Holomorphic Spaces
MSRI Publications
Volume **33**, 1998

Recent Progress in the Function Theory of the Bergman Space

HÅKAN HEDENMALM

Abstract. The recent developments in the function theory of the Bergman space are reviewed. Key ingredients are: factorization based on extremal divisors, an analog of Beurling's invariant subspace theorem, concrete examples of invariant subspaces of index higher than one, a partial description of zero sequences, characterizations of interpolating and sampling sequences, and some remarks on weighted Bergman spaces.

1. The Hardy and Bergman Spaces: A Comparison

The Hardy space H^2 consists of all holomorphic functions on the open unit disk $\mathbb{D}$ such that

$$\|f\|_{H^2} = \sup_{0<r<1} \left(\int_{\mathbb{T}} |f(rz)|^2 \, ds(z) \right)^{\frac{1}{2}} < +\infty, \tag{1–1}$$

where $\mathbb{T}$ stands for the unit circle and ds is arc length measure, normalized so that the mass of $\mathbb{T}$ equals 1. In terms of Taylor coefficients, the norm takes a more appealing form: if $f(z) = \sum_n a_n z^n$, then

$$\|f\|_{H^2} = \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}}.$$

The Bergman space L^2_a, on the other hand, consists of all holomorphic functions on $\mathbb{D}$ such that

$$\|f\|_{L^2_a} = \left(\int_{\mathbb{D}} |f(z)|^2 \, dS(z) \right)^{\frac{1}{2}} < +\infty,$$

where dS is area measure, normalized so that the mass of $\mathbb{D}$ equals 1. Though the integral expression of the norm is more straightforward than for the Hardy

Key words and phrases. Bergman spaces, canonical divisors.

This research was supported in part by the Swedish Natural Science Research Council.

space, it is more complicated in terms of Taylor coefficients: if $f(z) = \sum_n a_n z^n$, then

$$\|f\|_{L^2_a} = \left(\sum_n \frac{|a_n|^2}{n+1}\right)^{\frac{1}{2}}.$$

The Bergman space L^2_a contains H^2 as a dense subspace. It is intuitively clear from the definition of the norm in H^2 that functions in it have well-defined boundary values in $L^2(\mathbb{T})$. This is however not the case for L^2_a. In fact, there is a function in it which fails to have radial limits at every point of $\mathbb{T}$. This is a consequence of a more general statement due to MacLane [1962]; see also [Luzin and Privalov 1925; Cantor 1964]. Apparently the two spaces H^2 and L^2_a are very different from a function-theoretic perspective.

The Hardy space theory. The classical factorization theory for the Hardy spaces (these are the spaces H^p, with $0 < p \le +\infty$, which are defined by property (1–1), with 2 replaced by p), which relies on work due to Blaschke, Riesz, and Szegö, requires some familiarity with the concepts of a Blaschke product, a singular inner function, an inner function, and an outer function. Let H^∞ stand for the space of bounded analytic functions in $\mathbb{D}$, supplied with the supremum norm. Given a (finite or infinite) sequence $A = \{a_j\}_j$ of points in $\mathbb{D}$, one considers the product

$$B_A(z) = \prod_j \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z} \qquad \text{for } z \in \mathbb{D},$$

which converges to a function in H^∞ of norm 1 if and only if the Blaschke condition $\sum_j 1 - |a_j| < +\infty$ is fulfilled, in which case A is said to be a Blaschke sequence, and B_A is said to be a Blaschke product. We note that for Blaschke sequences A, B_A vanishes precisely on the A in $\mathbb{D}$, with appropriate multiplicities, depending on how many times a point is repeated in the sequence. Moreover, the function B_A has boundary values of modulus 1 almost everywhere, provided that the limits are taken in nontangential approach regions. We also note that if the sequence A fails to be Blaschke, the product B_A collapses to 0. Given a finite positive Borel measure μ on the unit circle $\mathbb{T}$, which is singular to arc length Lebesgue measure, one associates a singular inner function

$$S_\mu(z) = \exp\left(-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right) \qquad \text{for } z \in \mathbb{D},$$

which is in H^∞, and has norm 1 there. Also, S_μ has no zeros in $\mathbb{D}$, and its nontangential boundary values are almost all 1 in modulus. This is the general criterion for a function in H^∞ to be inner: to have boundary values of modulus 1 almost everywhere. A product of a unimodular constant, a Blaschke product, and a singular inner function, is still inner, and all inner functions are obtained this way. If h is a real-valued L^1 function on $\mathbb{T}$, the associated outer function is

$$O_h(z) = \exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, h(\zeta)\, ds(\zeta)\right) \qquad \text{for } z \in \mathbb{D},$$

which is an analytic function in $\mathbb{D}$ with $|O_h(z)| = \exp\big(h(z)\big)$ almost everywhere on the circle, the boundary values of O_h being thought of in the non-tangential sense. The function O_h is in H^2 if and only if $\exp(h)$ is in $L^2(\mathbb{T})$. The factorization theorem in H^2 then states that every nonidentically vanishing f in H^2 has the form

$$f(z) = \gamma B_A(z) S_\mu(z) O_h(z) \qquad \text{for } z \in \mathbb{D},$$

where γ is a unimodular constant, and $\exp(h) \in L^2(\mathbb{T})$. The natural setting for the factorization theory is a larger class of functions, known as the Nevanlinna class. To make a long story short it consists of all functions of the above type, where no additional requirement is made on h, and where the singular measure μ is allowed to take negative values as well. We denote the Nevanlinna class by N. It is well known that $f \in N$ if and only if the function f is holomorphic in $\mathbb{D}$, and

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+ |f(rz)|\, ds(z) < +\infty.$$

The Bergman space case: inner functions. The Bergman space L_a^2 contains H^2. How then does it relate to N? It turns out that there are functions in N that are not in L_a^2, and that there are functions in L_a^2 which are not in N. The latter statement follows from the fact alluded to above that there is a function in L_a^2 lacking nontangential boundary values altogether. The functions in N, on the other hand, all do have finite nontangential boundary values almost everywhere. The former statement follows from a much simpler example: take μ equal to a point mass at, say 1, and consider the function $1/S_\mu$. It is in the Nevanlinna class, but it is much too big near 1 to be in L_a^2.

The classical Nevanlinna factorization theory is ill-suited for the Bergman space. This is particularly apparent from the fact that there are zero sequences for L_a^2 that are not Blaschke. The question is which functions can replace the Blaschke products or more general inner functions in the Bergman space setting. There may be several ways to do this, but only one is canonical from the point of view of operator theory.

A subspace M of H^2 is invariant if it is closed and $zM \subset M$. It is well known that inner functions in H^2 are characterized as elements of unit norm in some $M \ominus zM$, where M is a nonzero invariant subspace. Following Halmos, we call $M \ominus zM$ the wandering subspace for M. For a collection L of functions in H^2, we let $[L]$ stand for the smallest invariant subspace containing L. We note that $u \in H^2$ is an inner function if and only if

$$h(0) = \int_{\mathbb{T}} h(z)\, |u(z)|^2\, ds(z) \qquad \text{for } h \in L_h^\infty(\mathbb{D}), \tag{1–2}$$

$L_h^\infty(\mathbb{D})$ being the Banach space of bounded harmonic functions on $\mathbb{D}$.

We take (1–2) as the starting point in our search for analogues of inner functions for the Bergman space L_a^2. We say that a function $G \in L_a^2$ is L_a^2-inner provided that

$$h(0) = \int_{\mathbb{D}} h(z)\,|G(z)|^2\,dS(z) \qquad \text{for } h \in L_h^\infty(\mathbb{D}). \tag{1–3}$$

A function G of unit norm in L_a^2 is L_a^2-inner if and only if it is in a wandering subspace $M \ominus zM$ for some nonzero invariant subspace M of L_a^2. In contrast with the H^2 case, where $M \ominus zM$ always has dimension 1 (unless M is the zero subspace), this time the dimension may take any value in the range $1, 2, 3, \dots, +\infty$. This follows from the dilation theory developed by Apostol, Bercovici, Foiaş, and Pearcy [Apostol et al. 1985]. The dimension of $M \ominus zM$ will be referred to as the index of the invariant subspace M.

For the space H^2, Beurling's invariant subspace theorem yields a complete description:

Theorem 1.1 [Beurling 1949]. *Let M be an invariant subspace of H^2, and let $M \ominus zM$ be the associated wandering subspace. Then $M = [M \ominus zM]$. If M is not the zero subspace, then $M \ominus zM$ is one-dimensional and spanned by an inner function, call it φ. It follows that $M = [\varphi] = \varphi H^2$.*

A natural question is whether the analogous statement $M = [M \ominus zM]$ (with the brackets referrring to the invariant subspace lattice of L_a^2 this time) holds for general invariant subspaces M of L_a^2. Pleasantly, and perhaps surprisingly, this turns out to be true [Aleman et al. 1996]. We shall return to this matter in Section 3.

2. Factorization of Zeros in the Bergman Space

The treatment of the subject matter of this section is taken from [Hedenmalm 1991; 1994a; Duren et al. 1993; 1994]. It should be mentioned that the first results on factorization in Bergman spaces were obtained by Horowitz [1974], and slightly later, but independently, by Korenblum [1975].

An example. We begin with a simple but illuminating example: a multiple zero at the origin of multiplicity n. Let M_n be the subspace of L_a^2 of all such functions, which is clearly invariant. The associated wandering subspace $M_n \ominus zM_n$ is one-dimensional, and spanned by the unit vector $G_n(z) = \sqrt{n+1}\,z^n$. According to the terminology introduced in the previous section, the function G_n is an L_a^2-inner function. Since it comes from a zero-based invariant subspace, it is a Bergman space analog of a (finite) Blaschke product. Let f be an arbitrary element of M_n, which then has a Taylor expansion $f(z) = \sum_{j=n}^\infty a_j z^j$. It can be factored $f = G_n g$, where $g(z) = (n+1)^{-1/2} \sum_{j=n}^\infty a_j z^{j-n}$. Let us compare the norms of f and g,

$$\|g\|_{L_a^2}^2 = \frac{1}{n+1} \sum_{j=n}^\infty \frac{|a_j|^2}{j-n+1} \le \sum_{j=n}^\infty \frac{|a_j|^2}{j+1} = \|f\|_{L_a^2}^2;$$

here, we used that

$$j+1 \le (n+1)(j-n+1) \qquad \text{for } j=n,\, n{+}1,\, \dots .$$

Since $g = f/G_n$, we see that division by the unit element G_n of the wandering subspace $M_n \ominus zM_n$ is contractive on M_n; in other words, multiplication by G_n is norm expansive on L^2_a.

General zero sets. Now let A be a zero sequence for the space L^2_a, counting multiplicities, and let M_A be the subspace of all functions in L^2_a that vanish on A, with at least the given multiplicity at each point. It is an invariant subspace, and its wandering subspace $M_A \ominus zM_A$ is one-dimensional. Let G_A denote a unit element of the wandering subspace. Let j be the multiplicity of the origin in the sequence A (which is 0 if the origin is not in A). By multiplying G_A by an appropriate unimodular constant, we may suppose that it solves the extremal problem

$$\sup\bigl\{\operatorname{Re} G^{(j)}(0) :\ G \in M_A,\ \|G\|_{L^2_a} = 1\bigr\}. \tag{2–1}$$

The above example with a multiple zero at the origin suggests that the function G_A may be a contractive divisor on M_A. This turns out to be the case. To begin with, we must rule out the possibility that the function G_A may have extraneous zeros.

More general invariant subspaces. We consider a more general situation with an invariant subspace M of index 1, and let G_M be a unit element of the one-dimensional wandering subspace $M \ominus zM$. We shall show that for $f \in H^2$, $G_M f$ is in L^2_a, and that

$$\|f\|_{L^2_a} \le \|G_M f\|_{L^2_a} \le \|f\|_{H^2}. \tag{2–2}$$

One of the inequalities states that multiplication by G_M expands the L^2_a norm of functions in H^2; this is what entails, after some work, that division by G_M is well-defined and norm contractive $M \to L^2_a$. As in the above case $M = M_A$, we may assume, by multiplying G_M by an appropriate unimodular constant, that it solves the extremal problem

$$\sup\bigl\{\operatorname{Re} G^{(j)}(0) :\ G \in M,\ \|G\|_{L^2_a} = 1\bigr\}, \tag{2–3}$$

where j is the multiplicity of the common zero at the origin of all the functions in M. For this reason, we shall refer to G_M as the extremal function for M. Since G_M is an L^2_a-inner function,

$$h(0) = \int_{\mathbb{D}} h(z)\,|G_M(z)|^2\,dS(z) \qquad \text{for } h \in L^\infty_h(\mathbb{D}),$$

and so

$$\int_{\mathbb{D}} h(z)\bigl(|G_M(z)|^2 - 1\bigr)\,dS(z) = 0 \qquad \text{for } h \in L^\infty_h(\mathbb{D}). \tag{2–4}$$

Equation (2–4) has the interpretation that $|G_M|^2 - 1$ annihilates the bounded harmonic functions on $\mathbb{D}$.

Potential theory. Consider the function Φ_M that solves the boundary value problem

$$\begin{cases} \Delta \Phi_M = |G_M|^2 - 1 & \text{on } \mathbb{D}, \\ \Phi_M = 0 & \text{on } \mathbb{T}, \end{cases} \tag{2–5}$$

where Δ is one quarter of the usual Laplacian (this is not so important, really, one can use the standard Laplacian, only later would we then have to use slightly different Green functions). If we play around with Green's formula, in the form

$$\int_{\mathbb{D}} \bigl(v \Delta u - u \Delta v \bigr)\, dS = \int_{\mathbb{T}} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} \right) \tfrac{1}{2}\, ds, \tag{2–6}$$

where the normal derivatives are taken in the outward direction, and forget about regularity requirements, then (2–4) can be reformulated as saying that the normal derivative of Φ_M vanishes on $\mathbb{T}$. If we add this condition to (2–5), this system becomes overdetermined. Elliptic equations of order $2m$ are determined by m boundary data, so we may raise the order of the partial differential equation to 4 and keep a unique solution. This is accomplished by applying a Laplacian to both sides, and we get, in view of $\Delta\bigl(|G_M|^2 - 1\bigr) = |G_M'|^2$, that

$$\begin{cases} \Delta^2 \Phi_M = |G_M'|^2 & \text{on } \mathbb{D}, \\ \Phi_M = 0 & \text{on } \mathbb{T}, \\ \dfrac{\partial}{\partial n} \Phi_M = 0 & \text{on } \mathbb{T}. \end{cases} \tag{2–7}$$

The Green function for Δ^2 is the function $U(z,\zeta)$ on $\mathbb{D} \times \mathbb{D}$ that solves for fixed $\zeta \in \mathbb{D}$

$$\begin{cases} \Delta^2 U(\,\cdot\,,\zeta) = \delta_\zeta & \text{on } \mathbb{D}, \\ U(\,\cdot\,,\zeta) = 0 & \text{on } \mathbb{T}, \\ \frac{\partial}{\partial n} U(\,\cdot\,,\zeta) = 0 & \text{on } \mathbb{T}, \end{cases}$$

and it is given explicitly as

$$U(z,\zeta) = |z-\zeta|^2 \Gamma(z,\zeta) + (1-|z|^2)(1-|\zeta|^2),$$

where

$$\Gamma(z,\zeta) = 2 \log \left| \frac{z-\zeta}{1-\bar{\zeta} z} \right|$$

is the Green function for Δ. These are the expressions obtained when it is agreed to identify locally integrable functions φ with the corresponding measures $\varphi\, dS$ (recall that dS involved some normalization) to interpret the functions as distributions.

By now it should not require too much of a leap of faith to believe that

$$\Phi_M(z) = \int_{\mathbb{D}} U(z,\zeta)\, |G_M'(\zeta)|^2\, dS(\zeta),$$

so that in view of the fact that $0 < U(z,\zeta)$ on $\mathbb{D}\times\mathbb{D}$, $0 < \Phi_M$ on $\mathbb{D}$, unless G_M is constant, in which case $\Phi_M = 0$. If we apply Green's formula (2–6) and recall the definition of Φ_M, it follows that

$$\int_{\mathbb{D}} \big(|G_M(z)|^2 - 1\big)\, |f(z)|^2 \, dS(z) = \int_{\mathbb{D}} \Phi_M(z)\, |f'(z)|^2 \, dS(z),$$

for, say, polynomials f. We rewrite this as

$$\begin{aligned} \int_{\mathbb{D}} |G_M(z) f(z)|^2 \, dS(z) \\ &= \int_D |f(z)|^2 \, dS(z) + \int_{\mathbb{D}} \Phi_M(z)\, |f'(z)|^2 \, dS(z) \\ &= \int_D |f(z)|^2 \, dS(z) + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta)\, |f'(z)|^2 |G_M'(\zeta)|^2 \, dS(z)\, dS(\zeta). \end{aligned} \tag{2–8}$$

Let Ψ solve

$$\begin{cases} \Delta\Psi = -1 & \text{on } \mathbb{D}, \\ \Psi = 0 & \text{on } \mathbb{T}; \end{cases}$$

the solution comes out to be $\Psi(z) = 1 - |z|^2$. The function $\Phi_M - \Psi$ is subharmonic, and has 0 boundary values, so inside $\mathbb{D}$ it must be ≤ 0. In view of what we have already shown, it follows that $0 \le \Phi_M(z) \le \Psi(z) = 1 - |z|^2$. It is well known that

$$\|f\|_{H^2}^2 = \|f\|_{L_a^2}^2 + \int_{\mathbb{D}} (1-|z|^2)\, |f'(z)|^2 \, dS(z) \qquad \text{for } f \in H^2,$$

so that by continuity, identity (2–8) extends to all $f \in H^2$, and (2–2) holds. Let $\mathcal{A}(G_M)$ be the space of all functions $f \in L_a^2$ with

$$\|f\|_{\mathcal{A}(G_M)}^2 = \|f\|_{L_a^2}^2 + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta)\, |f'(z)|^2 |G_M'(\zeta)|^2 \, dS(z)\, dS(\zeta) < +\infty,$$

and let $\mathcal{A}_0(G_M)$ be the closure of the polynomials in $\mathcal{A}(G_M)$. Then multiplication by G_M is an isometry $\mathcal{A}_0(G_M) \to M \subset L_a^2$, and $H^2 \subset \mathcal{A}_0(G_M) \subset \mathcal{A}(G_M) \subset L_a^2$. Moreover, the injection mappings $H^2 \to \mathcal{A}_0(G_M)$ and $\mathcal{A}(G_M) \to L_a^2$ are contractions. It follows that the invariant subspace generated by G_M, $[G_M]$, equals $G_M\mathcal{A}_0(G_M)$. A number of questions appear:

- Is $\mathcal{A}_0(G_M) = \mathcal{A}(G_M)$?
- Is $[G_M] = M$?
- Is $\mathcal{A}(G_M) = \{f \in L_a^2 : G_M f \in L_a^2\}$?

The answer to the first two questions is yes [Aleman et al. 1996] (see Section 3). The answer to the third question is no [Borichev and Hedenmalm 1995; 1997].

Extraneous zeros. Let λ be a point of $\mathbb{D}$, and let G_λ be the extremal function associated with M_λ, the invariant subspace of all functions vanishing at λ. In terms of the Bergman kernel function $k(z,\zeta) = (1-\bar{\zeta}z)^{-2}$, it has the form

$$G_\lambda(z) = \left(1 - \frac{1}{k(\lambda,\lambda)}\right)^{-\frac{1}{2}} \left(1 - \frac{k(z,\lambda)}{k(\lambda,\lambda)}\right),$$

and one quickly verifies that on $\mathbb{T}$, it has modulus bigger than 1, and in $\mathbb{D}$, it has a simple zero at λ, and nowhere else. This means that if $f \in L_a^2$ vanishes at λ, then $f/G_\lambda \in L_a^2$, and since multiplication by G_λ is norm expansive on L_a^2 (see (2–2) and (2–8) for $M = M_\lambda$), $\|f/G_\lambda\|_{L_a^2} \le \|f\|_{L_a^2}$. Now suppose that G_M has an extraneous zero at λ by which we mean that G_M vanishes at λ with a multiplicity higher than that of some element of M. By inspection of the extremal problem (2–3), to which G_M is the unique solution, we see that λ cannot be 0. If we divide G_M by G_λ, we get an element of L_a^2. If G_M/G_λ is in fact in M, then the function $\widetilde{G} = \gamma G_M/G_\lambda$, with $\gamma = \|G_M/G_\lambda\|_{L_a^2}^{-1}$, is a competing function with G_M in the extremal problem (2–3). It has norm 1, belongs to M, and the j-th derivative at 0 is $\widetilde{G}^{(j)}(0) = \gamma G_M^{(j)}(0)/G_\lambda(0)$, which is bigger than $G_M^{(j)}(0)$, as $1 < \gamma$ and $G_\lambda(0) < 1$. Hence $\widetilde{G}$ is more extremal than G_M itself, which leads to a contradiction. So, the assumption that G_M had an extraneous zero must be false. The weak point thus far is that we have not explained why G_M/G_λ was in M in the first place. Recall that $M \ominus zM$ was one-dimensional: from a perturbation argument it follows that $M \ominus (z-\lambda)M$ is one-dimensional for each $\lambda \in \mathbb{D}$. The subspace $(z-\lambda)M$ having codimension one in M means that it consists of all functions in M having an extra zero (or a zero of multiplicity one higher) at λ, so that G_M, having an extraneous zero at λ, must be in $(z-\lambda)M$. The conclusion that G_M/G_λ is in M follows.

Factorization of zeros. Let us see what kinds of conclusions we can draw from the above. For a finite zero sequence A, G_A has no extraneous zeros in $\mathbb{D}$, extends analytically across $\mathbb{T}$ to a rational function with poles at the reflected points in $\mathbb{T}$ of A, and the expansive multiplier property (2–2) implies that $1 \le |G_A|$ on $\mathbb{T}$. One shows that $\mathcal{A}_0(G_A) = L_a^2$ (the norms are different, though equivalent), so that G_A is an expansive multiplier on all of L_a^2. It follows that

$$\|f\|_{L_a^2}^2 = \|f/G_A\|_{L_a^2}^2 + \int_{\mathbb{D}}\int_D U(z,\zeta)\,|(f/G_A)'(z)|^2 |G_A'(\zeta)|^2\, dS(z)\, dS(\zeta), \qquad f \in M_A,$$

and if we go to the limit as A approaches an infinite zero sequence, Fatou's lemma yields a $\ge$ in place of the $=$ sign. In particular, $f/G_A \in \mathcal{A}(G_A) \subset L_a^2$, and division by G_A is norm contractive $M_A \to L_a^2$.

3. General Invariant Subspaces

Most of the results mentioned here are from [Aleman et al. 1996].

The following version of (2–8) will prove useful:

$$\|f\|_{L^2_a}^2 = \|f/G_M\|_{L^2_a}^2 + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta)\,|(f/G_M)'(z)|^2 |G_M'(\zeta)|^2\, dS(z)\, dS(\zeta) \qquad \text{for } f \in [G_M]. \tag{3–1}$$

A skewed projection operator. Let M be an invariant subspace in L^2, and suppose for the moment that it has index 1; G_M denotes the associated extremal function. For $\lambda \in \mathbb{D}$,

$$f = \frac{f(\lambda)}{G_M(\lambda)} G_M + \left(f - \frac{f(\lambda)}{G_M(\lambda)} G_M \right) \qquad \text{for } f \in M$$

offers a unique decomposition of M as a sum, $M = (M \ominus zM) + (z-\lambda)M$, and as the two summands are got by bounded (skewed) projection operators, the subspaces $M \ominus zM$ and $(z-\lambda)M$ are at a positive angle. Note that the first projection,

$$Q_\lambda f = \frac{f(\lambda)}{G_M(\lambda)} G_M \qquad \text{for } f \in M,$$

is well-defined for all $\lambda \in \mathbb{D}$ as we know that G_M has no extraneous zeros. In terms of Q_λ, identity (3–1) can be written as

$$\|f\|_{L^2_a}^2 = \int_{\mathbb{D}} \|Q_\lambda f\|_{L^2_a}^2\, dS(\lambda) + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta) \Delta_z \Delta_\zeta |Q_z f(\zeta)|^2\, dS(z)\, dS(\zeta) \qquad \text{for } f \in [G_M]. \tag{3–2}$$

This form lends itself to generalization to general invariant subspaces M, not necessarily of index 1. Namely, one shows that a skewed decomposition of the type $M = (M \ominus zM) + (z-\lambda)M$ holds in general, so that a corresponding projection $Q_\lambda : M \to M \ominus zM$ can be defined, and it depends analytically on $\lambda \in \mathbb{D}$. Moreover, (3–2) carries over, almost letter by letter:

$$\|f\|_{L^2_a}^2 = \int_{\mathbb{D}} \|Q_\lambda f\|_{L^2_a}^2\, dS(\lambda) + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta) \Delta_z \Delta_\zeta |Q_z f(\zeta)|^2\, dS(z)\, dS(\zeta) \qquad \text{for } f \in [M \ominus zM]. \tag{3–3}$$

Abel summation. Define the bounded linear operator $L : M \to M$ by declaring that $Lf = f/z$ for $f \in zM$, and $Lf = 0$ for $f \in M \ominus zM$. Also, let P be the orthogonal projection $M \to M \ominus zM$. If f is in M, we can decompose f as a sum of an element of $M \ominus zM$ and a "remainder term" by the formula $f = Pf + zLf$. Repeating this for Lf, we obtain $f = Pf + zPLf + z^2L^2f$. Continuing this process, we get the formal series

$$f = Pf + zPLf + z^2PL^2f + z^3PL^3f + \cdots,$$

each term of which is in $[M \ominus zM]$. If the series were to converge to f for each given $f \in M$, the assertion $M = [M \ominus zM]$ would be immediate. However, this is probably not the case in general, so we choose the second best thing: we form the Abel series

$$R_t f = Pf + tzPLf + t^2z^2PL^2f + t^3z^3PL^3f + \cdots \qquad \text{for } 0 \le t < 1,$$

which does converge to an element of $[M \ominus zM]$, in the hope that $R_t f \to f$ as $t \to 1$. The skewed projection Q_λ has a similar series expansion,

$$Q_\lambda f = Pf + \lambda PLf + \lambda^2 PL^2 f + \lambda^3 PL^3 f + \cdots \qquad \text{for } \lambda \in \mathbb{D},$$

which one can use to show that $Q_\lambda R_t = Q_{t\lambda}$. The operator R_t may also be thought of as given by $R_t f(z) = Q_{tz} f(z)$. As $t \to 1$, $Q_{tz} f(z) \to Q_z f(z)$, and $Q_z f(z) = f(z)$, because by the definition of Q_λ, $f(z) - Q_\lambda f(z)$ is zero when $z = \lambda$. If follows that $R_t f(z) \to f(z)$ as $t \to 1$ pointwise in $\mathbb{D}$. It remains to show that the convergence holds in norm, too.

Controlling the norm of the Abel sum. General functional analysis arguments show that we do not really need to show that $R_t f$ tends to f in norm, weak convergence would suffice. Moreover, weak convergence would follow if we only had a uniform bound of the norm of $R_t f$ as $t \to 1$. This is the crux of the problem. By (3–3) and the identity $Q_\lambda R_t = Q_{t\lambda}$,

$$\|R_t f\|^2_{L^2_a} = \int_{\mathbb{D}} \|Q_{t\lambda} f\|^2_{L^2_a}\, dS(\lambda) + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta)\Delta_z\Delta_\zeta |Q_{tz} f(\zeta)|^2\, dS(z)\, dS(\zeta) \qquad \text{for } f \in M. \tag{3–4}$$

Certain regularity properties of the biharmonic Green function $U(z,\zeta)$ can be used to show that, as $t \to 1$, the right hand side of (3–4) tends to

$$\int_{\mathbb{D}} \|Q_\lambda f\|^2_{L^2_a}\, dS(\lambda) + \int_{\mathbb{D}}\int_{\mathbb{D}} U(z,\zeta)\Delta_z\Delta_\zeta |Q_{tz} f(\zeta)|^2\, dS(z)\, dS(\zeta), \tag{3–5}$$

so if we can only bound this expression, we are done. A bound that works is $\|f\|^2_{L^2_a}$. The approach in [Aleman et al. 1996] is based on the identity

$$\|f\|^2_{L^2_a} = \int_{\mathbb{T}} \|Q_{r\lambda} f\|^2_{L^2_a}\, ds(\lambda) + \int_{\mathbb{D}}\int_{\mathbb{T}} (|z|^2 - r^2) \left| \frac{f(z) - Q_{r\lambda} f(z)}{z - r\lambda} \right|^2 ds(\lambda)\, dS(z),$$

for $0 < r < 1$. By cleverly applying Green's theorem to the above right hand side expression and obtaining estimates of “remainder terms” as $r \to 1$, Aleman, Richter, and Sundberg were able to show that the expression (3–5) is no bigger than $\|f\|^2_{L^2_a}$, whence the following analogue of Beurling's theorem follows.

THEOREM 3.1. *Let M be an invariant subspace of L^2_a. Then $M = [M \ominus zM]$.*

Invertibility and cyclicity. A function $f \in L^2_a$ is said to be cyclic if $[f] = L^2_a$. It has been a long standing problem whether there are noncyclic functions $f \in L^2_a$ that are invertible, that is, $1/f \in L^2_a$. A complicated construction of such functions was recently found by Borichev and Hedenmalm [1995; 1997]. The idea is that if the given function f grows maximally fast on a "big" set, then an H^∞ function must be small there and hence cannot lift the small values of f as required by cyclicity. This example has certain consequences for the uniqueness of inner-outer factorization in L^2_a [Aleman et al. 1996]. It is not clear whether the invariant subspace associated to the constructed invertible noncyclic function is in the closure of the collection of zero-based invariant subspaces, with respect to any of the topologies suggested by Korenblum [1993].

4. Zero Sequences

The first results on zero sequences for Bergman space functions were obtained by Horowitz [1974; 1977]. For instance, he showed that the union of two zero sequences need not be a zero sequence. If we consider the corresponding zero-based invariant subspaces, call them M_A and M_B, then $M_A \cap M_B = \{0\}$. Actually, this behavior of index one invariant subspaces is the raison d'être for the invariant subspaces of higher index [Richter 1987]. Also, look at the explicit constructions in Section 6, and [Hedenmalm 1993; Hedenmalm et al. 1996b].

The sharpest results so far were obtained by Seip [1994; 1995]. The tools were borrowed from the fundamental work of Korenblum [1975; 1977] on the topological algebra $A^{-\infty}$, which consists of all functions f holomorphic in $\mathbb{D}$ that meet the growth condition $|f(z)| \le C(1-|z|)^{-N}$ for some positive constants C and N.

For a finite subset F of $\mathbb{T}$, let $\mathbb{T} \setminus F = \cup_k I_k$ be the complementary arcs, and consider the Beurling-Carleson entropy of F,

$$\widehat{\varkappa}(F) = \sum_k \frac{|I_k|}{2\pi}\left(\log\frac{2\pi}{|I_k|} + 1\right).$$

Let $d(\,\cdot\,,\cdot\,)$ be the curvilinear metric $d(e^{it}, e^{is}) = \pi^{-1}|t-s|$ on $\mathbb{T}$, where it is assumed that $|t-s| \le \pi$. The Korenblum star associated with the finite set F is

$$G(F) = \bigl\{z \in \bar{\mathbb{D}} \setminus \{0\} : d(z/|z|, F) \le 1 - |z|\bigr\} \cup \{0\},$$

and for a sequence A of points in $\mathbb{D}$, let

$$\sigma(A, F) = \sum_{z \in A \cap G(F)} \log\frac{1}{|z|}$$

be the local "Blaschke sum". The Korenblum density $\delta(A)$ of the sequence A is the infimum over all β such that

$$\sup_F \bigl(\sigma(A,F) - \beta\widehat{\varkappa}(F)\bigr) < +\infty,$$

the supremum being taken over all finite subsets of $\mathbb{T}$.

THEOREM 4.1 ([SEIP 1994]). *Let A be a sequence of points in $\mathbb{D} \setminus \{0\}$. If A is the zero sequence of some function in L^2_a, then $\delta(A) \le \frac{1}{2}$. On the other hand, if $\delta(A) < \frac{1}{2}$, then A is the zero sequence of some L^2_a function.*

5. Interpolating and Sampling Sequences

A sequence $A = \{a_j\}_j$ of distinct points in $\mathbb{D}$ is said to be a sampling sequence for L^2_a if

$$\sum_j (1-|a_j|^2)^2 |f(a_j)|^2 \asymp \|f\|^2_{L^2} \qquad \text{for } f \in L^2_a,$$

where the $\asymp$ sign means that the left hand side is bounded from above and below by positive constant multiples of the right hand side. The reason why the factor $(1-|a_j|^2)^2$ is needed is that in a more general setting, one should use the reciprocal of the reproducing kernel, $k(a_j, a_j)^{-1}$. Similarly, A is interpolating for L^2_a provided that to every l^2 sequence $\{w_j\}_j$, there exists a function $f \in L^2_a$ such that

$$(1-|a_j|^2) f(a_j) = w_j \quad \text{for all } \ j.$$

Generally, sampling sequences are fat, and interpolating sequences are thin. Clearly, a sampling sequence cannot be a zero sequence for L^2_a. However, every interpolating sequence for L^2_a is also a zero sequence, for the following reason. Take an interpolant for the sequence $w_1 = 1$ and $w_j = 0$ for all other j, and multiply this function by $z - a_1$ to get a nonidentically vanishing function that vanishes on the sequence A. This actually only shows that A must be a subsequence of an L^2_a zero sequence, but it is well known that every subsequence of a zero sequence for L^2_a is itself a zero sequence for L^2_a [Horowitz 1974; Hedenmalm 1991]. Seip [1993] has obtained a complete description of the sampling and interpolating sequences for L^2_a. We shall try to describe the result, but in order to do so, we need some notation.

A sequence $A = \{a_j\}_j$ of points in $\mathbb{D}$ is said to be uniformly discrete if for some $\varepsilon > 0$,

$$\varepsilon \le \left| \frac{a_j - a_k}{1 - \bar{a}_k a_j} \right| \qquad \text{for } j \ne k.$$

For $\lambda \in \mathbb{D}$, let A_λ be the image of A under the conformal automorphism of the unit disk

$$\varphi_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda} z} \qquad \text{for } z \in \mathbb{D}.$$

Associate with A_λ the function $n(r, A_\lambda)$, which counts the number of points of A_λ contained within the disk $r\mathbb{D}$ $(0 < r < 1)$. We shall need the definite integral

$$N(r, A_\lambda) = \int_0^r n(t, A_\lambda)\, dt \qquad \text{for } 0 < r < 1.$$

If $A(r)$ stands for the function

$$A(r) = \log \frac{1+r}{1-r} \qquad \text{for } 0 < r < 1,$$

Seip defines the upper density of A as

$$D^+(A) = \limsup_{r \to 1} \sup_{\lambda \in \mathbb{D}} \frac{N(r, A_\lambda)}{A(r)},$$

and the lower density as

$$D^-(A) = \liminf_{r \to 1} \inf_{\lambda \in \mathbb{D}} \frac{N(r, A_\lambda)}{A(r)}.$$

For the standard Bergman space, his result is as follows.

THEOREM 5.1. *A sequence A of distinct points in $\mathbb{D}$ is sampling for L_a^2 if and only if it can be expressed as a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence A' for which $D^-(A') > \frac{1}{2}$.*

THEOREM 5.2. *A sequence A of distinct points in $\mathbb{D}$ is interpolating for L_a^2 if and only if it is uniformly discrete and $D^+(A') < \frac{1}{2}$.*

6. Invariant Subspaces of Index Two or Higher

Back in Section 1 it was mentioned that an invariant subspace M of the Bergman space L_a^2 may have index $1, 2, 3, \ldots$, whereas the most obvious examples have index 1. For instance, every zero based invariant subspace M_A has index 1, and so does every singly generated one. Invariant subspaces based on singular masses at the boundary have index 1, too [Hedenmalm et al. 1996a]. It is therefore a natural question to ask what these invariant subspaces of higher index look like. A simple constuction was found in [Hedenmalm 1993].

Let A and B be two disjoint zero sequences for L_a^2, and let $C = A \cup B$. Let $M = M_A \vee M_B$, the smallest invariant subspace containing both M_A and M_B; it is obtained as the closure of the sum $M_A + M_B$. It turns out that in this situation, either $M = L_a^2$ or M has index 2. Moreover, what determines which of these alternatives occurs is the fatness of the sequence C. If one of the sequences A and B fails to accumulate on an arc of $\mathbb{T}$, then C is not fat enough, and so $M = L_a^2$. On the other hand, if C is sampling, then M has index 2, because M_A and M_B are at a positive angle. To see this, let $C = \{c_j\}_j$, and use the sampling property

$$\|f\|_{L_a^2}^2 \asymp \sum_j (1 - |c_j|^2)^2 |f(c_j)|^2 \qquad \text{for } f \in L_a^2.$$

Let $f \in M_A$ and $g \in M_B$ be arbitrary. Then for every point c_j of C, we have

$$|f(c_j) + g(c_j)|^2 = |f(c_j)|^2 + |g(c_j)|^2,$$

so that

$$\|f+g\|^2_{L^2_a} \asymp \sum_j (1-|c_j|^2)^2 |f(c_j)+g(c_j)|^2$$
$$= \sum_j (1-|c_j|^2)^2 \big(|f(c_j)|^2+|g(c_j)|^2\big) \asymp \|f\|^2_{L^2_a}+\|g\|^2_{L^2_a}.$$

Consequently, the sum M_A+M_B is closed, and it is easy to show that M has index two: $zM = zM_A + zM_B$ [Hedenmalm 1993].

7. Green Functions for Weights and Factorization

The results discussed in this section are mostly from [Hedenmalm 1996].

It is clear from Section 2 that Green functions for certain elliptic operators of order 4 play an important role in the study of factorization in Bergman spaces. Let ω be a nonnegative sufficiently smooth function in $\mathbb{D}$, and let $L^2_a(\omega)$ be the corresponding weighted Bergman space of all holomorphic functions f in $\mathbb{D}$ with

$$\|f\|^2_{L^2_a(\omega)} = \int_{\mathbb{D}} |f(z)|^2 \omega(z)\, dS(z) < +\infty.$$

It is a Hilbert space if ω is not equal to 0 too frequently near $\mathbb{T}$; this is not so precise, but it is enough for here. For the extremal functions to have a chance to be good divisors, we need to ask of ω that

$$h(0) = \int_{\mathbb{D}} h(z)\omega(z)\, dS(z) \qquad \text{for } h \in L^\infty_h(\mathbb{D}).$$

The relevant Green function is that of the operator $\Delta\omega^{-1}\Delta$ on $\mathbb{D}$, and issue is whether it is positive (or at least nonnegative). It was shown in [Hedenmalm 1994b] that it is positive for the weights $\omega_\alpha(z) = (\alpha+1)\,|z|^{2\alpha}$, with $\alpha > -1$, and the issue at hand is whether this information can be used to tell us anything about weights that are convex combinations of these. For instance, is it true in general that if ω and μ are two weights, with associated Green functions U_ω and U_μ, that we have, with $\omega[t] = (1-t)\omega + t\mu$,

$$(1-t)U_\omega(z,\zeta)+tU_\mu(z,\zeta) \le U_{\omega[t]}(z,\zeta) \quad \text{for } (z,\zeta)\in\mathbb{D}\times\mathbb{D} \text{ and } 0<t<1\,? \tag{7–1}$$

This is probably not so, although I cannot supply an immediate counterexample. However, with some additional information given in terms of the Green functions, it is true. If we apply a Laplacian to $U_\omega(z,\zeta)$, we get

$$\Delta_z U_\omega(z,\zeta) = \omega(z)\big(\Gamma(z,\zeta)+H_\omega(z,\zeta)\big),$$

where $H_\omega(z,\zeta)$ is harmonic in z. We call H_ω the harmonic compensator. If $H_\mu(z,\zeta) \le H_\omega(z,\zeta)$ holds pointwise, then (7–1) holds, and $H_\mu(z,\zeta) \le H_{\omega_t}(z,\zeta)$ also holds pointwise. A consequence of this result is that if

$$\omega(z) = \int_{]-1,+\infty[} (\alpha+1)\,|z|^{2\alpha} d\rho(\alpha),$$

where ρ is a probability measure, then

$$0 \leq \int_{]-1,+\infty[} U_{\omega_\alpha}(z,\zeta)\, d\rho(\alpha) \leq U_\omega(z,\zeta) \qquad \text{for } (z,\zeta) \in \mathbb{D} \times \mathbb{D}.$$

For related work see, for instance, [Shimorin 1993; 1995].

References

[Aleman et al. 1996] A. Aleman, S. Richter, and C. Sundberg, "Beurling's theorem for the Bergman space", *Acta Math.* **177**:2 (1996), 275–310.

[Apostol et al. 1985] C. Apostol, H. Bercovici, C. Foiaş, and C. Pearcy, "Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra, I", *J. Funct. Anal.* **63**:3 (1985), 369–404.

[Beurling 1949] A. Beurling, "On two problems concerning linear transformations in Hilbert space", *Acta Math.* **81** (1949), 239–255.

[Borichev and Hedenmalm 1995] A. A. Borichev and P. J. H. Hedenmalm, "Cyclicity in Bergman-type spaces", *Internat. Math. Res. Notices* **5** (1995), 253–262.

[Borichev and Hedenmalm 1997] A. Borichev and H. Hedenmalm, "Harmonic functions of maximal growth: invertibility and cyclicity in Bergman spaces", *J. Amer. Math. Soc.* **10**:4 (1997), 761–796.

[Cantor 1964] D. G. Cantor, "A simple construction of analytic functions without radial limits", *Proc. Amer. Math. Soc.* **15** (1964), 335–336.

[Duren et al. 1993] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg, "Contractive zero-divisors in Bergman spaces", *Pacific J. Math.* **157**:1 (1993), 37–56.

[Duren et al. 1994] P. Duren, D. Khavinson, H. S. Shapiro, and C. Sundberg, "Invariant subspaces in Bergman spaces and the biharmonic equation", *Michigan Math. J.* **41**:2 (1994), 247–259.

[Hedenmalm 1991] H. Hedenmalm, "A factorization theorem for square area-integrable analytic functions", *J. Reine Angew. Math.* **422** (1991), 45–68.

[Hedenmalm 1993] P. J. H. Hedenmalm, "An invariant subspace of the Bergman space having the codimension two property", *J. Reine Angew. Math.* **443** (1993), 1–9.

[Hedenmalm 1994a] H. Hedenmalm, "A factoring theorem for the Bergman space", *Bull. London Math. Soc.* **26**:2 (1994), 113–126.

[Hedenmalm 1994b] P. J. H. Hedenmalm, "A computation of Green functions for the weighted biharmonic operators $\Delta\, |z|^{-2\alpha} \Delta$, with $\alpha > -1$", *Duke Math. J.* **75**:1 (1994), 51–78.

[Hedenmalm 1996] P. J. H. Hedenmalm, "Boundary value problems for weighted biharmonic operators", *Algebra i Analiz* **8**:4 (1996), 173–192.

[Hedenmalm et al. 1996a] H. Hedenmalm, B. Korenblum, and K. Zhu, "Beurling type invariant subspaces of the Bergman spaces", *J. London Math. Soc.* (2) **53**:3 (1996), 601–614.

[Hedenmalm et al. 1996b] H. Hedenmalm, S. Richter, and K. Seip, "Interpolating sequences and invariant subspaces of given index in the Bergman spaces", *J. Reine Angew. Math.* **477** (1996), 13–30.

[Horowitz 1974] C. Horowitz, "Zeros of functions in the Bergman spaces", *Duke Math. J.* **41** (1974), 693–710.

[Horowitz 1977] C. Horowitz, "Factorization theorems for functions in the Bergman spaces", *Duke Math. J.* **44**:1 (1977), 201–213.

[Korenblum 1975] B. Korenblum, "An extension of the Nevanlinna theory", *Acta Math.* **135**:3-4 (1975), 187–219.

[Korenblum 1977] B. Korenblum, "A Beurling-type theorem", *Acta Math.* **138**:3-4 (1977), 265–293.

[Korenblum 1993] B. Korenblum, "Outer functions and cyclic elements in Bergman spaces", *J. Funct. Anal.* **115**:1 (1993), 104–118.

[Luzin and Privalov 1925] N. N. Luzin and I. I. Privalov, "Sur l'unicité et la multiplicité des fonctions analytiques", *Ann. Sci. Ecole Norm. Sup.* **42** (1925), 143–191.

[MacLane 1962] G. R. MacLane, "Holomorphic functions, of arbitrarily slow growth, without radial limits", *Michigan Math. J.* **9** (1962), 21–24.

[Richter 1987] S. Richter, "Invariant subspaces in Banach spaces of analytic functions", *Trans. Amer. Math. Soc.* **304**:2 (1987), 585–616.

[Seip 1993] K. Seip, "Beurling type density theorems in the unit disk", *Invent. Math.* **113**:1 (1993), 21–39.

[Seip 1994] K. Seip, "On a theorem of Korenblum", *Ark. Mat.* **32**:1 (1994), 237–243.

[Seip 1995] K. Seip, "On Korenblum's density condition for the zero sequences of $A^{-\alpha}$", *J. Anal. Math.* **67** (1995), 307–322.

[Shimorin 1993] S. M. Shimorin, "Factorization of analytic functions in weighted Bergman spaces", *Algebra i Analiz* **5**:5 (1993), 155–177. In Russian; translation in *St. Petersburg Math. J.* **5**:5 (1994), 1005–1022.

[Shimorin 1995] S. M. Shimorin, "On a family of conformally invariant operators", *Algebra i Analiz* **7**:2 (1995), 133–158. In Russian; translation in *St. Petersburg Math. J.* **7**:2 (1996), 287–306.

Håkan Hedenmalm
Department of Mathematics
Lund University
Box 118
S-221 00 Lund, Sweden
haakan@math.lu.se