 0$ by Lemma 2.13. Because the principal curvatures have constant multiplicities, we can always find a local orthonormal principal frame when desired. We begin with the case $a^2 + 4c = 0$. Assume that there is a principal curvature $\lambda \neq a/2$ with corresponding unit principal vector $X \in W^\perp$. Set $Y = \varphi X$ so that in the notation of Theorem 6.26, $\mu = a/2$. Then

$X\mathfrak{m} = 0$ by Theorem 6.26. Also $Y\mathfrak{m} = 0$ for any $Y \in T_\mu$, by the first part of Theorem 6.26. We conclude that $\mathfrak{m}$ is constant.

Now consider the case $a^2 + 4c \neq 0$. Then $a/2$ does not occur as a principal curvature. Suppose that there is a unit principal vector field X such that $X\mathfrak{m} \neq 0$. Take $Y = \varphi X$ and use the setup of Lemma 6.26. A straightforward but tedious calculation using Corollary 2.3 and Lemma 6.26 yields

$$f(\lambda)X\mathfrak{m} = 0, \tag{6.9}$$

where f is the polynomial $a_0t^4 + a_1t^3 + a_2t^2 + a_3t + a_4$ whose coefficients are

$$\begin{aligned} a_0 &= 8a, \\ a_1 &= -4(a^2 + \mathfrak{m}a - 8c), \\ a_2 &= 2(2a^3 - \mathfrak{m}a^2 - 4ca - 16\mathfrak{m}c), \\ a_3 &= 2(\mathfrak{m}a^3 + 6ca^2 + 10\mathfrak{m}ca + 16c^2), \\ a_4 &= -2\mathfrak{m}ca^2. \end{aligned}$$

If $a = 0$, then $\lambda\mu = c$ by Corollary 2.3 and equation (6.9) reduces to

$$32c\lambda(\lambda^2 - \mathfrak{m}\lambda + c) = 0.$$

Direct substitution shows that this equation is also satisfied by μ. Thus $\lambda + \mu = \mathfrak{m}$. Suppose now that ν is a nonzero principal curvature distinct from λ and μ. Applying Lemma 6.26 again, we get

$$\begin{aligned} (2\lambda + 4\nu - 3\mathfrak{m})X\nu &= \nu X\mathfrak{m}, \\ (2\lambda + 4\sigma - 3\mathfrak{m})X\sigma &= \sigma X\mathfrak{m}, \end{aligned}$$

where σ is the principal curvature for $\varphi(T_\nu)$. Noting that $\nu\sigma = c$, we have

$$\begin{aligned} ((2\lambda - 3\mathfrak{m})\sigma + 4c)X\nu &= cX\mathfrak{m}, \\ ((2\lambda - 3\mathfrak{m})\nu + 4c)X\sigma &= cX\mathfrak{m}; \end{aligned}$$

these two equations can be added to yield $X(\nu + \sigma) = \frac{1}{2}X\mathfrak{m}$. Recalling that $a = 0$, we know that $\mathfrak{m}$ is the sum of n terms, each of which is either $\lambda + \mu$ or a term of the form $\nu + \sigma$. Thus

$$X\mathfrak{m} = kX\mathfrak{m} + \frac{n-k}{2}X\mathfrak{m},$$

where k is the multiplicity of λ. This implies that $n = k = 1$, a contradiction. We conclude that $a \neq 0$.

Now differentiate $f(\lambda) = 0$ with respect to X and use the first equality in the conclusion of Lemma 6.26 to get

$$g(\lambda)X\mathfrak{m} = 0,$$

where $g = b_0 t^4 + b_1 t^3 + b_2 t^2 + b_3 t + b_4$ with

$$\begin{aligned} b_0 &= 24a, \\ b_1 &= -8(2a^2 + \mathfrak{m}a - 4c), \\ b_2 &= 2(6a^3 - \mathfrak{m}a^2 + 12ca - 16\mathfrak{m}c), \\ b_3 &= 8c(a^2 + 4c), \\ b_4 &= 2\mathfrak{m}ca^2. \end{aligned}$$

Now $f(\lambda) = 0$ and $g(\lambda) = 0$ may be regarded as equations in $\mathfrak{m}$ of degree 1 from which $\mathfrak{m}$ can be eliminated to yield a degree 7 polynomial equation in λ with constant coefficients. Thus λ is constant. In particular, $X\lambda = 0$ and $X\mu = 0$ so that $X\mathfrak{m} = 0$ by Lemma 6.26. Again, this is a contradiction. We must conclude that $\mathfrak{m}$ is constant. □

Lemma 6.27 is the final ingredient required for the proof of Theorem 6.23. In view of the argument in the proof of Theorem 6.23, we now see that Lemma 6.27 is true even without the hypothesis of constant multiplicities.

A Riemannian manifold is said to be a *Ryan space* if $R \cdot S = 0$. As far as we can determine, this term (and variations of it) were first used by Ki, Nakagawa, and Suh [Ki et al. 1990b]. The spaces themselves were introduced in [Ryan 1971; 1972] and independently by R. L. Bishop and S. I. Goldberg [1972], and have been studied by many authors in the intervening years. Subsequently, the same spaces have been called *Ricci-semisymmetric*. See [Deszcz 1992], for example, where the term is part of a comprehensive naming scheme for a number of conditions, all related to the notion of *pseudosymmetry*. We will adopt the terminology used in the literature being surveyed.

A Riemannian manifold is a *cyclic-Ryan space* if the cyclic sum over tangent vectors X, Y, and Z of $(R(X,Y) \cdot S)Z$ vanishes. A real hypersurface in a complex space form is said to be *pseudo-Ryan* if $\langle (R(X_1, X_2) \cdot S)X_3, X_4 \rangle = 0$ provided all X_i lie in $W^\perp$. The Ryan condition is too strong to be satisfied by a real hypersurface, as we shall see in Theorem 6.29. We discuss the weaker cyclic-Ryan condition first. The following result was proved in [Ki et al. 1990b].

THEOREM 6.28. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If M satisfies the cyclic-Ryan condition, then M is a Hopf hypersurface.*

PROOF. Our initial discussion will be valid for any hypersurface in a complex space form. We begin by applying the Gauss equation and (1.11) to get

$$R(X,Y)(SZ) = \big(AX \wedge AY + c(X \wedge Y + \varphi X \wedge \varphi Y + 2\langle X, \varphi Y\rangle \varphi)\big) \times \big((2n+1)cZ - 3c\langle Z, W\rangle W - PZ\big),$$

where $P = A^2 - (\operatorname{trace} A)A = A^2 - \mathfrak{m}A$. The right side is the sum of the following terms:

$$
\begin{aligned}
&(1) \qquad (2n+1)c^2(X \wedge Y)Z,\\
&(2) \qquad -3c^2(\langle Z,W\rangle\langle Y,W\rangle X - \langle Z,W\rangle\langle X,W\rangle Y),\\
&(3) \qquad -c(X \wedge Y)PZ - c(\varphi X \wedge \varphi Y)PZ,\\
&(4) \qquad (2n+1)c^2(\varphi X \wedge \varphi Y)Z,\\
&(5) \qquad 2(2n+1)c^2\langle X,\varphi Y\rangle\varphi Z,\\
&(6) \qquad -2c\langle X,\varphi Y\rangle\varphi PZ,\\
&(7) \qquad (2n+1)c(AX \wedge AY)Z,\\
&(8) \qquad -3c\langle Z,W\rangle(AX \wedge AY)W,\\
&(9) \qquad (AX \wedge AY)PZ.
\end{aligned}
$$

Because of the first Bianchi identity, the cyclic sum of $(R(X,Y)\cdot S)Z$ is equal to the cyclic sum of $R(X,Y)(SZ)$. We look at the cyclic sums of each of the terms (1)–(9) above and conclude by straightforward calculation the following facts. Terms (1) and (2) and the first part of (3) all sum to 0, while cyclic sum of the remainder of (3) is equal to the cyclic sum of

$$-c\langle (P\varphi + \varphi P)Y, Z\rangle \varphi X.$$

The cyclic sum of (4) and (5) taken together is zero, as is the cyclic sum of (7). The cyclic sum of (8) is equal to that of

$$-3c(\langle Z,W\rangle\langle AY,W\rangle - \langle Y,W\rangle\langle AZ,W\rangle)AX.$$

The cyclic sum of (9) is zero. Here we need to use the fact the P commutes with A. The results of these observations is that the cyclic sum of $(R(X,Y)\cdot S)Z$ is equal to the cyclic sum of

$$\langle (P\varphi+\varphi P)Y, Z\rangle\varphi X + 2\langle X,\varphi Y\rangle\varphi PZ + 3(\langle Z,W\rangle\langle AY,W\rangle - \langle Y,W\rangle\langle AZ,W\rangle)AX, \tag{6.10}$$

multiplied by $-c$. Now suppose that the cyclic-Ryan condition holds. Take $X = W$ and compute the cyclic sum of (6.10) to obtain

$$
\begin{aligned}
0 = \langle \varphi PW, Y\rangle\varphi Z + \langle P\varphi Z, W\rangle\varphi Y + 2\langle Y,\varphi Z\rangle\varphi PW&\\
+ 3(\langle Z,W\rangle\langle AY,W\rangle - \langle Y,W\rangle\langle AZ,W\rangle)AW&\\
+ 3(\langle W,W\rangle\langle AZ,W\rangle - \langle Z,W\rangle\langle AW,W\rangle)AY&\\
+ 3(\langle Y,W\rangle\langle AW,W\rangle - \langle W,W\rangle\langle AY,W\rangle)AZ.&
\end{aligned} \tag{6.11}
$$

Apply φ to the right side of (6.11) and take the trace (as a linear map in Z) to get

$$\begin{aligned}0 = \langle \varphi PW, Y\rangle(-2n+3) - 2\langle \varphi PW, Y\rangle \\ + 3(\langle \varphi AW, W\rangle\langle AY, W\rangle - \langle Y, W\rangle\langle A\varphi AW, W\rangle) \\ + 3(\langle A\varphi AY, W\rangle - \langle \varphi AY, W\rangle\langle AW, W\rangle) \\ + 3(\langle Y, W\rangle\langle AW, W\rangle - \langle AY, W\rangle).\end{aligned} \tag{6.12}$$

Note that $\langle A\varphi AW, W\rangle = 0$ and $\mathrm{trace}(\varphi A) = 0$, so the equation reduces to

$$(2n-1)\varphi PW + 3A\varphi AW = 0. \tag{6.13}$$

Let $U = \varphi AW$ and rewrite (6.12) as

$$\begin{aligned}\langle AU, Y\rangle\varphi Z - \langle AU, Z\rangle\varphi Y + 2\langle Y, \varphi Z\rangle AU \\ = (2n-1)\big(\langle Z, W\rangle\langle AY, W\rangle - \langle Y, W\rangle\langle AZ, W\rangle\big)AW \\ + (2n-1)(\langle \varphi Z, U\rangle AY - \langle \varphi Y, U\rangle AZ),\end{aligned} \tag{6.14}$$

where we have used the fact that $(A - aI)W = -\varphi U$. Upon taking the inner product with U and using the fact that $\langle AW, U\rangle = 0$, we get

$$(n-1)(\langle \varphi Z, U\rangle\langle AY, U\rangle - \langle \varphi Y, U\rangle\langle AZ, U\rangle) = \langle Y, \varphi Z\rangle\langle AU, U\rangle. \tag{6.15}$$

Taking $Y = U$ gives

$$(n-2)\langle AU, U\rangle\varphi U = 0$$

and hence $\langle AU, U\rangle = 0$ provided that $n \geq 3$. If we put $Z = U$ in (6.14), we get

$$(2n-3)\langle \varphi Y, U\rangle AU - \langle AY, U\rangle\varphi U = 0.$$

Combining this with (6.15) yields

$$2(n-2)\langle \varphi Y, U\rangle AU = 0,$$

so that AU must be zero. (Again, we have used the hypothesis that $n \geq 3$.) Then $\varphi PW = 0$ by (6.13) and we can simplify (6.11). Put $Z = \varphi U$ in (6.11) to get

$$-\langle Y, W\rangle\langle A\varphi U, W\rangle AW + \langle A\varphi U, W\rangle AY + \langle (aI - A)W, Y\rangle A\varphi U = 0.$$

In other words,

$$\langle U, U\rangle(\langle Y, W\rangle AW - AY) = \langle \varphi U, Y\rangle A\varphi U. \tag{6.16}$$

We intend to show that $AW = aW$, that is, that $U = 0$. Suppose then, that $U \neq 0$ at some point of M. For the rest of this proof, we work in a neighborhood where U is nonzero. Let $\mathcal{V}$ be the two-dimensional subspace spanned by $\{W, \varphi U\}$ and let $\bar{U}$ be a unit vector in the direction of φU. We see immediately from (6.16) that $AY = 0$ for Y orthogonal to $\mathcal{V}$. Thus $\mathcal{V}$ is A-invariant, and if we write $AW = aW + b\bar{U}$ and $A\bar{U} = bW + d\bar{U}$, a short calculation reveals that

$PX = (b^2 - ad)X = pX$ (say) for $X \in \mathcal{V}$. In the cyclic sum of (6.10), substitute $X = \varphi U$ and $Y = U$, and take Z orthogonal to X, Y, and W to get

$$p\,|U|^2\varphi Z = 0.$$

So $p = 0$, and we conclude that $P = 0$. Now, let X and Y be principal vectors in $\mathcal{V}$ with corresponding principal curvatures λ and μ. Since $P = 0$, we have $\lambda^2 - \mathfrak{m}\lambda = \lambda^2 - (\lambda+\mu)\lambda = 0$ so that $\lambda\mu = 0$. Take μ to be the zero principal curvature. To see that this leads to a contradiction, take $Z \in \mathcal{V}^\perp$ and use Codazzi's equation to get

$$-A[Y,Z] = c\langle Y, W\rangle\varphi Z + 2\langle Y, \varphi Z\rangle W,$$

which reduces to

$$c\langle Y, W\rangle\,|Z|^2 = 0$$

upon taking the inner product with φZ. We have to use the fact that $A(\varphi Z) = 0$. We conclude that $\langle Y, W\rangle = 0$. This makes $\bar{U}$ and hence W principal, which is a contradiction. Thus M is a Hopf hypersurface. ☐

We can use Theorem 6.28 to get a direct proof that Ryan spaces cannot occur as hypersurfaces when $n \geq 3$.

THEOREM 6.29. *In a complex space form of constant holomorphic sectional curvature* $4c \neq 0$, *there exists no real hypersurface* M^{2n-1}, $n \geq 3$, *satisfying* $R \cdot S = 0$. *For* $n = 2$, *there are no Hopf hypersurfaces satisfying* $R \cdot S = 0$.

PROOF. Suppose that M^{2n-1}, where $n \geq 2$, satisfies $R \cdot S = 0$. The cyclic-Ryan condition is satisfied so M is Hopf by Theorem 6.28 if $n \geq 3$. Otherwise, it is Hopf by hypothesis. Let X be any principal vector orthogonal to W with associated principal curvature λ. Evaluating $(R(X,W) \cdot S)W$ using the Gauss equation and (1.11), we see that $R \cdot S = 0$ if and only if

$$(\lambda a + c)(3c + (\lambda - a)(\mathfrak{m} - \lambda - a)) = 0 \tag{6.17}$$

for all principal curvatures λ whose principal spaces are in $W^\perp$. If λ and μ are principal curvatures corresponding to X and φX respectively, we also must have

$$\lambda\mu = \frac{\lambda+\mu}{2}a + c, \tag{6.18}$$

by Corollary 2.3. Of the various ways in which these two conditions might be satisfied, $\lambda a + c = \mu a + c = 0$ is impossible since it implies $a \neq 0$, $\lambda \neq 0$, $\mu \neq 0$, $\lambda = \mu$, and $\lambda^2 = \lambda a + c$. On the other hand, if $\lambda a + c$ and $\mu a + c$ are both nonzero, then both λ and μ satisfy the quadratic equation

$$t^2 - \mathfrak{m}t + \mathfrak{m}a - a^2 - 3c = 0. \tag{6.19}$$

If λ and μ are distinct, any other principal curvature ρ (on $W^\perp$) must satisfy $\rho a + c = 0$, since it cannot satisfy (6.19). However, given the φ-invariance of

$W^\perp$, and the first remark, there can be no such ρ. Thus λ, μ, and a are the only principal curvatures.

Suppose that there is a point where two distinct principal curvatures exist as in (6.18). First check that neither λ nor μ is equal to $a/2$. With this possibility eliminated, there is a neighborhood in which $\mathfrak{m} = (n-1)(\lambda+\mu)+a$, $\mathfrak{m} = \lambda+\mu$, and

$$\lambda\mu = \mathfrak{m}a - a^2 - 3c = \frac{\mathfrak{m}a}{2} + c.$$

We now calculate that

$$(2n-3)a^2 + 8(n-2)c = 0.$$

If $c > 0$, this is already a contradiction. If $c < 0$, we continue, obtaining

$$\lambda\mu = \frac{2n+1}{2n-3}c.$$

Since the relevant examples of Hopf hypersurfaces with constant principal curvatures have $\lambda\mu > 0$, the situation in this paragraph cannot occur. (The coefficients of (6.19) are constant since $(n-2)\mathfrak{m} = -a$ and $a \neq 0$ by (6.18).)

A third possibility is that for all points and every principal $\lambda a + c = 0$ but $\mu a + c \neq 0$. Then μ satisfies (6.19) as does any further principal curvature ν. In addition, T_ν must be φ-invariant and $\nu^2 = a\nu + c$. Using (6.18) and (6.19), respectively, we can compute μ and ν in terms of a and c. If this situation holds at a point, then it also holds in some neighborhood, which is therefore a Hopf hypersurface with constant principal curvatures. This is clearly impossible since none of the standard examples have three distinct principal curvatures on $W^\perp$. We conclude that no such ν can exist.

The only remaining possibility is that there are two distinct constant principal curvatures on $W^\perp$,

$$\lambda = -\frac{c}{a} \qquad \text{and} \qquad \mu = -\frac{ca}{a^2+2c},$$

the latter having been calculated from (6.18). Because φ interchanges the principal subspaces, the only possible candidates among the standard examples are the Type B hypersurfaces. However, one can check that none of the Type B hypersurfaces in fact satisfy (6.17). This concludes the proof that a hypersurface satisfying the hypothesis cannot exist. □

Theorem 6.29 was proved in [Kimura and Maeda 1989] for $c > 0$. For any $c \neq 0$ it can be deduced from the (ii) ⇒ (i) implication of the next theorem (Theorem 6.30) by checking that none of the pseudo-Einstein hypersurfaces satisfy $R \cdot S = 0$. We do not prove that part of Theorem 6.30 here but refer to [Ki et al. 1990b]. Weakening the condition $R \cdot S = 0$ in either of two directions, we get two additional characterizations of the pseudo-Einstein hypersurfaces discussed in Theorems 6.1 and 6.2. In the first case, we look at the cyclic sum. In the second case, we restrict the condition to $W^\perp$.

Theorem 6.30. *For a real hypersurface M^{2n-1}, where $n \geq 3$, in a complex space form of constant holomorphic sectional curvature $4c \neq 0$, the following conditions are equivalent*:

(i) *M satisfies the pseudo-Einstein condition.*
(ii) *M satisfies the cyclic-Ryan condition.*
(iii) *M satisfies the pseudo-Ryan condition and is a Hopf hypersurface.*

Thus, if M satisfies any of these three conditions, it is one of the hypersurfaces listed in Theorems 6.1 *and* 6.2.

The equivalence of (i) and (ii) was established by Ki, Nakagawa, and Suh [Ki et al. 1990b]. Condition (iii) was studied by S.-B. Lee, N.-G. Kim, and S.-S. Ahn [Lee et al. 1990] who proved that condition (iii) implies that M is of type A or B. However, one can check that among these hypersurfaces only pseudo-Einstein ones actually satisfy (iii). Thus, the three conditions are equivalent.

We will give the proof of the equivalence of (i) and (iii). To simplify the proof, we set $TZ = \langle Z, W\rangle W$ and $P = A^2 - \mathfrak{m}A$. Then the Ricci tensor S can be expressed as $S = c_1 I + c_2 T - P$ for suitable constants c_1 and c_2; see (1.11). It is easy to check that $\langle (R(X,Y)T)Z_1, Z_2\rangle = 0$ for any X and Y, and for Z_1 and Z_2 in $W^\perp$. Consequently, for such arguments,

$$\langle (R(X,Y)S)Z_1, Z_2\rangle = -\langle (R(X,Y)P)Z_1, Z_2\rangle.$$

We now give an alternate characterization of pseudo-Ryan hypersurfaces, which follows immediately from this discussion.

Proposition 6.31. *A real hypersurface in a complex space form is pseudo-Ryan if and only if*

$$\langle R(X,Y)PZ_1, Z_2\rangle + \langle R(X,Y)PZ_2, Z_1\rangle = 0 \tag{6.20}$$

for all $X, Y, Z_1, Z_2 \in W^\perp$. ◁

Lemma 6.32. *Let M^{2n-1}, where $n \geq 2$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c$. Suppose that M satisfies the pseudo-Ryan condition at a point p. Let X and Y be unit principal vectors in $W^\perp$ at p with $AX = \lambda X$, $AY = \mu Y$, $PX = \alpha X$, and $PY = \beta Y$. Then*

$$(\alpha - \beta)(\lambda\mu + 2nc)\langle \varphi X, Y\rangle = 0$$

and

$$(\lambda - \mu)(\lambda + \mu - \mathfrak{m})(\lambda\mu + 2nc)\langle \varphi X, Y\rangle = 0.$$

Proof. From the definition of P, we see that $\alpha = \lambda^2 - \mathfrak{m}\lambda$ and $\beta = \mu^2 - \mathfrak{m}\mu$. Consequently,

$$\alpha - \beta = (\lambda - \mu)(\lambda + \mu - \mathfrak{m}). \tag{6.21}$$

By Proposition 6.31,

$$0 = (\alpha - \beta)\langle R(Z_1, Z_2)X, Y\rangle, \tag{6.22}$$

for all $Z_1, Z_2 \in W^{\perp}$. In particular, using the Gauss equation, we compute

$$\begin{aligned}0 &= (\alpha-\beta)\langle R(X,Y)Y,X\rangle\\ &= (\alpha-\beta)\big(c+\lambda\mu+c\langle(\varphi X\wedge\varphi Y)Y,X\rangle+2c\langle X,\varphi Y\rangle\langle\varphi Y,X\rangle\big)\\ &= (\alpha-\beta)\big(c+\lambda\mu+c\langle X,\varphi Y\rangle^2+2c\langle X,\varphi Y\rangle^2\big)\\ &= (\alpha-\beta)\big(c+\lambda\mu+3c\langle X,\varphi Y\rangle^2\big). \qquad (6.23)\end{aligned}$$

We also conclude that, for any unit vector $Z \in W^{\perp}$,

$$\begin{aligned}\langle R(\varphi Z, Z)X, Y\rangle\\ &= c\langle(\varphi Z\wedge Z)X,Y\rangle + c\langle(\varphi^2 Z\wedge\varphi Z)X,Y\rangle\\ &\quad + 2c\langle\varphi Z,\varphi Z\rangle\langle\varphi X,Y\rangle + \langle(A\varphi Z\wedge AZ)X,Y\rangle\\ &= (2c+\lambda\mu)(\langle Z,X\rangle\langle\varphi Z,Y\rangle - \langle\varphi Z,X\rangle\langle Z,Y\rangle) + 2c\langle\varphi X,Y\rangle. \qquad (6.24)\end{aligned}$$

Unless $\alpha = \beta$ (in which case, we are finished), the left hand side of (6.24) must be zero by (6.22). Let Z run through an orthonormal basis of $W^{\perp}$ and take the sum, to get

$$0 = -2(\lambda\mu + 2nc)\langle X, \varphi Y\rangle,$$

and the conclusion of the lemma follows by using (6.21). □

Lemma 6.32 does not require that $c \neq 0$. However, we will have no further use for the result in the case $c = 0$.

Corollary 6.33. *In Lemma* 6.32, *suppose that* $Y = \varphi X$. *If* $n \geq 3$ *and* $c \neq 0$, *then* $\alpha = \beta$.

Proof. Taking $Z = X$ and $Y = \varphi X$ in (6.24) yields $\lambda\mu + 4c = 0$. On the other hand, Lemma 6.32 gives $(\alpha-\beta)(\lambda\mu+2nc) = 0$. Subtracting these equations, we obtain $2c(n-2)(\alpha-\beta) = 0$, as required. □

Proof that (i) $\Longleftrightarrow$ (iii) in Theorem 6.30. Assuming (iii), we will prove that the principal curvatures must be constant. Then Theorem 4.13 limits the possibilities to those on Takagi's and Montiel's lists. Then we can pick from these lists the examples that actually satisfy the pseudo-Ryan condition.

First suppose that there is a point where P has two or more distinct eigenvalues on $W^{\perp}$. Then, there are orthonormal principal vectors X and Y as in the hypothesis of Lemma 6.32 with $\alpha \neq \beta$. Assuming that neither λ nor μ is equal to $a/2$, let $A\varphi X = \nu\varphi X$. By Corollary 6.33, $\lambda^2 - \mathfrak{m}\lambda = \nu^2 - \mathfrak{m}\nu$. Also, by (6.23), $\lambda\mu + c = \nu\mu + c = 0$ since $\operatorname{span}\{T_\lambda, \varphi T_\lambda\}$ and $\operatorname{span}\{T_\nu, \varphi T_\nu\}$ are orthogonal. Thus $\lambda = \nu$. This shows that λ (and similarly μ) satisfies the quadratic equation

$$t^2 - at - c = 0. \qquad (6.25)$$

Note that $a/2$ cannot occur as a principal curvature since an application of Corollary 2.3 would imply that $a^2 + 4c = 0$ and hence, by (6.25), that $\lambda = a/2$. On the other hand, the argument just completed shows that no further principal curvature is possible since it would have to satisfy (6.25). Thus, $W^{\perp}$ decomposes into two φ-invariant principal subspaces and the associated principal curvatures are constant. This shows that M is (at least locally) a Type A hypersurface.

The alternative possibility is that P is a multiple of the identity on $W^{\perp}$. In other words, for any two principal curvatures λ and μ (corresponding to principal vectors in $W^{\perp}$) at any point, we must have

$$(\lambda - \mu)(\lambda + \mu - \mathfrak{m}) = 0.$$

In particular, the number of distinct eigenvalues of A on $W^{\perp}$ is at most 2. Suppose that at some point there is a principal curvature $\lambda \neq a/2$ with associated unit principal vector $X \in W^{\perp}$ such that $A\varphi X = \mu\varphi X$, with $\mu \neq \lambda$. Then $\lambda^2 - \mathfrak{m}\lambda = \mu^2 - \mathfrak{m}\mu$ so that $\lambda + \mu = \mathfrak{m}$. This shows that there can be no other principal curvature on $W^{\perp}$. Each of λ and μ has multiplicity $n-1$. Thus, we have

$$(n-1)\lambda + (n-1)\mu + a = \mathfrak{m} = \lambda + \mu$$

and hence,

$$(n-2)(\lambda + \mu) + a = 0.$$

Again by Corollary 2.3,

$$\lambda\mu - (\lambda + \mu)a/2 - c = 0,$$

so that λ and μ satisfy the quadratic equation

$$t^2 + \frac{a}{n-2}t + c - \frac{a^2}{2(n-2)} = 0.$$

Again, A has two distinct constant curvatures on $W^{\perp}$. Because the principal spaces are interchanged by φ, we have locally a Type B hypersurface.

The existence of one point satisfying either of the conditions discussed in the preceding two paragraphs implies that the condition holds globally. The remaining possibility is that for all points and all principal curvatures $\lambda \neq a/2$ on $W^{\perp}$, the principal space T_λ is φ-invariant. If there is no such λ, then $A = (a/2)I$ on $W^{\perp}$ and our hypersurface is a horosphere (Type A0). Now suppose that there is at least one such λ. Then there can be at most one other principal curvature μ. Further, arguing as in the first part of this proof, any such μ satisfies $\mu \neq a/2$. We can ignore this case since it was covered in the first step of the proof (Type A2). If there is no such μ, then $A = \lambda I$ on $W^{\perp}$ and we have a geodesic sphere or tube over a complex hyperbolic hyperplane (Type A1).

It remains to determine which of the manifolds above actually satisfy the pseudo-Ryan condition. In this discussion, we look at (6.20) and assume that the vectors are taken from a principal orthonormal basis that is φ-invariant up to sign. We first note that if M is pseudo-Einstein, then P is a multiple of the identity on $W^{\perp}$, so that (iii) is satisfied. Thus all hypersurfaces of types A0 and A1 are pseudo-Ryan. Now look at the Type A2 hypersurfaces as described in

Section 3. In (6.20), we may choose Z_1 and Z_2 in distinct principal distributions (since the formula clearly holds otherwise). Then (6.20) reduces to

$$(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \mathfrak{m})\langle R(X,Y)Z_1, Z_2\rangle = 0. \tag{6.26}$$

If X and Y are in the same principal distribution, we calculate the curvature term using the Gauss equation (1.8) and find, noting φ-invariance, that it vanishes. Thus, we take X and Z_2 to be in the λ_1-distribution and Y and Z_1 to be principal for λ_2. Now, using the fact that $\lambda_1\lambda_2 + c = 0$, the Gauss equation for (X, Y) becomes

$$R(X,Y) = c(\varphi X \wedge \varphi Y) + 2c\langle X, \varphi Y\rangle\varphi,$$

so that the curvature term in (6.26) is $c(\langle \varphi Y, Z_1\rangle\langle \varphi X, Z_2\rangle)$, which vanishes unless $Z_1 = \pm\varphi Y$ and $Z_2 = \pm\varphi X$. In this case, however, the curvature term is nonzero and it is necessary that $(\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \mathfrak{m}) = 0$ in order for M to be pseudo-Ryan. This is precisely the condition for M to be pseudo-Einstein. We examine the Type B hypersurfaces in the same way, using the fact that the principal spaces are interchanged by φ and conclude that the only Type B hypersurfaces satisfying (iii) are the pseudo-Einstein ones.

Conversely, if (i) is assumed, P is a multiple of the identity on $W^\perp$. We can use (6.21) which is a valid expression for comparing the eigenvalues of P and does not depend on the pseudo-Ryan condition in the hypothesis of Lemma 6.32. Because of (6.2), we get $\alpha = \beta$ in (6.21) as required. □

7. A is W-Parallel

As we have seen, $\nabla A = 0$ is too strong a condition to be satisfied by a hypersurface in $\mathbb{C}P^n$ or $\mathbb{C}H^n$. However, if we merely ask that A be parallel in the direction of the structure vector W, interesting results are possible.

Theorem 7.1. *Let M^{2n-1}, where $n \geq 3$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Assume that the principal curvatures of M have constant multiplicities. If $\nabla_W A = 0$, then M is an open subset of a Type A hypersurface from Takagi's list or is a non-homogeneous hypersurface which is a tube of radius $r\pi/4$ over a certain Kähler submanifold of $\mathbb{C}P^n$.*

This classification was performed by Kimura and Maeda [1991]. The key to the theorem is the following proposition, which we generalize to the case $4c \neq 0$.

Proposition 7.2. *Let M^{2n-1}, where $n \geq 2$, be a real hypersurface in a complex space form of constant holomorphic sectional curvature $4c \neq 0$. If $\nabla_W A = 0$, then M is a Hopf hypersurface.*

The first step in the proof of Proposition 7.2 is to show that under the conditions of the proposition, AW must be principal.

LEMMA 7.3. *Under the assumptions of Proposition* 7.2, *at any point* $p \in M$, *either* $AW = 0$ *or* AW *is principal. If* $AW \neq 0$ *then* $\langle AW, W\rangle \neq 0$.

PROOF. Let X be any tangent vector field. Applying the Codazzi equation (1.9) to X and W yields

$$(\nabla_X A)W = -c\varphi X,$$

and consequently

$$\nabla_X(AW) = -c\varphi X + A\varphi AX. \tag{7.1}$$

On the other hand, applying the Codazzi equation to any tangent X and Y gives

$$A\varphi((\nabla_X A)Y - (\nabla_Y A)X) = -c(\langle X, W\rangle AY - \langle Y, W\rangle AX),$$

and thus

$$\begin{aligned}\nabla_X\nabla_Y(AW) - \nabla_{\nabla_X Y}(AW) = -c(\nabla_X\varphi)Y + (\nabla_X A)\varphi AY + A\varphi(\nabla_X A)Y\\ + \langle Y, AW\rangle A^2X - \langle AX, AY\rangle AW.\end{aligned}$$

The curvature tensor R now satisfies

$$R(X, Y)(AW) = (\nabla_X A)\varphi AY - (\nabla_Y A)\varphi AX + \langle Y, AW\rangle A^2X - \langle X, AW\rangle A^2Y.$$

On the other hand, using the Gauss equation, we can calculate that

$$\begin{aligned}R(X, Y)(AW) = c(\langle Y, AW\rangle X - \langle X, AW\rangle Y) + (\langle AY, AW\rangle AX - \langle AX, AW\rangle AY)\\ + c(\langle\varphi Y, AW\rangle\varphi X - \langle\varphi X, AW\rangle\varphi Y) + 2c\langle X, \varphi Y\rangle\varphi AW.\end{aligned}$$

The last two equations give us two expressions for $\langle R(X, Y)AW, W\rangle$, which we equate to get

$$\begin{aligned}0 = 2(\langle Y, AW\rangle\langle X, A^2W\rangle - \langle X, AW\rangle\langle Y, A^2W\rangle)\\ + \langle W, c\langle\varphi Y, AW\rangle\varphi X - c\langle\varphi X, AW\rangle\varphi Y + 2c\langle X, \varphi Y\rangle\varphi AW\rangle.\end{aligned}$$

Since Y was arbitrary, we conclude that

$$\langle A^2W, X\rangle AW = \langle X, AW\rangle A^2W.$$

Setting $X = W$ this becomes

$$|AW|^2 AW = \langle AW, W\rangle A^2W. \tag{7.2}$$

If AW is orthogonal to W, then $AW = 0$. On the other hand, if $AW \neq 0$, we can rewrite (7.2) as

$$A(AW) = \alpha AW \tag{7.3}$$

for some nonzero α, and the conclusion of the lemma then follows. □

Proof of Proposition 7.2. Suppose that there is a point p where W is not principal, that is, where $\varphi AW \neq 0$. Work in a neighborhood where this condition holds. By Lemma 7.3, we know that AW is a principal vector, with principal curvature

$$\alpha = \frac{|AW|^2}{\langle AW, W\rangle}.$$

Let $a = \langle AW, W\rangle$, which is nonzero by Lemma 7.3. Then we compute

$$\begin{aligned}\nabla_X(A^2W) &= (\nabla_X A)AW + A(\nabla_X A)W + A^2\varphi AX,\\ \nabla_X(\alpha AW) &= (X\alpha)AW - c\alpha\varphi X + \alpha A\varphi AX.\end{aligned}$$

By (7.3) the left sides of these two equations are equal. Equating the right sides, taking inner product with Y, and subtracting the same expression with X and Y interchanged, we get

$$\begin{aligned}(X\alpha)\langle AW, Y\rangle - (Y\alpha)\langle AW, X\rangle &= c\big(\langle X, W\rangle\langle\varphi Y, AW\rangle\\ &\quad -\langle Y, W\rangle\langle\varphi X, AW\rangle + 2\langle X, \varphi Y\rangle a\big) + \langle(A^2\varphi A + A\varphi A^2)X, Y\rangle\\ &\quad -c\langle(A\varphi + \varphi A)X, Y\rangle + 2c\alpha\langle\varphi X, Y\rangle - 2\alpha\langle A\varphi AX, Y\rangle. \end{aligned}\tag{7.4}$$

If we set $Y = W$ in this equality, we get

$$\begin{aligned}a(X\alpha) = W\alpha\langle AW, X\rangle &+ c\langle X, \varphi AW\rangle + c\langle\varphi AW, X\rangle\\ &- \langle A\varphi A^2W, X\rangle - \langle A^2\varphi AW, X\rangle + 2\alpha\langle A\varphi AW, X\rangle.\end{aligned}$$

Consequently,

$$a(\operatorname{grad}\alpha) = (W\alpha)AW + \alpha A\varphi AW + 2c\varphi AW - A^2\varphi AW. \tag{7.5}$$

Using (7.5), we can rewrite the left-hand side of (7.4) and let $X = AW$ in (7.4) to get

$$\begin{aligned}&\frac{1}{a}(-\alpha^2 aA\varphi AW - 2c\alpha a\varphi AW + \alpha aA^2\varphi AW)\\ &\qquad = -ca\varphi AW - 2ca\varphi AW - cA\varphi AW - c\alpha\varphi AW\\ &\qquad\qquad + \alpha A^2\varphi AW + \alpha^2 A\varphi AW + 2c\alpha\varphi AW - 2\alpha^2 A\varphi AW.\end{aligned}$$

Simplifying and noting that $c \neq 0$, we get

$$A\varphi AW = 3(\alpha - a)\varphi AW.$$

This further simplifies (7.5) and we obtain

$$a\operatorname{grad}\alpha = (W\alpha)AW + (2c - 3(3a - 2\alpha)(a - \alpha))\varphi AW. \tag{7.6}$$

We can also calculate directly that

$$Xa = 2\langle A\varphi AX, W\rangle = 6(a - \alpha)\langle\varphi AW, X\rangle.$$

Consequently, if we let $\sigma = 6(a - \alpha)$, we get

$$\operatorname{grad} a = \sigma\varphi AW \tag{7.7}$$

and

$$\operatorname{grad}\sigma = \frac{6}{a}(\rho\varphi AW - (W\alpha)AW),$$

where ρ is a scalar. Next, compute

$$\begin{aligned}\nabla_X(\operatorname{grad} a) = \frac{6}{a}(\rho\langle\varphi AW, X\rangle\varphi AW - (W\alpha)\langle AW, X\rangle\varphi AW)\\ + a\sigma AX - \sigma\langle A^2W, X\rangle W - c\sigma(-X + \langle X, W\rangle W) + \sigma\varphi A\varphi AX,\end{aligned}$$

and then

$$\begin{aligned}\langle\nabla_X(\operatorname{grad} a), Y\rangle = -\frac{6}{a}((W\alpha)\langle AW, X\rangle\langle\varphi AW\, Y\rangle) - \sigma\alpha\langle AW, X\rangle\langle W, Y\rangle\\ + \sigma\langle(\varphi A)^2X, Y\rangle + \text{terms symmetric in } X \text{ and } Y.\end{aligned}$$

Using the fact that

$$\langle\nabla_X(\operatorname{grad} a), Y\rangle = \langle\nabla_Y(\operatorname{grad} a), X\rangle$$

we see that

$$\begin{aligned}0 = -\frac{6(W\alpha)}{a}(\langle AW, X\rangle\langle\varphi AW, Y\rangle - \langle AW, Y\rangle\langle\varphi AW, X\rangle)\\ - \sigma\alpha(\langle AW, X\rangle\langle W, Y\rangle - \langle AW, Y\rangle\langle W, X\rangle) + \sigma\langle((\varphi A)^2 + (A\varphi)^2)X, Y\rangle = 0.\end{aligned}$$

Let $Y = W$ in this equality to get

$$(W\alpha)\varphi AW = (a-\alpha)(3a - 2\alpha)(AW - aW).$$

Taking the inner product of this with φAW and AW in turn yields

$$(W\alpha)\,|\varphi AW| = 0$$

and

$$(3a - 2\alpha)(a-\alpha)^2 a = 0.$$

This again allows us to simplify (7.6) to

$$a \operatorname{grad}\alpha = 2c\varphi AW, \tag{7.8}$$

while

$$\operatorname{grad} a = -6(a-\alpha)\varphi AW.$$

Now $a \neq \alpha$, since $0 \neq |AW - aW|^2 = \alpha a - 2a^2 + a^2 = (\alpha - a)a$, and hence $3a = 2\alpha$. Comparing (7.4) and (7.8) now gives

$$9a^2 + 4c = 0.$$

Thus a is constant and $\operatorname{grad} a = 0$, a contradiction. This completes the proof of Proposition 7.2 since, by (7.7), W is again principal. ☐

PROOF OF THEOREM 7.1. By Proposition 7.2, if $\nabla_W A = 0$, then M is a Hopf hypersurface and we can write $AW = aW$. By Theorem 3.1, a is constant. We can then divide the discussion into two cases, $a = 0$ and $a \neq 0$.

Suppose that $a = 0$. Then $A\varphi A = -c\varphi$, by (7.1). If no principal curvature is equal to $(1/r)\cot(\pi/4) = 1/r$, the focal map associated to the principal curvature a has constant rank and M lies on a tube of radius $r\pi/4$ over a complex submanifold. This is an application of [Cecil and Ryan 1982, Theorem 1, p. 489]. Even if $\lambda = 1/r$ is a principal curvature of constant multiplicity, we can make the same claim. In this case, T_λ can be easily seen to be φ-invariant. Whether or not M is a Type A hypersurface (e.g., a geodesic sphere of radius $r\pi/4$) or a nonhomogeneous hypersurface would depend on whether there were additional principal curvatures.

In the case $a \neq 0$, the following theorem of S. Maeda and S. Udagawa [1990] completes the proof. ☐

THEOREM 7.4. *Let M^{2n-1}, where $n \geq 3$, be a Hopf hypersurface in a complex space form of constant holomorphic sectional curvature $4c > 0$. Suppose that $a \neq 0$. If $\nabla_W A = 0$, then M is an open subset of a Type A hypersurface from Takagi's list.*

PROOF. Let X be any principal vector in $W^\perp$ with corresponding principal curvature λ. By (7.1),

$$0 = a\varphi AX + c\varphi X - A\varphi AX = a\lambda\varphi X + c\varphi X - \lambda A\varphi X. \tag{7.9}$$

Thus $\lambda A\varphi X = (\lambda a + c)\varphi X$. Comparing this with the formula in Corollary 2.3 yields $a(\lambda^2 - a\lambda - c) = 0$. This equation has two distinct roots, hence there are at most two distinct principal curvatures and they are locally constant. Each principal space is φ-invariant, as can be seen from (7.9). This rules out the Type B hypersurfaces as possibilities and completes the proof. ☐

Related results are found in [Pyo 1994a; 1994b]. In addition, some authors have studied similar conditions using Lie derivatives instead of covariant derivatives; see, for example, [Ki and Suh 1995; Ki et al. 1991; Ki et al. 1992; Ki et al. 1994; Ki et al. 1996; Kim et al. 1992a; Kimura and Maeda 1995; Pyo and Suh 1995].

8. Additional Topics

In this section we briefly discuss some topics related to the preceding material that we have not been able to include in this article.

Ruled Real Hypersurfaces. This class of hypersurfaces that does not belong to our list of "standard examples" but have occurred in some recent classification results. We introduce their definition and state a few of their properties.

Take a regular curve γ in $\tilde{M}$ ($\mathbb{C}P^n$ or $\mathbb{C}H^n$) with tangent vector field X. At each point of γ there is a unique complex projective or hyperbolic hyperplane cutting γ so as to be orthogonal not only to X but to JX. The union of these hyperplanes is called a *ruled real hypersurface*. It will be an embedded hypersurface locally although globally it will in general have self-intersections and singularities.

THEOREM 8.1. *Ruled real hypersurfaces have the following properties.*

(i) *The holomorphic distribution* $W^\perp$ *is integrable.*
(ii) *The structure vector* W *is not principal.*
(iii) *The principal curvatures are not all constant.*
(iv) *The shape operator has rank* 2 *and is* η*-parallel.*
(v) *The principal space for the zero principal curvature lies in* $W^\perp$. ◁

We will not discuss the related classification results here. Relevant references are [Kimura 1987a; Kimura and Maeda 1989; Maeda and Udagawa 1990; Suh 1992; 1995; Ahn et al. 1993; Taniguchi 1994; Pyo 1994c; Ki and Kim 1994; Ki and Suh 1995; 1996].

Isoparametric Hypersurfaces. In real space forms, isoparametric hypersurfaces are characterized by the fact that all their principal curvatures are constant. However, there are other equivalent characterizations. In the case of complex space forms, the analogous properties turn out not to be equivalent.

The following conditions can be considered.

(i) M has constant principal curvatures.
(ii) M is one of a parallel family of hypersurfaces of constant mean curvature.
(iii) M is one of a transnormal system, a system of parallel hypersurfaces with common normal geodesics.
(iv) The lifted hypersurface M' is an isoparametric hypersurface of $\tilde{M}'$.

Relevant references are [D'Atri 1979; Bolton 1973; Carter and West 1985; Li 1988; Park 1989; Wang 1982; 1983; 1987]. The new examples in $\mathbb{C}P^n$ are hypersurfaces M whose lifts M' are isoparametric but have more than two principal spaces that are not horizontal. Then the principal curvatures of M are need not be constant.

Real Hypersurfaces in Quaternionic Projective Space. Several authors have studied such hypersurfaces. Instead of the structure vector W, there is a three-dimensional distinguished subspace of the tangent space to be considered. Many basic questions analogous to those treated in the complex space forms have been asked and answered. Relevant references are [Berndt 1991; Berndt and Vanhecke 1992; 1993; Dong 1993; Hamada 1993; Ki et al. 1997; Martínez 1988; Martínez and Pérez 1986; Pak 1977; Pérez 1991; 1992; 1993a; 1993b; 1994; 1996a; 1996b; Pérez and Santos 1985; 1991; 1993; Pérez and Suh 1996a; 1996b].

9. Conclusion and Open Problems

In the preceding sections we have tried to present in an orderly fashion the central results concerning real hypersurfaces in $\mathbb{C}P^n$ and $\mathbb{C}H^n$. Because of the limitations of time and space, we have had to forego presenting the details of all the stated theorems, though we have done so for a good proportion of them. Two facts emerge from our study. First, some results hold for all dimensions $n \geq 2$, while many others require $n \geq 3$. Second, many results require the hypersurface to be a Hopf hypersurface ($AW = aW$) while some hold more generally. Some papers in the literature are vague about which of these conditions are being assumed, or rely on results that may require stronger hypotheses than those explicitly presented. We have attempted to be as explicit as possible in our presentation and in case of ambiguity in the literature have tried to err on the side of caution in those cases where we did not work through the complete proofs.

The following are questions and problems that appear to us to be open.

QUESTION 9.1. (See Proposition 1.4.) Although φA cannot vanish on an open set, are there examples for which φA vanishes at isolated points or on sets of lower dimension? The same question can be asked about umbilics (see Theorem 1.5) and the vanishing of $\varphi A + A\varphi$ (see Corollary 2.12).

QUESTION 9.2. Do Theorems 4.6 and 4.7 extend to $n = 2$? Are there hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ that have ≤ 2 principal curvatures, other than the standard examples?

QUESTION 9.3. (See Theorems 4.9 and 4.10.) $R \cdot A$ never vanishes for a Hopf hypersurface. Are there non-Hopf examples for which it does? $R \cdot A$ cannot vanish on an open set if $c > 0$. What about $c < 0$? Any counterexamples would have to satisfy $n \geq 3$ in view of Theorem 4.11.

QUESTION 9.4. (See Theorem 4.12.) A hypersurface in $\mathbb{C}P^n$ with three principal curvatures, all constants, must be a Hopf hypersurface, and hence one of the standard examples. Is the same true for $\mathbb{C}H^n$?

QUESTION 9.5. (See Theorems 6.1 and 6.2.) Classify the pseudo-Einstein hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$.

QUESTION 9.6. Does Theorem 6.2 still hold if the pseudo-Einstein hypothesis is relaxed to allow σ and ρ to be functions as is the case in Theorem 6.1?

QUESTION 9.7. (See Corollary 6.5.) Although there are some estimates involving $|\nabla S|$ in the literature, there do not seem to be any that are as simple as seen in Theorem 1.11 for $|\nabla A|$. Are there converses for any of the statements in Corollary 6.5?

QUESTION 9.8. (See Theorem 6.15.) Are there any Ricci-parallel hypersurfaces in $\mathbb{C}P^2$ or $\mathbb{C}H^2$? As we have seen, these could not be Hopf hypersurfaces.

QUESTION 9.9. Is Theorem 6.17 true for $c < 0$? (This theorem is concerned with the condition $\nabla_W S = 0$).

QUESTION 9.10. Many results have been proved for $n \geq 3$ but questions remain concerning the case $n = 2$. For example, Theorems 5.5, 6.18, 6.19, 6.20, 6.21, 6.23, and 6.30 can be considered from this point of view.

QUESTION 9.11. The question of classifying homogeneous real hypersurfaces in $\mathbb{C}H^n$ remains an outstanding open question. See [Berndt 1990].

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