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The Abstract Interpolation Problem and Applications

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Abstract. A number of classical interpolation problems can be reduced to the following scheme. One is interested in the finding Schur class operator functions $w(\zeta) : E_1 \to E_2$, with $\zeta \in \mathbb{D}$, that satisfy certain interpolation conditions. The data of the problem are encoded in the Lyapunov identity

$$D(T_2x, T_2y) - D(T_1x, T_1y) = \langle M_1x, M_1y\rangle - \langle M_2x, M_2y\rangle,$$

where x, y are elements of a vector space X, D is a positive semidefinite quadratic form on X, T_1 and T_2 are linear operators on X, and M_1, M_2 are linear operators from X to the separable Hilbert spaces E_1, E_2. After introducing the de Branges–Rovnyak function space H^w associated with w, we can formulate the interpolation conditions thus: w is a solution to the interpolation problem if and only if there exists a linear mapping $F : X \to H^w$ such that

$$\|Fx\|^2_{H^w} \le D(x, x)$$

and

$$(FT_1x)(t) = t(FT_2x)(t) - \begin{bmatrix} \mathbb{1} & w(t) \\ w(t)^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -M_2x \\ M_1x \end{bmatrix}$$

for a.e. t with $|t| = 1$. The solutions w turn out to be the scattering matrices of the unitary colligations that extend the isometric colligation defined by the Lyapunov identity. These extensions and their scattering functions can be described using a "universal" extension and its scattering operator function. The description formula for solutions looks like

$$w = s_0 + s_2(1 - \omega s)^{-1}\omega s_1,$$

where

$$S = \begin{bmatrix} s & s_1 \\ s_2 & s_0 \end{bmatrix} : N_2 \oplus E_1 \to N_1 \oplus E_2$$

is the scattering matrix of the "universal" extension and $\omega : N_1 \to N_2$ is an arbitrary parameter from the Schur class. The matrix S is defined essentially uniquely by the data of the problem and is called the scattering matrix of the problem. Using the functional model and the Fourier representation of the "universal" extension one can investigate analytic properties of the scattering matrices S for classes of interpolation problems.

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Contents

Lecture 1. The Abstract Interpolation Problem

I will begin with the formal setup of the Abstract Interpolation Problem, or AIP, then consider several examples and discuss the role of the AIP in their investigation.

The data of the AIP are encoded in the Fundamental Identity:

$$D(T_2x,T_2y)-D(T_1x,T_1y)=\langle M_1x,M_1y\rangle_{E_1}-\langle M_2x,M_2y\rangle_{E_2}, \tag{1–1}$$

where x,y are elements of a vector space X (which has no a priori topological structure), D is a positive-semidefinite quadratic form on X, T_1 and T_2 are linear operators on X, E_1 and E_2 are separable Hilbert spaces, and M_1,M_2 are linear mappings from X into E_1,E_2.

De Branges–Rovnyak function spaces. Let w be a Schur class operator function; that is, w associates to each ζ in the unit disk $\mathbb{D}=\{\zeta:|\zeta|\le 1\}$ a contraction $w(\zeta):E_1\to E_2$, varying analytically with ζ. The de Branges–Rovnyak space L^w consists of the functions $f:\mathbb{T}\to E_2\oplus E_1$ (where $\mathbb{T}=\{\zeta:|\zeta|=1\}$ is the unit circle) having the form

$$f=\begin{bmatrix}\mathbb{1}_{E_2} & w\\ w^* & \mathbb{1}_{E_1}\end{bmatrix}^{1/2} g \tag{1–2}$$

for some $g\in L^2(E_2\oplus E_1)$. We set

$$\|f\|_{L^w}\overset{\text{def}}{=}\inf\|g\|_{L^2},$$

where the infimum is taken over all $g\in L^2(E_2\oplus E_1)$ such that (1–2) is satisfied. All such preimages g differ by the addition of (arbitrary) elements of $\operatorname{Ker}\left[\begin{smallmatrix}1 & w\\ w^* & 1\end{smallmatrix}\right]$, and the infimum is attained when

$$g(t)\perp \operatorname{Ker}\begin{bmatrix}\mathbb{1} & w(t)\\ w(t)^* & \mathbb{1}\end{bmatrix}\quad\text{a.e. on }\mathbb{T}.$$

Let $\pi_w^{-1}f$ denote the particular g that achieves the infimum. Thus

$$\|f\|^2_{L^w}=\|\pi_w^{-1}f\|^2_{L^2}=\int_{\mathbb{T}}\big\|(\pi_w^{-1}f)(t)\big\|^2_{E_2\oplus E_1}\,m(dt),$$

where $m(dt)$ is Lebesgue measure. The inner product in L^w is defined by

$$\langle f, h\rangle_{L^w} = \left\langle \pi_w^{-1} f,\, \pi_w^{-1} h\right\rangle_{L^2}.$$

Now L^w is a Hilbert space. As a set it is contained in L^2. It might happen that a function $f \in L^w$ admits the representation

$$f(t) = \begin{bmatrix} \mathbb{1} & w(t) \\ w(t)^* & \mathbb{1} \end{bmatrix} \check{g}(t) \quad \text{a.e.},$$

where $\check{g}(t)$ need not be in L^2; then, for any $h \in L^w$,

$$\langle f, h\rangle_{L^w} = \left\langle \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix}^{1/2} \check{g}(t),\, \pi_w^{-1} h \right\rangle_{L^2} = \int_{\mathbb{T}} \left\langle \check{g}(t), h(t)\right\rangle_{E_2 \oplus E_1} m(dt).$$

We also define the space H^w as the subspace of L^w consisting of

$$f = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \quad \text{with } f_2 \in H^2_+(E_2) \text{ and } f_1 \in H^2_-(E_1),$$

where H^2_+ and H^2_- are the standard Hardy classes.

Setup of the AIP. The Schur class function $w : E_1 \to E_2$ is said to be a *solution of the AIP* (with the data specified above) if there exists a linear mapping $F : X \to H^w$ such that, for all $x \in X$, the following conditions are satisfied:

(i) $\|Fx\|^2_{H^w} \le D(x, x)$.

(ii) $tFT_2x - FT_1x = \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -M_2x \\ M_1x \end{bmatrix}$ for a.e. $t \in \mathbb{T}$.

Property (ii) is, in fact, an implicit formula for the mapping F. Sometimes it defines F uniquely, sometimes not; but any mapping F that possesses (ii) is very special. We will describe all such maps in Lecture 4.

If we write

$$Fx = \begin{bmatrix} F_+x \\ F_-x \end{bmatrix},$$

the conditions $Fx \in H^w$ and $\|Fx\|^2_{H^w} \le D(x, x)$ are equivalent to the conjunction of three conditions:

(a) $F_+x \in H^2_+(E_2)$.

(b) $F_-x \in H^2_-(E_1)$.

(c) $\|Fx\|^2_{L^w} \le D(x, x)$.

By the definition of the inner product in L^w, property (c) is actually an upper bound for the average boundary values of Fx.

My goal now is to explain why this abstract problem is an "interpolation" problem. Before passing to examples I would like to consider a special case of the data (which was considered earlier than the general case).

A special case. Assume that the operators $(\zeta T_2 - T_1)$ and $(T_2 - \bar{\zeta} T_1)$ are invertible for all $\zeta \in \mathbb{D}$ except possibly for a discrete set. Because the first components of the sides of relation (ii) above are H^2_+ functions, one can consider the analytic continuation of the relation inside the unit disk $\mathbb{D}$:

$$\zeta(F_+T_2x)(\zeta) - (F_+T_1x)(\zeta) = -M_2x + w(\zeta)M_1x. \tag{1–3}$$

We emphasize that the vectors $M_2x \in E_2$ and $M_1x \in E_1$ are independent of ζ. Fix $\zeta \in \mathbb{D}$. Because the mapping F is linear, we have

$$\zeta(F_+T_2x)(\zeta) = \bigl(F_+(\zeta T_2x)\bigr)(\zeta).$$

Actually, for any complex number μ,

$$\mu(F_+T_2x)(\zeta) = \bigl(F_+(\mu T_2x)\bigr)(\zeta);$$

in particular, this is true for $\mu = \zeta$. We can reexpress (1–3) now as

$$\bigl(F_+((\zeta T_2 - T_1)x)\bigr)(\zeta) = \bigl(-M_2 + w(\zeta)M_1\bigr)x.$$

Replacing x by $(\zeta T_2 - T_1)^{-1}x$, we end up with

$$(F^w_+x)(\zeta) = (w(\zeta)M_1 - M_2)(\zeta T_2 - T_1)^{-1}x. \tag{1–4}$$

One can see better now the interpolation meaning of the property $F_+x \in H^2_+$: the "zeros" of the numerator cancel the "zeros" of the denominator, which means that w obeys the interpolation constraint

$$w(\zeta)M_1x = M_2x$$

at certain "points" ζ.

In a similar way, the second components of the two sides of equality (ii) can be reexpresed, under the assumptions at hand, as

$$(F^w_-x)(\zeta) = \bar{\zeta}(M_1 - w(\zeta)^*M_2)(T_2 - \bar{\zeta}T_1)^{-1}x. \tag{1–5}$$

The interpolation meaning of the property $F^w_-x \in H^2_-$ is similar to the one considered above, but now for $w(\zeta)^*$.

The meaning of property (c) will be discussed in the examples.

Remark. Under the assumptions of this section, condition (ii) defines F *uniquely* and *explicitly* for any solution w. This allows us to write F^w instead of F. Thus, under these assumptions, one can give a more explicit setup for the AIP:

Let

$$F^wx = \begin{bmatrix} F^w_+x \\ F^w_-x \end{bmatrix}$$

be defined by the formulas (1–4) and (1–5). The Schur class function w is said to be a solution of the AIP if F^w possesses properties (a), (b), and (c).

Generally, condition (ii) does not define F in a unique way, but we will see in Lecture 4 how to describe all such mappings F.

References for Lecture 1 are [Katsnelson et al. 1987; Kheifets 1988a; 1988b; 1990b; Kheifets and Yuditskii 1994].

Lecture 2. Examples

Example 1: The Nevanlinna–Pick Problem. Let $\zeta_1, \dots, \zeta_n, \dots$ be a finite or infinite sequence of points in the unit disk $\mathbb{D}$; let $w_1, \dots, w_n, \dots$ be a sequence of complex numbers. One is interested in describing all the Schur class functions w such that

$$w(\zeta_k) = w_k.$$

The well-known solvability criterion is

$$\left[\frac{1 - \bar{w}_k w_j}{1 - \bar{\zeta}_k \zeta_j}\right]_{k,j=1}^{n} \geq 0 \quad \text{for all } n.$$

We define the data of the AIP associated with this problem. Because the functions w are scalar, $E_1 = E_2 = \mathbb{C}^1$. Consider the space X that consists of all sequences

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix}$$

that have only a finite number of nonvanishing components. No topology is assumed on X.

Define the sesquilinear form D on X by

$$D(x, y) = \sum_{k,j} \bar{y}_k \frac{1 - \bar{w}_k w_j}{1 - \bar{\zeta}_k \zeta_j}\, x_j, \quad \text{for } x, y \in X.$$

The (diagonal) linear operators

$$T = T_1 = \begin{bmatrix} \zeta_1 & & & \\ & \ddots & & \\ & & \zeta_n & \\ & & & \ddots \end{bmatrix}$$

and $T_2 = \mathbb{1}_X$ are well defined on the space X. We can now check the Fundamental Identity (1–1):

$$\begin{aligned} D(x, y) - D(Tx, Ty) &= \sum_{k,j} \bar{y}_k (1 - \bar{w}_k w_j) x_j \\ &= \sum_{k,j} \bar{y}_k x_j - \sum_{k,j} \bar{y}_k \bar{w}_k w_j x_j \\ &= \sum_{k} \bar{y}_k \cdot \sum_{j} x_j - \sum_{k} \bar{y}_k \bar{w}_k \cdot \sum_{j} w_j x_j. \end{aligned}$$

By defining $M_1x = \sum_j x_j$ and $M_2x = \sum_j w_jx_j$, we end up with

$$D(x,y) - D(Tx,Ty) = \overline{M_1y} \cdot M_1x - \overline{M_2y} \cdot M_2x.$$

The products on the right-hand side are actually the standard inner product in $\mathbb{C}^1$.

Consider now the AIP associated with these data. Let w be a solution of this AIP. This means that there exists a mapping $F : X \to H^w$ such that conditions (i) and (ii) on page 353 hold. Because $(\zeta\mathbb{1} - T)^{-1}$ exists for all ζ, such that $|\zeta| < 1$ and $\zeta \neq \zeta_j$, and because $(\mathbb{1} - \bar{\zeta}T)^{-1}$ exists for all ζ with $|\zeta| < 1$, we know that F has the following form (see Lecture 1, special case on page 354):

$$(F_+^w x)(\zeta) = (w(\zeta)M_1 - M_2)(\zeta\mathbb{1} - T)^{-1}x$$
$$(F_-^w x)(\zeta) = (M_1 - \overline{w(\zeta)}M_2)(\mathbb{1} - \bar{\zeta}T)^{-1}x.$$

Thus, F^w is defined uniquely for any solution w. It is easy to compute these expressions more explicitly for this example. Because

$$(\zeta\mathbb{1} - T)^{-1} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{1}{\zeta-\zeta_1}x_1 \\ \vdots \\ \frac{1}{\zeta-\zeta_n}x_n \\ \vdots \end{bmatrix},$$

we obtain

$$(F_+^w x)(\zeta) = w(\zeta)\sum_j \frac{1}{\zeta - \zeta_j}x_j - \sum_j \frac{w_jx_j}{\zeta - \zeta_j} = \sum_j \frac{w(\zeta) - w_j}{\zeta - \zeta_j}x_j;$$

in the same way

$$(F_-^w x)(\zeta) = \bar{\zeta}\sum_j \frac{1 - \overline{w(\zeta)}w_j}{1 - \bar{\zeta}\zeta_j}x_j.$$

The function w is a solution of the AIP if and only if $F^wx \in H^w$ and $\|F^wx\|^2_{H^w} \leq D(x,x)$ for all $x \in X$; that is, if and only if properties (a), (b), (c) of page 353 hold. One can see from the explicit formula that property (a) holds if and only if $w(\zeta_j) = w_j$. Property (b) holds automatically. To see what property (c) means, compute the left-hand side. On the boundary $(for|t| = 1)$, $F_+^w x$ and $F_-^w x$ look like

$$(F_+^w x)(t) = w(t)\sum_j \frac{x_j}{t - \zeta_j} - \sum_j \frac{w_jx_j}{t - \zeta_j},$$
$$(F_-^w x)(t) = \sum_j \frac{x_j}{t - \zeta_j} - \overline{w(t)}\sum_j \frac{w_jx_j}{t - \zeta_j}.$$

Or we can put these two formulas together:

$$(F^wx)(t) = \begin{bmatrix} \mathbb{1} & w(t) \\ w(t)^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -\sum_j \frac{w_jx_j}{t-\zeta_j} \\ \sum_j \frac{x_j}{t-\zeta_j} \end{bmatrix}.$$

Now,

$$\|F^w x\|^2_{H^w} = \left\langle \begin{bmatrix} F^w_+ x \\ F^w_- x \end{bmatrix}, \begin{bmatrix} -\sum_j \frac{w_j x_j}{t-\zeta_j} \\ \sum_j \frac{x_j}{t-\zeta_j} \end{bmatrix} \right\rangle_{L^2} = \left\langle F^w_- x, \sum_j \frac{x_j}{t-\zeta_j} \right\rangle_{L^2}$$

$$= \sum_j \bar{x}_j \frac{(F^w_- x)(\zeta_j)}{\bar{\zeta}_j} = \sum_j \bar{x}_j \sum_k \frac{1-\bar{w}_j w_k}{1-\bar{\zeta}_j \zeta_k} x_k = D(x,x).$$

The latter computation depends on property (a), $F^w_+ x \in H^2_+$. Thus, we can see that *w is a solution of the AIP with these data if and only if w is a solution of the Nevanlinna–Pick problem.* For this particular problem, condition (a) actually carries all the interpolation information, (b) holds automatically, and (c) follows from (a). Moreover, we emphasize that for any solution w of this problem we have the equality $\|F^w x\|^2_{H^w} = D(x,x)$ for all $x \in X$, instead of inequality.

To continue on to the second (more general) example, I will reformulate the first one. Consider the closed linear span of the functions

$$\left\{ \frac{1}{t-\zeta_j} \right\}$$

and denote by $K_{\bar{\theta}} \subseteq H^2_-$ the space $H^2_- \ominus \bar{\theta} H^2_-$, where θ is the Blaschke product with zeros ζ_j if they satisfy the Blaschke condition and $\theta \equiv 0$ otherwise (in this case $K_{\bar{\theta}} = H^2_-$). We can associate the function

$$\tilde{x}(t) = \sum_j \frac{x_j}{t-\zeta_j} \in K_{\bar{\theta}}$$

with any $x \in X$ of the first example. $K_{\bar{\theta}}$ is invariant under $P_- t$, where t is the independent variable, and P_- is the orthogonal projection onto H^2_-. It is easy to see that the operator T from the first example acts as $P_- t$ under this correspondence; in fact,

$$P_- t \sum_j \frac{x_j}{t-\zeta_j} = \sum_j \frac{\zeta_j x_j}{t-\zeta_j}.$$

Let W be the operator on the space X defined by

$$W \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = \begin{bmatrix} w_1 x_1 \\ \vdots \\ w_n x_n \\ \vdots \end{bmatrix}.$$

Obviously $WT = TW$. Under the correspondence,

$$\widetilde{Wx}(t) = \sum_j \frac{w_k x_j}{t-\zeta_j}.$$

A Schur class function w is a solution of the Nevanlinna–Pick problem if and only if $\widetilde{Wx} = P_- w\tilde{x}$. In fact,

$$P_- w \frac{1}{t-\zeta_j} = P_- \frac{w - w(\zeta_j)}{t-\zeta_j} + P_- \frac{w(\zeta_j)}{t-\zeta_j} = \frac{w(\zeta_j)}{t-\zeta_j}.$$

Observe that

$$M_1 x = \sum_j x_j = (\tilde{x})_{-1},$$

where the last notation stands for the Fourier coefficient of index -1 of the H^2_- function $\tilde{x}$. And $M_2 x = (\widetilde{Wx})_{-1}$. Since $\widetilde{Wx} = P_- w\tilde{x}$ for any solution w of the Nevanlinna–Pick problem, then $\|\widetilde{Wx}\|_{H^2_-} \le \|\tilde{x}\|_{H^2_-}$. We can reexpress the quadratic form D as

$$D(x,y) = \langle \tilde{x}, \tilde{y}\rangle_{H^2_-} - \langle \widetilde{Wx}, \widetilde{Wy}\rangle_{H^2_-}.$$

In fact, let

$$\tilde{x} = \frac{1}{t-\zeta_j} \quad \text{and} \quad \tilde{y} = \frac{1}{t-\zeta_k};$$

then

$$\widetilde{Wx} = \frac{w_j}{t-\zeta_j}, \qquad \widetilde{Wy} = \frac{w_i}{t-\zeta_i},$$

$$\langle \tilde{x}, \tilde{y}\rangle = \frac{1}{1-\bar{\zeta}_k\zeta_j}, \quad \langle \widetilde{Wx}, \widetilde{Wy}\rangle = \frac{\bar{w}_k w_j}{1-\bar{\zeta}_k\zeta_j}.$$

If $\tilde{x} = \sum_j \dfrac{x_j}{t-\zeta_j}$, then

$$\|\tilde{x}\|^2 - \|\widetilde{Wx}\|^2 = \sum_{k,j} \bar{x}_k \frac{1-\bar{w}_k w_j}{1-\bar{\zeta}_k\zeta_j} x_j = D(x,x).$$

Define the operator $\widetilde{W}$ on $K_{\bar{\theta}}$ by

$$\widetilde{W}(\tilde{x}) = \widetilde{Wx}.$$

This is well defined and $D \ge 0$ if and only if $\widetilde{W}$ is a contraction.

Example 2. Sarason's Problem. Now let θ be an arbitrary inner function (not necessarily a Blaschke product). Set $K_{\bar{\theta}} = H^2_- \ominus \bar{\theta} H^2_-$; this space is invariant under $P_- t$. Set $T = P_- t | K_{\bar{\theta}}$. Let W be a contractive operator on $K_{\bar{\theta}}$ with $WT = TW$. One is interested in finding all the Schur class functions w such that

$$Wx = P_- wx \quad \text{for all } x \in K_{\bar{\theta}}.$$

Associate the following AIP data to the Sarason problem: $X = K_{\bar{\theta}}$, $T_1 = T$, $T_2 = \mathbb{1}$, $D(x,x) = \|x\|^2 - \|Wx\|^2$, $M_1 x = (x)_{-1}$, $M_2 x = (Wx)_{-1}$; here $E_1 = E_2 = \mathbb{C}^1$. One can check the Fundamental Identity (see the beginning of Lecture 1). For this data the implicit definition (ii) of the mapping F yields an explicit formula for F:

$$F^w x = \begin{bmatrix} 1 & w \\ \bar{w} & 1 \end{bmatrix} \begin{bmatrix} -Wx \\ x \end{bmatrix} \quad \text{a.e. } t \in T,$$

for all $x \in X$ and any solution w of the AIP. Thus, F^w is unique for any solution w. Then again w is a solution of the AIP if and only if $F^w x \in H^w$ for all $x \in X$ and $\|F^w x\|^2_{H^w} \leq D(x,x)$, that is, if and only if conditions (a), (b) and (c) of page 353 hold.

For condition (a) we have

$$F^w_+ x = wx - Wx \in H^2_+ \iff P_-(wx - Wx) = 0 \iff P_- wx = Wx$$

for all $x \in X$. In other words, condition (a) is satisfied if w solves the Sarason problem.

Condition (b), $F^w_- x = x - \bar{w} \cdot Wx \in H^2_-$, holds automatically for any Schur class function w.

For condition (c) we get

$$\begin{aligned} \|F^w x\|^2_{L^w} &= \left\langle \begin{bmatrix} F^w_+ x \\ F^w_- x \end{bmatrix}, \begin{bmatrix} -Wx \\ x \end{bmatrix} \right\rangle_{L^2} = \langle F^w_- x,\, x\rangle_{L^2} \\ &= \langle x - \bar{w} \cdot Wx, x\rangle = \langle x, x\rangle - \langle Wx, wx\rangle \\ &= \langle x, x\rangle - \langle Wx, P_- wx\rangle = \langle x, x\rangle - \langle Wx, Wx\rangle = D(x,x). \end{aligned}$$

Thus, for this data w *is a solution of the AIP if and only if* w *is a solution of the Sarason problem.* Property (a) carries all the interpolation information, (b) holds automatically, and (c) follows from (a). And we again have the equality $\|F^w x\|^2_{H^w} = D(x,x)$ for all $x \in X$ and for all solutions w, instead of inequality.

Example 2′. We now associate another AIP to the same Sarason problem. This AIP is best considered as being different (nonequivalent) from the one in Example 2, though they have a common set of solutions. The reason is that the coefficient matrices in the description formulas for the solution sets (see the last theorem of Lecture 4) are different and the associated universal colligations (see Section 2 of Lecture 4) are nonequivalent.

Let θ be an inner function, and set $K_\theta = H^2_+ \ominus \theta H^2_+$ and $T^*_\theta x_2 = P_+ \bar{t} x_2$ for $x_2 \in K_\theta$. Let W^*_2 be a contractive operator on K_θ that commutes with $T^*_\theta : W^*_2 T^*_\theta = T^*_\theta W^*_2$. Find all the Schur class functions w such that

$$W^*_2 x_2 = P_+ \bar{w} x_2.$$

One can check that the solutions of this problem and of the one in the previous example coincide (if the operator W of Example 2 and the operator W_2 are properly connected). Consider the AIP with the data $X = K_\theta$, $T_1 = \mathbb{1}$, $T_2 = T^*_\theta$, $E_1 = E_2 = \mathbb{C}^1$, $M_1 x_2 = (W^* x_2)_0$, and $M_2 x_2 = (x_2)_0$, where the notation $(\,\cdot\,)_0$ stands for the Fourier coefficient of index 0 of an H^2_+ function. As in Example 2,

F can be expressed explicitly and uniquely for any solution w of the AIP with this data:

$$F^w x_2 = \begin{bmatrix} 1 & w \\ \bar{w} & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -W_2^* x_2 \end{bmatrix}.$$

Now consider properties (a), (b) and (c). For (a) we have

$$F_+^w x_2 = x_2 - w \cdot W_2^* x_2 \in H_+^2;$$

this holds automatically for any Schur class function w. Property (b) becomes

$$F_-^w x_2 = \bar{w} x_2 - W_2^* x_2 \in H_-^2;$$

this holds if and only if $P_+ \bar{w} x_2 = W_2^* x_2$; that is, if and only if w solves the Sarason problem. Finally, for (c) we can write

$$\|F^w x_2\|_{L^w}^2 = D(x_2, x_2) \quad \text{for all } x_2 \in X$$

for any solution w. Thus, for these data property (b) carries all the interpolation information, (a) holds automatically for any Schur class w, and (c) follows from (b). The equality

$$\|F^w x_2\|_{H^w}^2 = D(x_2, x_2) \quad \text{for all } x_2 \in X$$

holds for any solution w.

Example 3. The boundary interpolation problem. Property (c) dominates in this example and (a) and (b) follow. The equality in condition (c) holds for some solutions but not for all of them. We will need some preliminaries.

A Schur class function w in the unit disk $\mathbb{D}$ is said to have *an angular derivative in the sense of Carathéodory* at the point $t_0 \in \mathbb{T}$ if $w(\zeta)$ has a nontangential unimodular limit w_0 as ζ goes to t_0, $|w_0| = 1$, and

$$\frac{w(\zeta) - w_0}{\zeta - \zeta_0}$$

has a nontangential limit w_0' at t_0.

Theorem (Carathéodory). *A Schur class function $w(\zeta)$ has an angular derivative at $t_0 \in \mathbb{T}$ if and only if*

$$D_{w,t_0} \stackrel{\text{def}}{=} \liminf_{\zeta \to t_0} \frac{1 - |w(\zeta)|^2}{1 - |\zeta|^2} < \infty$$

(here $|\zeta| \le 1$ and $\zeta \to t_0$ in an arbitrary way). In this case $w_0' = D_{w,t_0} \cdot \dfrac{w_0}{t_0}$ and

$$\frac{1 - |w(\zeta)|^2}{1 - |\zeta|^2} \longrightarrow D_{w,t_0}$$

as ζ goes to t_0 nontangentially. D_{w,t_0} vanishes if and only if w is a constant of modulus 1.

Moreover, a Schur class function w has an angular derivative in the sense of Carathéodory at the point $t_0 \in \mathbb{T}$ if and only if there exists a unimodular constant w_0 such that

$$\left|\frac{w(t)-w_0}{t-t_0}\right|^2 + \frac{1-|w(t)|^2}{|t-t_0|^2} \in L^1;$$

that is, if and only if this function is integrable over $\mathbb{T}$ against Lebesgue measure. In particular, this guarantees that

$$\frac{w-w_0}{t-t_0} \in H^2_+,$$

because the denominator is an outer function. In that case

$$\int_{\mathbb{T}} \left(\left|\frac{w(t)-w_0}{t-t_0}\right|^2 + \frac{1-|w(t)|^2}{|t-t_0|^2}\right) m(dt) = D_{w,t_0}.$$

Now consider the following interpolation problem. Let t_0 be a point of the unit circle $\mathbb{T}$, let w_0 be a complex number with $|w_0| = 1$, and let $0 \le D < \infty$ be a given nonnegative number. One wants to describe all the Schur class functions w such that $w(\zeta) \to w_0$ as $\zeta \to t_0$ (nontangentially) and $D_{w,t_0} \le D$.

Associate to this problem the AIP data $X = \mathbb{C}^1$, $D(x,x) = \bar{x}Dx$, $T_1x = t_0x$, $T_2x = x$, $M_1x = x$, $M_2x = w_0x$, $E_1 = E_2 = \mathbb{C}^1$; then we can check the Fundamental Identity (1–1). The left-hand side of the identity is

$$\bar{x}Dx - \bar{x}\bar{t}_0Dt_0x = 0,$$

and the right-hand side is

$$\overline{M_1x}\cdot M_1x - \overline{M_2x}\cdot M_2x = \bar{x}x - \bar{x}\bar{w}_0w_0x = 0.$$

The mapping F of Lecture 1 (pages 353–354) is unique for any solution w and can be computed explicitly for this data:

$$F^w x = \begin{bmatrix} 1 & w \\ \bar{w} & 1 \end{bmatrix} \begin{bmatrix} -\frac{w_0}{t-t_0} \\ \frac{1}{t-t_0} \end{bmatrix} x$$

We can analyse now what conditions (a), (b), and (c) tell us. Condition (a) becomes

$$F_+x = \frac{w-w_0}{t-t_0}x \in H^2_+; \quad \text{that is,} \quad \frac{w-w_0}{t-t_0} \in H^2_+.$$

Condition (b) becomes

$$F_-x = \frac{1-\bar{w}w_0}{t-t_0}\,x = \bar{t}\,\frac{\bar{w}-\bar{w}_0}{\bar{t}-\bar{t}_0}\cdot\frac{w_0}{t_0}x \in H^2_-.$$

Hence (a) and (b) coincide. Finally, condition (c) becomes

$$\begin{aligned}\|Fx\|^2_{L^w} &= \int_{\mathbb{T}} \left\langle (Fx)(t), \begin{bmatrix} -\frac{w_0}{t-t_0} \\ \frac{1}{t-t_0} \end{bmatrix} x \right\rangle_{\mathbb{C}^2} m(dt) \\ &= \int_{\mathbb{T}} \bar{x} \frac{-\bar{w}_0(w-w_0)+(1-\bar{w}w_0)}{|t-t_0|^2} x\, m(dt) \\ &= \bar{x} \int_{\mathbb{T}} \frac{(\bar{w}-\bar{w}_0)(w-w_0)+1-\bar{w}w}{|t-t_0|^2} m(dt)\, x \\ &= \bar{x} \int_{\mathbb{T}} \left(\left| \frac{w-w_0}{t-t_0} \right|^2 + \frac{1-|w|^2}{|t-t_0|^2} \right) m(dt)\, x = \bar{x} D_{w,t_0} x.\end{aligned}$$

Thus, $\|Fx\|^2_{L^w} \le D$ if and only if $D_{w,t_0} \le D$; that is, w is a solution of the AIP with this data if and only if w is a solution of the boundary interpolation problem.

A reference for Examples 2 and 2′ is [Kheifets 1990a]. References for Example 3 are [Kheifets 1996; Sarason 1994].

Lecture 3. Solutions of the Abstract Interpolation Problem

Role of the AIP. In Lecture 2 we showed how some specific analytic problems can be included in the AIP scheme. The AIP, as formulated in Lecture 1, is very well adapted to this inclusion (actually it arose from the experience of treating a number of problems of this type). To include a specific problem in the scheme one has to realize (if it is possible) what the associated data is and to prove the coincidence of two solution sets: that of the original analytic problem and that of the AIP with the associated data. The mapping F suggests an algorithm for what exactly is to be checked to prove this coincidence.

In this lecture we will slightly reformulate the AIP to adapt it better to solving the problem (now the origin of the data is unimportant to a certain extent).

Thus, the AIP can be viewed as an intermediate link between specific interpolation problems and their resolution. It has two sides: one of them looks at the specific interpolation problem and is devoted to proving the equivalence of the original problem and associated AIP, the second concerns solutions. As soon as a specific problem is included in the scheme, the analysis of its solutions goes in a standard and universal way.

Isometric colligation associated with the AIP data. Reformulation of the AIP. Let $[x]$ stand for the equivalence class of x with respect to the quadratic form D. (The equivalence relation is defined as follows: $x \sim 0$ if and only if $D(x,y)=0$ for all $y \in X$, and $x_1 \sim x_2$ if and only if $x_1 - x_2 \sim 0$). Consider the linear space of equivalence classes $\{[x] : x \in X\}$ and define an inner product in it by

$$\langle [x],[y] \rangle \stackrel{\text{def}}{=} D(x,y).$$

This product is well defined. One can complete it and obtain a Hilbert space, which we denote by H_0.

Rewrite the Fundamental Identity (1–1) as

$$D(T_1x, T_1y) + \langle M_1x, M_1y\rangle_{E_1} = D(T_2x, T_2y) + \langle M_2x, M_2y\rangle_{E_2}.$$

Using the notations introduced above one can reexpress this as

$$\big\langle [T_1x], [T_1y]\big\rangle_{H_0} + \langle M_1x, M_1y\rangle_{E_1} = \big\langle [T_2x], [T_2y]\big\rangle_{H_0} + \langle M_2x, M_2y\rangle_{E_2}. \tag{3–1}$$

Set

$$d_v \stackrel{\text{def}}{=} \operatorname{Clos}\left\{ \begin{bmatrix} [T_1x] \\ M_1x \end{bmatrix} : x \in X \right\} \subseteq H_0 \oplus E_1$$

and

$$\Delta_v \stackrel{\text{def}}{=} \operatorname{Clos}\left\{ \begin{bmatrix} [T_2x] \\ M_2x \end{bmatrix} : x \in X \right\} \subseteq H_0 \oplus E_2.$$

Define a mapping $V : d_v \to \Delta_v$ by the formula

$$V : \begin{bmatrix} [T_1x] \\ M_1x \end{bmatrix} \longrightarrow \begin{bmatrix} [T_2x] \\ M_2x \end{bmatrix}. \tag{3–2}$$

Because of (3–1), V is an isometry. This implies that V is well-defined. In fact, if

$$\begin{bmatrix} [T_1x'] \\ M_1x' \end{bmatrix} = \begin{bmatrix} [T_1x''] \\ M_2x'' \end{bmatrix}, \quad \text{that is,} \quad \begin{bmatrix} [T_1(x'-x'')] \\ M_1(x'-x'') \end{bmatrix} = 0,$$

then (3–1) implies that

$$\begin{bmatrix} [T_2(x'-x'')] \\ M_2(x'-x'') \end{bmatrix} = 0.$$

Now, if x' and x'' generate the same vector on the left-hand side of (3–2), they generate the same vector on the right-hand side of (3–2), which shows that V is well-defined.

Let w be a solution of the AIP; that is, suppose there exists a mapping $F : X \to H^w$ possessing properties (i) and (ii) of page 353. Property (i) says that

$$\|Fx\|^2_{H^w} \le D(x, x) \equiv \big\|[x]\big\|^2_{H_0}.$$

Hence, Fx depends only on the equivalence class $[x]$, not on the representative x. This means that F generates a mapping G of the equivalence classes

$$G[x] \stackrel{\text{def}}{=} Fx, \tag{3–3}$$

and

$$\|G[x]\|^2_{H^w} \equiv \|Fx\|^2_{H^w} \le \big\|[x]\big\|^2_{H_0}.$$

Thus G is a contraction. Since $\big\{[x] : x \in X\big\}$ is dense in H_0 and G is a contraction, it can be extended to a contraction of H_0 into H^w. Thus, *any* $F : X \to H^w$ *with properties* (i) *and* (ii) (actually up to now we used only (i)) *generates a contraction* $G : H_0 \to H^w$ *such that* $G[x] = Fx$ *for all* $x \in X$.

We now try to interpret property (ii). To this end we reexpress (ii) as

$$FT_2x + \bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} M_2x = \bar{t}\left(FT_1x + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} M_1x\right),$$

or

$$G[T_2x] + \bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} M_2x = \bar{t}\left(G[T_1x] + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} M_1x\right) \tag{3–4}$$

Our further goal is to realize the latter equality as one of the form

$$\begin{bmatrix} G[T_2x] \\ M_2x \end{bmatrix} = A^w \begin{bmatrix} G[T_1x] \\ M_1x \end{bmatrix}, \tag{3–5}$$

where $A^w : H^w \oplus E_1 \to H^w \oplus E_2$ is a linear operator. As we know, G maps H_0 into H^w, M_1 maps H_0 into E_1, and M_2 maps H_0 into E_2. The relation (3–4) is of the type

$$f'' + \bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} e_2 = \bar{t}\left(f' + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} e_1\right), \tag{3–6}$$

where $f', f'' \in H^w$ and $e_1 \in E_1$, $e_2 \in E_2$. Observe that the three subspaces

$$\left\{\begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} e_1 : e_1 \in E_1\right\}, \quad H^w, \quad \left\{\bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} e_2 : e_2 \in E_2\right\}$$

are mutually orthogonal in L^w (see page 352 for the definition of L^w). Let P^w be the orthogonal projection from L^w onto H^w; then, for any

$$f = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} \in L^w,$$

we have

$$P^w \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} - \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} P_- f_2 \\ P_+ f_1 \end{bmatrix} \tag{3–7}$$

Obviously this difference is in L^w. It can be rewritten as

$$P^w \begin{bmatrix} f_2 \\ f_1 \end{bmatrix} = \begin{bmatrix} P_+ f_2 - wP_+ f_1 \\ P_- f_1 - w^* P_- f_2 \end{bmatrix}.$$

Hence, it is in H^w. Because

$$\begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} P_- f_2 \\ P_+ f_1 \end{bmatrix}$$

is orthogonal to H^w, P^w is really the orthogonal projection.

Because

$$f' = \begin{bmatrix} f'_+ \\ f'_- \end{bmatrix} \in H^w,$$

we can obtain from (3–6) and (3–7)

$$\begin{aligned} \bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} e_2 &= (\mathbb{1}_{L^w} - P^w)\left\{\bar{t}\left(f' + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} e_1\right)\right\} = \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} P_-\bar{t}(f'_+ + we_1) \\ P_+\bar{t}(f'_- + e_1) \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} \bar{t}(f'_+(0) + w(0))e_1 \\ 0 \end{bmatrix} = \bar{t}\begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} \big(f'_+(0) + w(0)e_1\big). \end{aligned}$$

Thus

$$e_2 = f'_+(0) + w(0)e_1. \tag{3–8}$$

Now, from (3–6) and (3–8),

$$f'' = \bar{t}(f' + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} e_1) - \bar{t} \begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} e_2 = \bar{t}(f' + \begin{bmatrix} w \\ \mathbb{1} \end{bmatrix} e_1) - \begin{bmatrix} \mathbb{1} \\ w^* \end{bmatrix} \big(f'_+(0) + w(0)e_1\big).$$

That is,

$$f'' = \bar{t}\left(f' - \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} f'_+(0) \\ 0 \end{bmatrix}\right) + \bar{t} \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -w(0) \\ \mathbb{1} \end{bmatrix} e_1. \tag{3–9}$$

Putting (3–8) and (3–9) together we obtain

$$\begin{bmatrix} f'' \\ e_2 \end{bmatrix} = A^w \begin{bmatrix} f' \\ e_1 \end{bmatrix}, \quad A^w = \begin{bmatrix} A^w_{\text{in}} & (A^w_1)^* \\ A^w_2 & A^w_{12} \end{bmatrix} : \begin{bmatrix} H^w \\ E_1 \end{bmatrix} \to \begin{bmatrix} H^w \\ E_2 \end{bmatrix}$$

where

$$\begin{aligned} A^w_{\text{in}} f &= P^w \bar{t} f = \bar{t}\left(f - \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} f_+(0) \\ 0 \end{bmatrix}\right) : H^w \to H^w, \\ (A^w_1)^* e_1 &= \bar{t} \begin{bmatrix} \mathbb{1} & w \\ w^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -w(0) \\ \mathbb{1} \end{bmatrix} : E_1 \to H^w, \\ A^w_2 f &= f_+(0) : H^w \to E_2, \\ A^w_{12} e_1 &= w(0) e_1 : E_1 \to E_2. \end{aligned}$$

Thus, condition (ii) of the AIP, equivalently (3–4) or (3–5), can be expressed as

$$\begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} \begin{bmatrix} [T_2 x] \\ M_2 x \end{bmatrix} = A^w \begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \begin{bmatrix} [T_1 x] \\ M_1 x \end{bmatrix}. \tag{3–10}$$

According to the definition (3–2) of the isometry V,

$$\begin{bmatrix} [T_2 x] \\ M_2 x \end{bmatrix} = V \begin{bmatrix} [T_1 x] \\ M_1 x \end{bmatrix}.$$

Combining this with (3–10) we conclude that

$$\begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} V \mid d_v = A^w \begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \mid d_v. \tag{3–11}$$

To give (3–11) a further interpretation we digress to recall some basic facts related to unitary colligations, their characteristic functions, and functional models.

Digression on unitary colligations, characteristic functions, and functional models. Let H, E_1, E_2 be separable Hilbert spaces. A unitary mapping A of $H \oplus E_1$ onto $H \oplus E_2$ is said to be a *unitary colligation.* The space H is called the *state space* of the colligation, E_1 is called the *input space*, and E_2 is called the *response space*. Both E_1 and E_2 are called *exterior spaces*.

The operator-valued function $w(\zeta) : E_1 \to E_2$ defined by the formula

$$w(\zeta) = P_{E_2} A(\mathbb{1}_{H \oplus E_1} - \zeta P_H A)^{-1} \mid E_1$$

is called the *characteristic function* of the colligation A. Because A is unitary, $w(\zeta)$ is well defined for $\zeta \in \mathbb{D}$; w is a contractive-valued analytic operator-function in $\mathbb{D}$.

Using the block decomposition

$$A = \begin{bmatrix} A_{\text{in}} & A_1 \\ A_2 & A_{12} \end{bmatrix} : \begin{bmatrix} H \\ E_1 \end{bmatrix} \to \begin{bmatrix} H \\ E_2 \end{bmatrix},$$

the characteristic function can be reexpressed as

$$w(\zeta) = A_{12} + \zeta A_2(\mathbb{1}_H - \zeta A_{\text{in}})^{-1} A_1.$$

The unitary colligation A is said to be *simple* with respect to the exterior spaces E_1 and E_2 if there is no nonzero reducing subspace for A in H; that is, if there is no nonzero subspace $H_{\text{res}} \subseteq H$ that is invariant for A and A^*. We shall call the maximal reducing subspace for A in H the *residual subspace* of the colligation A. Thus a unitary colligation is simple if the residual subspace is trivial.

Let $H_{\text{res}} \subseteq H$ be the residual subspace of A. Let $H_{\text{simp}} = H \ominus H_{\text{res}}$. Then $A : H_{\text{simp}} \oplus E_1 \to H_{\text{simp}} \oplus E_2$ is a simple unitary colligation, and $A \,|\, H_{\text{res}}$ is a unitary operator on H_{res}.

Two unitary colligations

$$A : H \oplus E_1 \to H \oplus E_2 \quad \text{and} \quad A' : H' \oplus E_1 \to H' \oplus E_2$$

with the same exterior spaces are said to be *unitarily equivalent* if there exists a unitary mapping $\mathcal{G} : H \to H'$ such that

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A = A' \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix}.$$

THEOREM. *Two simple unitary colligations A and A' are unitarily equivalent if and only if their characteristic functions coincide.*

Let w be a Schur class operator function, $w(\zeta) : E_1 \to E_2$. The unitary colligation A^w considered above is simple and w is the characteristic function of this colligation. Thus any Schur class operator function is the characteristic function of a unitary colligation.

Let $A : H \oplus E_1 \to H \oplus E_2$ be a simple unitary colligation, with characteristic function w. Then a unitary mapping $\mathcal{G} : H \to H^w$ that performs an equivalence between A and A^w is defined as follows:

$$(\mathcal{G}h)(\zeta) = \begin{bmatrix} (\mathcal{G}_+h)(\zeta) \\ (\mathcal{G}_-h)(\zeta) \end{bmatrix} = \begin{bmatrix} P_{E_2}A(\mathbb{1}_{H\oplus E_1} - \zeta P_H A)^{-1}h \\ \bar{\zeta}P_{E_1}A^*(\mathbb{1}_{H\oplus E_2} - \bar{\zeta}P_H A^*)^{-1}h \end{bmatrix} \tag{3–12}$$

The colligation A^w is called a *functional model* of A and the mapping $\mathcal{G}$ is called the *Fourier representation* of H.

If A is not simple, the Fourier representation $\mathcal{G}$ defined by the same formula (3–12) is a unitary mapping from H_{simp} onto H^w and performs an equivalence between the simple parts of A and A^w. It vanishes on H_{res}:

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A = A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix}.$$

Solutions of the AIP as characteristic functions. We are ready to proceed with the analysis of the AIP solutions. We begin with (3–11). Assume for simplicity that we have the equality

$$\|Fx\|^2_{H^w} = D(x,x) \quad \text{for all } x \in X.$$

We noticed in Lecture 2 that for some problems this is the case for any solution w. In other words, our assumption means that the map G defined in (3–3) is an isometry:

$$\|Gh_0\|^2_{H^w} = \|h_0\|^2_{H_0}.$$

Set $H_1 = H^w \ominus GH_0$ and $H = H_0 \oplus H_1$. Define a mapping $\mathcal{G} : H \to H^w$ by setting

$$\mathcal{G} \mid H_0 = G, \qquad \mathcal{G} \mid H_1 = \mathbb{1}_{H_1},$$

and observe that $\mathcal{G}$ is a unitary mapping of H onto H^w. We can write

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} V \mid d_V = A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \mid d_V, \tag{3–13}$$

instead of (3–11), because

$$d_V \subseteq H_0 \oplus E_1 \subseteq H \oplus E_1,$$
$$\Delta_V \subseteq H_0 \oplus E_2 \subseteq H \oplus E_2.$$

Finally, we obtain

$$V = \begin{bmatrix} \mathcal{G}^* & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \mid d_V. \tag{3–14}$$

Define the operator A by

$$A \overset{\text{def}}{=} \begin{bmatrix} \mathcal{G}^* & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix}. \tag{3–15}$$

A is a simple unitary colligation from $H \oplus E_1$ onto $H \oplus E_2$, and is unitarily equivalent to A^w. Hence, the characteristic function of A is w. By definition,

$H_0 \subseteq H$ and $A \,|\, d_V = V$ (see (3–14)); that is, A is a *unitary extension* of V. Thus *any solution w of the AIP is the characteristic function of a unitary extension A of the isometry V associated with the data:*

$$A : H \oplus E_1 \to H \oplus E_2 \quad \text{with} \quad H \supseteq H_0, \quad A \,|\, d_V = V.$$

In general, the equality $\|Fx\|^2_{H^w} = D(x,x)$ for all $x \in X$ does not hold, only the inequality $\|Fx\|^2_{H^w} \le D(x,x)$ for all $x \in X$. Nevertheless, the conclusion is still valid, but it is more difficult to prove. (A short and simple proof was recently found by J. Ball and T. Trent [Ball and Trent $\ge$ 1997].)

REMARK. I would like to emphasize one basic difference between the general case (inequality) and the special case (equality) considered in this section. We need one more definition: a unitary extension $A : H \oplus E_1 \to H \oplus E_2$ of an isometric colligation $V : d_V \to \Delta_V$, with $d_V \subseteq H_0 \oplus E_1$, $\Delta_V \subseteq H_0 \oplus E_2$, $H_0 \subseteq H$ is said to be a *minimal extension* if A has no nonzero reducing subspace in $H \ominus H_0$. If an extension A is nonminimal we can discard the reducing subspace in $H \ominus H_0$ and end up with a unitary colligation that has the same characteristic function and that *still extends* V, so we can consider minimal extensions only. But a minimal extension need not be a simple colligation. The absence of a reducing subspace for A in $H \ominus H_0$ does not mean that A has no reducing subspace in H at all. By discarding such a reducing subspace, we end up with a unitary colligation that has the same characteristic function but that *no longer extends* V. In the case of equality, $\|Fx\|^2_{H^w} = D(x,x)$ for all $x \in X$, the corresponding minimal extension is a simple colligation, as it is equivalent to A^w (see (3–14)); in the case of inequality, there exists $x \in X$ such that $\|Fx\|^2_{H^w} < D(x,x)$, and it is not simple.

A natural question arises now: does an arbitrary unitary extension A of the isometry V produce a solution of the AIP? The answer is yes, and it is easy to prove.

Let $A : H \oplus E_1 \to H \oplus E_2$ be a unitary extension of V. Let $\mathcal{G}$ be the Fourier representation associated with the colligation A; see (3–12). Then $\mathcal{G}$ maps H into H^w, where w is the characteristic function of A. Thus $\mathcal{G}$ is a contractive operator and

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A = A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix}. \tag{3–16}$$

Define the mapping $F : X \to H^w$ by

$$Fx \overset{\text{def}}{\equiv} G[x] \overset{\text{def}}{\equiv} \mathcal{G}[x].$$

Recall that $[x] \in H_0 \subseteq H$. Since $\mathcal{G}$ is a contraction, we have

$$\|Fx\|^2_{H^w} \le \big\|[x]\big\|^2_{H^0} = D(x,x);$$

that is, F satisfies (i). Since A extends V, (3–16) yields

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} V \mid d_V = A^w \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \mid d_V.$$

Since $d_V \subseteq H_0 \oplus E_1$ and $\Delta_V \subseteq H_0 \oplus E_2$, we can replace $\mathcal{G}$ with G:

$$\begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} V \mid d_V = A^w \begin{bmatrix} G & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix} \mid d_V.$$

As we have seen, the latter is equivalent to (ii). Thus w, the characteristic function of A, is a solution of the AIP.

Remark. Starting with an extension A of V we do not need the unitarity of $\mathcal{G}$ to check that the characteristic function of A is a solution. Neither the characteristic function nor the mapping $\mathcal{G}$, much less F, feels the residual part. Starting with the solution w in the general case we actually are given information on the simple part of the corresponding extension of V only. But we have to restore the residual part also in order to obtain the extension.

I will finish this lecture with the following summary of the discussion.

Theorem. *Let V be the isometric colligation associated with the AIP data*

$$V : d_V \to \Delta_V, \quad d_V \subseteq H_0 \oplus E_1, \quad \Delta_V \subseteq H_0 \oplus E_2.$$

(*no special assumptions are made on the data at present*). *Let $A : H \oplus E_1 \to H \oplus E_2$ be a minimal unitary extension of V, where $H \supseteq H_0$, so $A \,|\, d_V = V$. Let $w(\zeta)$ be the characteristic function of the colligation A*:

$$w(\zeta) \stackrel{\text{def}}{=} P_{E_2} A(\mathbb{1}_{H\oplus E_1} - \zeta P_H A)^{-1} \mid E_1.$$

Then w is a solution of the AIP. The corresponding mapping $F : X \to H^w$ is defined by

$$Fx = \mathcal{G}[x],$$

where

$$(\mathcal{G}h)(\zeta) = \begin{bmatrix} (\mathcal{G}_+ h)(\zeta) \\ (\mathcal{G}_- h)(\zeta) \end{bmatrix} = \begin{bmatrix} P_{E_2} A(\mathbb{1}_{H\oplus E_1} - \zeta P_H A)^{-1} h \\ \bar{\zeta} P_{E_1} A^* (\mathbb{1}_{H\oplus E_2} - \bar{\zeta} P_H A^*)^{-1} h \end{bmatrix} \quad \textit{for } h \in H.$$

All solutions of the AIP and the corresponding mappings F that satisfy (i) *and* (ii) *are of this form.*

References to Lecture 3 are [Katsnelson et al. 1987; Kheifets 1988a; 1988b; 1990b; Kheifets and Yuditskii 1994].

Lecture 4. Description of the Solutions of the AIP

In a sense, the description of all the solutions of the AIP and the corresponding mappings F was given in the last theorem of Lecture 3. The goal of this lecture is to give a description that separates explicitly the common part (related to the data) and the free parameters. The first step is the description of the unitary extensions of the given isometric colligation V in this fashion.

Unitary extensions of isometric colligations Let V be an isometric colligation,

$$V : d_V \to \Delta_V \quad \text{with} \quad d_V \subseteq H_0 \oplus E_1, \quad \Delta_V \subseteq H_0 \oplus E_2.$$

Let A be a minimal unitary extension of V,

$$A : H \oplus E_1 \to H \oplus E_2 \quad \text{with} \quad H \supseteq H_0, \quad A \,|\, d_V = V.$$

Let $d_V^\perp$ and $\Delta_V^\perp$ be the orthogonal complements of d_V in $H_0 \oplus E_1$ and Δ_V in $H_0 \oplus E_2$. Let $H_1 = H \ominus H_0$. Then the orthogonal complement of d_V in $H \oplus E_1$ is $H_1 \oplus d_V^\perp$ and the orthogonal complement of Δ_V in $H \oplus E_2$ is $H_1 \oplus \Delta_V^\perp$. Since A is a unitary operator mapping d_V onto Δ_V (since $A \,|\, d_V = V$), A has to map the orthogonal complement $H_1 \oplus d_V^\perp$ onto the orthogonal complement $H_1 \oplus \Delta_V^\perp$. Denote by A_1 the restriction of A to $H_1 \oplus d_V^\perp$. Thus, $A_1 : H_1 \oplus d_V^\perp \to H_1 \oplus \Delta_V^\perp$ is a unitary colligation. Since A is a minimal extension, A_1 is a simple colligation.

Conversely, take an arbitrary simple unitary colligation A_1 with the same exterior spaces $d_V^\perp$ and $\Delta_V^\perp$ and an arbitrary admissible state space H_1:

$$A_1 : H_1 \oplus d_V^\perp \to H_1 \oplus \Delta_V^\perp. \tag{4–1}$$

Let $H = H_0 \oplus H_1$ and define an extension of V by

$$A \,\Big|\, d_V = V, \qquad A \,\Big|\, H_1 \oplus d_V^\perp = A_1.$$

The result is a minimal unitary extension of V. Thus, the free parameter of a minimal unitary extension of V is an arbitrary simple unitary colligation A_1 with fixed exterior spaces $d_V^\perp$ and $\Delta_V^\perp$ and arbitrary admissible state space H_1. The word "admissible" means here that there exists a unitary colligation with this state space. For example: If $d_V^\perp$ and $\Delta_V^\perp$ are finite dimensional but their dimensions are different, then H_1 cannot be of finite dimension; if the dimensions of $d_V^\perp$ and $\Delta_V^\perp$ are equal then H_1 can be of arbitrary dimension. To give a full explanation it is enough to consider model colligations. Let $\omega : d_V^\perp \to \Delta_V^\perp$ be an arbitrary Schur class operator function (contractive-valued and analytic in the unit disc). One can take as A_1 the (model) unitary colligation

$$A^\omega = \begin{bmatrix} A^\omega_{\mathrm{in}} & (A^\omega_1)^* \\ A^\omega_2 & A^\omega_{12} \end{bmatrix} : \begin{bmatrix} H^\omega \\ d_V^\perp \end{bmatrix} \to \begin{bmatrix} H^\omega \\ \Delta_V^\perp \end{bmatrix},$$

where H^ω is the de Branges–Rovnyak space corresponding to the operator function ω (see page 352) and

$$\begin{aligned} A_{\mathrm{in}}^{\omega} f &= P^{\omega}\bar{t}f = \bar{t}\left(f - \begin{bmatrix} \mathbb{1} & \omega \\ \omega^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} f_+(0) \\ 0 \end{bmatrix}\right) : H^\omega \to H^\omega, \\ (A_1^\omega)^* &= \bar{t} \begin{bmatrix} \mathbb{1} & \omega \\ \omega^* & \mathbb{1} \end{bmatrix} \begin{bmatrix} -\omega(0) \\ \mathbb{1} \end{bmatrix} : d_V^\perp \to H^\omega, \\ A_2^\omega f &= f_+(0) : H^\omega \to \Delta_V^\perp, \\ A_{12}^\omega &= \omega(0) : d_V^\perp \to \Delta_V^\perp. \end{aligned}$$

Let $A_1 : H_1 \oplus d_V^\perp \to H_1 \oplus \Delta_V^\perp$ and $A_1' : H_1' \oplus d_V^\perp \to H_1' \oplus \Delta_V^\perp$ be two simple unitary colligations. If they are unitarily equivalent, that is, if there exists a unitary mapping $\mathcal{G}_1 : H_1 \to H_1'$ such that

$$\begin{bmatrix} \mathcal{G}_1 & 0 \\ 0 & \mathbb{1}_{\Delta_V^\perp} \end{bmatrix} A_1 = A_1' \begin{bmatrix} \mathcal{G}_1 & 0 \\ 0 & \mathbb{1}_{d_V^\perp} \end{bmatrix},$$

the corresponding minimal unitary extensions of V are unitarily equivalent colligations:

$$\begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_2} \end{bmatrix} A = A' \begin{bmatrix} \mathcal{G} & 0 \\ 0 & \mathbb{1}_{E_1} \end{bmatrix},$$

with $\mathcal{G} \,|\, H_1 = \mathcal{G}_1$ and $\mathcal{G} \,|\, H_0 = \mathbb{1}_{H_0}$.

We emphasize here that this is more than just equivalence of unitary colligations, as $\mathcal{G}$ is the identity on H_0. We will call such extensions *equivalent extensions*. The following proposition follows from the previous discussion.

Claim 1. *Two minimal unitary extensions A and A' of V are equivalent extensions if and only if the corresponding simple unitary colligations A_1 and A_1' are unitarily equivalent colligations.*

The next claim is a straightforward consequence of the formulas of the last theorem of Lecture 3.

Claim 2. *Two equivalent minimal extensions of V generate the same solution w and the same mapping $F : X \to H^w$.*

Thus, a solution w of the AIP and a corresponding mapping $F : X \to H^w$ represent equivalence classes of simple unitary colligations $A_1 : H_1 \oplus d_V^\perp \to H_1 \oplus \Delta_V^\perp$. Hence, the latter may serve as free parameters. But we know from the digression on page 366 that the only invariant of this equivalence class is the characteristic function of A_1. In other words:

Claim 3. *The free parameter of pairs (w, F) (consisting of a solution w and a corresponding mapping F) is an arbitrary Schur class (contractive-valued and analytic in $\mathbb{D}$) operator function $\omega(\zeta) : d_V^\perp \to \Delta_V^\perp$.*

Universal unitary colligations associated with isometric colligations. In the previous section we extracted the free parameter ω that the solution w and the corresponding mapping F depend on. It is clear that the common part of all the extensions of the isometry V is the isometry V itself. To obtain nice and explicit formulas it is convenient to associate a *universal unitary colligation* to the isometry V. Let V be an isometric colligation,

$$V : d_V \to \Delta_V \quad \text{with} \quad d_V \subseteq H_0 \oplus E_1, \quad \Delta_V \subseteq H_0 \oplus E_2,$$

and let $d_V^\perp$ and $\Delta_V^\perp$ be the orthogonal complements of d_V and Δ_V. Let N_1 be an isomorphic copy of $d_V^\perp$ (that is, there exists a unitary and surjective mapping $u_1 : d_V^\perp \to N_1$). Let N_2 be an isomorphic copy of $\Delta_V^\perp$ (that is, there exists a unitary and surjective mapping $u_2 : \Delta_V^\perp \to N_2$). Define a unitary colligation $A_0 : H_0 \oplus E_1 \oplus N_2 \to H_0 \oplus E_2 \oplus N_1$ by setting

$$\begin{aligned} &A_0 \,|\, d_V = V \qquad (d_V \subseteq H_0 \oplus E_1), \\ &A_0 \,|\, d_V^\perp = u_1, \qquad A_0 \,|\, N_2 = u_2^*. \end{aligned}$$

Remark. A_0 is not a unitary extension of V in the sense considered earlier, because we now do not extend the state space H_0, only the exterior spaces: $E_1 \oplus N_2$ instead of E_1 and $E_2 \oplus N_1$ instead of E_2. Of course, A_0 extends V but in a different sense. We will call this extension a *universal extension*; the name is motivated by category theory. We will see the role of this colligation below.

From now on we fix the unitary mappings u_1 and u_2, and also their images N_1 and N_2. Also we fix the notation A_1 for a simple unitary colligation $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$ (this class of colligations is obviously related to the colligations of the form (4–1) defined earlier, and is denoted by the same symbol). Thus, now the equivalence classes of the simple unitary colligations $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$ will be free parameters of the solutions w and the corresponding mappings F; that is, the Schur class functions $\omega(\zeta) : N_1 \to N_2$ are free parameters now.

Coupling of unitary colligations and unitary extensions of isometric colligations. Let V be an isometric colligation,

$$V : d_V \to \Delta_V \quad \text{with} \quad d_V \subseteq H_0 \oplus E_1, \quad \Delta_V \in H_0 \oplus E_2.$$

Let A_0 be the universal colligation associated with V:

$$A_0 : H_0 \oplus E_1 \oplus N_2 \to H_0 \oplus E_2 \oplus N_1,$$

with $A_0 \,|\, d_V = V$, $A_0(d_V^\perp) = N_1$, $A_0(N_2) = \Delta_V^\perp$. Let A be a minimal unitary extension of V:

$$A : H \oplus E_1 \to H \oplus E_2 \quad \text{with} \quad H \supseteq H_0, \quad A \,|\, d_V = V.$$

We have seen that one can associate a simple unitary colligation $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$ with A, where $H_1 = H \ominus H_0$:

$$A_1 = \begin{bmatrix} \mathbb{1}_{H_1} & 0 \\ 0 & (A_0^*) \,|\, \Delta_V^\perp \end{bmatrix} (A \,|\, H_1 \oplus d_V^\perp) \begin{bmatrix} \mathbb{1}_{H_1} & 0 \\ 0 & (A_0^*) \,|\, N_1 \end{bmatrix} \tag{4–2}$$

An arbitrary simple unitary colligation A_1 with the exterior spaces N_1 and N_2 arises this way.

I will now give a procedure for recovering A from A_0 (fixed colligation) and A_1 (arbitrary parameter). Let $A_1 : H_1 \oplus N_1 \to H_1 \oplus N_2$ be an arbitrary simple unitary colligation. Let

$$\begin{bmatrix} h_0' \\ e_2 \\ n_1 \end{bmatrix} = A_0 \begin{bmatrix} h_0 \\ e_1 \\ n_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} h_1' \\ n_2' \end{bmatrix} = A_1 \begin{bmatrix} h_1 \\ n_1' \end{bmatrix}. \tag{4–3}$$

We can choose vectors on the right-hand sides of these relations in an arbitrary way and compute the vectors on the left-hand sides, or we can choose vectors on the left-hand sides in an arbitrary way and compute the vectors on the right-hand sides (because A_0 and A_1 are unitary operators).

Let us consider, in addition to the relations above, two more relations

$$n_1 = n_1' \quad \text{and} \quad n_2 = n_2', \tag{4–4}$$

and see what can be chosen now in an arbitrary way. Observe that the colligation A_0 possesses the property

$$P_{N_1} A_0 \,|\, N_2 = 0,$$

because A_0 sends N_2 onto $\Delta_V^\perp \subseteq H_0 \oplus E_2$, which is orthogonal to N_1. This property of A_0 guarantees (although it is not necessary) that we can choose h_0, h_1, e_1 in an arbitrary way and compute h_0', h_1', e_2 from the system of equations (4–3), (4–4) in a unique way, along with $n_1 = n_1'$ and $n_2 = n_2'$ as well. Take the result of this computation as the definition of the new linear operator A:

$$A \begin{bmatrix} h_1 \\ h_0 \\ e_1 \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} h_1' \\ h_0' \\ e_2 \end{bmatrix}. \tag{4–5}$$

It is easy to see that A is a unitary colligation, $A : H_1 \oplus H_0 \oplus E_1 \to H_0 \oplus H_0 \oplus E_2$. Moreover, A extends V. To see this, take an arbitrary vector

$$\begin{bmatrix} h_0 \\ e_1 \end{bmatrix} \in d_V;$$

then take

$$\begin{bmatrix} h_0' \\ e_2 \end{bmatrix} = V \begin{bmatrix} h_0 \\ e_1 \end{bmatrix} \in \Delta_V,$$

and take also

$$h_1 = h_1' = 0, \quad n_1 = n_1' = 0, \quad n_2 = n_2' = 0.$$

This collection of vectors satisfies the system of equations (4–3), (4–4): it obviously fits (4–4) and the second equality in (4–3); it satisfies the first equality in (4–3) because $A_0 \,|\, d_V = V$; i.e.,

$$\begin{bmatrix} V\left[\begin{smallmatrix} h_0 \\ e_1 \end{smallmatrix}\right] \\ 0 \end{bmatrix} = A_0 \begin{bmatrix} h_0 \\ e_1 \\ 0 \end{bmatrix} \quad \text{if } \begin{bmatrix} h_0 \\ e_1 \end{bmatrix} \in d_V.$$

Hence, by the definition of the colligation A, these vectors (actually, a part of them) satisfy (4–5):

$$\begin{bmatrix} 0 \\ V\left[\begin{smallmatrix} h_0 \\ e_1 \end{smallmatrix}\right] \end{bmatrix} = A \begin{bmatrix} 0 \\ h_0 \\ e_1 \end{bmatrix} \quad \text{if } \begin{bmatrix} h_0 \\ e_1 \end{bmatrix} \in d_V.$$

This means that A extends V.

We will say that the colligation A is the *feedback coupling* of A_0 with A_1 (or the *loading* of A_0 with A_1). It is also easy to check that these two procedures—the extraction of A_1 from the extension A and the feedback coupling of A_0 with A_1—are mutually inverse: performing the two successively we come back to the colligation that we started with.

Our next goal is to express the characteristic function w and the Fourier representation $\mathcal{G}$ of the colligation A in terms of the characteristic functions and Fourier representations of the colligations A_0 and A_1. But we need a digression first.

Digression: Unitary colligations, characteristic functions, Fourier representation, and discrete time dynamics. Let $A : H \oplus E_1 \to H \oplus E_2$ be a unitary colligation. One can associate with A the discrete-time dynamical system

$$\begin{bmatrix} h(k+1) \\ e_2(k) \end{bmatrix} = A \begin{bmatrix} h(k) \\ e_1(k) \end{bmatrix}, \tag{4–6}$$

where k is a nonnegative integer, $h(0)$ is an arbitrary vector from H, and $\{e_1(k)\}_{k=0}^{\infty}$ is an arbitrary input signal.

Let $\zeta \in \mathbb{D}$ be the corresponding spectral parameter (complex frequency). Let $\tilde{h}^+(\zeta)$, $\tilde{e}_1^+(\zeta)$, and $\tilde{e}_2^+(\zeta)$ be the discrete Laplace transforms of $h(k)$, $e_1(k)$, and $e_2(k)$, respectively; that is,

$$\tilde{h}^+(\zeta) = \sum_{k=0}^{\infty} \zeta^k h(k), \quad \tilde{e}_1^+(\zeta) = \sum_{k=0}^{\infty} \zeta^k e_1(k), \quad \tilde{e}_2^+(\zeta) = \sum_{K=0}^{\infty} \zeta^k e_2(k).$$

If the input signal $\{e_1(k)\}$ is square summable,

$$\sum_{k=0}^{\infty} \left\| e_1(k) \right\|^2 < \infty,$$

then the state evolution $\{h(k)\}_{k=0}^{\infty}$ and the output signal $\{e_2(k)\}_{k=0}^{\infty}$ possess the same property.

Multiplying (4–6) by ζ^k and taking the summation over k, one obtains the spectral form of the dynamics equation:

$$\begin{aligned}\tilde{h}^+(\zeta) &= (\mathbb{1}_H - \zeta A_{in})^{-1} \cdot h(0) + \mathcal{G}_-(\zeta)^* \tilde{e}_1^+(\zeta)\\ \tilde{e}_2^+(\zeta) &= \mathcal{G}_+(\zeta) \cdot h(0) + w(\zeta) \cdot \tilde{e}_1^+(\zeta),\end{aligned} \tag{4–7}$$

where $w(\zeta)$ is the characteristic function of the colligation A, and $\mathcal{G}_+, \mathcal{G}_-$ are the components of the Fourier representation $\mathcal{G}$ (see (3–12)).

One can also rewrite (4–6) as

$$A^* \begin{bmatrix} h(k+1) \\ e_2(k) \end{bmatrix} = \begin{bmatrix} h(k) \\ e_1(k) \end{bmatrix}. \tag{4–8}$$

Now consider the negative integers $k \le -1$. (This actually means inverting time and exchanging the roles of the input and the output). Considering the past Laplace transform,

$$\tilde{h}^-(\zeta) = \sum_{k=-\infty}^{-1} \bar{\zeta}^{|k|} h(k), \quad \tilde{e}_2^-(\zeta) = \sum_{k=-\infty}^{-1} \bar{\zeta}^{|k|} e_2(k), \quad \tilde{e}_1^-(\zeta) = \sum_{k=-\infty}^{-1} \bar{\zeta}^{|k|} e_1(k),$$

one can rewrite (4–8) as

$$\begin{aligned}\tilde{h}^-(\zeta) &= \bar{\zeta}(\mathbb{1} - \bar{\zeta} A_{\text{in}}^*)^{-1} h(0) + \mathcal{G}_+(\zeta)^* \tilde{e}_2^-(\zeta),\\ \tilde{e}_1^-(\zeta) &= \mathcal{G}_-(\zeta) \cdot h(0) + w(\zeta)^* \tilde{e}_2^-(\zeta).\end{aligned} \tag{4–9}$$

The second formulas in (4–7) and (4–9) are the most convenient for our purposes:

$$\begin{aligned}\tilde{e}_2^+(\zeta) &= \mathcal{G}_+(\zeta) h(0) + w(\zeta) \tilde{e}_1^+(\zeta)\\ \tilde{e}_1^-(\zeta) &= \mathcal{G}_-(\zeta) h(0) + w(\zeta)^* \tilde{e}_2^-(\zeta).\end{aligned} \tag{4–10}$$

Formulas describing the solutions w of the AIP and the corresponding mappings F. Let A be the feedback coupling of A_0 with A_1. Denote the characteristic function of A_0 by $S(\zeta) : E_1 \oplus N_2 \to E_2 \oplus N_1$. Denote also the entries of S corresponding to this decomposition of the spaces as follows:

$$S = \begin{bmatrix} s_0 & s_2 \\ s_1 & s \end{bmatrix} : \begin{bmatrix} E_1 \\ N_2 \end{bmatrix} \longrightarrow \begin{bmatrix} E_2 \\ N_1 \end{bmatrix}.$$

Let the characteristic function of A_1 be $\omega(\zeta) : N_1 \to N_2$. Now write the dynamics related to A_0 and the dynamics related to A_1 in the form (4–10). I am going to consider the "+" parts only now; the treatment of the "−" parts is analogous. We have

$$\tilde{n}_2^+(\zeta) = \mathcal{G}_+^1(\zeta) h_1(0) + \omega(\zeta) \tilde{n}_1^+(\zeta),$$

$$\begin{bmatrix} \tilde{e}_2^+(\zeta) \\ \tilde{n}_1'^+(\zeta) \end{bmatrix} = \mathcal{G}_+^0(\zeta) h_0(0) + S(\zeta) \begin{bmatrix} \tilde{e}_1^+(\zeta) \\ \tilde{n}_2'^+(\zeta) \end{bmatrix}.$$

The coupling condition is

$$\tilde{n}_1^+ = \tilde{n}_1'^+, \qquad \tilde{n}_2^+ = \tilde{n}_2'^+.$$

Express $\tilde{e}_2^+(\zeta)$ in terms of

$$\begin{bmatrix} h_1(0) \\ h_0(0) \end{bmatrix}$$

and $\tilde{e}_1^+(\zeta)$, excluding $\tilde{n}_1^+ = \tilde{n'}_1^+$ and $\tilde{n}_2^+ = \tilde{n'}_2^+$. What we obtain now has to coincide (see (4–10)) with

$$\tilde{e}_2^+(\zeta) = \mathcal{G}_+(\zeta) \begin{bmatrix} h_1(0) \\ h_0(0) \end{bmatrix} + w(\zeta)\tilde{e}_1^+(\zeta).$$

This leads to the formulas

$$\begin{aligned} w(\zeta) &= s_0(\zeta) + s_2(\zeta)\omega(\zeta)(\mathbb{1}_{N_1} - s(\zeta)\omega(\zeta))^{-1} s_1(\zeta) \\ \mathcal{G}_+(\zeta) \begin{bmatrix} h_1(0) \\ h_0(0) \end{bmatrix} &= \bigl[\psi(\zeta)\omega(\zeta),\ \mathbb{1}_{E_2}\bigr] \mathcal{G}_+^0(\zeta)h_0(0) + \psi(\zeta)\mathcal{G}_+^1(\zeta)h_1(0), \end{aligned} \tag{4–11}$$

where

$$\psi(\zeta) = s_2(\zeta)(\mathbb{1}_{N_2} - \omega(\zeta)s(\zeta))^{-1}. \tag{4–12}$$

Remark. By the definition of the characteristic function,

$$S(0) = P_{E_2 \oplus N_1} A_0 \,|\, (E_1 \oplus N_2).$$

In particular, $s(0) = P_{N_1} A \,|\, N_2 = 0$. This guarantees that the formulas (4–11) and (4–12) make sense, since $\|s(\zeta)\| \le |\zeta|$ when $|\zeta| < 1$ by Schwarz's lemma.

Considering the "–" parts of (4–10) for A_0, A_1, and A we obtain

$$\mathcal{G}_-(\zeta) \begin{bmatrix} h_1(0) \\ h_0(0) \end{bmatrix} = \bigl[\varphi(\zeta)^*\omega(\zeta)^*,\ \mathbb{1}_{E_1}\bigr] \mathcal{G}_-^0(\zeta)h_0(0) + \varphi(\zeta)^*\mathcal{G}_-^1(\zeta)h_1(0), \tag{4–13}$$

where

$$\varphi(\zeta) = (\mathbb{1}_{N_1} - s(\zeta)\omega(\zeta))^{-1} s_1(\zeta). \tag{4–14}$$

Combining the expressions (4–11) and (4–12) for $\mathcal{G}_+$ and $\mathcal{G}_-$ we arrive at

$$\mathcal{G} \begin{bmatrix} h_1 \\ h_0 \end{bmatrix} = \begin{bmatrix} \psi\omega & \mathbb{1}_{E_2} & 0 & 0 \\ 0 & 0 & \varphi^*\omega^* & \mathbb{1}_{E_1} \end{bmatrix} \mathcal{G}^0 h_0 + \begin{bmatrix} \psi & 0 \\ 0 & \varphi^* \end{bmatrix} \mathcal{G}^1 h_1. \tag{4–15}$$

As we saw in the last theorem of Lecture 3,

$$Fx = \mathcal{G}[x] \quad \text{for all } x \in X,$$

where $[x] \in H_0$ is the equivalence class generated by the quadratic form D. Hence

$$Fx = \begin{bmatrix} \psi\omega & \mathbb{1}_{E_2} & 0 & 0 \\ 0 & 0 & \varphi^*\omega^* & \mathbb{1}_{E_1} \end{bmatrix} \mathcal{G}^0[x].$$

The following theorem summarizes this lecture.

Theorem. *Let V be the isometric colligation associated with the AIP data* (*page* 362). *Let A_0 be the unitary colligation associated with V* (*page* 372). *Let S be the characteristic function of A_0*:

$$S(\zeta) = P_{E_2 \oplus N_1} A_0 (\mathbf{1}_{H_0 \oplus E_1 \oplus N_2} - \zeta P_{H_0} A_0)^{-1} \mid (E_1 \oplus N_2),$$
$$S(\zeta) = \begin{bmatrix} s_0(\zeta) & s_2(\zeta) \\ s_1(\zeta) & s(\zeta) \end{bmatrix} : \begin{bmatrix} E_1 \\ N_2 \end{bmatrix} \longrightarrow \begin{bmatrix} E_2 \\ N_1 \end{bmatrix}.$$

Then the solutions w of the AIP and the corresponding mappings $F : X \to H^w$ are described by

$$w = s_0 + s_2 \omega (\mathbf{1}_{N_1} - s\omega)^{-1} s_1$$
$$Fx = \begin{bmatrix} \psi\omega & \mathbf{1}_{E_2} & 0 & 0 \\ 0 & 0 & \varphi^* \omega^* & \mathbf{1}_{E_1} \end{bmatrix} \mathcal{G}^0[x] \quad \textit{for } x \in X$$

where ω is an arbitrary Schur class function $\omega(\zeta) : N_1 \to N_2$, for $|\zeta| < 1$,

$$\psi = s_2 (\mathbf{1}_{N_2} - \omega s)^{-1}, \qquad \varphi = (\mathbf{1}_{N_1} - s\omega)^{-1} s_1,$$
$$\mathcal{G}^0(\zeta) h_0 = \begin{bmatrix} \mathcal{G}^0_+(\zeta) h_0 \\ \mathcal{G}^0_-(\zeta) h_0 \end{bmatrix} = \begin{bmatrix} P_{E_2 \oplus N_1} A_0 (\mathbf{1} - \zeta P_{H_0} A_0)^{-1} h_0 \\ \bar{\zeta} P_{E_1 \oplus N_2} A_0^* (\mathbf{1} - \bar{\zeta} P_{H_0} A_0^*)^{-1} h_0 \end{bmatrix}.$$

S and $\mathcal{G}^0$ depend on the data of the AIP only, *whereas ω is arbitrary.*

We can see that the parameter ω defines uniquely not only the solution w but also the corresponding mapping $F : X \to H^w$. This suggests denoting the mappings F by $F^\omega : X \to H^w$. In particular cases, when the mapping F is unique for a solution w, that is, when the F^ω coincide if the corresponding solutions w coincide, it can be denoted by F^w.

References to Lecture 4 are [Arov and Grossman 1983; Katsnelson et al. 1987; Kheifets 1988a; 1988b; 1990b; Kheifets and Yuditskii 1994].

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