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Taut Immersions into Complete Riemannian Manifolds

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Abstract. The main purpose of this paper is to propose a natural generalization of the notion of a taut immersion into a complete Riemannian manifold. We explain the motivation behind our definition, give many examples, note some interesting topological and geometric properties of such immersions, and remark on many intriguing relations to other well-known topics in geometry such as transformation groups, Morse theory, Blaschke manifolds and tori without conjugate points.

1. Introduction

Chern and Lashof started the study of tight immersions of compact manifolds into $\mathbb{R}^n$ in the 1950's. Recall that the *total absolute curvature* $\tau(M,\phi)$ of an immersion $\phi : M \to \mathbb{R}^n$ is the volume of the normal bundle map ξ from the unit normal bundle $\nu^1(M)$ to S^{n-1} (defined by $\xi(v) = v$). Since $-d\xi_v$ is equal to the shape operator A_v of M in the direction of v, we can write

$$\tau(M,\phi) = \frac{1}{c_{n-1}} \int_{\nu^1(M)} |\det(A_v)|\, dv,$$

where dv is the natural volume element on $\nu^1(M)$ and c_{n-1} is the volume of the unit sphere S^{n-1}. Chern and Lashof [1957; 1958] proved that

$$\tau(M,\phi) \geq \sum_i b_i(M),$$

where $b_i(M)$ is the i-th Betti number of M with respect to $\mathbb{Z}_2$. The number

$$\tau(M) = \inf_{\phi} \tau(M,\phi)$$

is clearly a differential invariant of M. An immersion $\phi : M \to \mathbb{R}^n$ is called *an immersion with minimal total absolute curvature* if $\tau(M,\phi) = \tau(M)$. The main problem studied by Chern and Lashof was to characterize such immersions. For

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example, they proved that an immersion of S^{n-1} into $\mathbb{R}^n$ with minimal total absolute curvature must be a convex hypersurface.

Kuiper [1959] reformulated the study of immersions with minimal total absolute curvature in terms of the Morse theory of height functions. First recall that the Morse number $\mu(M)$ of M is defined to be the minimum number of critical points a Morse function on M can have. It follows from the Morse inequalities that $\mu(M) \geq \sum_i b_i(M)$. Chern and Lashof's proof in fact gives $\tau(M) \geq \mu(M)$. Sharpe [1988] proved that $\tau(M) = \mu(M)$ under mild assumptions on the dimensions.

If $\mu(M) = \sum_i b_i(M)$, then for an immersion $\phi : M \to \mathbb{R}^n$ the following statements are equivalent:

(i) ϕ has minimal total absolute curvature.
(ii) $\tau(M, \phi) = \sum_i b_i(M)$.
(iii) For generic $a \in \mathbb{R}^n$ the height function $h_a : M \to \mathbb{R}$ defined by $h_a(x) = \phi(x) \cdot a$ is a perfect Morse function.

(Recall that a Morse function on M is called perfect if the number of index k critical points is equal to the k-th Betti number of M for all k. Here, as always in this paper, homology groups have coefficients in $\mathbb{Z}_2$).

An immersion satisfying conditions (i)–(iii) is called *tight*. If M is a tight submanifold in $\mathbb{R}^n$ and M lies in S^{n-1}, the restriction of the distance squared function $f_a(x) = \|x - a\|^2$ to M is also a perfect Morse function for generic $a \in \mathbb{R}^n$. Banchoff [1970] proved that tight surfaces contained in a round sphere satisfy the so-called spherical two-piece property. This motivated Carter and West [1972] to define a notion of tautness, as follows. A submanifold M of $\mathbb{R}^n$ is *taut* if, for generic $a \in \mathbb{R}^n$, the distance squared function $f_a : M \to \mathbb{R}$ defined by $f_a(x) = \|x - a\|^2$ is perfect, and a submanifold M in S^n is taut if M is taut in $\mathbb{R}^{n+1}$. This is equivalent to saying that M in S^n is taut if a squared spherical distance function f_a of M is a perfect Morse function for a generic a.

There have been much progress and many beautiful results in the study of tight and taut immersions in space forms; see for example [Cecil and Ryan 1985] and the articles in this volume. But there has not been a notion of tautness for submanifolds in arbitrary Riemannian manifolds. One reason is that the function $f_p : N \to \mathbb{R}$, defined by setting $f_p(x)$ equal to the square of the distance from p to x, is not differentiable if the cut locus of p is not empty [Wolter 1979]. This is not a problem if N is S^n, since a submanifold M does not meet the cut locus of p for generic p. But this is not the case in general, and there is no simple and direct way to generalize the notion of tautness.

To explain our definition of a taut immersion into a complete Riemannian manifold, we first review the definitions of focal points, the energy functional on path spaces, and the relations between them.

Let (N, g) be a Riemannian manifold, $\phi : M \to N$ an immersion, and $\nu(M)$ the normal bundle of M. We will assume throughout the paper that all immersions are proper. The *endpoint map* $\eta : \nu(M) \to N$ of M is by definition the restriction of the exponential map exp to $\nu(M)$. If $v \in \nu(M)_x$ is a singular point of η and the dimension of the kernel of $d\eta_v$ is m, then v is called a *focal normal of multiplicity* m and $\exp(v)$ is called a *focal point of multiplicity* m of M with respect to x in N. When $M = \{p\}$ is a single point, a focal point of multiplicity k of M is called a *conjugate point of order* k. The *focal data*, $\Gamma(M)$, is defined to be the set of all pairs (v, m) such that v is a focal normal of multiplicity m of M. The *focal variety* $\mathcal{V}(M)$ is the set of all pairs $(\eta(v), m)$ with $(v, m) \in \Gamma(M)$.

For $B \subset N \times N$, let $P(N, B)$ denote the set of all H^1-paths γ in N such that $(\gamma(0), \gamma(1)) \in B$. (A path is H^1 if it is absolutely continuous and the norm of its derivative is square integrable.) For a fixed $p \in N$, let $\pi : P(N,\, N{\times}p) \to N$ be the fibration defined by $\pi(\gamma) = \gamma(0)$, and let $P(N,\, \phi{\times}p)$ denote $\phi^*(P(N,\, N{\times}p))$, that is, the space of pairs (q, γ) such that $q \in M$ and γ a H^1-path $\gamma : [0, 1] \to N$ such that $(\gamma(0), \gamma(1)) = (\phi(q), p)$. The space $P(N,\, \phi{\times}p)$ is a Hilbert manifold [Palais 1963]. If $M \subset N$ and ϕ is the inclusion map, then $P(N,\, \phi{\times}p)$ is diffeomorphic to the space $P(N,\, M{\times}p)$. Let

$$E_p : P(N,\, \phi{\times}p) \to \mathbb{R}, \qquad E_p(q, \gamma) = \int_0^1 \|\gamma'(t)\|^2\, dt$$

be the energy functional. Then it is well-known that $(q, \gamma) \in P(N,\, \phi{\times}p)$ is a critical point of E_p if and only if γ is a geodesic normal to $\phi(U)$ at $\phi(q) = \gamma(0)$ parametrized proportional to arc length, where U is a neighborhood of q on which ϕ is injective. It is also well-known that E_p is a Morse function if and only if p is not a focal point of M. Notice that we do not require in this paper that the levels of critical points of a Morse function are different, only that all critical points are nondegenerate. The Morse index theorem says that the index of E_p at a critical point (q, γ) is the sum of the integers m such that $\gamma(t)$ is a focal point of multiplicity m of M with respect to q with $0 < t < 1$.

Let μ_k denote the number of critical points of index k of E_p in $P(N,\, \phi{\times}p)$, and let b_k denote the k-th Betti number of $P(N,\, \phi{\times}p)$. It is known that E_p is bounded below and satisfies the Palais–Smale condition [Palais and Smale 1964]. So μ_k is finite for all k, and the weak Morse inequalities say that $\mu_k \geq b_k$ for all k. The function E_p is called *perfect* if $\mu_k = b_k$ for all k.

We will prove in Section 2 that, if $N = \mathbb{R}^n$ or S^n, a submanifold $\phi : M \to N$ is taut if and only if the energy functional $E_p : P(N,\, \phi{\times}p) \to \mathbb{R}$ is perfect for generic $p \in N$. This leads to a natural generalization of the notion of a taut immersion into any complete Riemannian manifold N, namely:

Definition 1.1. An immersion $\phi : M \to N$ of (N, g) is called *taut* if the energy functional $E_p : P(N,\, \phi{\times}p) \to \mathbb{R}$ is perfect for every p in N that is not a focal point of M. In particular, a point $q \in N$ is called a *taut point* if $\{q\}$ is a taut submanifold of N, that is, if $E_p : P(N,\, q{\times}p) \to \mathbb{R}$ is perfect for every $p \in N$ that is not conjugate to q along some geodesic.

Taut immersions of M in S^n can also be defined in terms of the lower bound of volumes of certain images under the endpoint map. In fact, if M is an immersed submanifold of S^n then

$$\operatorname{vol}(\nu_k(M)) \geq b_k(M) \operatorname{vol}(S^n)$$

for all k, and the equalities hold if and only if M is taut in S^n, where $\nu_k(M)$ is the set of $v \in \nu(M)$ such that the foot point of v is a nondegenerate critical point of $f_{\exp(v)}$ with index k and $\nu_k(M)$ is equipped with the metric induced from N via η. A similar statement is true for an immersion $\phi : M \to N$ if N is a Riemannian manifold with finite volume: $\operatorname{vol}(\nu_k(M)) \geq b_k \operatorname{vol}(N)$, and the equalities hold if and only if M is taut. Here b_k is the k-th Betti number of $P(N, \phi \times p)$ and $\nu_k(M)$ is the set of $v \in \nu(M)$ such that $\gamma(t) = \exp(tv)$ is a nondegenerate critical point of $E_{\exp(v)}$ with index k.

Before we explain our results concerning taut immersions into arbitrary complete Riemannian manifolds, we review some fundamental properties of taut immersions in $\mathbb{R}^n$:

(1) A taut immersion is an embedding.
(2) Ozawa's theorem: If M is a taut submanifold in $\mathbb{R}^n$ and $a \in \mathbb{R}^n$ is a focal point of M then $f_a : M \to \mathbb{R}$ is a perfect Morse–Bott function. By a *Morse–Bott function* we mean a function with the property that every connected component of the set of critical points is a nondegenerate critical submanifold.
(3) A distance sphere in $\mathbb{R}^n$ is taut, and conversely a taut immersion of S^{n-1} into $\mathbb{R}^n$ must be a distance sphere.
(4) Orbits of the isotropy representation of a symmetric space (s-representation) are taut in Euclidean space. Modeled on these orbits, isoparametric and weakly isoparametric submanifolds of $\mathbb{R}^n$ are introduced and shown to be taut [Terng 1985; 1987; Hsiang et al. 1988]. Recall that a submanifold of $\mathbb{R}^n$ is *isoparametric* [Harle 1982; Carter and West 1985; Terng 1985] if its normal bundle is flat and the principal curvatures along any parallel normal field are constant. A submanifold of $\mathbb{R}^n$ is called *weakly isoparametric* [Terng 1987] if its normal bundle is flat, the principal curvatures along any parallel normal field have constant multiplicities, and the lines of curvatures are standard circles. In fact, principal orbits of s-representations are isoparametric, and isoparametric submanifolds are weakly isoparametric.

The goal of this paper is to investigate whether a taut immersion of M into an arbitrary Riemannian manifold has the above properties. We will explain our results item by item:

Property (1): It will be proved in section 2 that if N is simply connected and $\phi : M \to N$ is a taut immersion, then ϕ is an embedding. If N is not simply connected, ϕ may have self-intersections. For example, let N be a flat n-dimensional torus and $r > 0$ a number slightly bigger than the injectivity radius.

Then the restriction of the exponential map at a point $p \in T^n$ to the sphere of radius r gives a taut immersion of S^{n-1} into T^n with self-intersections.

Property (2): We prove an analogue of Ozawa's theorem in Section 2: If $\phi : M \to N$ is taut then for any focal point p of M in N the energy functional E_p is a perfect Morse–Bott function on $P(N, \phi\times p)$. This is a very useful tool for the study of taut immersions because it allows us to use Kuiper's top set technique and Bott's technique of using the critical submanifolds of E_p to obtain geometric and topological properties of M and N.

Property (3): Assume that N is an n-dimensional simply connected, complete Riemannian manifold that is not a rational homology sphere. We prove in Section 6 that if S^{n-1} is a taut hypersurface in N and S^{n-1} is null-homotopic, then S^{n-1} is a distance sphere. In particular, a null-homotopic taut S^{n-1} in a simply connected symmetric space N^n is a distance sphere.

We also prove that the distance sphere of radius r centered at p is taut in N if and only if p is a taut point of N. So the question whether a distance sphere centered at p in N is taut is equivalent to the question whether p is a taut point in N.

It is obvious that for $\mathbb{R}^n$ and S^n all points are taut. The questions we study in Section 6 are: Is every point of N taut? Is there some taut point in N? The answers are definitely no for both questions for an arbitrary Riemannian manifold N, because if p is a taut point of N then E_p is perfect on the loop space $\Omega(N) = P(N, p\times p)$. This gives a lot of restrictions on the structure of conjugate points of p. But there are many Riemannian manifolds for which all points are taut. For example, we will prove that all points in symmetric spaces and Blaschke manifolds are taut. Roughly speaking, a simply connected manifold N is Blaschke at a point p if the conjugate point data of geodesics starting in p is the same as that of a compact rank one symmetric space. In particular, if N is *Blaschke* at p then there exist $l > 0$ and an integer a such that the first conjugate point along any geodesic ray at p occurs at length l with multiplicity a [Besse 1978]. N is called a *Blaschke manifold* if it is Blaschke at every point. For example, a simply connected rank-one symmetric space is Blaschke. In fact, $a = n-1$ for S^n, $a = 1$ for $\mathbb{CP}^m$ with $n = 2m$, $a = 3$ for $\mathbb{HP}^m$ with $n = 4m$ and $a = 7$ for $\mathrm{Ca}\mathbb{P}^2$ with $n = 16$. (Here $\mathbb{HP}^m$ is the quaternionic projective space and $\mathrm{Ca}\mathbb{P}$ is the Cayley plane.) Notice that if N is a compact, simply connected rank-one symmetric space then the first three nonzero Betti numbers of $\Omega(N) = P(N, p\times p)$ are b_0, b_a, b_{a+n-1}, and they are all equal to 1.

So the following questions arise naturally: What can one say about the geometry and topology of a Riemannian manifold with all points taut? We have some results concerning this question:

(i) If the first three nonzero Betti numbers of the loop space $\Omega(N)$ of the compact Riemannian manifold N are b_0, b_a, b_{a+n-1}, and they are equal to 1 for some

$1 \leq a \leq n-1$ and N has a taut point p, then N is Blaschke at p. So it follows from a theorem of Warner [1967] that homologically N is a rank-one symmetric space.

(ii) If the loop space of N is as in (i) and all points of N are taut then N is Blaschke. By work of Sato [1984] and Yang [1990], it follows that N is homeomorphic to a compact rank-one symmetric space.

(iii) If g is a Riemannian metric on S^n such that all points are taut then g must be the standard metric.

(iv) If g is a Riemannian metric on the n-torus T^n such that all points are taut then (T^n, g) is flat.

Property (4): An isometric G-action is called *hyperpolar* if there exists a closed, flat submanifold Σ that meets every G-orbit and meets orthogonally at every intersection point. Using results from [Bott and Samelson 1958; Conlon 1971], it follows that orbits of a hyperpolar action are taut. In fact, the isotropy representations of symmetric spaces are essentially all the hyperpolar actions on Euclidean spaces [Dadok 1985]. There are many hyperpolar actions on symmetric spaces. Hence there exist many homogeneous taut submanifolds in symmetric spaces.

A principal orbit of a hyperpolar action on a symmetric space N has flat normal bundle and its focal data is invariant under the parallel translation with respect to the induced normal connection. Motivated by these orbit examples, we introduced equifocal and weakly equifocal submanifolds in compact symmetric spaces, and proved that they are again taut [Terng and Thorbergsson 1995]. Recall that a submanifold M of a compact symmetric space N is called *equifocal* if the normal bundle is flat and abelian and the focal data is invariant under normal parallel translations. M is called *weakly equifocal* if the normal bundle is flat and abelian, the multiplicity of the k-th focal point along the normal geodesic ray $\exp(tv(x))$ is independent of $x \in M$ for a parallel normal field v, and the focal radius on a focal leaf of multiplicity one is constant. It was proved in [Terng and Thorbergsson 1995] that principal orbits of a hyperpolar action are equifocal, and equifocal submanifolds share many of the geometric and topological properties of the principal orbits of hyperpolar actions. It follows from the definitions that equifocal submanifolds are weakly equifocal.

A hypersurface M in $\mathbb{R}^n$ is *Dupin* if all its lines of curvatures are standard circles, and is *proper Dupin* if it is Dupin and all its principal curvatures have constant multiplicities. Pinkall [1986] proved that taut hypersurfaces are Dupin, and Thorbergsson [1983] proved that proper Dupin hypersurfaces are taut. In this paper, we also define a notion of Dupin submanifold in an arbitrary complete Riemannian manifold. We generalize the above results to symmetric spaces.

This is the beginning of our project on taut immersions into a complete Riemannian manifold. We have obtained some basic results, but many interesting questions remain open.

Our paper is organized as follows: In Section 2, we give general results concerning taut immersions into a complete Riemannian manifold N obtained from Morse theory. In Section 3, we review results on taut submanifolds that are orbits of some isometric actions. In Section 4, we review results on equifocal and weakly equifocal submanifolds in symmetric spaces, and their relation to tautness. In Section 5, we study Dupin submanifolds in complete Riemannian manifolds and submanifolds in Hilbert spaces. In Section 6, we study manifolds with taut points and taut spheres. In the Appendix, we use infinite-dimensional Morse theory to prove some of the results in Section 2.

2. Taut Immersions

In this section, we will prove that our new Definition 1.1 is equivalent to the original definition of tautness if the ambient space is $\mathbb{R}^n$ and S^n. If $\phi : M \to N$ is taut and the ambient space N is simply connected, or more generally if the path space $P(N,\, \phi\times p)$ is connected, then we will show that ϕ is injective. We will also prove an analogue of Ozawa's theorem for taut immersions into arbitrary Riemannian manifolds. It will again turn out in our more general situation that this result is one of the most important tools in dealing with taut submanifolds, see in particular Section 6.

We start with two propositions that prove that the two definitions of tautness are equivalent when the ambient space is $\mathbb{R}^n$ or S^n. We may assume that the submanifolds are embedded because a taut immersion in $\mathbb{R}^n$ or S^n is an embedding under either definition (see Theorem 2.5 for the case of the new definition).

PROPOSITION 2.1. *Let M be a submanifold of $\mathbb{R}^n$ and let $a \in \mathbb{R}^n$. Then*

(i) *$P(\mathbb{R}^n,\, M\times a)$ is homotopy equivalent to M, and*

(ii) *M is taut in $\mathbb{R}^n$ with respect to the original definition if and only if M is taut in $\mathbb{R}^n$ with respect to the new definition.*

PROOF. Given $b \in \mathbb{R}^n$, let $l_b(t) = (1-t)b + ta$ denote the line segment joining b to a. Then $\phi(\gamma) = l_{\gamma(0)}$ defines a deformation retract of $P(\mathbb{R}^n,\, M\times a)$ to $M^* = \{l_x : x \in M\}$, which is diffeomorphic to M. This proves (i). To prove (ii), we note that $E_a(l_x) = \|x-a\|^2 = f_a(x)$ and γ is a critical point of E_a of index k if and only if $\gamma(0)$ is an index k critical point of f_a. □

PROPOSITION 2.2. *Let M^m be a submanifold of S^n, set $\tilde{M} = P(S^n,\, M\times p)$, and let b_k be the k-th Betti number of M. Then*

(i) *the Poincaré polynomial of $\tilde{M}$ is*

$$\left(\sum_{j=0}^{m} b_j t^j\right)\left(\sum_{k=0}^{\infty} t^{k(n-1)}\right);$$

(ii) *M is taut in S^n with respect to the old definition if and only if M is taut in S^n with respect to the new definition.*

Proof. Let $\pi : P(S^n, S^n \times p) \to S^n$ denote the projection $\pi(\gamma) = \gamma(0)$, and take $p_0 \in S^n \setminus M$. Then $S^n \setminus \{p_0\}$ is contractible, which implies that π is trivial over $S^n \setminus \{p_0\}$. So

$$\tilde{M} = \pi^{-1}(M) \simeq M \times \pi^{-1}(p_0) = M \times P(S^n, p_0 \times p_0).$$

Now (i) follows since the Poincaré polynomial for $P(S^n, p_0 \times p_0)$ is $\sum_{k=0}^{\infty} t^{k(n-1)}$ [Milnor 1963].

It is known that $q \in M$ is a critical point of f_p for $p \in S^n$, where f_p is a squared spherical distance function, if and only if there exists $v \in \nu(M)$ such that $\|v\| < \pi$ and $p = \exp_q(v)$. Given a critical point q of f_p, of index k, and a natural number j, let $\gamma_{q,j}$ denote the geodesic starting from q by going j times around the normal circle in the direction of v and then continuing to p. Then $\gamma_{q,j}$ is a critical point of E_p with index $k + j(n-1)$. Moreover, all critical points of E_p arise this way. By (i), this proves that the number of index r critical points of E_p is equal to $b_r(\tilde{M})$, which proves (ii). $\square$

Hence there is no ambiguity in the definition of taut immersions, and from now on we will always use the new definition. The following proposition follows immediately from the definition.

Proposition 2.3. *Suppose $\phi : M \to N$ is a taut immersion, and that $f : N \to N$ is an isometry. Then $f \circ \phi : M \to N$ is also a taut immersion.*

Note that the set of focal points of M in N has measure zero and E_p has at least $b_k(P(N, \phi \times p))$ critical points of index k if p is not a focal point of M in N. Hence the following proposition is an easy consequence of the definition.

Proposition 2.4. *Let $\phi : M \to N$ be an immersion, $\eta : \nu(M) \to N$ the endpoint map, and let $\nu_k(M)$ denote the set of $v \in \nu(M)$ such that $\gamma(t) = \exp(tv)$ is a nondegenerate critical point of index k of $E_{\exp(v)}$. If N has finite volume, then*

$$\operatorname{vol}(\nu_k(M)) \geq b_k \operatorname{vol}(N),$$

where b_k is the k-th Betti number of $P(N, \phi \times p)$ and $\nu_k(M)$ is equipped with the metric induced from N via η. Moreover, equality holds for all k if and only if M is taut.

Theorem 2.5. *Suppose $\phi : M \to N$ is a taut immersion. If $P(N, \phi \times p)$ is connected, then ϕ is injective. In particular, if N is simply connected and M connected, then ϕ is injective.*

Proof. Assume that $P(N, \phi \times p)$ is connected and that ϕ is not injective. Then there is a ball $B_\varepsilon(p)$ in N such that $\phi^{-1}(B_\varepsilon(p))$ is disconnected. We can choose p such that E_p is a Morse function. It follows that E_p has at least two critical points of index zero. This implies that E_p is not perfect, since $P(N, \phi \times p)$ being connected implies that the Betti number b_0 is equal to one. This is a contradiction, so ϕ is injective. The homotopy sequence of the fibration $P(N, \phi \times p) \to M$;

$(q,\gamma) \to q$ implies that $P(N,\, \phi{\times}p)$ is connected if N is simply connected and M connected. $\square$

REMARK 2.6. We will show in Section 6 that if N is a complete Riemannian manifold without conjugate points then $\exp : S_r(0) \to N$ is taut for every $r > 0$, where $S_r(0)$ is the sphere in TM_p centered at 0 with radius r. If N is not simply connected we will thus have noninjective taut immersions of spheres into N. Notice that $P(N,\, \phi{\times}p)$ can be connected, although N is not simply connected. An example is given by letting M be a projective subspace of $N = RP^n$ and ϕ the inclusion.

To simplify the notation we will often let $\mathcal{M}_p$ denote the path space $P(N,\, \phi{\times}p)$, where $\phi : M \to N$ is an immersion and $p \in N$. We also set

$$\mathcal{M}_p^r = \{(q,\gamma) \in \mathcal{M}_p : E_p(q,\gamma) \le r\},$$
$$\mathcal{M}_p^{r-} = \{(q,\gamma) \in \mathcal{M}_p : E_p(q,\gamma) < r\}.$$

We will denote the set of regular values of E_p by $R(E_p)$.

Before we state the next proposition we review some well-known facts from Morse theory. Let us assume that $p \in N$ is not a focal point of the immersion $\phi : M \to N$. Then E_p is a Morse function. Let κ be a critical value of E_p and $\varepsilon > 0$ so small that $(\kappa{-}\varepsilon,\, \kappa) \subset R(E_p)$. Then we have

$$H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-\varepsilon}) = \mathbb{Z}_2^i,$$

where i is the number of critical points of E_p with index k and value κ. The nonzero homology classes of $H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-\varepsilon})$ can be represented by the local unstable manifolds of the critical points of E_p with index k and value κ. We will think of the local unstable manifolds as maps of k-dimensional closed disks with boundaries below the level $\kappa - \varepsilon$.

Now assume that $\phi : M \to N$ is a taut immersion. Then E_p is a perfect Morse function and the maps

$$H_k(\mathcal{M}_p^{\kappa-\varepsilon}) \to H_k(\mathcal{M}_p^\kappa) \qquad \text{and} \qquad H_k(\mathcal{M}_p^{\kappa-}) \to H_k(\mathcal{M}_p^\kappa)$$

are injective for all k. It follows from the homology sequence that

$$H_k(\mathcal{M}_p^\kappa) \to H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-\varepsilon})$$

is surjective for all k. Hence there is for every unstable manifold U of a critical point of E_p with index k and value κ a k-cycle z in $\mathcal{M}_p^\kappa$ that is homologous to U in $\mathcal{M}_p^\kappa$ modulo $\mathcal{M}_p^{\kappa-\varepsilon}$. By choosing ε smaller if necessary we can deform z into U in a Morse coordinate chart. We have thus seen that the local unstable manifold U can be completed to a cycle z in $\mathcal{M}_p^\kappa$. It is clear that such a z cannot be homologous within $\mathcal{M}_p^\kappa$ to a cycle in $\mathcal{M}_p^{\kappa-}$ since it maps onto a nontrivial cycle U in $H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-}) = H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-\varepsilon})$. Notice also that z cannot be homologous within $\mathcal{M}_p$ to a cycle in $\mathcal{M}_p^{\kappa-}$ since the maps $H_k(\mathcal{M}_p^{\kappa-}) \to H_k(\mathcal{M}_p^\kappa) \to H_k(\mathcal{M}_p)$ are injective and the class of z does not lie in the image of the first map.

In the next proposition we prove that these remarks hold true at a nondegenerate critical point even if E_p is not a Morse function.

PROPOSITION 2.7. *Let N be a Riemannian manifold and $\phi : M \to N$ a taut immersion. Let p be a point in N and $(q,\gamma) \in \mathcal{M}_p$ a nondegenerate critical point of E_p. Then the local unstable manifold at (q,γ) can be completed to a cycle z in $\mathcal{M}_p^\kappa$, where $\kappa = E_p(q,\gamma)$. Furthermore, if V is a Morse chart around (q,γ), the cycle z is not homologous within $\mathcal{M}_p^\kappa \cup V$ to any cycle in $\mathcal{M}_p^{\kappa-}$.*

PROOF. Let k denote the index of (q,γ). Since (q,γ) is a nondegenerate critical point there is a closed neighborhood U of p in N and differentiable functions $U \to M$ taking r to q_r and $U \to TM$ taking r to v_r such that $(q_p,\gamma_p) = (q,\gamma)$ and (q_r,γ_r) lies in $\mathcal{M}_r$ and is a nondegenerate critical point with index k of E_r, where $\gamma_r(t) = \exp tv_r$. There is a differentiable map $\Phi : U \times B^k \to P(N, N\times N)$ such that $U_r := \Phi(r, B^k)$ is a local unstable manifold in $\mathcal{M}_r$ of E_r at (q_r,γ_r) where B^k is a closed k-dimensional Euclidean ball.

By choosing U smaller if necessary, we can find an $\varepsilon > 0$ such that the boundaries of the unstable manifolds U_r of E_r at (q_r,γ_r) lie below the $\kappa - \varepsilon$ level of E_r for all $r \in U$. Set $\kappa(r) = E_r(q_r,\gamma_r)$. Again by choosing U smaller if necessary, we can assume that $\kappa(r) > \kappa - \varepsilon$.

Let $F_r : \mathcal{M}_r \to \mathcal{M}_p$ be the map that sends (s,f) to $(s,\tilde{f})$, where $\tilde{f}$ is the curve that one gets by adding the geodesic from r to p on f and then linearly reparametrizing so that it is still parametrized on $[0,1]$. By choosing U smaller if necessary we can assume that $F_r(\mathcal{M}_r^{\kappa-\varepsilon}) \subset \mathcal{M}_p^{\kappa-\varepsilon/2}$. Again by choosing U smaller if necessary we can assume that $F_r(U_r)$ lies in $\mathcal{M}_p^\kappa \cup V$ for all $r \in U$ where V is some fixed Morse coordinate chart around (q,γ).

Now notice that there is a point s in every neighborhood of p such that E_s is a Morse function. Let s be such a point in U. Then by the observations before this proposition there is a cycle $\tilde{z}$ in $\mathcal{M}_s^{\kappa(s)}$ that agrees with the unstable manifold U_s above the E_s-level $\kappa - \varepsilon$. Then $F_r(\tilde{z})$ is a cycle in $\mathcal{M}_p^\kappa \cup V$ that can be deformed within the Morse chart V into a cycle z in $\mathcal{M}_p^\kappa$ that agrees with U_p above the E_p-level $\kappa - \varepsilon/2$. It follows that the homology class of z maps into the nontrivial homology class of U_p in $H_k(\mathcal{M}_p^\kappa, \mathcal{M}_p^{\kappa-})$ under the map induced by the inclusion. Notice that there is a deformation retraction of $\mathcal{M}_p^\kappa \cup V$ onto $\mathcal{M}_p^\kappa$. It follows that z cannot be homologous within $\mathcal{M}_p^\kappa \cup V$ to a class in $\mathcal{M}_p^{\kappa-}$. This finishes the proof of the proposition. □

We now come to the generalization of the theorem of Ozawa [1986] on the distance functions of taut submanifolds in Euclidean spaces and in spheres. In the proof we will use the main idea of [Ozawa 1986].

THEOREM 2.8. *Let N be a Riemannian manifold and $\phi : M \to N$ a taut immersion. Then for every $p \in N$ the energy functional $E_p : P(N, \phi\times p) \to \mathbb{R}$ is a Morse–Bott function.*

PROOF. We will assume that M is embedded and work with $P(N, M\times q)$ instead of $P(N, \phi\times q)$ to simplify the notation. The general case does not require any new ideas.

We will work with finite-dimensional approximations of the path spaces $\mathcal{M}_p^{r-}$ as in [Milnor 1963]. Let $r > 0$ be some positive number. Let $i(N)$ denote the injectivity radius of N. Let n be a natural number greater than $r/i(N)^2$. We denote by $\mathcal{P}_p = \mathcal{P}_p(r, n)$ the space of continuous curves $\gamma : [0, 1] \to N$ that start in M and end in p and have the property that $\gamma \mid \big[(j-1)/n,\, j/n\big]$ is a geodesic of length less than $i(N)$ for all $j = 1, \dots, n$. Then $E_p(\gamma) < r$. Notice that $\mathcal{P}_p$ is a finite-dimensional open manifold. We denote its dimension by d.

There is a deformation retraction of $\mathcal{M}_p^{r-}$ onto $\mathcal{P}_p$ [Milnor 1963]. The critical points of E_p in $\mathcal{M}_p^{r-}$ are of course contained in $\mathcal{P}_p$. Conversely, if γ is a critical point of the restriction of E_p to $\mathcal{P}_p$, then it is also a critical point of E_p on $\mathcal{M}_p^{r-}$. Their indices and nullity in $\mathcal{P}_p$ are the same as in $\mathcal{M}_p^{r-}$. A connected component of the set of critical points in $\mathcal{M}_p^{r-}$ is nondegenerate if and only if it is nondegenerate in $\mathcal{P}_p$.

Let γ be a geodesic in $\mathcal{P}_p$ whose index as a critical point of E_p we denote by i. We assume that the nullity of γ as a critical point is $n_0 > 0$. This means that p is a focal point of M along γ.

We now start following the arguments in [Ozawa 1986]. There are coordinates $x = (x_1, \dots, x_d)$ around γ in $\mathcal{P}_p$ such that

$$E_p(x) = E_p(\gamma) - x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_{d-n_0}^2 + O(\|x\|^3)$$

in these coordinates and γ corresponds to 0.

For $c = (c_1, \dots, c_{n_0})$ we set

$$A(c) = \{x_{d-n_0+j} = c_j \text{ for all } j = 1, \dots, n_0\}.$$

Then the function $E_p \mid A(c)$ has a nondegenerate critical point of index i in $(0, c)$ with value $E_p(0, c) = E_p(\gamma) + O(\|(0, c)\|^3)$. We set $\kappa = E(\gamma)$ and $\kappa_c = E_p(0, c)$.

Our goal is to show that $\kappa_c = \kappa$ for c small. It then follows that $(0, c)$ is a critical point of E_p and hence that γ lies in a nondegenerate critical submanifold of dimension n_0. We do this in two steps. First we show that $\kappa_c \le \kappa$ for c small. Then we prove that $\kappa_c \ge \kappa$ for c small.

We introduce the stable and unstable manifolds of $E_p \mid A(c)$ at $(0, c)$ before we start with the proof of the two steps explained above.

We can parametrize the family of local stable manifolds of $E_p \mid A(c)$ at $(0, c)$ by a differentiable map Φ into $\mathcal{P}_p$ depending on c and the elements of a closed ball $B_r(0)$ in $\mathbb{R}^{d-n_0-i}$. We set $S_c = \Phi(c, B_r(0))$. Then $S_c \in A(c)$ is the stable manifold of $E_p \mid A(c)$ at $(0, c)$.

Similarly we parametrize the family of local unstable manifolds of $E_p \mid A(c)$ at $(0, c)$ by a differentiable map Ψ depending on c and the elements of a closed ball $B_r(0)$ in $\mathbb{R}^i$. We set $U_c = \Psi(c, B_r(0))$.

We will use the notation $\mathcal{P}_p^s$ for the set of curves in $\mathcal{P}_p$ with E_p-value less than or equal to s and $\mathcal{P}_p^{s-}$ for the set of curves in $\mathcal{P}_p$ with E_p-value strictly less than s.

We choose $\varepsilon > 0$ and $\delta > 0$ so small that we have the following situation: $\|c\| \le \delta$ implies that the local stable manifold S_c in $A(c)$ is a nontrivial cycle in $(\mathcal{P}_p - \mathcal{P}_p^{\kappa_c-\varepsilon}) \cap A(c) \mod (\mathcal{P}_p - \mathcal{P}_p^{\kappa_c+\varepsilon}) \cap A(c)$. This means that E_p is strictly greater than $\kappa_c + \varepsilon$ on ∂S_c. By choosing δ smaller if necessary we can assume that $\|c\| \le \delta$ implies that E_p is strictly greater than $\kappa + \varepsilon$ on ∂S_c.

Furthermore, we assume ε and δ chosen such that the local unstable manifold U_c in $A(c)$ is a nontrivial cycle in $\mathcal{P}_p^{\kappa_c+\varepsilon} \cap A(c) \mod \mathcal{P}_p^{\kappa_c-\varepsilon} \cap A(c)$. This means that E_p is strictly smaller than $\kappa_c - \varepsilon$ on ∂U_c. After choosing δ smaller if necessary, we can assume that E_p is strictly smaller than $\kappa - \varepsilon$ on ∂U_c.

We can now begin the proof of the two cases.

(i) We continue the geodesic γ beyond p to a geodesic $\tilde{\gamma}$ defined on $[0, 1{+}t_0]$, where $t_0 > 0$ is so small that there is no focal point of M between p and $q := \tilde{\gamma}(1 + t_0)$ and the length of $\tilde{\gamma}$ between $(n-1)/n$ and $1 + t_0$ is less than $i(N)$, the injectivity radius of N. Denote by $\tilde{\mathcal{P}}_q = \tilde{\mathcal{P}}_q(r, n)$ the path space defined as above except that curves are parametrized between 0 and $1 + t_0$ instead of between 0 and 1 (but still with breaks at $1/n, \ldots, (n-1)/n$). The dimension of $\tilde{\mathcal{P}}_q$ is equal to that of $\mathcal{P}_p$, which was denoted by d. We assume that t_0 is so small that $\tilde{\gamma} \in \tilde{\mathcal{P}}_q$.

Notice that $\tilde{\gamma}$ is a nondegenerate critical point of E_q of index $i + n_0$. By tautness, the local unstable manifold of E_q at $\tilde{\gamma}$ can be completed to a cycle z in $\tilde{\mathcal{P}}_q$ under the $E_q(\tilde{\gamma})$-level; see Proposition 2.7.

Now we make a further restriction on t_0. We assume it is so small that $\tilde{f} \in \tilde{\mathcal{P}}_q$ for every $f \in S_c$ with $\|c\| \le \delta$, where $\tilde{f}$ is the path we get by replacing the segment $f \,|\, [(n{-}1)/n, 1]$ of f by the geodesic segment between $f((n-1)/n)$ and q parametrized on the interval $[(n{-}1)/n, 1{+}t_0]$. This gives us a differentiable map of S_c into $\tilde{\mathcal{P}}_q$ that is also differentiable in the parameter c. We denote the images by $\tilde{S}_c$.

By choosing t_0 smaller if necessary we can arrange that the boundaries of $\tilde{S}_c$ for all $\|c\| \le \delta$ have E_q-values strictly above $\kappa + \varepsilon$ and that $\kappa + \varepsilon > E_q(\tilde{\gamma})$. For t_0 small enough we have that $E_q \,|\, \tilde{S}_0$ has a nondegenerate absolute minimum in $\tilde{\gamma}$ and the absolute minimum is not reached in any other point. It follows that the intersection number of z and the relative cycle $\tilde{S}_0 \mod \tilde{\mathcal{P}}_q - \tilde{\mathcal{P}}_q^{\kappa+\varepsilon}$ is equal to one. (Notice that the dimensions of z and $\tilde{S}_0$ are complementary).

Now assume that there is a c_0 with $\|c_0\| \le \delta$ such that $\kappa_{c_0} > \kappa$. Again by choosing t_0 smaller if necessary we can assume that E_q is strictly larger than $E_q(\tilde{\gamma})$ on $\tilde{S}_{c_0}$. Let $c(t)$ be a path between 0 and c_0 such that $\|c(t)\| \le \delta$. Then $\tilde{S}_{c(t)}$ gives a homotopy between the cycles $\tilde{S}_0$ and $\tilde{S}_{c_0}$ keeping the boundaries above the E_q-level $\kappa + \varepsilon$. The intersection number of z and $\tilde{S}_{c_0}$ is equal to 0. This is a contradiction. It follows that $\kappa_c \le \kappa$ for all $\|c\| < \delta$.

(ii) We now let $\tilde{\gamma}$ denote the restriction of the geodesic γ to the interval $[0, 1-t_0]$, where $t_0 > 0$ is so small that there is no focal point of M between $q := \gamma(1 - t_0)$ and p. Denote by $\tilde{\mathcal{P}}_q = \tilde{\mathcal{P}}(r, n)$ the path space defined as above except that curves are parametrized between 0 and $1 - t_0$ instead of between 0 and 1 (but still with breaks at $1/n, \ldots, (n-1)/n$; that is, we also assume $t_0 < 1/n$). Notice that the dimension of $\tilde{\mathcal{P}}_q$ is again equal to d and that $\tilde{\gamma} \in \mathcal{P}_q$ and that $\tilde{\gamma}$ is a nondegenerate critical point of E_q with index i.

As above we make further restrictions on t_0. We assume it is so small that $\tilde{f} \in \tilde{\mathcal{P}}_q$ for every $f \in U_c$ with $\|c\| < \delta$, where $\tilde{f}$ is the path we get by replacing the segment $f \,|\, [(n-1)/n, 1]$ of f by the geodesic segment between $f((n-1)/n)$ and q parametrized on the interval $[(n-1)/n, 1-t_0]$. This gives us a differentiable map of U_c into $\tilde{\mathcal{P}}_q$ that is also differentiable in the parameter c. We denote the images by $\tilde{U}_c$.

By choosing t_0 smaller if necessary we can arrange that the boundaries of $\tilde{U}_c$ for all $\|c\| \le \delta$ have E_q-values strictly below $\kappa - \varepsilon$ and that $\kappa - \varepsilon < E_q(\tilde{\gamma})$. For t_0 small enough we have that $E_q \,|\, \tilde{U}_0$ has a nondegenerate absolute maximum in $\tilde{\gamma}$ and that $E_q \,|\, \tilde{U}_0$ does not have any further critical points. It follows that $\tilde{U}_0$ is homologous to the unstable manifold of E_q at $\tilde{\gamma} \mod \tilde{\mathcal{P}}_q^{\kappa-\varepsilon}$. Proposition 2.7 now implies that $\tilde{U}_0$ can be completed to a cycle z below the E_q level $\kappa - \varepsilon$ and that z is not homologous within $\tilde{\mathcal{P}}_q^{\tilde{\kappa}} \cup V$ to any cycle strictly below the level $\tilde{\kappa} := E_q(\tilde{\gamma})$, where V is some Morse chart around $\tilde{\gamma}$; see Proposition 2.7.

We now fix a Morse chart V around $\tilde{\gamma}$. By choosing δ smaller if necessary, we can assume that $\tilde{U}_c$ lies in $\tilde{\mathcal{P}}_q^{\tilde{\kappa}} \cup V$ for all $\|c\| \le \delta$.

We now assume that there is a c_0 with $\|c_0\| \le \delta$ such that $\kappa_{c_0} < \kappa$. By choosing t_0 smaller if necessary we can assume that E_q is strictly smaller than $E_q(\tilde{\gamma})$ on $\tilde{U}_{c_0}$. Let $c(t)$ be a path between 0 and c_0 such that $\|c(t)\| \le \delta$. Then $\tilde{U}_{c(t)}$ gives a homotopy between the cycles $\tilde{U}_0$ and $\tilde{U}_{c_0}$ keeping the boundaries below the E_q level $\kappa - \varepsilon$. This induces a homotopy of z within $\tilde{\mathcal{P}}_q^{\tilde{\kappa}} \cup V$ that deforms z below the E_q-level $E_q(\tilde{\gamma})$. This contradicts Proposition 2.7. It follows that $\kappa_c = \kappa$ for all $\|c\| < \delta$, thus finishing the proof. □

THEOREM 2.9. *Let $\phi : M \to N$ be a taut immersion. Then, given any $p \in N$, the map between the homology groups*

$$H_*(\mathcal{M}_p^r) \to H_*(\mathcal{M}_p)$$

induced by the inclusion of $\mathcal{M}_p^r$ into $\mathcal{M}_p$ is injective for all $r \ge 0$. In particular, the energy function E_p is a perfect Morse–Bott function.

PROOF. If E_p is a Morse function, it is perfect by the definition of tautness, and the claim of the theorem follows by standard Morse theory (see the remarks before Proposition 2.7). We therefore assume that E_p is not a Morse function. Then it is a Morse–Bott function by Theorem 2.8. We have to show that it is a perfect Morse–Bott function. Now assume that $H_*(\mathcal{M}_p^r) \to H_*(\mathcal{M}_p)$ is not injective. Let z be a nontrivial cycle in $\mathcal{M}_p^r$ that is homologous to zero in $\mathcal{M}_p$.

Let w be a chain in $\mathcal{M}_p$ such that $\partial w = z$. Let q be a point close to p such that E_q is a Morse function. Let $F_p : \mathcal{M}_p \to \mathcal{M}_q$ be the map that sends $(r, \gamma) \in \mathcal{M}_p$ to $(r, \tilde{\gamma})$, where $\tilde{\gamma}$ is the curve we get by adding onto γ the geodesic segment between p and q and then reparametrizing it between 0 and 1. We define $F_q : \mathcal{M}_q \to \mathcal{M}_p$ similarly. It follows, since E_p is a Morse–Bott function, that the critical levels of E_p are isolated. We assume that q is so close to p that there is an $\varepsilon > 0$ such that $(r,\, r{+}3\varepsilon) \subset R(E_p)$ and $F_p(\mathcal{M}_p^r) \subset \mathcal{M}_q^{r+\varepsilon}$ and $F_q(\mathcal{M}_q^{r+\varepsilon}) \subset \mathcal{M}_p^{r+2\varepsilon}$. There is an obvious continuous deformation of $F_q(F_p(\gamma))$ into γ since these curves only differ up to parametrization by paths that go back and forth between p and q. We assume that q is so close to p that the deformation of $F_q(F_p(\mathcal{M}_p^r))$ into $\mathcal{M}_p^r$ takes place within $\mathcal{M}_p^{r+3\varepsilon}$. Now let $\tilde{z} = F_p(z)$ and $\tilde{w} = F_p(w)$. Since $\tilde{z} = \partial \tilde{w}$, we see that $\tilde{z}$ is homologous to zero in $\mathcal{M}_p$. The injectivity of $H_*(\mathcal{M}_q^{r+\varepsilon}) \to H_*(\mathcal{M}_q)$ implies that there is a chain $\tilde{y}$ in $\mathcal{M}_q^{r+\varepsilon}$ such that $\tilde{z} = \partial \tilde{y}$. Set $y = F_q(\tilde{y})$. Notice that y lies in $\mathcal{M}_p^{r+2\varepsilon}$. Notice also that $F_q(\tilde{z})$ and z are homologous within $\mathcal{M}_p^{r+3\varepsilon}$. It follows that z is homologous to zero in $\mathcal{M}_p^{r+3\varepsilon}$ since $\partial y = F_q(\tilde{z})$, and z and $F_q(\tilde{z})$ are homologous in $\mathcal{M}_p^{r+3\varepsilon}$. This is a contradiction since $\mathcal{M}_p^r$ is a deformation retract of $\mathcal{M}_p^{r+3\varepsilon}$. This finishes the proof of the theorem. $\square$

COROLLARY 2.10. *Suppose (N, g) is a simply connected complete Riemannian manifold, and M is a connected taut submanifold of N. If γ_0 is an index* 0 *critical point of E_p with nullity k on $P(N,\, M{\times}p)$ then $E_p(\gamma_0)$ is the absolute minimum of E_p, and $C = E_p^{-1}(E_p(\gamma_0))$ is a connected k-dimensional critical submanifold of E_p.*

3. Variationally Complete and Polar Actions

DEFINITION 3.1. Let G act on a complete Riemannian manifold N isometrically. A Jacobi field J is called *G-transversal* if it is the variational field of a family of geodesics that are perpendicular to the orbits. The G-action is called *variationally complete* if any G-transversal Jacobi field that is tangent to orbits at two points is the restriction of some Killing field on N induced by the action.

One of the main results (Theorem I) of the paper [Bott and Samelson 1958], reformulated in our terminology, is the following theorem:

THEOREM 3.2. *Suppose G acts on N by isometries, and the G-action is variationally complete. Then the orbits of G are taut.*

Let (G, K) be a symmetric pair, and let $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the corresponding Cartan decomposition. There are three natural actions associated to the symmetric pair:

(i) K acts on G/K by $g \cdot (hK) = (gh)K$.

(ii) The adjoint representation of G on $\mathcal{G}$ restricted to K leaves $\mathcal{P}$ invariant. So K acts on $\mathcal{P}$ by $\mathrm{Ad}(K)$, which is also the isotropy representation of G/K at eK.

(iii) The group $K \times K$ acts on G by $(k_1, k_2) \cdot g = k_1 g k_2^{-1}$.

Bott and Samelson apply the preceding theorem to symmetric spaces to prove the next result [Bott and Samelson 1958, Theorem II]:

THEOREM 3.3. *Let (G, K) be a symmetric pair, and let $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the corresponding Cartan decomposition. Then the action of $K \times K$ on G, the action of K on G/K, and the action of K on $\mathcal{P}$ are variationally complete. Hence the orbits of these actions are taut.*

Bott had earlier [1956] proved important special cases of this theorem. The action of a compact Lie group G on itself by conjugation is variationally complete, and the same is true for the adjoint representation of G on its Lie algebra $\mathcal{G}$. In these two cases everything is much simpler and one does not really need Theorem 3.2 to prove the tautness of the orbits because all indices of critical points are even in these cases.

Let G be a compact Lie group equipped with a bi-invariant metric, and L a closed subgroup of $G \times G$. Then L acts on G isometrically by $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. Hermann [1960] generalized the first part of Theorem 3.3 as follows:

THEOREM 3.4. *Let (G, H) and (G, K) be two symmetric pairs of the compact Lie group G. Then the action of $H \times K$ on G and the action of H on G/K are variationally complete.*

Conlon [1971] found a geometric condition on isometric actions that implies variational completeness. We will use a terminology that differs somewhat from his.

DEFINITION 3.5 [Palais and Terng 1987]. Let G be a compact Lie group acting on the complete Riemannian manifold N by isometries. The G-action on N is said to be *polar* if there is a closed submanifold Σ of N that meets all orbits of G, and every intersection between Σ and an orbit is perpendicular. Such Σ is called a *section*. If the section is flat, the action is said to be *hyperpolar*.

It is easy to see that sections of a polar action are totally geodesic [Palais and Terng 1987]. So polar actions on flat Riemannian manifolds are hyperpolar.

THEOREM 3.6 [Conlon 1971]. *A hyperpolar action is variationally complete.*

COROLLARY 3.7. *The orbits of hyperpolar actions are taut.*

REMARK 3.8. All the variationally complete examples in Theorem 3.3 and 3.4 are hyperpolar.

EXAMPLE 3.9 [Heintze et al. 1995]. Recall that the *cohomogeneity* of a G-action on N is defined to be the codimension of the principal orbits. If an isometric G-action on N is of cohomogeneity one and the normal geodesics of principal orbits are closed, then the G-action is hyperpolar. Now suppose G/K is a rank-2 symmetric space, $\mathcal{G} = \mathcal{K} + \mathcal{P}$ is the corresponding Cartan decomposition, and the dimension of G/K is n. Then the K action on $\mathcal{P}$ leaves the unit sphere S^{n-1}

of $\mathcal{P}$ invariant, and the induced action of K on S^{n-1} is hyperpolar. Moreover, the action of $K \times \mathrm{SO}(n-1)$ on $\mathrm{SO}(n)$ is also hyperpolar. These examples of hyperpolar actions are different from those given in Theorem 3.3 and 3.4.

The following three problems arise naturally in the study of taut orbits in symmetric spaces:

PROBLEM 3.10. Classify all cohomogeneity one actions on symmetric spaces.

PROBLEM 3.11. Classify all hyperpolar actions on symmetric spaces.

PROBLEM 3.12. Classify all taut orbits in symmetric spaces.

When the symmetric space N is $\mathbb{R}^n$ or S^n, Problem 3.11 is solved by Dadok's Theorem [Dadok 1985]. In fact, he proved that if $\rho : H \to \mathrm{SO}(n)$ is polar then there exist a symmetric space G/K and a linear isometry $A : \mathbb{R}^n \to \mathcal{P}$ such that A maps H-orbits onto K-orbits, where $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is the corresponding Cartan decomposition. When the symmetric space N is the hyperbolic space H^n, Problem 3.11 is solved by Wu [1992].

Although there exist many examples, Problems 3.10 and 3.11 are far from being solved for general symmetric spaces. Next we explain the reduction of these problems to problems concerning Lie algebras. To explain this, we note that, since the group of isometries of a simply connected, compact symmetric space G/K is G, to classify hyperpolar actions on G/K it suffices to find all closed subgroups H of G such that the action of H on G/K is hyperpolar. It was proved in [Heintze et al. 1995] that for a closed subgroup H of G, the action of H on G/K is hyperpolar if and only if the action of $H \times K$ on G is hyperpolar. So to classify hyperpolar actions on compact symmetric spaces, it suffices to classify hyperpolar actions on simply connected, compact Lie groups. In fact, this was further reduced to a problem on Lie algebras:

THEOREM 3.13 [Heintze et al. 1995]. *Let G be a simply connected, compact Lie group equipped with the bi-invariant metric defined by the negative of the Killing form on $\mathcal{G}$, and let H be a closed subgroup of $G \times G$. Then the following statements are equivalent*:

(i) *The H-action on G is hyperpolar.*
(ii) *$g_0^{-1}\nu(H \cdot g_0)$ is abelian for some principal orbit $H \cdot g_0$.*
(iii) *There exists $g_0 \in G$ such that the orthogonal complement of*

$$\{g_0 x g_0^{-1} - y : (x, y) \in \mathcal{H}\}$$

is an abelian subalgebra of $\mathcal{G}$.

The problem of finding all $\mathcal{H}$ satisfying condition (iii) is still unsolved. For further results on hyperpolar actions, see [Alekseevskiĭ and Alekseevskĭ 1992; 1993; Heintze et al. 1994; 1995].

Problem 3.12 is not even solved for $\mathbb{R}^n$. In fact, it is not known what all the variationally complete actions on $\mathbb{R}^n$ are. Proposition 2.10 of [Terng 1991] asserted that an isometric action of G on $\mathbb{R}^n$ is variationally complete if and only if it is polar, but this must now be regarded as unproven: The proof depended on the main theorem of [Carter and West 1990], which stated that a totally focal submanifold of $\mathbb{R}^n$ is isoparametric, and which in turn is unsettled by the recent discovery of a gap in the demonstration of Theorem 5.1 of the same paper. (A submanifold M of $\mathbb{R}^n$ is *totally focal* if $\eta^{-1}(C)$ consists of all critical points of the endpoint map $\eta : \nu(M) \to \mathbb{R}^n$, where C is the set of all singular values of η. In other words, M is a totally focal submanifold of $\mathbb{R}^n$ if for any $a \in \mathbb{R}^n$ the critical points of the distance squared function f_a are either all nondegenerate or all degenerate.)

EXAMPLE 3.14. There are examples of orthogonal representations all of whose orbits are taut although the representations are neither polar nor variationally complete. For example, the orbits of the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n \times \mathbb{R}^n$, for $n \geq 3$, defined by $g \cdot (v, w) = (gv, gw)$ are all taut by arguments as given in [Pinkall and Thorbergsson 1989]. First note that principal orbits of this action are diffeomorphic to the Stiefel manifold of orthonormal two-frames in $\mathbb{R}^n$, and the singular orbits are S^{n-1} and 0. To prove all orbits are taut, we note:

(i) If $\{v, w\}$ is orthonormal, then the $\mathrm{SO}(n)$-orbit through (v, w) is the standard embedding of the Stiefel manifold of orthonormal two-frames of $\mathbb{R}^n$ as a singular orbit of the isotropy representation of the symmetric space $\mathrm{Gr}(2, n)$. So it is taut.

(ii) If v and w are linearly independent and $v = ae_1 + be_2$ and $w = ce_1 + de_2$, where e_1, e_2 is an orthonormal two-frame, then the $\mathrm{SO}(n)$-orbit $M_{(v,w)}$ through (v, w) is the image of the orbit through (e_1, e_2) under the linear transformation $(x, y) \mapsto (ax + by, cx + dy)$. Since tightness is invariant under linear transformations and a taut submanifold in Euclidean space is tight, the orbit $M_{(v,w)}$ is tight. But $M_{(v,w)}$ lies in a sphere, so it is taut.

(iii) If (v, w) has rank one, then the orbit $M_{(v,w)}$ is a standard S^{n-1}, hence taut.

It is proved in [Heintze et al. 1994] that a polar representation cannot have repeated irreducible factors. So this action is not polar. To see this action is not variationally complete, we first note that a focal submanifold of an isoparametric submanifold is not totally focal. So $M_{(e_1,e_2)}$ is not totally focal. Now if the action of $\mathrm{SO}(n)$ on $\mathbb{R}^{2n}$ is variationally complete then its principal orbits must be totally focal [Terng 1991], which then implies that $M_{(e_1,e_2)}$ is totally focal, a contradiction.

Using a similar argument, we see that all orbits of the action of $\mathrm{SO}(n)$ on k copies of $\mathbb{R}^n$ by $g \cdot (v_1, \ldots, v_k) = (gv_1, \ldots, gv_k)$ with $k \leq n$ are taut. Similar constructions also work for other classical groups too. In fact, all orbits of k copies of the standard representation of $\mathrm{SU}(n)$ with $k < n$ on $\mathbb{C}^{kn}$ are taut, and all orbits of k (with $k \leq n$) copies of the standard representation of $\mathrm{Sp}(n)$ on $\mathbb{H}^{kn}$ are taut. (Here $\mathbb{H}$ denotes the quaternions.)

Example 3.15. Examples of inhomogeneous isoparametric hypersurfaces in spheres were first given in [Ozeki and Takeuchi 1975], and then in a more systematic way in [Ferus et al. 1981]. It is shown in the latter paper that there is even an inhomogeneous isoparametric hypersurface in a sphere with a homogeneous focal manifold M_-. One sees easily that M_- cannot be an orbit of a polar representation. It is not difficult to show that

$$M_- = \{(u,v) \in \mathbb{H}^n \times \mathbb{H}^n : \|(u,v)\| = 1 \text{ and } u = \alpha v \text{ for some } \alpha \in \mathrm{Sp}(1)\}.$$

Clearly, M_- is an orbit of $\mathrm{Sp}(1)\times\mathrm{Sp}(n)$ acting on $\mathbb{H}^n \times \mathbb{H}^n$ by $(\alpha, A)\cdot(u_1,u_2) = (Au_1, \alpha A u_2)$.

The above examples explain the complexity of the following open problem:

Problem 3.16. Classify all orthogonal representations all of whose orbits are taut.

It follows from the definition of a hyperpolar action that the cohomogeneity of such an action on a rank-k symmetric space has to be at most k. In particular, this implies that a hyperpolar action on S^n must be of cohomogeneity one. A polar action on S^n in general need not be variationally complete. To see this, first we recall that the set of focal points of a principal orbit of a variationally complete action is the set of all singular points of the action [Bott and Samelson 1958; Terng 1991]. Now suppose $\rho : G \to O(n)$ is irreducible, polar, and of cohomogeneity $k \geq 3$, and that M is a principal orbit in S^{n-1}. Then the action of G on S^{n-1} is polar and of cohomogeneity $k-1 \geq 2$. Moreover, $\tau(M)$ is again a principal orbit in S^{n-1}, where $\tau : S^{n-1} \to S^{n-1}$ is the antipodal map. But $\tau(x)$ is a focal point of multiplicity $k-2$ of M with respect to x for all $x \in M$. So the set of all focal points of M as a submanifold of S^{n-1} is the union of the set of singular points and $\tau(M)$. This implies that the action of G on S^{n-1} is not variationally complete. But G-orbits are taut in S^{n-1}. These examples lead us naturally to the following question:

Question 3.17. Are orbits of polar actions on a symmetric space taut?

4. Equifocal and Weakly Equifocal Submanifolds

The notions of equifocal and weakly equifocal submanifolds in symmetric spaces were introduced in [Terng and Thorbergsson 1995]. These submanifolds give new examples of nonhomogeneous taut submanifolds, and are geometric analogues of the principal orbits of hyperpolar actions on symmetric spaces. In this section, we will review some results on these submanifolds proved in [Terng and Thorbergsson 1995].

First, we summarize some geometric and topological properties of principal orbits of hyperpolar actions. Suppose the action of G on a compact Riemannian manifold N is hyperpolar, and M is a principal G-orbit in N. Then [Bott and Samelson 1958; Palais and Terng 1987] we can say that:

(a) $\nu(M)$ is flat and has trivial holonomy. In fact, given $v \in \nu(M)_p$, let $\tilde{v}(g \cdot p) = g_*(v)$. Then $\tilde{v}$ gives a well-defined equivariant normal vector field on M, and $\tilde{v}$ is parallel with respect to the induced normal connection.

(b) $\exp(\nu(M)_x)$ is a closed flat submanifold of N for all $x \in M$.

(c) $v \in \nu(M)_p$ is a focal normal of multiplicity k of M with respect to p if and only if $\tilde{v}(x)$ is a focal normal of multiplicity k of M with respect to x for all $x \in M$.

(d) $y \in N$ is a focal point of multiplicity k of M if and only if $G \cdot y$ is a singular orbit and $k = \dim(G \cdot p) - \dim(G \cdot y)$.

(e) Let $\eta : \nu(M) \to N$ denote the endpoint map, take $v \in \nu(M)_p$, and let

$$M_v = \{\eta(\tilde{v}(x)) : x \in M\}.$$

Then $M_v = G \cdot \exp(v)$. Moreover, the map $\eta_v : M \to M_v$ defined by

$$\eta_v(x) = \exp(\tilde{v}(x)) = \exp(g_*(v)) = g \cdot \exp(v)$$

is a fibration, and the fiber $\eta_v^{-1}(y)$ is diffeomorphic to a principal orbit of the slice representation of G at y. $\big($Recall that the slice representation at y is the representation of G_y on $\nu(G \cdot y)_y$ defined by $g * v = g_*(v)\big)$.

(f) A point $q \in N$ is called *subregular* if there is no singular point $x \in N$ such that $G_x \subset G_q$ and $G_x \neq G_q$. Suppose $q = \exp(v)$ for some $v \in \nu(M)_x$ and q is subregular. Then there exists an integer m_q such that q is a focal point of multiplicity m_q of M with respect to all $y \in \eta_v^{-1}(q)$, and $\eta_v^{-1}(q)$ is diffeomorphic to the sphere S^{m_q}. This follows since the slice representation at q is polar with only one nontrivial orbit type.

(g) Bott and Samelson proved that E_a on $P(N, M \times a)$ is perfect for generic $a \in N$ (see Section 3). By definition, therefore, M is taut in N. They proved this by constructing a linking cycle at every critical point of E_a. We now give a geometric sketch of their construction. Assume that $a \in N$ is not a focal point of M and all focal points on each critical point γ of E_a on $P(N, M \times a)$ are subregular. Let γ be a critical point of E_a, and let $\gamma(0) = p \in M$. Then there exists $v \in \nu(M)_p$ such that $\gamma(t) = \exp(tv)$ and $\gamma(1) = a$. Let $p_1 = \gamma(t_1)$, $\ldots$, $p_r = \gamma(t_r)$ be focal points on γ with $0 < t_r < \cdots < t_2 < t_1 < 1$, and let $m_i = \dim(G_{p_i}) - \dim(G_p)$ be the dimension of $\eta_{t_i v}^{-1}(p_i)$. Now construct an iterated sphere bundle ξ_r as follows:

$$\xi_r = \{(g_1 p, g_2 g_1 p, \ldots, g_r g_{r-1} \cdots g_1 p) : g_j \in G_{p_j},\ 1 \leq j \leq r\}.$$

Define a smooth map $\phi : \xi_r \to P(N, M \times a)$ by setting $\phi(y_1, \ldots, y_r)$ to the curve that restricted to the interval $\big[i/n, (i+1)/n\big]$ is the image of $\gamma \,|\, \big[i/n, (i+1)/n\big]$ under $g_{i+1} g_i \cdots g_1$. The image of ϕ lies on a constant energy level E_a. By cutting off corners of the broken geodesics in $\phi(\xi_r)$ we can deform $\phi(\xi_r)$ into a linking cycle of E_a at γ, i.e., $\phi(\xi_r)$ a completion of a local unstable manifold at γ below the energy level γ; see Section 2.

We realized that at least in symmetric spaces the properties (d)–(g) only depend on properties (a)–(c). This led us to the definition of equifocal submanifolds as being those submanifolds of symmetric spaces that have these three properties.

To make the definition more precise, we recall that an *r-flat* in a rank-k symmetric space $N = G/K$ is an r-dimensional, totally geodesic, flat submanifold. Let $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ be the corresponding Cartan decomposition. Then every flat is contained in some k-flat, and every k-flat is of the form $\pi(g \exp(\mathcal{A}))$, where $g \in G$ and $\mathcal{A}$ is a maximal abelian subalgebra in $\mathcal{P}$. If N is a compact Lie group of rank k, then a k-flat in N is just a maximal torus. But an r-flat need not be closed in general.

Let M be an immersed submanifold of a symmetric space N. The normal bundle $\nu(M)$ is called *abelian* if $\exp(\nu(M)_x)$ is contained in some flat of N for each $x \in M$. It is called *globally flat* if the induced normal connection is flat and has trivial holonomy.

Let v be a globally defined normal field on M, and $\eta_v : M \to N$ denote the endpoint map associated to v defined by setting $\eta_v(x) = \exp(v(x))$.

Definition 4.1. A connected, immersed submanifold M in a symmetric space N is called *equifocal* if

(1) $\nu(M)$ is globally flat and abelian, and
(2) if v is a parallel normal field on M such that $\eta_v(x_0)$ is a focal point of multiplicity k of M with respect to x_0, then $\eta_v(x)$ is a focal point of multiplicity k of M with respect to x for all $x \in M$. (Equivalently, the focal data $\Gamma(M)$ is "invariant under normal parallel translation").

Examples 4.2. (i) Principal orbits of a hyperpolar action on a compact symmetric space are equifocal since they satisfy properties (a), (b), and (c) of page 199.

(ii) A distance sphere in an irreducible compact symmetric space N is equifocal if and only if N has rank one. This follows from the fact that a geodesic normal to an equifocal hypersurface in an irreducible compact symmetric space is closed (see Theorem 4.8(b) below). As a consequence, if a distance sphere is equifocal, then all geodesics in N are closed and the rank of N is one. However, we will see in Section 6 that distance spheres in compact symmetric spaces are always taut.

Remarks 4.3. (i) It is proved in [Terng 1985] that if M is isoparametric in $\mathbb{R}^n$ then $\nu(M)$ is globally flat. Given a unit vector $v \in \nu(M)_x$, then $t_0 v$ is a focal normal of multiplicity k if and only if $1/t_0$ is a principal curvature of multiplicity k of M in the direction v. So it follows that a submanifold M in $\mathbb{R}^n$ is isoparametric if and only if it is equifocal.

(ii) A hypersurface M in a sphere S^n is called *isoparametric* if it has constant principal curvatures. It turns out that M is isoparametric in S^n if and only if M is equifocal in S^n. The study of isoparametric hypersurfaces in S^n has a long history, and these hypersurfaces have many remarkable properties [Münzner 1980; 1981]. We will make some remarks about equifocal hypersurfaces and isoparametric hypersurfaces towards the end of this section.

(iii) Notice that an isoparametric hypersurface of the real hyperbolic space H^n is equifocal, but the converse is not true. In fact, an equifocal hypersurface in H^n can be characterized by the property that the principal curvatures whose absolute values are greater than one are constant.

Ewert [1997] has proved the following two results:

THEOREM 4.4. *A complete hypersurface in a symmetric space of noncompact type is equifocal if and only if it is a tube around a submanifold without focal points. Furthermore, such an equifocal hypersurface is taut.*

THEOREM 4.5. *Suppose M is a submanifold of a simply connected complete Riemannian manifold N such that M has no focal points. Then M is taut in N.*

Wu [1994] defined a submanifold M in N to be *hyper-isoparametric* if it satisfies the following conditions:

(1) M is curvature adapted, i.e., the operator $B_v(u) = R(v,u)(v)$ leaves TM invariant and commutes with the shape operator A_v for all $v \in \nu(M)$,
(2) $\nu(M)$ is globally flat and abelian,
(3) the principal curvatures along any parallel normal field are constant.

Note that M is hyper-isoparametric if and only if M is curvature adapted and equifocal. Wu independently obtained some of our results by using the method of moving frames. But an equifocal submanifold is in general neither curvature adapted nor has constant principal curvatures. For example, there are many such equifocal hypersurfaces in $\mathbb{CP}^n$ [Wang 1982].

Henceforth, we will assume that $N = G/K$ is a compact, rank-k symmetric space of semisimple type, that $\mathcal{G} = \mathcal{K} + \mathcal{P}$ is a Cartan decomposition, and that N is equipped with the G-invariant metric given by the restriction of the negative of the Killing form of $\mathcal{G}$ to $\mathcal{P}$.

To simplify the terminology we make the following definition:

DEFINITION 4.6. Let M be a submanifold in N, and $v \in \nu(M)_x$. Then t_0 is called a *focal radius* of M with multiplicity m along v if $\exp_x(t_0 v)$ is a focal point of multiplicity m of M with respect to v.

Then a submanifold M with globally flat abelian normal bundle of a compact symmetric space N is equifocal if the focal radii of M along any parallel normal field are constant.

A smooth normal field v on a submanifold M is called a *focal normal field* if $v/\|v\|$ is parallel and there exists an integer k such that $\exp(v(x))$ is a focal point of multiplicity k of M with respect to x for all $x \in M$. If v is a smooth focal normal field of an equifocal submanifold M in N, then $\|v\|$ is constant on

M, and the endpoint map $\eta_v : M \to N$ has constant rank. So the kernel of $d\eta_v$ defines an integrable distribution $\mathcal{F}_v$ with $\eta_v^{-1}(y)$ as leaves, and $M_v = \eta_v(M)$ is an immersed submanifold of N. We will call $\mathcal{F}_v$, $\eta_v^{-1}(y)$ and M_v respectively the *focal distribution*, *focal leaf* and the *focal manifold* defined by the focal normal field v.

THEOREM 4.7 [Terng and Thorbergsson 1995]. *Let v be a parallel normal field on an equifocal submanifold M of a compact, simply connected, symmetric space N. Then $M_v = \eta_v(M)$ is an embedded submanifold. Moreover*:

(a) *M_v is taut*;
(b) *if $\exp(v(x))$ is not a focal point then M_v is again equifocal and the endpoint map $\eta_v : M \to M_v$ is a diffeomorphism*; *and*
(c) *if $\exp(v(x))$ is a focal point then $\eta_v : M \to M_v$ is a fibration and the fiber $\eta_v^{-1}(y)$ is diffeomorphic to a finite dimensional isoparametric submanifold in the Euclidean space $\nu(M_v)_y$.*

If M is a principal orbit of a hyperpolar G-action on N, then (b) and (c) are consequences of the following facts:

(i) If v is a parallel normal field on M, then $\eta_v(g \cdot x_0) = g \cdot \exp(v(x_0))$.
(ii) The focal points of M are the set of singular points of the G-action.
(iii) The slice representation of a polar action is hyperpolar. So in particular, the orbits of the slice representation are isoparametric.

The basic ideas in the proof of Theorem 4.7(a) are:

(1) There is a geometric analogue of subregular points for equifocal submanifolds.
(2) The focal leaves corresponding to "subregular" points are diffeomorphic to standard spheres.
(3) There is a construction of linking cycles for critical points of E_a that is similar to the one sketched in (g) on page 199.

We also associated to each equifocal submanifold M of N an affine Weyl group W and a marked affine Dynkin diagram. We proved that the critical points of $E_p : P(N, M\times p) \to \mathbb{R}$ and their indices can be described in terms of W and the marked Dynkin diagram. We describe this situation more precisely in the following theorem. Notice that the results are to a large extent analogous to the rich structure theory of isoparametric submanifolds in Euclidean spaces [Terng 1985].

THEOREM 4.8. *Suppose M is a codimension-r equifocal submanifold of a simply connected, compact symmetric space N. Then*:

(a) *For a focal normal field v, the leaf of the focal distribution $\mathcal{F}_v$ through $x \in M$ is diffeomorphic to an isoparametric submanifold in $\nu(M_v)_{\eta_v(x)}$.*
(b) *$\exp(\nu(M)_x) = T_x$ is an r-dimensional flat torus in N for all $x \in M$.*

(c) *There exists an affine Weyl group* W *with* $r+1$ *nodes in its affine Dynkin diagram such that*, *for* $x \in M$,

(i) W *acts isometrically on* $\nu(M)_x$, *and the set of singular points of the* W-*action on* $\nu(M)_x$ *is the set of all* $v \in \nu(M)_x$ *such that* $\exp(v)$ *is a focal point of* M *with respect to* x, *and*

(ii) $M \cap T_x = \exp_x(W \cdot 0)$.

(d) *Let* D_x *denote the Weyl chamber of the* W-*action on* $\nu(M)_x$ *containing* 0, *and let* $\triangle_x = \exp(D_x)$. *Then*

(i) $\exp_x$ *maps the closure of* D_x *isometrically onto the closure of* $\triangle_x$, *and*

(ii) *there is a labeling of the open faces of* $\triangle_x$ *by* $\sigma_1(x)$, $\ldots$, $\sigma_{r+1}(x)$ *and integers* m_1, $\ldots$, m_{r+1} *independent of* x *such that if* $y \in \partial\triangle_x$, *then* y *is a focal point with respect to* x *of multiplicity* m_y, *where* m_y *is the sum of* m_i *such that* y *is in the closure of* $\sigma_i(x)$.

(e) *Let* $p \in N$, *let* v *be a parallel normal field on* M, *and let* E *be the energy functional on the path space* $P(M, p{\times}M_v)$. *Then the* $\mathbb{Z}_2$-*homology of* $P(M, p{\times}M_v)$ *can be computed explicitly in terms of* W *and* m_1, $\ldots$, m_{r+1}; *moreover*,

(i) *if* p *is not a focal point of* M *then* E *is a perfect Morse function*, *and*

(ii) *if* p *is a focal point of* M *then* E *is nondegenerate in the sense of Bott and perfect*.

Remark 4.9. Theorems 4.7 and 4.8 are invalid when N is not simply connected. To see this, let N be the real projective space $\mathbb{RP}^n$ and M a distance sphere in N centered at x_0. Then M is certainly equifocal. Let v be a unit normal field on M. Then there exists $t_0 \in \mathbb{R}$ such that $\exp(t_0 v(x)) = x_0$ for all $x \in M$. Let T_x be the normal circle at a point x in M. Then D_x is an interval, and $\triangle_x = T_x \setminus \{x_0\}$. Moreover, there exists t_1 such that the parallel set M_{t_1} is the cut locus of the center x_0, which is a $\mathbb{Z}_2$-quotient of M, i.e., a projective hyperplane. Notice that the focal variety of M consists of only one point $(x_0, n{-}1)$ and M_{t_1} is not diffeomorphic to M. In fact M_{t_1} has the same dimension as M and satisfies all the conditions in the definition of an equifocal submanifold except that the normal bundle does not have trivial holonomy. Although a parallel manifold M_v of M in a simply connected compact symmetric space N is either equifocal or a focal submanifold, this need not be the case if N is not simply connected.

Definition 4.10. A connected, compact, immersed submanifold M with a globally flat and abelian normal bundle in a symmetric space N is called *weakly equifocal* if, given a parallel normal field v on M, the following conditions are satisfied:

(1) The multiplicities of the focal radius functions along v are constant, i.e., the focal radius functions t_j are smooth functions on M that can be ordered,

$$\cdots < t_{-2}(x) < t_{-1}(x) < 0 < t_1(x) < t_2(x) < \cdots,$$

and the multiplicities m_j of the focal radii $t_j(x)$ are constant on M.

(2) the focal radius function t_j is constant on $\eta_{t_j v}^{-1}(x)$ for all $x \in M_{t_j v}$.

REMARKS 4.11. (i) It is proved in [Terng and Thorbergsson 1995] that condition (2) on the focal radii in the definition of weakly equifocal submanifolds is always satisfied if the dimension of $\eta_{t_j v}^{-1}(x)$ is at least two.

(ii) It follows from the definitions that a (weakly) equifocal submanifold in a rank k symmetric space has codimension less than or equal to k, and that equifocal implies weakly equifocal.

In the following theorem we bring our main results on weakly equifocal submanifolds.

THEOREM 4.12 [Terng and Thorbergsson 1995]. *Suppose M is an immersed, weakly equifocal compact submanifold of a simply connected symmetric space N of compact type. Then*

(a) *M is embedded*,

(b) *M is taut*,

(c) *for a focal normal field v, the set $\eta_v^{-1}(x)$ is diffeomorphic to a taut submanifold of a finite-dimensional Euclidean space for all $x \in M$.*

We now discuss how equifocal and weakly equifocal submanifolds relate to submanifolds of Euclidean space.

DEFINITION 4.13 [Terng 1987]. A submanifold M in $\mathbb{R}^n$ is called *weakly isoparametric* if

(1) $\nu(M)$ is globally flat,

(2) the multiplicities of the principal curvatures λ along a parallel normal field v are constant, and

(3) if $d\lambda(X) = 0$ for X in the eigenspace $E_\lambda(v)$ corresponding to the principal curvature λ.

A submanifold of $\mathbb{R}^n$ is therefore clearly weakly isoparametric if and only if it is weakly equifocal. Pinkall [1985] called a hypersurface in $\mathbb{R}^n$ or S^n *proper Dupin* if the multiplicities of the principal curvatures are constant and $d\lambda(X) = 0$ for X in the eigenspace $E_\lambda(v)$ corresponding to the principal curvature λ. A hypersurface M in $\mathbb{R}^n$ or S^n is therefore weakly equifocal if and only if it is proper Dupin. It was proved in [Thorbergsson 1983] that proper Dupin hypersurfaces are taut. In [Terng 1987] this was generalized to weakly isoparametric submanifolds. These results are of course special cases of Theorem 4.12.

Assume that M is an isoparametric hypersurface of S^n with g distinct constant principal curvatures $\lambda_1 > \cdots > \lambda_g$ along the unit normal field v with multiplicities $m_1, \dots, m_g$. Let E_j denote the curvature distribution defined by λ_j, i.e., $E_j(x)$ is equal to the eigenspace of $A_{v(x)}$ with respect to the eigenvalue $\lambda_j(x)$. Then the focal distributions of M are the curvature distributions E_j. It follows from the structure equations of S^n that there exists $0 < \theta < \pi/g$ such that the principal curvatures are $\lambda_j = \cot(\theta + (j-1)\pi/g)$ with $j = 1, \ldots, g$, and the parallel set $M_t = M_{tv}$ for $-\pi/g + \theta < t < \theta$ is again an isoparametric hypersurface. The focal sets $M^+ = M_{\theta v}$ and $M^- = M_{\theta - \pi/g}$ are embedded submanifolds of S^n with codimension $m_1 + 1$ and $m_g + 1$, respectively, and the focal variety of M in S^n is equal to

$$\{(x, m_1) : x \in M^+\} \cup \{(x, m_g) : x \in M^-\}.$$

Another consequence of the structure equations is that the leaves of each E_j are standard spheres. Using topological methods, Münzner proved that

(1) g has to be $1, 2, 3, 4$ or 6,
(2) $m_i = m_1$ if i is odd, and $m_i = m_2$ if i is even,
(3) S^n can be written as the union $D_1 \cup D_2$, where D_1 is the normal disk bundle of M^+, D_2 is the normal disk bundle of M^- and $D_1 \cap D_2 = M$, and
(4) the $\mathbb{Z}_2$-homology of M can be given explicitly in terms of g and m_1, m_2; in particular, the sum of the $\mathbb{Z}_2$-Betti numbers of M is $2g$.

It is proved in [Thorbergsson 1983] that proper Dupin hypersurfaces have the above properties (1)–(4).

We end this section by restricting ourselves to equifocal hypersurfaces in symmetric spaces to see to which extent our results on equifocal submanifolds generalize the theory of isoparametric hypersurfaces in spheres that we have been sketching.

Assume that M is an immersed compact equifocal hypersurface in a simply connected, compact, semisimple symmetric space N. Then the following statements follow from Theorems 4.7 and 4.8:

(a) The normal geodesics to M are circles of constant length, which will be denoted by l.
(b) There exist integers m_1, m_2, an even number $2g$ and $0 < \theta < l/(2g)$ such that
    (1) the focal points on the normal circle $T_x = \exp(\nu(M)_x)$ are
    $$x(j) = \exp((\theta + (j-1)l/(2g))v(x)) \quad \text{for } 1 \le j \le 2g,$$
    with multiplicity m_1 if j is odd and m_2 if j is even, and
    (2) the group generated by reflections in pairs of focal points $x(j)$, $x(j+g)$ on the normal circle T_x is isomorphic to the dihedral group W with $2g$ elements, and hence W acts on T_x.
(c) $M \cap T_x = W \cdot x$.
(d) Let $\eta_{tv} : M \to N$ denote the endpoint map defined by tv, where v is a unit normal field, and let $M_t = \eta_{tv}(M) = \{\exp(tv(x)) : x \in M\}$ denote the set parallel to M at distance t; then M_t is an equifocal hypersurface and η_{tv} maps M diffeomorphically onto M_t if $t \in (-l/(2g)+\theta,\ \theta)$.

(e) $M^+ = M_\theta$ and $M^- = M_{-l/(2g)+\theta}$ are embedded submanifolds of codimension m_1+1 and m_2+1 in N, and the maps $\eta_{\theta v} : M \to M^+$ and $\eta_{(-l/(2g)+\theta)v} : M \to M^-$ are S^{m_1}- and S^{m_2}-bundles, respectively.
(f) The focal variety $\mathcal{V}(M)$ equals $(M^+, m_1) \cup (M^-, m_2)$.
(g) $\{M_t : t \in [-l/(2g)+\theta,\, \theta]\}$ gives a singular foliation of N, which is analogous to the orbit foliation of a cohomogeneity one isometric group action on N.
(h) $N = D_1 \cup D_2$ and $D_1 \cap D_2 = M$, where D_1 and D_2 are diffeomorphic to the normal disk bundles of M^+ and M^-, respectively.
(i) M_t is taut in N for all $t \in \mathbb{R}$.
(j) The $\mathbb{Z}_2$-homology of $P(N,\, p \times M_t)$ can be computed explicitly in terms m_1 and m_2.

This generalizes most of the theory of isoparametric hypersurfaces in spheres to equifocal hypersurfaces in simply connected compact symmetric spaces. There is only one important result that we have not been able to generalize: Münzner's celebrated restriction on the possible values of g. Bing-le Wu [1995] solves this problem for the rank-one symmetric spaces except the Cayley plane:

Theorem 4.14 [Wu 1995]. *Suppose M is an equifocal hypersurface of a projective space $\mathbb{CP}^n$ or $\mathbb{HP}^n$, and g is the number of focal points along a normal geodesic of M. Then g is either* 2, 4 *or* 6.

5. Dupin Submanifolds in a Complete Riemannian Manifold

In this section, we will introduce the notion of Dupin submanifolds in a general Riemannian manifold and study its relation to tautness. First we review some results concerning Dupin submanifolds in $\mathbb{R}^n$ and in Hilbert spaces. Then we explain a linearization technique developed in [Terng and Thorbergsson 1995], which lifts submanifolds in symmetric spaces to submanifolds in Hilbert space. This lifting technique allows us to apply the extensive theory developed for taut submanifolds in Hilbert spaces to taut submanifolds in symmetric spaces. These results then motivate our definition of Dupin submanifolds in an arbitrary Riemannian manifold.

The spectral theory of the shape operators and the Morse theory of the Euclidean distance squared functions of submanifolds in $\mathbb{R}^n$ are closely related, and they play essential roles in the study of the geometry and topology of submanifolds in $\mathbb{R}^n$. Given a submanifold M of $\mathbb{R}^n$, the shape operator A is a smooth bundle morphism from $\nu(M)$ to the bundle of self-adjoint operators $L_s(TM, TM)$. So there is an open dense subset $\mathcal{V}_0$ of $\nu^1(M)$ such that the principal curvatures are differentiable functions on $\mathcal{V}_0$ with locally constant multiplicities. Reckziegel [1979] proved that given any $v \in \mathcal{V}_0 \cap \nu(M)_{x_0}$ and an eigenspace E_0 of A_v with respect to a nonzero eigenvalue, there exist a connected submanifold S of M through x_0 and a parallel normal field ξ of M along S such that $\xi(x_0) = v$, $TS_{x_0} = E_0$, and TS_x is equal to an eigenspace of $A_{\xi(x)}$ for all $x \in S$. Moreover, if $d = \dim S > 1$, then S is a standard sphere. However, for $v \notin \mathcal{V}_0$ and E_0 an eigenspace of A_v of dimension > 1, such S may not exist and even when it exists it need not be a standard sphere. This leads to the definition in [Reckziegel 1979] that a connected submanifold S of M is called a *curvature surface* if there is a parallel normal field ξ of M along S such that at each point x of S the tangent plane TS_x is a full eigenspace of $A_{\xi(x)}$. A one dimensional curvature surface is called a *line of curvature*.

DEFINITION 5.1 [Pinkall 1985]. A submanifold M of $\mathbb{R}^n$ is called *Dupin* if every line of curvature is a standard circle.

Recall that a Dupin hypersurface is proper Dupin if the multiplicities of the principal curvatures are constant; see Section 4.

THEOREM 5.2 [Pinkall 1986]. *A taut submanifold M in $\mathbb{R}^n$ is Dupin*, *i.e.*, *the lines of curvature of M are standard circles.*

QUESTION 5.3. Let M be a complete Dupin submanifold of $\mathbb{R}^n$. Is M taut?

THEOREM 5.4 [Pinkall 1986]. *Let M be a compact submanifold of $\mathbb{R}^n$, and let*

$$\nu_\varepsilon(M) = \{x + v : v \in \nu(M)_x,\ \|v\| = \varepsilon\}$$

denote the ε-tube of M. For small ε, the hypersurface $\nu_\varepsilon(M)$ is taut if and only if M is taut. Moreover, the set of focal points of $\nu_\varepsilon(M)$ is $\Gamma(M) \cup M$, where $\Gamma(M)$ is the set of focal points of M in $\mathbb{R}^n$.

If M is Dupin with constant multiplicities, then $\nu_\varepsilon(M)$ is a proper Dupin hypersurface. So, by [Thorbergsson 1983], $\nu_\varepsilon(M)$ and hence also M are taut. Notice that a weakly isoparametric submanifold of $\mathbb{R}^n$ is taut, but the multiplicities of the principal curvatures are in general not constant.

The three basic local invariants (the induced metric, the second fundamental form, and the induced normal connection), the endpoint map, and the focal points and focal radii of an immersed Hilbert manifold M in a Hilbert space V are defined exactly the same way as for immersed submanifolds in $\mathbb{R}^n$. But the spectral theory for the shape operators of M is complicated and the infinite-dimensional differential topology and Morse theory cannot be applied easily without further restrictions. In order to develop a good theory of submanifold geometry in Hilbert space, proper Fredholm submanifolds were introduced in [Terng 1989]. Recall that an immersed finite codimension submanifold M of a Hilbert space V is called *proper Fredholm* (PF) if the endpoint map η restricted to any finite radius normal disk bundle is proper and Fredholm [Terng 1989]. Properness implies that the map $f_a : M \to \mathbb{R}$ defined by $f_a(x) = \|x - a\|^2$ satisfies the Palais–Smale condition C, so we can apply Morse theory to these functions. The Fredholm condition allows us to use infinite-dimensional differential topology of Fredholm maps. Note that, if $V = \mathbb{R}^n$, then M is PF if and only if the immersion is a proper map.

A PF submanifold M of V has the following general properties [Terng 1989]:

(a) The shape operators are compact.
(b) All nonzero principal curvatures have finite multiplicities.
(c) The set of nonfocal points of M is open and dense in V (by the transversality theorem for Fredholm maps).
(d) The set of focal points along the normal ray $\{x + tv : t \in R\}$ is locally finite for any $v \in \nu(M)_x$.

A submanifold M of V is *taut* if it is PF and f_a is perfect for generic $a \in V$. Curvature surfaces of a PF submanifold M in V can be defined exactly the same way as in the finite-dimensional case. Using the same proof as in that case we obtain the following:

THEOREM 5.5. *Suppose M is a PF submanifold of V. Then there exists an open dense subset $\mathcal{V}_0$ of $\nu^1(M)$ such that given any $v_0 \in \mathcal{V}_0 \cap \nu(M)_{x_0}$ and multiplicity k principal curvature λ_0 of M in the direction of v_0 there exists a curvature surface S through x_0 satisfying the following conditions*:

(i) *TS_{x_0} is the eigenspace of A_{v_0} with eigenvalue λ_0.*
(ii) *If $k > 1$ then S is a standard sphere.*
(iii) *If M is taut and $k = 1$, then S is a standard circle.*

It is easy to see that linear subspaces of V with finite codimension are PF and taut. But if the dimension of V is infinite, the unit sphere S centered at 0 is not PF. This is because $\eta^{-1}(0) = S$ and S is not compact, contradicting the condition that η is proper. Since S is contractible, the Euclidean distance squared functions are not perfect.

Using exactly the same proof as in [Pinkall 1986], we can generalize Theorem 5.4 to Hilbert spaces:

THEOREM 5.6. *Suppose M is a PF submanifold of a Hilbert space V. Then the ε-tube $\nu_\varepsilon(M)$ is taut if and only if M is taut.*

A PF submanifold M of a Hilbert space V is called *isoparametric* if $\nu(M)$ is globally flat and the principal curvatures along any parallel normal field are constant. If the multiplicities of principal curvatures (but not necessarily the curvatures themselves) are constant and lines of curvatures are circles, we call *M weakly isoparametric*. A theory of isoparametric and weakly isoparametric submanifolds of Hilbert spaces is developed in [Terng 1989]. In particular, the next result is proved there:

THEOREM 5.7. *If M is a (weakly) isoparametric submanifold of a Hilbert space V, then M is taut.*

To get geometrically and topologically interesting taut submanifolds in V, we need to use infinite-dimensional transformation groups. First we review some definitions. Let G be a Hilbert Lie group and M a Riemannian Hilbert manifold. A smooth isometric G-action on M is called *proper* if $g_n \cdot x_n \to x_0$ and $x_n \to x_0$

in M implies that $\{g_n\}$ has a convergent subsequence in G; and the G-action is called *Fredholm* if every orbit map $G \to G \cdot x$ is a Fredholm map. An isometric, proper Fredholm (PF) action is called *polar* if there exists a smooth, closed submanifold Σ of M such that every orbit meets Σ, and all intersections between Σ and orbits are perpendicular. Such a Σ is called a *section* for the action.

THEOREM 5.8 [Terng 1989]. *The orbits of a polar action on a Hilbert space are isoparametric submanifolds or focal manifolds of isoparametric submanifolds, and hence taut.*

THEOREM 5.9 [Terng 1995]. *Let G be a compact Lie group, H a closed subgroup of $G \times G$, $P(G,H)$ the group of H^1-paths $g : [0,1] \to G$ such that $(g(0), g(1)) \in H$, and $V = H^0([0,1], \mathcal{G})$. Suppose the action of H on G is hyperpolar. Then the action of $P(G,H)$ on V by gauge transformations*

$$g \cdot u = gug^{-1} - g_x g^{-1}$$

is polar.

Therefore the hyperpolar actions in Section 3 provide many examples of taut orbits in Hilbert space. The simplest example of this kind comes from the action of the diagonal group $H = \triangle(\mathrm{SU}(2))$ on $G = \mathrm{SU}(2)$, i.e., the Adjoint action. Let M be a principal Adjoint orbit of SU(2). Then a principal orbit of $P(G,H)$ is a hypersurface of V with Poincaré polynomial

$$1 + 2\sum_{i>0} t^{2i},$$

and is diffeomorphic to $P(\mathrm{SU}(2), e \times M)$. There are exactly two singular orbits in V, which are codimension 3 submanifolds of V with Poincaré polynomial

$$\sum_{i \geq 0} t^{2i}.$$

The singular orbits are diffeomorphic to the group of based loops in SU(2).

Since f_a satisfies condition C, we can prove an analogue of Ozawa's theorem [Ozawa 1986] (see also Section 2):

THEOREM 5.10 [Terng and Thorbergsson 1995]. *Suppose M is a taut submanifold in a Hilbert space V, and take $a \in V$. Then the distance squared function $f_a : M \to \mathbb{R}$ is a perfect Morse–Bott function. Moreover, if x_0 is a critical point of f_a with nullity k, the k-dimensional critical submanifold of f_a at x_0 is a taut submanifold in some finite-dimensional affine subspace of V.*

Next we will explain the relation between tautness in symmetric spaces and in Hilbert spaces. Let $\pi : G \to G/K$ denote the natural fibration defined by $\pi(g) = gK$, and $\phi : L^2([0,1], \mathcal{G}) \to G$ the parallel translation from 0 to 1, i.e., $\phi(u) = g(1)$, where g satisfies $g^{-1} g_x = u$ and $g(0) = e$. Now for a submanifold M of G/K, we set $M^* = \pi^{-1}(M)$ and $\tilde{M} = \phi^{-1}(\pi^{-1}(M))$. One of our main steps in developing the theory of equifocal submanifolds in symmetric spaces is to relate the focal structures of M, M^* and $\tilde{M}$. In fact, we proved in [Terng and Thorbergsson 1995] that the following three statements are equivalent:

(i) M is (weakly) equifocal in G/K.
(ii) $M^* = \pi^{-1}(M)$ is (weakly) equifocal in G.
(iii) $\tilde{M} = \phi^{-1}(M^*)$ is (weakly) isoparametric in $L^2([0,1], \mathcal{G})$.

Hence we can apply the theory of (weakly) isoparametric submanifolds in Hilbert space to obtain many of the properties described in Theorems 4.8 and 4.12. This trick of lifting submanifolds of a symmetric space G/K to submanifolds in Hilbert space can be viewed as a useful linearization process. In fact, we can do the same for tautness:

THEOREM 5.11. *Let $V = L^2([0,1], \mathcal{G})$, and let $\phi : V \to G$ the parallel translation from* 1 *to* 0, *i.e.*, $\phi(u) = g(0)$, *where g is the solution of $g^{-1}g_x = u$ and $g(1) = e$. Let M be a submanifold of G, and set $\tilde{M} = \phi^{-1}(M)$. Then M is taut in G if and only if $\tilde{M}$ is taut in V.*

PROOF. Suppose $p = e^{-a}$. Given $g \in P(G, M\times p)$, let $v = g^{-1}g_x$ and define $\tilde{g}(x) = g(x)e^{ax}$. Then $\tilde{g} \in P(G, M\times e)$ and

$$\tilde{g}^{-1}\tilde{g}_x = e^{-ax}g^{-1}g_x e^{ax} + a = e^{-ax}v(x)e^{ax} + a \in \tilde{M}.$$

Let $F_a : V \to V$ be the isometry defined by $F_a(v) = e^{-ax}ve^{ax} + a$, and $\psi : P(G, G\times p) \to V$ the diffeomorphism defined by $\psi(g) = g^{-1}g_x$. Then $(F_a \circ \psi)(P(G, M\times p)) = \tilde{M}$ and $E_p = f_a \circ F_a \circ \psi$. Since both F_a and ψ are diffeomorphisms, E_p is perfect if and only if f_a is perfect. □

THEOREM 5.12. *Let G/K be a symmetric space, $\pi : G \to G/K$ the natural projection, M a submanifold of G/K, and $M^* = \pi^{-1}(M)$. Then M is taut in G/K if and only if M^* is taut in G.*

PROOF. Fix $g_0 \in G$, and set $p_0 = \pi(g_0)$. Let F denote the diffeomorphism from $P(G/K, p_0\times M)\times P(K, e\times K)$ to $P(G, g_0\times M^*)$ defined by $F(x,k)(t) = \tilde{x}(t)k(t)$, where $\tilde{x}(t)$ is the horizontal lift of $x(t)$ to G with $\tilde{x}(0) = g_0$. Note that if α is a horizontal curve then $\alpha^{-1}\alpha_x \in \mathcal{P}$. Since $\mathcal{K} \perp \mathcal{P}$,

$$\|(\tilde{x}k)'\|^2 = \|\tilde{x}'k + \tilde{x}k'\|^2 = \|\tilde{x}^{-1}\tilde{x}' + k'k^{-1}\|^2 = \|\tilde{x}'\|^2 + \|k'\|^2.$$

So we have

$$E_{g_0}(F(x,k)) = E_{p_0}(x) + E_e(k).$$

Notice that $P(K, e\times K)$ is contractible and that the only critical point of $E_e : P(K, e\times K) \to \mathbb{R}$ is the constant path. This shows that $E_e : P(K, e\times K) \to \mathbb{R}$ is perfect. So E_{p_0} on $P(G/K, p_0\times M)$ is perfect if and only if E_{g_0} is perfect on $P(G, g_0\times M^*)$. □

If M is a taut submanifold of $N = \mathbb{R}^n$ and if C is a critical submanifold of the distance squared function f_a, then C is also taut in $\mathbb{R}^n$ [Ozawa 1986]. But it is not known whether this is true for general N.

QUESTION 5.13. Suppose M is a taut submanifold of a complete Riemannian manifold N and $\tilde{C}$ is a critical submanifold of the energy functional E_p on $P(N, p{\times}M)$, and let $C = \{\gamma(1) : \gamma \in \tilde{C}\}$. Is C taut in N?

Now we are ready to define the notion of Dupin submanifolds in any complete Riemannian manifold.

DEFINITION 5.14. Let M be a submanifold of a complete Riemannian manifold N, let $v_0 \in \nu(M)_{x_0}$ be a focal normal of multiplicity k of M, and set $p_0 = \exp(v_0)$. A k-dimensional submanifold S of N is called a *focal leaf* of M at x_0 if there exists a parallel normal field v of M along S such that

(i) $\exp(v(x)) = p_0$ for all $x \in S$,
(ii) $v(x_0) = v_0$,
(iii) $v(x)$ is a focal normal of multiplicity k of M for all $x \in S$.

DEFINITION 5.15. A submanifold M of a nonnegatively curved complete Riemannian manifold N is called *Dupin* if there exists an open dense subset $\mathcal{V}_0$ of the set $\mathcal{V}$ of all focal normals of M such that given any multiplicity-k focal normal $v_0 \in \mathcal{V}_0 \cap \nu(M)_{x_0}$ there exists a focal leaf S of M through x_0 such that S is a k-dimensional submanifold of the distance sphere centered at $\exp(v_0)$ with radius $\|v_0\|$, and S is diffeomorphic to S^k.

DEFINITION 5.16. A submanifold M of a nonnegatively curved complete Riemannian manifold N is called *proper Dupin* if it is Dupin and the multiplicities are locally constant on the set of focal normals of M.

REMARKS 5.17. (i) When $N = \mathbb{R}^n$, Definitions 5.14, 5.15 and 5.16 agree with the original definitions.

(ii) We restrict ourselves to nonnegatively curved manifolds in Definitions 5.15 and 5.16 for the following reason: If the ambient space N is the real hyperbolic space, then the conditions on M in the definitions would not imply anything for principal curvatures whose absolute value is less than or equal to one. Consequently, the definitions would not agree with the existing definition of Dupin and proper Dupin submanifolds in hyperbolic space. However, Definition 5.16 does make sense for arbitrary complete Riemannian manifold regardless of the sign of the curvature, and "Dupin" in our sense is closely related with the notion of tautness.

(iii) The local theory of Dupin submanifolds in hyperbolic space is equivalent to the one in Euclidean space and the sphere. The reason for this is that "stereographic projections" from the sphere and the hyperbolic space into the Euclidean space respects the Dupin property. The theory of compact Dupin submanifolds is therefore also the same in the real space forms. In hyperbolic space the local theory coincides with the theory of complete Dupin submanifolds for the following reason: The image of the hyperbolic space under the "stereographic projection" is a ball in the Euclidean space. Hence a local Dupin submanifold in hyperbolic space can be mapped to Euclidean space where by a homothety one can arrange that its boundary lies outside of the ball model. Now we can map it back to a complete Dupin submanifold in hyperbolic space.

From Theorems 5.10, 5.11, and 5.12 follows:

PROPOSITION 5.18. *Suppose M is a taut submanifold of the compact symmetric space G/K, and $v_0 \in \nu(M)_{x_0}$ is a focal normal of multiplicity k of M in G/K. Then there exists a k-dimensional focal leaf S through x_0. Moreover, S lies in the distance sphere of radius $\|v_0\|$ centered at* $\exp(v_0)$.

The following proposition follows from Theorems 5.5, 5.11 and 5.12.

PROPOSITION 5.19. *If M is a taut submanifold of a nonnegatively curved symmetric space, then M is Dupin.*

CONJECTURE 5.20. If M is taut in a nonnegatively curved complete Riemannian manifold N, then M is Dupin.

CONJECTURE 5.21. If M is a compact submanifold of a complete Riemannian manifold N, then there exists a subset $\mathcal{F}_0$ of $\nu(M)$ such that

(i) $\mathcal{F}_0$ is an open and dense subset of the set of all focal normals of M, and
(ii) if $v_0 \in \nu(M)_{x_0} \cap \mathcal{F}_0$ is a focal normal of multiplicity k and $k > 1$, there exists a focal leaf S through x_0 such that S is a k-dimensional submanifold of the distance sphere centered at $\exp(v_0)$ with radius $\|v_0\|$ and S is diffeomorphic to S^k.

If Conjecture 5.21 is true and M is a proper Dupin submanifold of N, then using a construction similar to the one explained in (g) on page 199, we can construct a linking cycle for each critical point of the energy functional E_p on $P(N, M \times p)$, i.e., M is taut in N.

The study of equifocal submanifolds (Section 4) indicates that the following more general class of submanifolds in complete Riemannian manifolds might have very rich geometric and topological properties.

DEFINITION 5.22. A submanifold M of a complete Riemannian manifold N is said to have *parallel focal structure* if

(i) $\nu(M)$ is globally flat, and
(ii) given a parallel normal field v on M such that $v(x_0)$ is a focal normal of multiplicity k of M, the vector $v(x)$ is a focal normal of multiplicity k of M for all $x \in M$.

CONJECTURE 5.23. Suppose M is a submanifold of a complete Riemannian manifold N and M has parallel focal structure. Then M is taut.

We end this section with a question of a somewhat different nature.

QUESTION 5.24. Is a totally geodesic submanifold in a compact symmetric space taut? Assume M is taut in N_1 which is totally geodesic in the compact symmetric space N_0. Is then M taut in N_0? If the answer to the second question is yes, it is sufficient to study whether maximal totally geodesic submanifolds are taut in the first question. Notice that it is very easy to find counterexamples to both questions if the ambient space is not symmetric.

6. Taut Points and Spheres

One of the main topics of this section is to study the geometric and topological properties of a Riemannian manifold containing taut points. We discuss the relation between Blaschke manifolds and manifolds all of whose points are taut. We also show that a point is taut if and only if a distance sphere centered at that point is taut. At the end of the section, we study the question whether a taut S^{n-1} in an n-dimensional complete Riemannian manifold is a distance sphere.

Let (N, g) be a complete Riemannian manifold, $p \in N$, and $\gamma_v(t) = \exp(tv)$ for $v \in TN_p$. Let

$$\mu(v) = \sup\{r > 0 : \gamma_v \mid [0, r] \text{ is a minimizing geodesic}\}.$$

If $\mu(v)$ is finite, $\exp(\mu(v)v)$ is called the *cut point* of p along γ_v.

PROPOSITION 6.1. *Suppose (N, g) is a complete simply connected Riemannian manifold of dimension n, with a taut point p. Then the first conjugate point of p along a geodesic coincides with the cut point of p along that geodesic and vise versa.*

PROOF. Since N is simply connected, $P(N, p\times q)$ is connected. By Corollary 2.10, any critical point of index 0 of E_q is an absolute minimum.

Suppose $\gamma_v(t) = \exp(tv)$, $\mu(v) = t_0$ and $\gamma(t_1)$ is the first conjugate point along γ. We will prove that $t_0 = t_1$. First notice that $t_0 \le t_1$ since a geodesic is not minimizing after its first conjugate point. Assume that $t_1 > t_0$. By definition of $\mu(v)$, for $t_2 \in (t_0, t_1)$, $\gamma = \gamma_v \mid [0, t_2]$ does not minimize the distance between p and $\gamma_v(t_2)$. Since there are no conjugate points on γ, γ is a critical point of $E_{\gamma(t_2)}$ with index zero. By the tautness of p, γ is therefore an absolute minimum, a contradiction. This proves that $t_0 = t_1$.

Assume that $\gamma(t_0)$ is a cut point of p along the geodesic γ and that there is no conjugate point of p along γ. Then, for $s > t_0$, $\gamma \mid [0, s]$ is not a minimum point of $E_{\gamma(s)}$ on $P(n, p\times\gamma(s))$. But $\gamma \mid [0, s]$ has index 0. Since p is taut, $E_{g(s)}(\gamma \mid [0, s])$ is an absolute minimum, a contradiction. So there must be conjugate point along γ, say at $\gamma(t_1)$. So $t_0 = t_1$. $\square$

Corollary 6.2. *Suppose (N, g) is a compact simply connected Riemannian manifold of dimension n with a taut point p. Then there is a conjugate point along every geodesic starting in p.*

Proof. Since N is compact, every geodesic γ starting at p has a cut point of p. By Proposition 6.1, this cut point is a conjugate point along γ. □

Theorem 6.3. *If (N, g) is a symmetric space, then every point of N is taut.*

Proof. Suppose $N = G/K$. By Theorem 3.3, orbits of the K-action on N are taut. But the K-orbit at eK is $\{eK\}$. So eK is a taut point. Using Proposition 2.3 and the fact that N is homogeneous, all points in N are taut. □

This gives rise to the following question.

Conjecture 6.4. Suppose all points of (N, g) are taut, and N is homotopy equivalent to a compact symmetric space. Then (N, g) is a symmetric space.

We will prove this conjecture when N is a torus or a sphere by using deep results in Riemannian geometry. Then we will show that it is equivalent to the Blaschke conjecture when N is homotopy equivalent to a compact rank-one symmetric space. So this conjecture is more general than the Blaschke conjecture which is still not settled.

Theorem 6.5. *Let (N, g) be a complete Riemannian manifold without conjugate points. Then all points in (N, g) are taut. Now let $\tilde{g}$ be another Riemannian metric on N with respect to which all points are taut. Then also $\tilde{g}$ has no conjugate points.*

Proof. Let p and q be points in N. Then all critical points of the energy functional E_p on $P(N, p{\times}q)$ have index zero and are nondegenerate since there are no conjugate points. It follows that all points are taut. It also follows that all connected components of $P(N, p{\times}q)$ are contractible. Now let $\tilde{g}$ be a metric on N such that all points are taut. Let γ be a geodesic starting in p. If there is a conjugate point on γ, then we can find a q on γ such that E_q has a nondegenerate critical point with positive index i. By tautness, $P(N, p{\times}q)$ would have a nonvanishing Betti number in a positive dimension contradicting the contractibility of the connected components of $P(N, p{\times}q)$. Hence $\tilde{g}$ has no conjugate points. □

E. Hopf [1948] proved that a Riemannian metric on a two-torus without conjugate points is flat. The generalization of this result to higher dimensions was one of the well-known open problems in Riemannian geometry. It was finally solved by Burago and Ivanov [1994].

Theorem 6.6 [Burago and Ivanov 1994]. *Suppose (T^n, g) has no conjugate points. Then g is flat.*

It follows from Theorem 6.5 that all points on a flat torus are taut. As a consequence of Theorems 6.6 and 6.5, we have:

THEOREM 6.7. *Let g be a metric on T^n so that all points are taut. Then g is flat.*

We now come to Blaschke manifolds [Besse 1978]. We first review some definitions. Let p and q be distinct points in a complete Riemannian manifold M. Set $d = d(p, q)$. Then the *link from p to q* is defined to be the set $\Lambda(p, q)$ of unit vectors $X = \gamma'(d)$ in T_qM where $\gamma : [0, d] \to M$ is a length-minimizing geodesic between p and q. A compact Riemannian manifold is said to be a *Blaschke manifold at p* if for every point q in the cut locus $C(p)$ of p the link $\Lambda(p, q)$ is a totally geodesic sphere in the unit sphere of T_qM. A Riemannian manifold is said to be *Blaschke* if it is Blaschke at all of its points. One says that a compact n-dimensional Riemannian manifold M is *Allamigeon–Warner* at a point $p \in M$ if there is a number $l > 0$ and an integer k between 1 and $n - 1$ such that the first conjugate point on any geodesic starting in p comes after distance l and has multiplicity k. It is proved in [Besse 1978, Theorem 5.43, p. 137] that if (M, g) is Blaschke at p, then M is Allamigeon–Warner at p. Moreover, if M is simply connected, then M is Blaschke at p if and only if it is Allamigeon–Warner at p.

It is well-known that if (N^n, g) is a simply connected compact, rank-one symmetric space, it is Blaschke and hence also Allamigeon–Warner at every point. The integer k in the definition of N being Allamigeon–Warner at a point is 1, 3, 7 and $n - 1$ if N is a complex projective space, a quaternionic projective space, a Cayley plane and a standard sphere respectively. Moreover, the first three nonvanishing Betti numbers of the space of based loops in N are b_0, b_k, b_{k+n-1}.

BLASCHKE CONJECTURE 6.8. Every Blaschke manifold is isometric to a compact symmetric space of rank one.

In spite of the name, the Blaschke conjecture was never made by Blaschke in this generality. It was solved for the two-sphere by L. Green [1963] and for the n-sphere by Berger and Kazdan:

THEOREM 6.9 [Besse 1978]. *If (S^n, g) is a Blaschke manifold, then (S^n, g) is the standard sphere with constant sectional curvature.*

The Blaschke Conjecture is still not settled for the other simply connected, compact rank-one symmetric spaces. But a lot of progress has been made; see the references in [Reznikov 1994]. For example, if (N, g) is a Blaschke manifold, then:

(i) There exists $l > 0$ such that all geodesics of N are closed with length l [Besse 1978].
(ii) N is homeomorphic to a compact rank-one symmetric space [Sato 1984; Yang 1990].
(iii) We may assume that all geodesics of N are closed with length 2π by rescaling the metric. Then the number $i(N) = \mathrm{vol}(N)/\mathrm{vol}(S^n)$ is an integer [Weinstein 1974].

(iv) If N is homeomorphic to a rank-one symmetric space N_0 then $i(N) = i(N_0)$ [Reznikov 1994].

We now want to apply our generalization of Ozawa's theorem 2.8 to study compact manifolds with taut points. In fact, we will show that under certain assumptions on the Betti numbers of the space of based loops, these manifolds are Blaschke manifolds. First we consider the sphere case.

THEOREM 6.10. *Let (N, g) be a simply connected compact Riemannian manifold of dimension n such that the first two nonvanishing Betti numbers of the based loop space are b_0 and b_{n-1}. If $p \in N$ is a taut point, then (N, g) is Blaschke at p and N is homeomorphic to S^n.*

PROOF. Let γ be a geodesic starting in p. By Proposition 6.1 and Corollary 6.2 there is a point q on γ that is the first conjugate point on γ and is also a cut point. Since the path space $P(N, p\times q)$ has trivial homology in dimensions $0 < i < n-1$, it follows from the tautness of p that the multiplicity of q as a conjugate point along γ is $n-1$. We assume that $\gamma(1) = q$. Then $\gamma_0 = \gamma \,|\, [0,1]$ is a critical point of E_q on $P(N, p\times q)$ with index 0 and nullity $n-1$. It now follows from Theorem 2.8 that there is an $(n-1)$-dimensional nondegenerate critical manifold in $P(N, p\times q)$ through γ_0. As a consequence, all geodesics starting in p first meet in q after constant distance l, have their first conjugate point at constant distance l and this conjugate point is q. It follows easily that N is homeomorphic to a sphere. $\square$

As consequence of Theorems 6.9 and 6.10, we have:

COROLLARY 6.11. *Suppose all points of (N^n, g) are taut, and N is homotopy equivalent to S^n. Then (N, g) is isometric to the standard sphere.*

COROLLARY 6.12. *If all points of (S^n, g) are taut, then g is the standard metric.*

REMARK 6.13. There are Riemannian metrics on spheres with some, but not all, points taut. For example, let (S^2, g) be a surface of revolution whose curvature is not constant. Then the SO(2)-action is hyperpolar, and the north and the south poles are fixed points of the action. So both the north and the south poles are taut. Not all points of (S^2, g) are taut, because otherwise g would be the standard metric on S^2 by Corollary 6.12.

THEOREM 6.14. *Let (N, g) be a simply connected compact Riemannian manifold of dimension n. Suppose the first three nonvanishing Betti numbers b_i of the based loop space of N are b_0, b_a and b_{a+n-1} for some $1 \le a < n-1$ and $b_a = 1$. If p is taut in N, then (N, g) is Blaschke at p.*

PROOF. Let $\gamma : [0, \infty) \to N$ be a unit-speed geodesic starting at p. By Corollary 6.2, γ has a conjugate point. Let q be the first conjugate point along γ. Using the tautness of p and the assumption on the Betti numbers of the based loop space, the multiplicity of q must be a. It now follows again from the assumption on the Betti numbers of the loop space and the tautness of p that the multiplicity of the second conjugate point q_2 on γ must be $n-1$. It follows from Theorem 2.8 that all geodesics starting in p will meet at q_2 after distance l, where l is the arc length of γ from p to q_2. Furthermore, all the geodesics of length l from p to q_2 have index a with q_2 as a second conjugate point of multiplicity $n-1$. We claim that $q_2 = p$. To see this, notice that the energy functional $E_p : P(N, p{\times}p) \to \mathbb{R}$ has a critical point γ of index a. It follows that γ is degenerate, since otherwise both γ and γ^{-1} would be nondegenerate critical points of index a contradicting the tautness of p and $b_a = 1$. This implies that $q_2 = p$. Using Theorem 2.8, we obtain that every geodesic γ starting in p will be back to p after length l, the first conjugate point has multiplicity a, and the second conjugate point along γ is $\gamma(l) = p$ with multiplicity $n-1$.

We would now like to show that the first conjugate point on γ comes at distance $l/2$. Assume it comes earlier at distance t_0. We look at γ^{-1} between p and $q = \gamma(t_0 + \varepsilon)$, where ε is a small number such that neither γ nor γ^{-1} have a conjugate point in q and $t_0 + \varepsilon \leq l/2$. Then γ^{-1} from p to q is not a minimizing geodesic between p and q. By tautness it must therefore have a conjugate point. This conjugate point must have multiplicity a by what we have already proved. It follows that $E_q : P(N, p{\times}q) \to \mathbb{R}$ has at least two nondegenerate critical points of index a contradicting tautness and $b_a = 1$.

Now assume that the first conjugate point on γ comes later than $l/2$, say at $\gamma(t_0) = q$. Then $\gamma \,|\, [0, t_0]$ is an index 0 critical point of E_q. So γ is a minimizing geodesic from p to q by tautness of p. But γ^{-1} from p to q is shorter, a contradiction. It follows that the first conjugate point of a geodesic starting in p comes at distance $l/2$ and has multiplicity a.

So we have proved that every geodesic starting in p has its first conjugate point after distance $l/2$ and its multiplicity is a. This is exactly the definition of N being Allamigeon–Warner at p. Since N is simply connected it follows from [Besse 1978, Theorem 5.43] that N is Blaschke at p. This finishes the proof. ☐

Corollary 6.15. *Suppose all points of* (N, g) *are taut, and the first three nonzero Betti numbers of the based loop space of* N *are* b_0, b_a, b_{a+n-1} *for some* $1 \leq a \leq n-1$ *and* $b_a = 1$. *Then* (N, g) *is Blaschke. In particular,* N *is homeomorphic to a compact rank-one symmetric space.*

Corollary 6.16. *Suppose* N *is homotopy equivalent to a simply connected rank-one symmetric space, and* g *is a Riemannian metric on* N *such that all points of* (N, g) *are taut. Then* (N, g) *is Blaschke.*

Next we recall a definition in [Besse 1978]:

Definition 6.17. A Riemannian manifold (N, g) is called a L^p_l-manifold if all geodesics starting at p return to p after length l.

It follows from the proof of Theorem 6.14 that:

Corollary 6.18. *Let (N, g) be as in Theorem* 6.14. *If (N, g) has a taut point p, then (N, g) is a L_l^p-manifold.*

One can say a lot about the topology of L_l^p-manifolds. Using Corollary 6.18 and results from [Bott 1954; Samelson 1963] (compare [Besse 1978, Theorem 7.23]) on L_l^p-manifolds, we have:

Theorem 6.19. *Let (N, g) be as in Theorem* 6.14. *Assume that N has a taut point. Then a must be* 1, 3, 7, *or* $n-1$. *Moreover*:

(1) *If $a = 1$, then $n = 2m$ and N has the homotopy type of $\mathbb{CP}^m$.*
(2) *If $a = 3$, then $n = 4m$ and N has the integral cohomology ring of $\mathbb{HP}^n$.*
(3) *If $a = 7$, then $n = 16$ and N has the integral cohomology ring of CaP^2.*
(4) *If $a = n - 1$, then N is homeomorphic to S^n.*

The next theorem gives a converse of Theorem 6.14, which shows that Conjecture 6.4 is more general than the Blaschke Conjecture 6.8.

Theorem 6.20. *Let N be a simply connected Blaschke manifold. Then all points in N are taut.*

Proof. There is a number l such that every unit-speed geodesic in N is a simple closed geodesic with least period l [Besse 1978, Corollary 5.42]. Here simple means that there is no self-intersection in one period. Moreover, for every p and a point q in the cut locus of p we have $d(p, q) = l/2$ [Besse 1978, Proposition 5.39]. The conjugate points on a closed geodesics come after distance $l/2$, l, $3l/2$ and so on. Now fix p and let $q \neq p$ be a point in N that is not at distance $l/2$ from p. Then every geodesic between p and q has image on the same closed geodesic. It follows that $E_q : P(N, p{\times}q) \to \mathbb{R}$ is a Morse function whose critical points have indices a, $a+n-1$, $2a+n-1$, $2(a+n-1)$ and so on. It is clear that E_q is perfect if $a > 1$ (and hence $n > 2$), since there are no critical points with indices differing by 1. If $a = 1$, then by [Besse 1978, Theorem 7.23], N is homotopy equivalent to a $\mathbb{CP}^m$, where $2m = n$. The loop spaces of N and $\mathbb{CP}^m$ are therefore also homotopy equivalent. It follows that the loop space of N has nontrivial homology in dimensions a, $a+n-1$, $2a+n-1$, $2(a+n-1)$ and so on, and hence that E_q is perfect. □

Combining Corollary 6.16 and Theorem 6.20 we get:

Theorem 6.21. *Let (N, g) be a simply connected compact Riemannian manifold that is homotopy equivalent to a compact rank one symmetric space. Then (N, g) is Blaschke if and only if all points of (N, g) are taut.*

We assume that N is topologically or homotopically a rank-one symmetric space in the discussion above for simplicity. Of course, the following problem is one of our main interests:

Problem 6.22. Suppose (N, g) is a simply connected complete Riemannian manifold such that all points of N are taut. What can we say about the geometry and topology of (N, g)?

The last topic we will deal with in this section is tautness of spheres. First we prove a theorem that relates tautness of a point p and tautness of the distance spheres centered at p.

Theorem 6.23. *Let N be a complete Riemannian manifold and p a point in N. Set $\phi = \exp_p : S_\varepsilon(0) \to N$, where $S_\varepsilon(0)$ is the sphere of radius ε around the origin in TM_p and ε is smaller than the conjugate radius of N at p. Then ϕ is taut if and only if p is taut.*

Proof. Let B denote the geodesic disk $B_\varepsilon(p)$. Since B is contractible, the fibration $P(N, B\times p) \to B$ defined by $\gamma \mapsto \gamma(0)$ is trivial. Hence the fibration restricts to $P(N, \phi\times p)$ is trivial, and we have

$$P(N, \phi\times p) \sim S \times P(N, p\times p).$$

So the Poincaré polynomial of $P(N, \phi\times p)$ is

$$(1 + t^{n-1}) \sum b_k t^k, \tag{6.1}$$

where $n = \dim N$ and b_k is the k-th Betti number of $P(N, p\times p)$. Now let q be some point in N. Then the critical points of $E_q : P(N, \phi\times q) \to \mathbb{R}$ are pairs (r, γ) where γ is a geodesic starting perpendicularly to $\phi(S_\varepsilon(p))$ in $\phi(r)$ and ending in q. There are two possibilities: either after adding to γ or deleting from γ a geodesic segment of length ε we end up in p. A critical point of $E_q : P(N, p\times q) \to \mathbb{R}$ of index k therefore gives rise two critical points of $E_p : P(N, \phi\times q) \to R$ of index k and $k + (n-1)$ respectively, and vise versa. The equivalence of the tautness of S and p now follows from Equation (6.1). $\square$

As a consequence of Theorems 6.3 and 6.23, we have:

Corollary 6.24. *The distance spheres in a symmetric space N are taut if their radius is smaller than the conjugate radius of N.*

Next we study whether a null-homotopic taut sphere S^{n-1} in an n-dimensional manifold (N, g) must be a distance sphere. First remember that if S is an embedded hypersurface of N that is null-homotopic, then $N \setminus S$ has two components. If one of the components is diffeomorphic to an n-dimensional ball, then we say that S *bounds a ball on one side.*

Theorem 6.25. *Let $S = S^{n-1}$ be a null-homotopic embedded taut hypersurface in a complete Riemannian manifold N of dimension n. Assume that S^{n-1} bounds a ball on one side. Then S is a distance sphere.*

Proof. We denote by B the component of $M \setminus S$ that is diffeomorphic to an n-ball. We look at the parallel hypersurfaces of S in B. There must be a singularity in one of them. There is therefore a geodesic γ starting perpendicularly in S, going into B and having a first focal point p in B at distance l. We assume l is minimal with this property. We denote the multiplicity of this focal point by k and want to prove that it is equal to $n-1$. By Theorem 2.8, there is a k-dimensional critical submanifold C of minima of E_p through γ in $P(N,\, S{\times}p)$, which, by tautness represents a nontrivial homology class of $P(N,\, S{\times}p)$. Now C is contained in $P(B, S{\times}p)$. This implies that C represents a nontrivial homology class of $P(B, S\times p)$. Since $P(B,\, S{\times}p)$ does not have any nontrivial homology class of positive dimension less than $n-1$, it follows that $k=n-1$. The focal point p is therefore of index $n-1$ and it follows that C is an $(n-1)$-dimensional family of geodesics starting perpendicularly to S, going into B and meeting in p at distance l. Hence all geodesics starting perpendicularly to S and going into B meet in p at distance l. It follows that S is a geodesic sphere. □

The Differentiable Schoenfliess Theorem says that if $n \geq 5$ an embedded S^{n-1} in S^n always bounds a ball [Milnor 1965, Proposition D, p. 112]. The question whether an embedded and null-homotopic S^{n-1} in an arbitrary smooth n-manifold N bounds a ball is more complicated. It was answered by Ruberman [Ruberman 1997]:

THEOREM 6.26. *Suppose that $i : S^{n-1} \to N$ is a null-homotopic smooth embedding, where N has dimension n. Let $S = i(S^{n-1})$. Then one of the following statements must hold*:

(1) *S bounds a ball on one side.*
(2) *N is a rational homology sphere, the fundamental groups of both components of $N \setminus S$ are finite, and at least one of them is trivial.*

It follows from [Chern and Lashof 1957] that any taut hypersurface in S^n that is homeomorphic to a sphere, is a distance sphere. Together with Theorems 6.25 and 6.26, this implies the following results:

COROLLARY 6.27. *A null-homotopic, embedded taut hypersphere in a symmetric space must be a distance sphere.*

COROLLARY 6.28. *Suppose N is homotopy equivalent to a compact symmetric space which is not a sphere. Then a null-homotopic, embedded, taut hypersphere of (N, g) must be a distance sphere.*

THEOREM 6.29. *Suppose the n-dimensional manifold N is not a rational homology sphere. Then a null-homotopic, embedded taut hypersphere S^{n-1} of (N, g) must be a distance sphere.*

Appendix: Applications of Infinite-Dimensional Morse Theory to Section 2

Let $\phi : M \to N$ be an immersion, and take $p \in N$. It is known that $E_p : P(N, \phi\times p) \to \mathbb{R}$ satisfies the Palais–Smale condition. So we can apply infinite-dimensional Morse theory to E_p. As in Section 2 we set

$$\mathcal{M}_p = P(N, \phi\times p) \ \text{ and } \ \mathcal{M}_p^r = \{\gamma \in \mathcal{M}_p : E_p(\gamma) \le r\},$$

and let $R(E_p)$ and $C(E_p)$ denote the set of regular values and the set of singular values of E_p. In this appendix, we will explain how one can use infinite-dimensional Morse theory to prove the following theorem.

Theorem A.1. *If M is an immersed taut submanifold of N and $p \in N$ and r is a regular value of E_p, then the map*

$$i_* : H_*(\mathcal{M}_p^r) \to H_*(\mathcal{M}_p)$$

induced by the inclusion of $\mathcal{M}_p^r$ in $\mathcal{M}_p$ is injective.

The proof follows the same method as the proof of [Terng 1989, Proposition 5.8], except that here the function E_p changes its domain as p changes in N. But it is easy to see that it suffices to prove analogues of [Terng 1989, Propositions 5.6 and 5.7], here called Lemmas A.3 and A.4, and the rest of the proof is exactly the same.

We will make some remarks at the end of this appendix on how we can prove injectivity of i_* for all r with these methods if we use Čech homology instead of singular homology. Notice that we proved in Section 2 that i_* is injective for all r in singular homology, but that proof relies on Theorem 2.8 and is therefore more difficult. It would be nice to have a proof of Theorem 2.8 that does not use finite-dimensional approximations. The computations below should also be useful for writing such a proof.

Recall that the tangent space of $\mathcal{M}_p$ at γ is the set of all absolutely continuous vector fields v along γ such that $v(0) \in TM_{\gamma(0)}$ and $v(1) = 0$. A Riemannian metric is defined on $\mathcal{M}_p$ by

$$\langle u, v\rangle_1 = \int_0^1 (\nabla_{\gamma'} u(t), \nabla_{\gamma'} v(t))\, dt.$$

The gradient $\nabla E_p(\gamma) \in T(\mathcal{M}_p)_\gamma$ of E_p is implicitly defined by

$$d(E_p)_\gamma(u) = \langle \nabla E_p(\gamma), u\rangle_1$$

for all $u \in T(\mathcal{M}_p)_\gamma$. We next prove a formula for $\nabla E_p(\gamma)$.

Proposition A.2. *Let $\gamma \in \mathcal{M}_p$, and let $\{e_i\}$ be a parallel orthonormal frame field along γ such that $e_1(0), \dots, e_m(0)$ span $TM_{\gamma(0)}$. Write $\gamma'(t) = \sum_i a_i(t)e_i(t)$.*

Then

$$\|\nabla E_p(\gamma)\|_1^2 = \sum_{i\le m}\int_0^1 a_i^2\,dt + \sum_{i>m}\int_0^1 (a_i-\alpha_i)^2 = E(\gamma) - \sum_{i>m}\alpha_i^2,$$

where $\alpha_i = \int_0^1 a_i(t)\,dt$ *denotes the mean value of* a_i.

PROOF. Let F denote $\nabla E_p(\gamma)$, and write $F=\sum_i f_i e_i$. Note that

$$\begin{aligned} d(E_p)_\gamma(u) &= \int_0^1 (\nabla_{\gamma'}u,\gamma')\,dt = \int_0^1 (\gamma',u)'\,dt - \int_0^1 (\nabla_{\gamma'}\gamma',u)\,dt \\ &= \langle F,u\rangle_1 = \int_0^1 (\nabla_{\gamma'}F,\nabla_{\gamma'}u)\,dt \\ &= \int_0^1 (\nabla_{\gamma'}F,u)'\,dt - \int_0^1 (\nabla_{\gamma'}\nabla_{\gamma'}F,u)\,dt, \end{aligned}$$

for all $u\in T(\mathcal{M}_p)_\gamma$. So we have $f_i''=a_i'$, $f_i(1)=0$ for all i, $f_i(0)=0$ for all $i>m$, and

$$\int_0^1 \sum_i (f_i'-a_i)'(t)\,dt = 0.$$

The proposition now follows by a direct computation. □

LEMMA A.3. *If* $[r,s]\subset R(E_p)$, *there exists* $\delta>0$ *such that* $[r,s]\subset R(E_q)$ *if the distance* $d(p,q)<\delta$.

PROOF. If the claim is not true, there exist a sequence $q_n\in N$ converging to p and critical points γ_n of E_{q_n} on $P(N,\,M\times q_n)$ such that $E_{q_n}(\gamma_n)\in[r,s]$. Since γ_n is a critical point of E_{p_n}, there is a $v_n\in TN_{q_n}$ such that

$$\gamma_n(t)=\exp((1-t)v_n) \quad\text{and}\quad \|v_n\|^2=E(\gamma_n).$$

But $q_n\to p$ and $\|v_n\|\le\sqrt{s}$. So there exists a subsequence, still denoted by v_n, converging to some $v_0\in TN_p$. Set $\gamma(t)=\exp((1-t)v_0)$. Then γ is a critical point of E_p and $E(\gamma)\in[r,s]$, a contradiction. □

LEMMA A.4. *If* $[r,s]\subset R(E_p)$, *then there exist* $\delta_1>0,\delta_2>0$ *such that* $d(p,q)<\delta_1$ *and* $E_q(\gamma)\in[r,s]$ *imply that* $\|\nabla E_q(\gamma)\|_1\ge\delta_2$.

PROOF. Suppose the claim is false. First choose a $\delta>0$ as in Lemma A.3 that is also less than the injectivity radius at p. Then there exist a sequence $q_n\in N$ and $\gamma_n\in\mathcal{M}_{q_n}$ such that $d(q_n,p)<\delta$, $q_n\to p$, $E_{q_n}(\gamma_n)\in[r,s]$ and $\|\nabla E_{q_n}(\gamma_n)\|_1\to 0$.

Set $\delta_n=d(p,q_n)$ and let $\beta_n:[1-\delta_n,\,1]\to N$ denote the geodesic joining q_n to p parametrized by arc length, and

$$\tilde\gamma_n(t)=\begin{cases}\gamma_n\!\left(\dfrac{t}{1-\delta_n}\right), & \text{if } t\in[0,\,1-\delta_n],\\ \beta_n(t), & \text{if } t\in[1-\delta_n,\,1].\end{cases}$$

Let $e_i(t)$ be a parallel orthonormal frame along $\tilde{\gamma}_n$ as in Proposition A.2. Write $\gamma_n' = \sum_i a_i^n e_i$, and $\tilde{\gamma}_n' = \sum_i \tilde{a}_i^n e_i$ and $\beta_n' = \sum_i d_i^n e_i$. Then the d_i^n are constant and $\sum_i (d_i^n)^2 = 1$,

$$\tilde{a}_i^n(t) = \begin{cases} \dfrac{1}{1-\delta_n} a_i^n\Big(\dfrac{t}{1-\delta_n}\Big), & \text{if } t \in [0, 1-\delta_n], \\ d_i^n, & \text{if } t \in [1-\delta_n, 1]. \end{cases}$$

A direct computation shows that the mean $\tilde{\alpha}_i^n$ of $\tilde{a}_i^n$ is related to the mean α_i^n of a_i^n by

$$\tilde{\alpha}_i^n = \alpha_i^n + \delta_n d_i^n.$$

Using Proposition A.2, we obtain

$$\|\nabla E_p(\tilde{\gamma}_n)\|_1^2 = \frac{\|\nabla E_{q_n}(\gamma_n)\|_1^2}{1-\delta_n} + \delta_n + \sum_{i>m} \Big(\frac{\delta_n}{1-\delta_n}(d_i^n)^2 - 2\alpha_i^n d_i^n \delta_n - (d_i^n)^2 \delta_n^2\Big).$$

But $\|\nabla E_{q_n}(\gamma_n)\|_1 \to 0$, $\|d^n\| = 1$ and $\delta_n \to 0$. So we have $\|\nabla E_p(\tilde{\gamma}_n)\|_1 \to 0$. Since E_p satisfies condition C, there is a convergent subsequence $\tilde{\gamma}_{n_k} \to \gamma_0$ and γ_0 is a critical point of E_p. It is clear that $E_p(\gamma_0) \in [r, s]$, contradicting the assumption that $[r, s] \subset R(E_p)$. □

Remark A.5. We now explain how Theorem A.1 can be proved for all values r, not only regular ones, if we replace singular homology by Čech homology. We will denote Čech homology by $\check{H}_*$ and singular homology by H_*.

Let r be any real number and let (r_k) be a decreasing sequence of regular values of E_p that converge to r. Notice that $\check{H}_*(\mathcal{M}_p) = H_*(\mathcal{M}_p)$ and $\check{H}_*(\mathcal{M}_p^{r_k}) = H_*(\mathcal{M}_p^{r_k})$ for all k since $\mathcal{M}_p$ and $\mathcal{M}_p^{r_k}$ are manifolds (with or without boundary). By Theorem A.1 we have

$$\check{H}_*(\mathcal{M}_p^{r_k}) \to \check{H}_*(\mathcal{M}_p)$$

is injective for all k. We have

$$\check{H}_*(\mathcal{M}_p^{r}) = \lim_{k\to\infty} \check{H}_*(\mathcal{M}_p^{r_k})$$

by continuity of Čech homology. We also know that $\check{H}_*(\mathcal{M}_p^r) = \check{H}_*(\mathcal{M}_p^{r_k})$ for $k > k_0$ for some k_0 since $\check{H}_*(\mathcal{M}_p^{r_l}) \to \check{H}_*(\mathcal{M}_p^{r_m})$ is injective for $l > m$ and $H_*(\mathcal{M}_p^{r_k})$ is finite-dimensional for all k so that the sequence $(H_*(\mathcal{M}_p^{r_k}))$ must stabilize for big k. It now follows that

$$\check{H}_*(\mathcal{M}_p^{r}) \to \check{H}_*(\mathcal{M}_p)$$

is injective as we wanted to show.

The above argument is quite typical in the theory of tight and taut immersions; see [Kuiper 1980], for example.

It follows from Theorem 2.8 that $\mathcal{M}_p^r$ is homotopy equivalent to a CW-complex for all r. Consequently, Čech and singular homology coincide, and we have injectivity in singular homology in Theorem A.1 for all r (see Section 2).

Acknowledgemnt

We thank Professor D. Ruberman for proving a topological result we need in this paper: A null-homotopic embedded S^{n-1} in N^n bounds a ball on one side if N is not a rational homology sphere (Theorem 6.26). The proof is given in [Ruberman 1997] in this volume.

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