Holomorphic Spaces
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Harmonic Bergman Spaces

KAREL STROETHOFF

Abstract. We present a simple derivation of the explicit formula for the harmonic Bergman reproducing kernel on the ball in euclidean space and give an elementary proof that the harmonic Bergman projection is L^p-bounded, for $1 < p < \infty$. We furthermore discuss duality results.

1. Introduction

Throughout the paper n is a positive integer greater than 1. We will be working with functions defined on all or part of $\mathbb{R}^n$. Let D_j denote the partial derivative with respect to the j-th coordinate variable. Recall that $\nabla u(x) = (D_1 u(x), \cdots, D_n u(x))$. The Laplacian of u is $\Delta u(x) = D_1^2 u(x) + \cdots + D_n^2 u(x)$. A real- or complex-valued function u is harmonic on an open subset Ω of $\mathbb{R}^n$ if $\Delta u \equiv 0$ on Ω. The purpose of this article is to present an elementary treatment of some known results for the harmonic Bergman spaces consisting of all harmonic functions on the unit ball in $\mathbb{R}^n$ that are p-integrable with respect to volume measure. Several properties of these spaces are analogous to those of the Bergman spaces of analytic functions on the unit ball in $\mathbb{C}^n$. As in the analytic case, there is a reproducing kernel and associated projection. Duality results follow once we know that the projection is L^p-bounded. Coifman and Rochberg [1980] used deep results from harmonic analysis to establish L^p-boundedness of the harmonic Bergman projection.

An explicit formula for the harmonic Bergman reproducing kernel has only been determined recently; see [Axler et al. 1992]. We give a simple derivation for such a formula in Section 2. In Section 3 we give an elementary proof that the harmonic Bergman projection is L^p-bounded for $1 < p < \infty$. Our proof is similar to Forelli and Rudin's proof of L^p-boundedness of the analytic Bergman projection [Forelli and Rudin 1974/75; Rudin 1980], but as in Axler's argument [1988]

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in the context of the analytic Bergman spaces on the unit disk, we will avoid the use of the binomial theorem, the gamma function and Stirling's formula.

In Section 4 we discuss the dual and the predual of the Bergman space of integrable harmonic functions on the unit ball in $\mathbb{R}^n$. Analogous to the analytic case, the dual and predual are identified with the harmonic Bloch and little Bloch space, which are spaces of harmonic functions determined by growth conditions on the gradient. As in the analytic case, these duality results follow from the fact that the harmonic Bergman projection maps L^∞ onto the harmonic Bloch space.

The remainder of this section establishes some of the notation and contains the prerequisites from harmonic function theory needed in the paper.

We will repeatedly make use of Green's identity, which states that if Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and u and v are continuously twice-differentiable functions in an open set containing $\bar{\Omega}$, the closure of Ω, then

$$\int_\Omega (u\Delta v - v\Delta u)\, dV = \int_{\partial\Omega} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\, ds, \tag{1–1}$$

where V denotes volume measure on $\mathbb{R}^n$, s denotes surface area measure on $\partial\Omega$, and the symbol $D_{\mathbf{n}}$ denotes differentiation with respect to the outward unit normal vector $\mathbf{n}$: $D_{\mathbf{n}}w = \nabla w \cdot \mathbf{n}$. A special case of (1–1) is obtained by taking $v \equiv 1$: if u is harmonic in an open neighborhood of $\bar{\Omega}$, then

$$\int_{\partial\Omega} D_{\mathbf{n}}u\, ds = 0. \tag{1–2}$$

For $y \in \mathbb{R}^n$ and $r > 0$ we write $B(y,r) = \{x \in \mathbb{R}^n : |x-y| < r\}$ and $\bar{B}(y,r)$ for its closure. We use B to denote the unit ball $B(0,1)$ and write S for its boundary, the unit sphere $S = \{x \in \mathbb{R}^n : |x| = 1\}$. The area of S is easily determined: if we take $\Omega = B$, $u \equiv 1$ and $v(x) = |x|^2$ in (1–1), then $\Delta v \equiv 2n$ and $D_{\mathbf{n}}v \equiv 2$, and we obtain $2nV(B) = 2s(S)$; thus $s(S) = nV(B)$. It will be convenient to work with normalized surface area on S, which we denote by σ; thus $ds = nV(B)\, d\sigma$ on S and $\sigma(S) = 1$.

If u is harmonic on an open neighborhood of $\bar{B}(y,r)$, the chain rule gives $(d/dr)u(y+r\zeta) = \nabla u(y+r\zeta)\cdot\zeta = D_{\mathbf{n}}u(y+r\zeta)$, where the normal derviative $D_{\mathbf{n}}$ is taken with respect to $B(y,r)$, so that

$$\frac{d}{dr}\int_S u(y+r\zeta)\, d\sigma(\zeta) = \int_S D_{\mathbf{n}}u(y+r\zeta)\, d\sigma(\zeta) = 0,$$

by (1–2). Hence $\int_S u(y+r\zeta)\, d\sigma(\zeta)$ does *not* depend on r, so that the function u satisfies the following so-called *mean value property*

$$\int_S u(y+r\zeta)\, d\sigma(\zeta) = u(y). \tag{1–3}$$

In particular, if u is harmonic on an open set containing $\bar{B}$, we have

$$u(0) = \int_S u(\zeta)\, d\sigma(\zeta).$$

In fact, for any $y \in B$, $u(y)$ is a weighted average of u over S, namely $u(y) = \int_S u(\zeta)\, P(\zeta, y)\, d\sigma(\zeta)$, where $P(\zeta, y)$ is the so called *Poisson kernel* for the ball B. Since this Poisson kernel will in the sequel play a fundamental role, we will derive an explicit formula for P.

Fix $y \in B \setminus \{0\}$, choose any $0 < r < 1 - |y|$, and set

$$\Omega = \{x \in \mathbb{R}^n : r < |x - y| < 1\}.$$

We will only consider here the case $n > 2$. Put $v(x) = |x - y|^{2-n}$. It is easy to verify that v is harmonic on $\mathbb{R}^n \setminus \{y\}$ and $\nabla v(x) = (2-n)|x-y|^{-n}(x-y)$; consequently, $D_{\mathbf{n}} v = (2-n) r^{1-n}$ on $\partial B(y, r)$. For $\zeta \in S$ it is easy to verify that $|\zeta - y|^2 = |y|^2 \left|\zeta - y/|y|^2\right|^2$; thus, on S the function v coincides with the function $w(x) = |y|^{2-n} \left|x - y/|y|^2\right|^{2-n}$. Using that w is harmonic on $\bar{B}$, an application of (1–1) yields $\int_S u D_{\mathbf{n}} w\, ds = \int_S w D_{\mathbf{n}} u\, ds = \int_S v D_{\mathbf{n}} u\, ds$. It follows that

$$\int_S (u D_{\mathbf{n}} v - u D_{\mathbf{n}} w)\, ds = \int_S (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u)\, ds.$$

By (1–1)

$$\begin{aligned} \int_S (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u)\, ds &= \int_{\partial B(y,r)} (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u)\, ds \\ &= (2-n) r^{1-n} \int_{\partial B(y,r)} u\, ds = (2-n) n V(B) u(y). \end{aligned}$$

We conclude that $u(y) = \int_S u P_y\, d\sigma$, where $P_y = (2-n)^{-1}(D_{\mathbf{n}} v - D_{\mathbf{n}} w)$. A simple calculation shows that $P_y(\zeta) = P(\zeta, y)$ is given by the formula

$$P(\zeta, y) = \frac{1 - |y|^2}{|\zeta - y|^n}.$$

In particular we have

$$\int_S \frac{1 - |y|^2}{|y - \zeta|^n}\, d\sigma(\zeta) = 1 \quad \text{for } y \in B. \tag{1–4}$$

We extend the Poisson kernel P to a function on $B \times B$ as follows: if $x, y \in B$, then we set $P(x, y) = P(x/|x|, |x| y)$. This is called the *extended Poisson kernel*. Thus

$$P(x, y) = \frac{1 - |x|^2 |y|^2}{(1 - 2x \cdot y + |x|^2 |y|^2)^{n/2}} \quad \text{for } x, y \in B.$$

For each fixed $y \in B$ the function $x \mapsto P(x, y)$ is harmonic, and by symmetry, for each fixed $x \in B$ the function $y \mapsto P(x, y)$ is harmonic.

It will be convenient to use polar coordinates to integrate functions over balls. Heuristically, we have

$$\frac{d}{dr}\int_{B(0,r)} f(x)\,dV(x) = \int_{rS} f\,ds,$$

and because the sphere rS has area $nV(B)r^{n-1}$ this is equal to

$$nV(B)r^{n-1}\int_S f(r\zeta)\,d\sigma(\zeta).$$

Integrating with respect to r we obtain the following formula

$$\int_{B(0,\rho)} f(x)\,dV(x) = nV(B)\int_0^\rho r^{n-1}\int_S f(r\zeta)\,d\sigma(\zeta)\,dr. \tag{1–5}$$

If u is harmonic on B, then we define its *radial derivative* $\mathcal{R}u$ by $\mathcal{R}u(x) = \nabla u(x)\cdot x$. It is easy to verify that $\mathcal{R}u$ is also harmonic on B. In fact, this follows at once from the formula $\Delta(\mathcal{R}u) = 2\Delta u + \mathcal{R}(\Delta u)$, which is easy to verify.

The reader interested in learning more harmonic function theory should consult [Axler et al. 1992].

2. The Harmonic Bergman Spaces

For $1 \le p < \infty$ we denote by $b^p(B)$ the set of all harmonic functions u on B for which

$$\|u\|_p = \left(\int_B |u(x)|^p\,dV(x)\right)^{1/p} < \infty.$$

The spaces $b^p(B)$ are called *harmonic Bergman spaces*. The space $b^2(B)$ is a linear subspace of $L^2(B)$ with inner product given by

$$\langle f, g\rangle = \int_B f(x)\overline{g(x)}\,dV(x) \quad \text{for } f, g \in L^2(B).$$

If u is harmonic on B and $y \in B$ is fixed, it follows from (1–3) and Cauchy-Schwarz's inequality that $|u(y)|^2 \le \int_S |u(y + r\zeta)|^2\,d\sigma(\zeta)$ for $0 < r < 1 - |y|$. Applying (1–5) to the function $f(x) = |u(y + x)|^2$ with $\rho = 1 - |y|$ we see that

$$nV(B)\int_0^\rho r^{n-1}\int_S |u(y + r\zeta)|^2\,d\sigma(\zeta)\,dr = \int_{B(y,\rho)} |u|^2\,dV \le \|u\|_2{}^2,$$

and conclude that

$$|u(y)| \le \frac{1}{V(B)^{1/2}(1 - |y|)^{n/2}}\|u\|_2 \quad \text{for } u \in b^2(B). \tag{2–1}$$

It follows from (2–1) that $b^2(B)$ is a closed subspace of $L^2(B)$, and thus it is a Hilbert space. Inequality (2–1) implies that for each fixed $y \in B$ the linear functional $u \mapsto u(y)$ is bounded. By the Riesz representation theorem there is a unique function $R_y \in b^2(B)$, called *the reproducing kernel at* y, such that $u(y) = \langle u, R_y\rangle$, for all $u \in b^2(B)$. By considering real and imaginary parts, it is easily seen that the function reproducing the value at y for the real Bergman space, also reproduces the value at y for complex-valued functions of $b^2(B)$, and thus (by uniqueness) we conclude that R_y is real-valued. We write $R(x,y) = R_y(x)$, and call this the *Bergman reproducing kernel* of $b^2(B)$. Because the Bergman reproducing kernel is real–valued we have $R(y,x) = \langle R_x, R_y\rangle = \langle R_y, R_x\rangle = R_y(x) = R(x,y)$, for all $x, y \in B$. Thus R is a symmetric function on $B \times B$.

The Bergman kernel for the ball. In this subsection we will derive an explicit formula for the Bergman kernel R of B. We will not make use of so-called zonal and spherical harmonics used in [Axler et al. 1992] to calculate the Bergman kernel R, but instead use Green's identity to relate R to the extended Poisson kernel P.

Suppose u is harmonic on $\bar{B}$ and fix $y \in B$. Let v also be harmonic on $\bar{B}$, and define the function w by $w(x) = (|x|^2 - 1)v(x) = |x|^2 v(x) - v(x)$. Observe that the function Δw is harmonic on $\bar{B}$, for

$$\begin{aligned}\Delta w(x) &= \Delta(|x|^2)\, v(x) + 2\nabla|x|^2 \cdot \nabla v(x) + |x|^2 \Delta v(x) \\ &= 2nv(x) + 4x \cdot \nabla v(x) = 2nv(x) + 4\Re v(x).\end{aligned} \tag{2–2}$$

Note that $\nabla w(x) = 2v(x)\, x + (|x|^2 - 1)\nabla v(x)$, so that $D_{\mathbf{n}} w(x) = \nabla w(x) \cdot x/|x| = 2v(x)\,|x| + (|x|^2-1)D_{\mathbf{n}} v(x)$. In particular, $D_{\mathbf{n}} w \equiv 2v$ on S. Since $\Delta u \equiv 0$ on B and $w \equiv 0$ on S, it follows from (1–1) (applied to u and w) that

$$\int_B u\Delta w\, dV = \int_S u D_{\mathbf{n}} w\, ds = 2\int_S uv\, ds = 2nV(B)\int_S uv\, d\sigma. \tag{2–3}$$

It is clear that our choice for v should be the extended Poisson kernel: $v(x) = P(x,y)$ for $x \in B$. Then

$$\int_B u\Delta w\, dV = 2nV(B)\int_S u(\zeta)\, P(\zeta, y)\, d\sigma(\zeta) = 2nV(B)u(y).$$

We conclude that the harmonic function $\Delta w/(2nV(B))$ is the reproducing kernel at y. Using (2–2) we obtain the following formula for the Bergman kernel

$$R(x,y) = \frac{1}{nV(B)}(nP(x,y) + 2x \cdot \nabla_x P(x,y)). \tag{2–4}$$

By elementary calculus, we get the formula

$$R(x,y) = \frac{1}{nV(B)(1 - 2x\cdot y + |x|^2|y|^2)^{n/2}}\left(\frac{n(1-|x|^2|y|^2)^2}{1 - 2x\cdot y + |x|^2|y|^2} - 4|x|^2|y|^2\right). \tag{2–5}$$

In the next section we will need an estimate on $|R(x,y)|$. By Cauchy-Schwarz $x\cdot y \le |x||y|$. It follows that $(1-|x||y|)^2 = 1+|x|^2|y|^2-2|x||y| \le 1-2x\cdot y+|x|^2|y|^2$, and thus $(1-|x|^2|y|^2)^2 = (1+|x||y|)^2(1-|x||y|)^2 \le 4(1-2x\cdot y+|x|^2|y|^2)$. Therefore we have

$$|R(x,y)| \le \frac{4}{V(B)(1-2x\cdot y + |x|^2|y|^2)^{n/2}}. \tag{2–6}$$

3. L^p-Boundedness of the Bergman Projection and Duality

Let Q denote the orthogonal projection of $L^2(B)$ onto $b^2(B)$. If $f \in L^2(B)$ and $y \in B$, then $Q[f](y) = \langle Qf, R_y \rangle = \langle f, R_y \rangle$ and we have the following formula:

$$Q[f](y) = \int_B R(x, y)\, f(x)\, dV(x). \tag{3–1}$$

For fixed $y \in B$ the function $R(\cdot, y)$ is bounded, so that we can use the formula above for $Q[f]$ to extend the domain of Q to $L^p(B)$, where $1 \leq p < \infty$. Our goal is to prove the following theorem.

Theorem 3.1. *Let $1 < p < \infty$. Then Q maps $L^p(B)$ boundedly onto $b^p(B)$.*

Proof. We will use the so-called Schur Test. Specifically, we will show the existence of a positive function h and a constant C such that

$$\int_B h(x)^q\, |R(x, y)|\, dV(x) \leq C h(y)^q \quad \text{for all } y \in B, \quad \text{and} \tag{3–2}$$

$$\int_B h(y)^p\, |R(x, y)|\, dV(y) \leq C h(x)^p \quad \text{for all } x \in B, \tag{3–3}$$

where q denotes the conjugate index of p, that is, $q = p/(p-1)$. That this will imply the result is then proved as follows. Given $f \in L^p(B, dV)$, applying Hölder's inequality and (3–2) we have

$$\begin{aligned} |Q[f](y)| &\leq \int_B \frac{|f(x)|}{h(x)}\, h(x)\, |R(x, y)|\, dV(x) \\ &\leq \left(\int_B \frac{|f(x)|^p}{h(x)^p}\, |R(x, y)|\, dV(x) \right)^{1/p} \left(\int_B h(x)^q\, |R(x, y)|\, dV(x) \right)^{1/q} \\ &\leq C^{1/q} h(y) \left(\int_B \frac{|f(x)|^p}{h(x)^p}\, |R(x, y)|\, dV(x) \right)^{1/p}. \end{aligned}$$

Thus, applying Fubini's theorem to reverse the order of integration, and using (3–3) we obtain

$$\begin{aligned} \int_B |Q[f](y)|^p\, dV(y) &\leq C^{p/q} \int_B h(y)^p \left(\int_B \frac{|f(x)|^p}{h(x)^p}\, |R(x, y)|\, dV(x) \right) dV(y) \\ &= C^{p/q} \int_B \frac{|f(x)|^p}{h(x)^p} \left(\int_B h(y)^p\, |R(x, y)|\, dV(y) \right) dV(x) \\ &\leq C^{p/q} \int_B \frac{|f(x)|^p}{h(x)^p} \left(C h(x)^p \right) dV(x) = C^p \int_B |f(x)|^p\, dV(x), \end{aligned}$$

proving the theorem.

We claim that the function $h(x) = (1 - |x|^2)^{-1/(pq)}$ works, that is, satisfies (3–2) and (3–3). By symmetry in p and q, it will suffice to find a constant C_p for which

$$\int_B (1 - |x|^2)^{-1/p} |R(x, y)|\, dV(y) \leq C_p (1 - |y|^2)^{-1/p} \quad \text{for all } y \in B. \tag{3–4}$$

Fix $y \in B \backslash \{0\}$. For $0 < r < 1$ and $\zeta \in S$ it follows from (2–6) that

$$|R(r\zeta, y)| \le \frac{4}{V(B)} \frac{1}{(1 - ry \cdot \zeta + r^2|y|^2)^{n/2}} = \frac{4}{V(B)} \frac{1}{|\zeta - ry|^n}.$$

Using (1–5) and (1–4) we have

$$\begin{aligned}\int_B \frac{|R(x,y)|}{(1-|x|^2)^{1/p}}\, dV(x) &= nV(B)\int_0^1 r^{n-1}(1-r^2)^{-1/p}\int_S |R(r\zeta,y)|\, d\sigma(\zeta)\, dr\\ &\le 4n \int_0^1 r^{n-1}(1-r^2)^{-1/p}\int_S \frac{1}{|\zeta - ry|^n}\, d\sigma(\zeta)\, dr\\ &\le 2n\int_0^1 2r(1-r^2)^{-1/p}\frac{1}{1-r^2|y|^2}\, dr\\ &= 2n\int_0^1 (1-t)^{-1/p}(1-t|y|^2)^{-1}\, dt.\end{aligned}$$

Now

$$\int_0^{|y|^2} (1-t)^{-1/p}(1-t|y|^2)^{-1}\, dt \le \int_0^{|y|^2} (1-t)^{-1-1/p}\, dt \le p(1-|y|^2)^{-1/p}.$$

Also (recall that $1 - 1/p = 1/q$)

$$\begin{aligned}\int_{|y|^2}^1 (1-t)^{-1/p}(1-t|y|^2)^{-1}\, dt &\le (1-|y|^2)^{-1}\int_{|y|^2}^1 (1-t)^{-1/p}\, dt\\ &= (1-|y|^2)^{-1} q(1-|y|^2)^{1-1/p} = q(1-|y|^2)^{-1/p}.\end{aligned}$$

Addition yields

$$\int_0^1 (1-t)^{-1/p}(1-t|y|^2)^{-1}\, dt \le (p+q)(1-|y|^2)^{-1/p}.$$

Thus (3–4) is proved with $C_p = 2n(p+q)$. This concludes the proof of the L^p-boundedness of Q. In fact, from the Schur Test, the estimates above and the observation that $C_q = C_p$, we obtain the following bound on the norm of Q as an operator from $L^p(B)$ onto $b^p(B)$: $\|Q\| \le 2np^2/(p-1)$. $\square$

Remark 1. The norm estimate given above is far from being sharp; it can be improved by estimating the integrals using binomial series, as in [Forelli and Rudin 1974/75; Rudin 1980].

Remark 2. As we will see in the next section, Theorem 3.1 does *not* hold for $p = 1$ or $p = \infty$.

Duality. It is a consequence of the L^p-boundedness of the Bergman projection that the spaces $b^p(B)$ and $b^q(B)$ are dual to each other: if $v \in b^q(B)$, then the function φ defined by $\varphi(u) = \langle u, v\rangle$ ($u \in b^p(B)$) defines a bounded linear functional on $b^p(B)$, and every bounded linear functional on $b^p(B)$ is of the form above. To prove the latter statement, suppose that $\varphi \in b^p(B)^*$. By the Hahn-Banach theorem, φ extends to a bounded linear functional ψ on $L^p(B)$. There exists a $g \in L^q(B)$ such that $\psi(f) = \langle f, g\rangle$ for all $f \in L^p(B)$. In particular, if $u \in b^p(B)$, then $\varphi(u) = \langle u, g\rangle$. Note that $v = Q[g] \in Q(L^q(B)) = b^q(B)$. Using Fubini's theorem to reverse the order of integration it can then be shown that $\langle u, v\rangle = \langle u, g\rangle$, and we obtain that $\varphi(u) = \langle u, v\rangle$, for all $u \in b^p(B)$.

4. The Bloch Space and the Dual of $b^1(B)$

A harmonic function u on B is said to be a *Bloch function* if

$$\|u\|_{\mathcal{B}} = \sup_{x\in B}(1-|x|^2)|\nabla u(x)| < \infty.$$

The *harmonic Bloch space* $\mathcal{B}$ is the set of all harmonic Bloch functions on B. We will show that $\mathcal{B}$ is the dual of the Bergman space $b^1(B)$. We will first prove that $\mathcal{B} = Q[L^\infty(B)]$. The hard part of the proof will be to show that each function $u \in \mathcal{B}$ is of the form $u = Q[g]$, where g is a bounded function on B. If $u \in \mathcal{B}$, then it follows from the inequality $|\mathcal{R}u(x)| \le |\nabla u(x)|$ that the function $(1-|x|^2)\mathcal{R}u$ is bounded on B. Thus we will try to relate $Q[(1-|x|^2)\mathcal{R}u]$ to u. We will first find an expression for $Q[(1-|x|^2)u]$. Combining (2–4) and (3–1) we have

$$nV(B)Q[(1-|x|^2)u](y) = 2\big\langle(1-|x|^2)u,\, \mathcal{R}P_y\big\rangle + n\big\langle(1-|x|^2)u,\, P_y\big\rangle.$$

To rewrite $\big\langle(1-|x|^2)u,\, \mathcal{R}P_y\big\rangle$ we use the same idea as the derivation of formula (2–4): assuming u to be harmonic on an open set containing $\bar{B}$, apply identity (1–1) with u and $v(x) = (1-|x|^2)^2P(x,y)$. The outward normal derivative $D_{\mathbf{n}}v$ is 0 on S; thus (1–1) gives us

$$\int_B u(x)\Delta_x\big((1-|x|^2)^2P(x,y)\big)\,dV(x) = 0. \tag{4–1}$$

It is easily verified that

$$\Delta_x\big((1-|x|^2)^2P(x,y)\big) = 8|x|^2P(x,y)+4n(|x|^2-1)P(x,y)+8(|x|^2-1)\mathcal{R}_xP(x,y).$$

Thus (4–1) shows that

$$\int_B |x|^2u(x)P(x,y)\,dV(x)$$
$$= \tfrac{1}{2}n\int_B u(x)(1-|x|^2)P(x,y)\,dV(x) + \int_B u(x)(1-|x|^2)\,\mathcal{R}_xP(x,y)\,dV(x),$$

and using (3–1) and (2–4) we can write the equation above as

$$\int_B |x|^2 u(x) P(x,y)\, dV(x) = \tfrac{1}{2} n V(B) Q[(1-|x|^2)u](y). \tag{4–2}$$

It follows from (1–1) that $\int_S u(r\zeta) D_{\mathbf{n}} v(r\zeta)\, d\sigma(\zeta) = \int_S D_{\mathbf{n}} u(r\zeta) v(r\zeta)\, d\sigma(\zeta)$ for harmonic functions u and v on B and $0 < r < 1$. Multiplication by $nV(B)r^{n+1}$ and integration over r yields the formula

$$\int_B |x|^2 u(x) \mathcal{R} v(x)\, dV(x) = \int_B |x|^2 \mathcal{R} u(x) v(x)\, dV(x). \tag{4–3}$$

Applying (4–2) to the function $\mathcal{R}u$, and making use of (4–3), we also have

$$\int_B |x|^2 u(x) \mathcal{R}_x P(x,y)\, dV(x) = \tfrac{1}{2} n V(B) Q[(1-|x|^2)\mathcal{R}u](y). \tag{4–4}$$

Combining (4–2), (4–4) and (2–4) we arrive at

$$Q[|x|^2 u](y) = \tfrac{1}{2} n Q[(1-|x|^2)u](y) + Q[(1-|x|^2)\mathcal{R}u](y),$$

and, writing $u = Q[(1-|x|^2)u] + Q[|x|^2 u]$ we obtain the formula in the following theorem.

THEOREM 4.1. *If $u \in \mathcal{B}$, then*

$$u = Q[(1-|x|^2)\mathcal{R}u + (\tfrac{1}{2}n + 1)(1-|x|^2)u]. \tag{4–5}$$

PROOF. We have shown the stated result for u harmonic on an open set containing $\bar{B}$. To get the result for general $u \in \mathcal{B}$, let $0 < r < 1$ and consider the dilate u_r of u, defined by $u_r(x) = u(rx)$, $x \in B$. Since u_r is harmonic on the set $\{x \in \mathbb{R}^n : |x| < 1/r\}$, equation (4–5) holds for u_r. It is easily seen that $(\mathcal{R}u)_r = \mathcal{R}u_r$. We leave it as an exercise to show that $(1-|x|^2)\mathcal{R}u_r \to (1-|x|^2)\mathcal{R}u$ and $(1-|x|^2)u_r \to (1-|x|^2)u$ in $L^2(B)$ as $r \to 1^-$. The boundedness of Q and the continuity of point evaluation at y imply that $Q[(1-|x|^2)\mathcal{R}u_r](y) \to Q[(1-|x|^2)\mathcal{R}u](y)$ and $Q[(1-|x|^2)u_r](y) \to Q[(1-|x|^2)u](y)$ as $r \to 1^-$, and since $u_r(y) \to u(y)$ formula (4–5) follows. □

COROLLARY 4.2. $\mathcal{B} = Q[L^\infty(B)]$.

PROOF. If $g \in L^\infty(B)$, and $u = Q[g]$, then we claim that $u \in \mathcal{B}$. Differentiating $u(x) = \int_B g(y) R(x,y)\, dV(y)$ we obtain

$$D_j u(x) = \int_B g(y) \frac{\partial}{\partial x_j} R(x,y)\, dV(y),$$

and consequently

$$|\nabla u(x)| \le \|g\|_\infty \int_B |\nabla_x R(x,y)|\, dV(y). \tag{4–6}$$

Using (2–5) we have

$$\nabla_x R(x,y) = \frac{n(y-|y|^2x)}{(1-2x\cdot y+|x|^2|y|^2)^{1+n/2}}\left(\frac{n(1-|x|^2|y|^2)^2}{(1-2x\cdot y+|x|^2|y|^2)} - 4|x|^2|y|^2\right)$$
$$+\frac{1}{(1-2x\cdot y+|x|^2|y|^2)^{n/2}}\left(\frac{-4n(1-|x|^2|y|^2)|y|^2x}{(1-2x\cdot y+|x|^2|y|^2)}+\right.$$
$$\left.+\frac{2n(1-|x|^2|y|^2)^2(y-x|y|^2)}{(1-2x\cdot y+|x|^2|y|^2)^2} - 8|y|^2x\right).$$

Noting that $|y-|y|^2x|^2 = |y|^2-2|y|^2x\cdot y+|y|^4|x|^2 = |y|^2(1-2x\cdot y+|y|^2|x|^2)$, we see that $|y-|y|^2x| \le (1-2x\cdot y+|y|^2|x|^2)^{1/2}$. Recalling that also $1-|x|^2|y|^2 \le (1-2x\cdot y+|y|^2|x|^2)^{1/2}$ we obtain

$$|\nabla_x R(x,y)| \le \frac{C}{\big(1-2x\cdot y+|x|^2|y|^2\big)^{(n+1)/2}}. \tag{4–7}$$

Thus

$$\int_B |\nabla_x R(x,y)|\,dV(x) \le CnV(B)\int_0^1 r^{n-1}\int_S \frac{1}{|\zeta-ry|^{n+1}}\,d\sigma(\zeta)\,dr.$$

Now,

$$\int_S \frac{1}{|\zeta-ry|^{n+1}}\,d\sigma(\zeta) = \frac{1}{1-r^2|y|^2}\int_S \frac{1}{|\zeta-ry|}\,P(ry,\zeta)\,d\sigma(\zeta)$$
$$\le \frac{1}{1-r^2|y|^2}\,\frac{1}{(1-r|y|)}\int_S P(ry,\zeta)\,d\sigma(\zeta) \le \frac{1}{(1-r|y|)^2}.$$

Multiply by $nV(B)r^{n-1}$ and integrate with respect to r to obtain

$$\int_B \frac{1}{(1-2x\cdot y+|x|^2|y|^2)^{(1+n)/2}}\,dV(x) \le nV(B)\frac{1}{(1-|y|)}. \tag{4–8}$$

Using that $1/(1-|y|) = (1+|y|)/(1-|y|^2) \le 2/(1-|y|^2)$, we conclude from (4–6), (4–7) and (4–8) that $(1-|y|^2)|\nabla u(y)| \le 2nV(B)C\|g\|_\infty$ for all $y \in B$, establishing our claim that $u \in \mathcal{B}$. This proves the inclusion $Q[L^\infty(B)] \subset \mathcal{B}$.

The other inclusion $\mathcal{B} \subset Q[L^\infty(B)]$ follows from Theorem 4.1, for if $u \in \mathcal{B}$, the function $g = (1-|x|^2)\mathcal{R}u + (\frac{1}{2}n+1)(1-|x|^2)u$ is bounded on B. □

This corollary can be used to show that Theorem 3.1 does *not* hold for $p=1$ or $p=\infty$. It is not difficult to construct unbounded harmonic Bloch functions (the function $u(x) = \log((1-x_1)^2+{x_2}^2)$ provides an example), so Corollary 4.2 shows that the operator Q is not L^∞-bounded. By duality it follows that Q is not L^1-bounded either.

The dual of $b^1(B)$. We will show how Corollary 4.2 can be used to prove that $\mathcal{B}$ is the dual of $b^1(B)$. Our first concern is to define a "pairing": given a function $v \in \mathcal{B}$ we could try to define φ by $\varphi(u) = \langle u, v\rangle$ ($u \in \mathcal{B}$), but we have to be careful here: the function v need not be bounded, so there is no guarantee that $u\bar{v}$ is integrable for all $u \in b^1(B)$ (in fact, if v is unbounded, some standard functional analysis can be used to prove that there must exist a function $u \in b^1(B)$ for which $u\bar{v}$ is not integrable). This problem is easily overcome by using dilates. If u is a harmonic function on B and $0 < r < 1$, the dilate u_r, defined by $u_r(x) = u(rx)$, $x \in B$, is bounded on B. If $u \in C(\bar{B})$, then u is uniformly continuous on B, and it follows that $u_r \to u$ uniformly on B as $r \to 1^-$. Using the fact that $C(\bar{B})$ is dense in $L^1(B)$ it is easy to verify that if $u \in b^1(B)$, then $u_r \to u$ in $b^1(B)$ as $r \to 1^-$.

We are now ready to extend the usual pairing. Given $u \in b^1(B)$ and $v \in \mathcal{B}$ we claim that $\lim_{r\to 1^-}\langle u_r, v\rangle$ exists. To prove this, write $v = Q[g]$, where $g \in L^\infty(B)$. Since $u_r \in b^2(B)$ we have $\langle u_r, v\rangle = \langle u_r, Q[g]\rangle = \langle u_r, g\rangle$. The inequality $|\langle u_r, v\rangle - \langle u_s, v\rangle| = |\langle u_r - u_s, g\rangle| \le \|u_r - u_s\|_1 \|g\|_\infty$ together with the fact that $\|u_r - u_s\|_1 \to 0$ as $r, s \to 1^-$ (because $\|u_r - u_s\|_1 \le \|u_r - u\|_1 + \|u - u_s\|_1$) implies that $\lim_{r\to 1^-}\langle u_r, v\rangle$ exists. It is clear that the function φ defined by

$$\varphi(u) = \lim_{r\to 1^-}\langle u_r, v\rangle \quad \text{for } u \in b^1(B)$$

is a linear functional on $b^1(B)$. Using $|\langle u_r, v\rangle| \le \|u_r\|_1 \|g\|_\infty$ we see that $|\varphi(u)| \le \|u\|_1 \|g\|_\infty$, so $\varphi \in b^1(B)^*$.

We claim that every element of $b^1(B)^*$ is of the form above. For let $\varphi \in b^1(B)^*$. By the Hahn-Banach theorem, φ extends to a bounded linear functional ψ on $L^1(B, dV)$. There exists a $g \in L^\infty(B)$ such that $\psi(f) = \langle f, g\rangle$ for all $f \in L^1(B)$. In particular, $\varphi(u) = \langle u, g\rangle$ for all $u \in b^1(B)$. Now, if $u \in b^1(B)$, then $u_r \to u$ in $b^1(B)$ as $r \to 1^-$, and thus $\varphi(u) = \lim_{r\to 1^-}\varphi(u_r)$. Since $u_r \in b^2(B)$ we have $\langle u_r, g\rangle = \langle Q[u_r], g\rangle = \langle u_r, Q[g]\rangle$. Thus, if we set $v = Q[g]$, then $\varphi(u) = \lim_{r\to 1^-}\langle u_r, v\rangle$, as was to be proved. □

The predual of $b^1(B)$. In this subsection we will identify the predual of $b^1(B)$ under the above pairing. We define the space $\mathcal{B}_0$, called the *harmonic little Bloch space*, to be the set of all harmonic functions u on B for which $(1-|x|^2)|\nabla u(x)| \to 0$ as $|x| \to 1^-$. Clearly, $\mathcal{B}_0 \subset \mathcal{B}$. We leave it as an exercise for the reader to show that if $u \in \mathcal{B}_0$, then $\|u_r - u\|_{\mathcal{B}} \to 0$ as $r \to 1^-$ (the converse is also true). Now suppose that $u \in \mathcal{B}_0$ and $v \in b^1(B)$. We claim that $\lim_{r\to 1^-}\langle u_r, v\rangle$ exists. To show this, we use Corollary 4.2 (or rather its proof) to conclude that

$$\begin{aligned}|\langle u_r, v\rangle - \langle u, v_r\rangle| &\le |\langle u_r - u,\, v\rangle| + |\langle u,\, v - v_r\rangle|\\ &\le C\|u_r - u\|_{\mathcal{B}}\|v\|_1 + C\|u\|_{\mathcal{B}}\|v - v_r\|_1,\end{aligned}$$

and the statement follows. So if $v \in b^1(B)$, then $\varphi(u) = \lim_{r\to 1^-}\langle u_r, v\rangle$ defines a bounded linear functional on $\mathcal{B}_0$.

Our claim is that all bounded linear functionals on $\mathcal{B}_0$ arise this way. Suppose φ is a bounded linear functional on $\mathcal{B}_0$. Then $\psi(g) = \varphi(Q[g])$ defines a bounded linear functional on $C(\bar{B})$: $|\psi(g)| = |\varphi(Q[g])| \le \|\varphi\|\,\|g\|_2 \le \|\varphi\|\,\|g\|_\infty$. By the Riesz representation theorem $\psi(g) = \int_{\bar{B}} g\,d\mu$, for all $g \in C(\bar{B})$, where μ is a finite complex Borel measure on $\bar{B}$. If $u \in \mathcal{B}_0$, then $g(x) = (1-|x|^2)\{\mathcal{R}u(x) + (\frac{1}{2}n+1)u(x)\}$ is in $C(\bar{B})$, and, using Theorem 4.1, we have $\varphi(u) = \psi(g) = \int_{\bar{B}} g(y)\,d\mu(y)$. Define

$$v(x) = \int_{\bar{B}} (1-|y|^2)\{\mathcal{R}_x R(x,y) + (\tfrac{1}{2}n+1)R(x,y)\}\,d\bar{\mu}(y) \quad \text{for } x \in B.$$

The function v is clearly harmonic in B. We claim that in fact $v \in b^1(B)$. To show this, use Fubini's theorem to get

$$\int_B |v(x)|\,dV(x) \le \int_{\bar{B}} (1-|y|^2) \int_B \{|\mathcal{R}_x R(x,y)| + (\tfrac{1}{2}n+1)|R(x,y)|\}\,dV(x)\,d|\mu|(y)$$

In the proof of Corollary 4.2 we have seen that there exists a positive constant C such that $\int_B |\mathcal{R}_x R(x,y)|\,dV(x) \le C/(1-|y|^2)$. Also, using (2–6) and (1–5), we have

$$\begin{aligned}\int_B |R(x,y)|\,dV(x) &\le 4n \int_0^1 r^{n-1} \int_S \frac{1}{|\zeta - ry|^n}\,d\sigma(\zeta)\,dr\\ &= 4n \int_0^1 r^{n-1} \frac{1}{1-r^2|y|^2}\,dr\\ &\le 4n \int_0^1 r^{n-1} \frac{1}{1-|y|^2} = \frac{4}{1-|y|^2}.\end{aligned}$$

Thus,

$$\int_B |v(x)|\,dV(x) \le \int_{\bar{B}} (1-|y|^2) \frac{C'}{1-|y|^2}\,d|\mu|(y) = C' \int_{\bar{B}} d|\mu|(y) = C'\|\mu\| < \infty,$$

and our claim that $v \in b^1(B)$ is proved. Now, assuming u to be harmonic on $\bar{B}$, applying Fubini's theorem we have

$$\langle u, v\rangle = \int_{\bar{B}} (1-|y|^2) \int_B u(x)\{\mathcal{R}_x R(x,y) + (\tfrac{1}{2}n+1)R(x,y)\}\,dV(x)\,d\mu(y).$$

Similarly to (4–3) we have $\int_B u(x)\mathcal{R}_x R(x,y)\,dV(x) = \int_B \mathcal{R}u(x)R(x,y)\,dV(x) = \mathcal{R}u(y)$, thus

$$\int_B u(x)\{\mathcal{R}_x R(x,y) + (\tfrac{1}{2}n+1)R(x,y)\}\,dV(x) = \mathcal{R}u(y) + (\tfrac{1}{2}n+1)u(y)$$

and we obtain

$$\langle u, v\rangle = \int_{\bar{B}} (1-|y|^2)\left\{\mathcal{R}u(y) + (\tfrac{1}{2}n+1)u(y)\right\}\,d\mu(y) = \int_{\bar{B}} g(y)\,d\mu(y) = \varphi(u).$$

Hence $\varphi(u_r) = \langle u_r, v\rangle$, and since $u_r \to u$ in $\mathcal{B}_0$ and φ is continuous on $\mathcal{B}_0$ we have $\varphi(u) = \lim_{r\to 1^-}\langle u_r, v\rangle$. Note that this pairing coincides with the one we saw earlier: if $u \in \mathcal{B}_0$ and $v \in b^1(B)$, then $\lim_{r\to1^-}\langle u_r, v\rangle = \lim_{r\to1^-}\langle u, v_r\rangle$. $\square$

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Karel Stroethoff
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812–1032
United States
ma_kms@selway.umt.edu