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Commuting Operators and Function Theory on a Riemann Surface

VICTOR VINNIKOV

Abstract. In the late 70's M. S. Livšic has discovered that a pair of commuting nonselfadjoint operators in a Hilbert space, with finite nonhermitian ranks, satisfy a polynomial equation with constant (real) coefficients; in particular the joint spectrum of such a pair of operators lies on a certain algebraic curve in the complex plane, the so called discriminant curve of the pair of operators. More generally, it turns out that much in the same way as the study of a single nonselfadjoint operator is intimately related to the function theory on the complex plane, more specifically on the upper half-plane, the study of a system of commuting nonselfadjoint operators, at least with finite nonhermitian ranks, is related to the function theory on a compact Riemann surface of a higher genus, more specifically on a compact real Riemann surface. From a different perspective, while the study of a single nonselfadjoint operator leads to one-variable continuous time linear systems, the study of a pair of commuting nonselfadjoint operators leads to two-variable continuous time systems, which are necessarily overdetermined, hence must be considered together with an additional structure of compatibility conditions at the input and at the output. In this survey we give an introduction to the spectral theory of commuting nonselfadjoint operators and its interplay with system theory and the theory of Riemann surfaces and algebraic curves, including some recent results and open problems.

Introduction

It is fair to say that until the 1940's operator theory was mostly concerned with selfadjoint or unitary operators; several commuting selfadjoint or unitary operators do not present any essential new problems because such operators possess commuting resolutions of the identity. Starting with the work of Livšic and his associates in the 1940's and 1950's [Brodskiĭ and Livšic 1958; Brodskiĭ 1969], and later that of Sz.-Nagy and Foiaş [1967] and of de Branges and Rovnyak [1966a; 1966b], a comprehensive study of nonselfadjoint and nonunitary operators began, especially for operators that are not "too far" from being selfadjoint

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or unitary (i.e., the nonhermitian part $A - A^*$ or the defect operators $I - AA^*$, $I - A^*A$ are of finite rank, or trace class). This study has revealed deep connections with the theory of bounded analytic functions on the upper half-plane or on the unit disk, and more generally with the theory of matrix-valued and operator-valued functions on these domains possessing metric properties such as contractivity, and with system theory. The main point is that there is a relation between invariant subspaces of an operator and factorizations of its so-called characteristic function. The characteristic function turns out to be the transfer function of a certain associated linear time-invariant conservative dynamical system.

Initial attempts to generalize these results to several commuting nonselfadjoint or nonunitary operators ran into serious difficulties, since it was natural now to define the characteristic function as a function of several complex variables, which therefore does not admit a good factorization theory. However, it was discovered by Livšic in the late 1970's that a pair of commuting nonselfadjoint operators with finite nonhermitian ranks satisfy a polynomial equation with constant (real) coefficients. Therefore the joint spectrum of such a pair of operators lies on a (real) algebraic curve in $\mathbb{C}^2$, called the discriminant curve; and it seems natural that their spectral study would lead to function theory on the corresponding compact real Riemann surface (i.e., compact Riemann surface endowed with an antiholomorphic involution coming from the complex conjugation on the curve) rather than to function theory of two independent complex variables. This is indeed the case, and the proper analogue of the notion of the characteristic function of a single nonselfadjoint operator is the so-called joint characteristic function of a pair of operators, which is a mapping of certain vector bundles (or more generally, of certain sheaves) on the discriminant curve. It turns out again to have a system-theoretic interpretation as the transfer function of the associated linear time-invariant conservative dynamical system, which is now two-dimensional: the input, the state, and the output depend on two (continuous) parameters rather than one.

The objective of this survey is to give an introduction to the spectral theory of commuting nonselfadjoint operators and its interplay with system theory and the theory of Riemann surfaces and algebraic curves, including some recent results and open problems. In Section 1 we review the basic constructions as established by Livšic [1979; 1980; 1983; 1986a]; see also [Livšic 1987; Waksman 1987; Kravitsky 1983; Vinnikov 1992]. Section 2 discusses the joint characteristic function introduced in [Livšic 1986b] and further investigated in [Vinnikov 1992; 1994]; see also [Ball and Vinnikov 1996]. Much of the material discussed in Sections 1 and 2 appears in [Livšic et al. 1995]. In Section 3 we discuss semicontractive and semiexpansive functions on a compact real Riemann surface—the analogues of contractive and expansive functions on the upper half-plane or unit disk—and their canonical factorizations and we present functional models for commuting nonselfadjoint operators constructed by Alpay and Vinnikov [1994; a].

All the results obtained up to now deal with commuting operators close to self-adjoint. Of course, commuting operators close to unitary have to be studied as well. System-theoretically this means studying conservative discrete-time, rather than continuous-time, systems in two or more dimensions. It turns out to be quite nontrivial how to transfer various notions from the nonselfadjoint to the nonunitary case. As an indication (see Section 1 for a motivation), it is obvious what to require of the matrices σ_1, σ_2, γ if we want the curve $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = 0$ to be invariant under the anti-holomorphic involution $(\lambda_1, \lambda_2) \mapsto (\bar{\lambda}_1, \bar{\lambda}_2)$ (namely, the matrices should be self-adjoint); but what if we consider instead the anti-holomorphic involution

$$(\lambda_1, \lambda_2) \mapsto \left(\frac{1}{\bar{\lambda}_1}, \frac{1}{\bar{\lambda}_2}\right)?$$

Some progress on the corresponding proper framework for the study of commuting nonunitary operators has been achieved recently in joint work with J. Ball. Once the basic notions are fixed, given two commuting contractions A_1, A_2 with finite defects, the corresponding compact real Riemann surface X should be necessarily dividing (see Section 3 for the definition), and the functional model would also yield an $H^\infty(X_+)$ functional calculus for A_1, A_2. In particular if X is the double of a finitely connected planar domain S, and we denote by Z the global planar coordinate on $S = X_+$, then $T = Z(A_1, A_2)$ is an operator with spectral set S. This should provide a link to the work of Abrahamse and Douglas [1976], and may be also a useful approach to the well-known question whether an operator with a multiply connected spectral set admits a normal boundary dilation [Agler 1985].

1. Commuting Nonselfadjoint Operators, Two-Dimensional Systems, and Algebraic Curves

It is well-known (see, for example, [Brodskiĭ 1969; Livšic and Yantsevich 1971; Ball and Cohen 1991]) that the most natural object to consider in the study of a single (bounded) nonselfadjoint operator A in a Hilbert space H is not the operator A itself, but rather an *operator colligation* (or *node*) $\mathcal{C} = (A, H, \Phi, E, \sigma)$. Here E is an auxiliary Hilbert space called the external space of the colligation (H is called the inner space), $\Phi : H \to E$ and $\sigma : E \to E$ are bounded linear mappings with $\sigma^* = \sigma$, and

$$\frac{1}{i}\,(A - A^*) = \Phi^*\sigma\Phi. \tag{1–1}$$

We shall be considering only operators with a finite nonhermitian rank ($\dim(A - A^*)H < \infty$), so we assume $\dim E = M < \infty$. Note that a given operator A in H (with a finite nonhermitian rank) can be always embedded in a colligation by setting

$$E = (A - A^*)H, \qquad \Phi = P_E, \qquad \sigma = \frac{1}{i}\,(A - A^*)\,\Big|_E, \tag{1–2}$$

where P_E is the orthogonal projection of H onto E. (Another possible embedding is obtained by setting

$$\Phi = \left| \frac{A - A^*}{i} \Big|_E \right|^{1/2}, \qquad \sigma = \operatorname{sign} \frac{A - A^*}{i} \Big|_E,$$

where the absolute value and the sign functions are understood in the sense of the usual functional calculus for self-adjoint operators; this is used more often in single-operator theory because of the added convenience of $\sigma^2 = I$, but it does not admit a good generalization to the two-operator case.)

The advantage of the notion of colligation is that it allows us to "isolate" the nonhermitian part of the operator. In particular, given two colligations

$$\mathcal{C}' = (A', H', \Phi', E, \sigma) \quad \text{and} \quad \mathcal{C}'' = (A'', H'', \Phi'', E, \sigma),$$

with the same external part (E, σ), we define their *coupling*

$$\mathcal{C} = \mathcal{C}' \vee \mathcal{C}'' = (A, H, \Phi, E, \sigma),$$

where $H = H' \oplus H''$ and

$$A = \begin{pmatrix} A' & 0 \\ i\,\Phi''^* \sigma \Phi' & A'' \end{pmatrix}, \qquad \Phi = \begin{pmatrix} \Phi' & \Phi'' \end{pmatrix}, \tag{1–3}$$

the operators being written in the block form with respect to the direct sum decomposition $H = H' \oplus H''$. The coupling procedure allows us to construct operators with a more complicated spectral data out of operators with a simpler one, while preserving the nonhermitian part. Note that H'' is an invariant subspace of A. Conversely, if $H'' \subset H$ is an invariant subspace of the operator A in a colligation $\mathcal{C} = (A, H, \Phi, E, \sigma)$ and $H' = H \ominus H''$, it is easy to see that we can write $\mathcal{C} = \mathcal{C}' \vee \mathcal{C}''$, where $\mathcal{C}'$, $\mathcal{C}''$ are the *projections* of $\mathcal{C}$ onto the subspaces H', H'' respectively, given by

$$\mathcal{C}' = (P'A|_{H'}, H', \Phi|_{H'}, E, \sigma), \tag{1–4}$$

$$\mathcal{C}'' = (A|_{H''}, H'', \Phi|_{H''}, E, \sigma). \tag{1–5}$$

Here P' is the orthogonal projection of H onto H'.

The notion of a colligation has also a system-theoretic significance: a colligation $\mathcal{C} = (A, H, \Phi, E, \sigma)$ defines a (linear time-invariant) conservative system

$$i\,\frac{df}{dt} + Af = \Phi^* \sigma u, \tag{1–6}$$

$$v = u - i\,\Phi f. \tag{1–7}$$

Here $f = f(t)$ is the state, with values in the inner space H, and $u = u(t)$, $v = v(t)$ are respectively the input and the output, with values in the external space E. Conservativeness means that the difference in energy between the input and the output equals the change in the energy of the state; here we use the inner product as the energy form on the state (inner) space H and the hermitian form induced by σ as the (possibly indefinite) energy form on the input/output (external) space E. Thus the conservation law is

$$\frac{d}{dt}(f,f) = (\sigma u, u) - (\sigma v, v). \tag{1–8}$$

The coupling of colligations corresponds to the cascade connection of systems, i.e., forming a new system by feeding the output of the first one as the input into the second one: if u', f' and v' and u'', f'' and v'' are the input, the state and the output of the first and the second system respectively, then the input, the state and the output u, f and v of the new system are given by

$$u = u', \qquad f = \begin{pmatrix} f' \\ f'' \end{pmatrix}, \qquad v = v'',$$

while setting $u'' = v'$. A substitution into the system equations shows that we get exactly the formula (1–3) for the coupling.

We pass now to the study of a pair A_1, A_2 of (bounded) commuting nonselfadjoint operators in a Hilbert space H, with finite nonhermitian ranks. As a first try we may consider a *commutative* (*two-operator*) *colligation*

$$\mathcal{C} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2);$$

here again E is another Hilbert space (the external space, whose dimension we assume is $M < \infty$), $\Phi : H \to E$ and $\sigma_1, \sigma_2 : E \to E$ are bounded linear mappings with $\sigma_1^* = \sigma_1$, $\sigma_2^* = \sigma_2$, and

$$\frac{1}{i}\,(A_k - A_k^*) = \Phi^* \sigma_k \Phi \quad \text{for } k = 1, 2. \tag{1–9}$$

However, the notion of a commutative colligation does not possess enough structure: there is nothing in it to reflect the interplay between the two operators A_1, A_2. More concretely, the coupling of two commutative colligations with the same external part (E, σ_1, σ_2) (defined as in (1–3) except that one uses σ_1 and σ_2 instead of σ in the formulas for A_1 and A_2 respectively) is in general *not* commutative. In fact, even in a finite dimensional Hilbert space H it is not at all clear how to construct commuting nonselfadjoint operators with given nonhermitian parts.

It turns out that the correct object to consider in the study of a pair of commuting nonselfadjoint operators is a (*commutative two-operator*) *vessel* $\mathcal{V} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$. Here $(A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2)$ is a commutative two-operator colligation as in (1–9), and $\gamma, \tilde{\gamma} : E \to E$ are (bounded) self-adjoint operators such that

$$\sigma_1 \Phi A_2^* - \sigma_2 \Phi A_1^* = \gamma \Phi, \tag{1–10}$$

$$\sigma_1 \Phi A_2 - \sigma_2 \Phi A_1 = \tilde{\gamma} \Phi, \tag{1–11}$$

$$\tilde{\gamma} - \gamma = i\,(\sigma_1 \Phi \Phi^* \sigma_2 - \sigma_2 \Phi \Phi^* \sigma_1). \tag{1–12}$$

(The term "vessel" was coined in [Livšic et al. 1995]; earlier papers use instead the term "regular colligation".) Upon multiplying (1–10) and (1–11) by Φ^* on the left and using (1–9) and the obvious identities

$$(A_1 - A_1^*)A_2^* - (A_2 - A_2^*)A_1^* = A_1 A_2^* - A_2 A_1^*, \tag{1–13}$$

$$(A_1 - A_1^*)A_2 - (A_2 - A_2^*)A_1 = A_2^* A_1 - A_1^* A_2, \tag{1–14}$$

which follow from the commutativity of A_1 and A_2, we obtain

$$\tfrac{1}{i}\,(A_1 A_2^* - A_2 A_1^*) = \Phi^* \gamma \Phi, \tag{1–15}$$

$$\tfrac{1}{i}\,(A_2^* A_1 - A_1^* A_2) = \Phi^* \tilde{\gamma} \Phi. \tag{1–16}$$

Therefore the self-adjoint operators γ and $\tilde{\gamma}$ are related to the nonhermitian parts of $A_1 A_2^*$ and $A_2^* A_1$ respectively, and thus carry information about the interaction of A_1 and A_2. In the case when $\Phi : H \to E$ is onto, equations (1–15)–(1–16) are equivalent to (1–10)–(1–12), but in general the stronger relations (1–10)–(1–12) are needed for subsequent development. Note that analogously to (1–2), a given pair A_1, A_2 of commuting operators in H (with finite nonhermitian ranks) can be always embedded in a commutative vessel by setting

$$\begin{aligned} E &= (A_1 - A_1^*)H + (A_2 - A_2^*)H, \qquad & \Phi &= P_E, \\ \sigma_1 &= \tfrac{1}{i}\,(A_1 - A_1^*)|_E, & \sigma_2 &= \tfrac{1}{i}\,(A_2 - A_2^*)|_E, \\ \gamma &= \tfrac{1}{i}\,(A_1 A_2^* - A_2 A_1^*)|_E, & \tilde{\gamma} &= \tfrac{1}{i}\,(A_2^* A_1 - A_1^* A_2)|_E; \end{aligned} \tag{1–17}$$

the subspace E is invariant under $A_1 A_2^* - A_2 A_1^*$ and $A_2^* A_1 - A_1^* A_2$ because of (1–13)–(1–14).

Given a commutative vessel $\mathcal{V} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$, we define a polynomial in two complex variables λ_1, λ_2 by setting

$$p(\lambda_1, \lambda_2) = \det(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma). \tag{1–18}$$

We assume that $p(\lambda_1, \lambda_2) \not\equiv 0$, so that $p(\lambda_1, \lambda_2)$ is a polynomial with real coefficients of degree $M = \dim E$ at most. We call $p(\lambda_1, \lambda_2)$ the *discriminant polynomial* of the vessel $\mathcal{V}$, and the real (affine) plane algebraic curve C_0 with an equation $p(\lambda_1, \lambda_2) = 0$—the (*affine*) *discriminant curve*. To state the first fundamental result discovered by Livšic we have to introduce the principal subspace $\hat{H} \subseteq H$ of the vessel $\mathcal{V}$,

$$\hat{H} = \bigvee_{k_1, k_2 = 0}^{\infty} A_1^{k_1} A_2^{k_2} \Phi^*(E) = \bigvee_{k_1, k_2 = 0}^{\infty} A_1^{*k_1} A_2^{*k_2} \Phi^*(E). \tag{1–19}$$

Then $\hat{H}$ is reducing for A_1 and A_2, and the restrictions of A_1 and A_2 to $H \ominus \hat{H}$ are self-adjoint operators (the restriction of Φ to $H \ominus \hat{H}$ is 0); hence it is enough, at least in principle, to consider the restriction of our operators to the principal subspace.

THEOREM 1.1 (GENERALIZED CAYLEY–HAMILTON THEOREM [Livšic 1979; 1983]). *The operator $p(A_1, A_2)$ vanishes on the principal subspace $\hat{H}$.*

This theorem contains as special cases the classical Cayley–Hamilton Theorem and the theorem of Burchnall and Chaundy [1928] stating that a pair of commuting linear differential operators satisfy a polynomial equation with constant coefficients—a result that plays an important role in the study of finite-zone solutions of the KdV equation and other completely integrable nonlinear PDEs [Dubrovin 1981]. (To be precise, Theorem 1.1 implies the theorem of Burchnall and Chaundy only for formally self-adjoint differential operators; the general case follows from a more general version of Theorem 1.1, due to Kravitsky [1983].)

Theorem 1.1 implies that *the joint spectrum of the operators A_1 and A_2, restricted to the principal subspace $\hat{H}$, lies on the affine discriminant curve C_0.* (The joint spectrum is the set of all points $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that there exists a sequence $h_1, h_2, \ldots$ of vectors of unit length in H satisfying

$$\lim_{n\to\infty} (A_k - \lambda_k I)h_n = 0 \quad \text{for } k = 1, 2;$$

it was proved in [Livšic and Markus 1994] that for a pair of commuting operators with finite-dimensional (or more generally, compact) nonhermitian parts this is equivalent to any other reasonable definition of the joint spectrum; see [Harte 1972; Taylor 1970].) This is a first indication that the spectral analysis of a pair of commuting nonselfadjoint operators with finite nonhermitian ranks should be developed on a compact real Riemann surface (the normalization of the projective closure of C_0) rather than on a domain in $\mathbb{C}^2$.

In the definition (1–18) of the discriminant polynomial we have discriminated in favour of γ and against $\tilde{\gamma}$. However, we have the following remarkable equality.

THEOREM 1.2 [Livšic 1979; 1983]. $\det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma})$.

The proof is based on the theory of characteristic functions; we will give a system-theoretic explanation of why Theorem 1.2 is true in Section 2 below. We see that associated to the vessel $\mathcal{V}$ we have the discriminant polynomial $p(\lambda_1, \lambda_2)$ and two *self-adjoint determinantal representations* of it, $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ and $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma}$ (called, for system-theoretic reasons, the input and the output determinantal representation respectively); more geometrically, we have the affine discriminant curve C_0 (or rather its projective closure) and a pair of sheaves on it, given by the kernels of the matrices $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ and $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma}$. This will turn out to provide a proper algebro-geometrical framework for the study of the pair of operators A_1, A_2.

We now proceed to a system-theoretic interpretation. We start with a commutative two-operator colligation $\mathcal{C} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2)$ as in (1–9), and we write the corresponding (linear time-invariant) commutative conservative two-dimensional system

$$i\,\frac{\partial f}{\partial t_1} + A_1 f = \Phi^* \sigma_1 u, \tag{1–20}$$

$$i\,\frac{\partial f}{\partial t_2} + A_2 f = \Phi^* \sigma_2 u, \tag{1–21}$$

$$v = u - i\,\Phi f. \tag{1–22}$$

Here, as in (1–6)–(1–7), $f = f(t_1, t_2)$, $u = u(t_1, t_2)$ and $v = v(t_1, t_2)$ are the state, the input and the output respectively; the difference is that now we have a two-dimensional parameter $t = (t_1, t_2)$. One may think of t_1 as a time variable and of t_2 as a spatial variable, so that (1–20)–(1–22) describes a continuum of interacting temporal systems distributed in space; see [Livšic 1986a]. The energy conservation law is

$$\left(\xi_1 \frac{\partial}{\partial t_1} + \xi_2 \frac{\partial}{\partial t_2}\right)(f, f) = ((\xi_1\sigma_1 + \xi_2\sigma_2)u, u) - ((\xi_1\sigma_1 + \xi_2\sigma_2)v, v); \tag{1–23}$$

that is, the system conserves energy in any direction (ξ_1, ξ_2) in the (t_1, t_2) plane, where the (possibly indefinite) energy form in the input/output space in the direction (ξ_1, ξ_2) is induced by $\xi_1\sigma_1 + \xi_2\sigma_2$.

Unlike the usual one-dimensional system (1–6)–(1–7), the system (1–20)–(1–22) is overdetermined, the compatibility conditions arising from the equality of the mixed partials $\dfrac{\partial^2 f}{\partial t_1 \partial t_2} = \dfrac{\partial^2 f}{\partial t_2 \partial t_1}$. The commutativity $A_1 A_2 = A_2 A_1$ means precisely that the system is consistent for an arbitrary initial state $f(0,0) = f_0$ and the identically zero input. For an arbitrary input u the system (1–20)–(1–22) will not in general be consistent; using the system equations twice in the equality of the mixed partials we obtain

$$\Phi^* \left(\sigma_2 \frac{\partial u}{\partial t_2} - \sigma_1 \frac{\partial u}{\partial t_1}\right) + iA_2 \Phi^* \sigma_1 u - iA_1 \Phi^* \sigma_2 u = 0. \tag{1–24}$$

We see thus that if we assume the vessel condition (1–10), then a necessary and sufficient condition for the input to be compatible is given by

$$\Phi^* \left(\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + i\gamma\right) u = 0. \tag{1–25}$$

In particular we get a sufficient condition for the compatibility of the input entirely in terms of the external data of the system

$$\left(\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + i\gamma\right) u = 0. \tag{1–26}$$

If we use (1–22) to express the input in terms of the output and the state, and substitute into (1–26), we obtain

$$\begin{aligned}&\left(\sigma_2 \frac{\partial}{\partial t_1} - \sigma_1 \frac{\partial}{\partial t_2} + i\,(\gamma + i\,\sigma_1 \Phi \Phi^* \sigma_2 - i\,\sigma_2 \Phi \Phi^* \sigma_1)\right) v \\ &\qquad + i\,(\sigma_1 \Phi A_2 - \sigma_2 \Phi A_1 - (\gamma + i\,\sigma_1 \Phi \Phi^* \sigma_2 - i\,\sigma_2 \Phi \Phi^* \sigma_1)\Phi) f = 0.\end{aligned} \tag{1–27}$$

So we see that if we assume in addition the vessel conditions (1–11) and (1–12), then corresponding to (1–26) there is a sufficient condition for the compatibility of the output

$$\left(\sigma_2\frac{\partial}{\partial t_1}-\sigma_1\frac{\partial}{\partial t_2}+i\tilde{\gamma}\right)v=0. \tag{1–28}$$

Therefore *a commutative two-operator vessel is a commutative two-dimensional system* (1–20)–(1–22) *together with the compatibility PDEs* (1–26) *and* (1–28) *at the input and at the output respectively.*

To illustrate how well the notion of vessel suits the needs of the theory, let us consider the problem of coupling of two commutative two-operator colligations, i.e., cascade connection of two commutative two-dimensional systems. As we have noticed, the result will not in general be commutative. Now assume that we have two commutative vessels

$$\begin{aligned}\mathcal{V}' &= (A_1', A_2', H', \Phi', E, \sigma_1, \sigma_2, \gamma', \tilde{\gamma}'),\\ \mathcal{V}'' &= (A_1'', A_2'', H'', \Phi'', E, \sigma_1, \sigma_2, \gamma'', \tilde{\gamma}'')\end{aligned}$$

with the same (E, σ_1, σ_2), and we want their coupling

$$\mathcal{V} = \mathcal{V}' \vee \mathcal{V}'' = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$$

to be a commutative vessel, where, as in (1–3), $H = H' \oplus H''$ and

$$A_k = \begin{pmatrix} A_k' & 0 \\ i\,\Phi''^{*}\sigma_k\Phi' & A_k'' \end{pmatrix} \quad \text{for } k=1,2, \qquad \Phi = \begin{pmatrix} \Phi' & \Phi'' \end{pmatrix}. \tag{1–29}$$

Since, when forming the cascade connection, the output of the first system is fed into the second, the procedure makes sense only when the output compatibility PDE of the first system coincides with the input compatibility PDE of the second (and in this case the input compatibility PDE of the new system coincides with the input compatibility PDE of the first system, and the output compatibility PDE of the new system coincides with the output compatibility PDE of the second system). This explains the following result.

THEOREM 1.3 (MATCHING THEOREM [Livšic 1979; 1983]). $\mathcal{V}$ (*with* $\gamma = \gamma'$ *and* $\tilde{\gamma} = \tilde{\gamma}''$) *is a commutative vessel if and only if* $\tilde{\gamma}' = \gamma''$.

We can consider now the following inverse problem. Suppose we are given a real polynomial $p(\lambda_1, \lambda_2)$ defining a real (affine) plane curve C_0, a self-adjoint determinantal representation $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ of $p(\lambda_1, \lambda_2)$, and a subset $\mathcal{S}$ of C_0 which is closed and bounded in $\mathbb{C}^2$ and all of whose accumulation points are real points of C_0. We want *to construct, up to the unitary equivalence on the principal subspace, all commutative two-operator vessels with discriminant polynomial* $p(\lambda_1, \lambda_2)$, *input determinantal representation* $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$, *and the operators* A_1, A_2 *in the vessel having, on the principal subspace, joint spectrum* $\mathcal{S}$. Here two commutative two-operator vessels $\mathcal{V}^{(\alpha)} = (A_1^{(\alpha)}, A_2^{(\alpha)}, H^{(\alpha)}, \Phi^{(\alpha)}, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$ ($\alpha = 1, 2$) are said to be unitary equivalent on their principal subspaces $\hat{H}^{(1)}$

and $\hat{H}^{(2)}$ respectively if there is an isometric mapping U of $\hat{H}^{(1)}$ onto $\hat{H}^{(2)}$ such that

$$A_k^{(2)}|_{\hat{H}^{(2)}} = UA_k^{(1)}|_{\hat{H}^{(1)}}U^{-1} \quad (k = 1, 2), \qquad \Phi^{(2)}|_{\hat{H}^{(2)}} = \Phi^{(1)}|_{\hat{H}^{(1)}}U. \tag{1–30}$$

Suppose $p(\lambda_1, \lambda_2)$ is an irreducible polynomial (and is of degree $M = \dim E$, so that there are no factors "hidden" at infinity: this amounts to the condition $\det(\xi_1\sigma_1 + \xi_2\sigma_2) \not\equiv 0$). Suppose also that C_0—more precisely, its projective closure C—is a smooth (irreducible) curve (of degree M). Then a complete and explicit solution of the inverse problem stated above was obtained in [Vinnikov 1992]; see [Livšic et al. 1995, Chapter 12] for a detailed elementary exposition in the simplest nontrivial case $M = 3$. (The assumption that C is a smooth curve may be replaced by the assumptions that $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ is a maximal determinantal representation of $p(\lambda_1, \lambda_2)$ (i.e., its kernel has a maximal possible dimension at the singular points of C; see Section 2 below), and that the prescribed set $\mathcal{S}$ does not contain any singular points.) This solution leads to triangular models for the corresponding pairs of operators A_1, A_2 with finite nonhermitian ranks, generalizing the well-known triangular models (see [Brodskiĭ and Livšic 1958; Brodskiĭ 1969], for example) for a single nonselfadjoint operator. The solution is based on first constructing elementary objects—vessels with one-dimensional inner space corresponding to the points of the joint spectrum, and then coupling them using Theorem 1.3. It follows from the vessel condition (1–12) that in a vessel with one-dimensional inner space the output determinantal representation is determined by the input determinantal representation and the spectral data; the successive matching of output and input determinantal representations in Theorem 1.3 then gives a system of nonlinear difference (for the discrete part of the spectrum) and differential (for the continuous part of the spectrum) equations for self-adjoint determinantal representations of the polynomial $p(\lambda_1, \lambda_2)$. The geometric assumptions on the curve C imply that self-adjoint determinantal representations can be parametrized by certain points in the Jacobian variety of C [Vinnikov 1993]; and it turns out that passing from a self-adjoint determinantal representation to the corresponding point in the Jacobian variety linearises the systems of nonlinear difference and differential equations alluded to above. Actually, the system can be even solved explicitly using the theta functions, yielding explicit formulas for the operators A_1, A_2 in a triangular model. These formulas contain as a special case the "algebro-geometrical" formulas for finite-zone solutions of completely integrable nonlinear PDEs [Dubrovin 1981].

The fact that triangular models give us all the solutions to the inverse problem, i.e., that every commutative two-operator vessel with a smooth irreducible discriminant curve is unitarily equivalent (on its principal subspace) to a triangular model vessel (on its principal subspace) is related to the fact that the operators A_1, A_2 in the given vessel possess a "sufficiently nice" maximal chain of joint invariant subspaces; compare [Brodskiĭ 1969], for example, for the single-operator case. More important, this is related to the canonical factorization of the (normalized) joint characteristic function of the vessel — we will return to this point in Section 3.

We end this section by discussing several generalizations. First, even though we restrict our attention in this paper to pairs of commuting nonselfadjoint operators, the same framework can be applied to l-tuples for any l; much work here remains to be done, and we shall just review briefly the basic constructions, following [Livšic et al. 1995, Chapters 2,3,4,7]. We start with l commuting operators $A_1, \dots, A_l$ in a Hilbert space H (with finite nonhermitian ranks), and consider a *commutative l-operator vessel*

$$\mathcal{V} = \big(A_k\ (k = 1, \dots, l),\ H,\ \Phi,\\ E,\ \sigma_k\ (k = 1, \dots, l),\ \gamma_{kj}\ (k, j = 1, \dots, l),\ \tilde{\gamma}_{kj}\ (k, j = 1, \dots, l)\big),$$

where again E is the external space of the vessel (of dimension $M < \infty$), and $\Phi : H \to E$, $\sigma_k, \gamma_{kj}, \tilde{\gamma}_{kj} : E \to E$ are bounded linear mappings with $\sigma_k^* = \sigma_k$, $\gamma_{kj}^* = \gamma_{kj}$, $\tilde{\gamma}_{kj}^* = \tilde{\gamma}_{kj}$, $\gamma_{kj} = -\gamma_{jk}$, $\tilde{\gamma}_{kj} = -\tilde{\gamma}_{jk}$, and

$$\tfrac{1}{i}\,(A_k - A_k^*) = \Phi^* \sigma_k \Phi, \tag{1–31}$$

$$\sigma_k \Phi A_j^* - \sigma_j \Phi A_k^* = \gamma_{kj} \Phi, \tag{1–32}$$

$$\sigma_k \Phi A_j - \sigma_j \Phi A_k = \tilde{\gamma}_{kj} \Phi, \tag{1–33}$$

$$\tilde{\gamma}_{kj} - \gamma_{kj} = i\,(\sigma_k \Phi \Phi^* \sigma_j - \sigma_j \Phi \Phi^* \sigma_k), \tag{1–34}$$

for $k, j = 1, \dots, l$. Analogously to the two-operator case, $\mathcal{V}$ defines a (linear time-invariant) commutative conservative lD system together with appropriate compatibility PDEs at the input and at the output. We define the *input discriminant ideal* $\mathcal{I}$ of the vessel $\mathcal{V}$ to be the ideal in the polynomial ring $\mathbb{C}[\lambda_1, \dots, \lambda_l]$ generated by all polynomials of the form

$$p(\lambda_1, \dots, \lambda_l) = \det\bigg(\sum_{k,j=1}^{l} M^{kj}(\lambda_k \sigma_j - \lambda_j \sigma_k + \gamma_{kj})\bigg), \tag{1–35}$$

where $M^{kj} = -M^{jk}$ are arbitrary operators on E; the *(affine) input discriminant variety* $\mathcal{D}$ is the zero variety of the ideal $\mathcal{I}$, or what turns out to be the same, the set of all points $(\lambda_1, \dots, \lambda_l) \in \mathbb{C}^l$ such that

$$\bigcap_{k,j=1}^{l} \ker(\lambda_k \sigma_j - \lambda_j \sigma_k + \gamma_{kj}) \neq \{0\}. \tag{1–36}$$

The *output discriminant ideal* $\tilde{\mathcal{I}}$ of $\mathcal{V}$ and the *(affine) output discriminant variety* $\tilde{\mathcal{D}}$ are defined similarly replacing γ_{kj} by $\tilde{\gamma}_{kj}$. The generalized Cayley–Hamilton theorem states that

$$p(A_1^*, \dots, A_l^*) = 0 \tag{1–37}$$

for all $p \in \mathcal{I}$ and

$$p(A_1, \dots, A_l) = 0 \tag{1–38}$$

for all $p \in \tilde{\mathcal{I}}$, on the appropriately defined principal subspace $\hat{H}$. The analogue of Theorem 1.2 in general fails, i.e., we may have $\mathcal{I} \neq \tilde{\mathcal{I}}$; however—see Livšic and Markus [1994]—the discriminant varieties $\mathcal{D}$ and $\tilde{\mathcal{D}}$ may differ only by a finite number of isolated points, which have to be nonreal joint eigenvalues of finite multiplicity of either $A_1^*, \dots, A_l^*$ or $A_1, \dots, A_l$.

At least if we assume a nondegeneracy condition $\det\bigl(\sum_{k=1}^{l} \xi_k \sigma_k\bigr) \not\equiv 0$, it follows from (1–36) that the discriminant varieties $\mathcal{D}$ and $\tilde{\mathcal{D}}$ cannot contain components of (complex) dimension greater than 1 (e.g., if $\det \sigma_1 \neq 0$, then it follows that the projection onto the first coordinate λ_1 is (at most) finite to one). Hence $\mathcal{D}$ and $\tilde{\mathcal{D}}$ consist of one and the same (affine) algebraic curve in $\mathbb{C}^l$, and two possibly distinct finite collections of isolated points. It seems that these isolated points are related to various well-known pathologies for l commuting nonselfadjoint or nonunitary operators with $l > 2$, such as the failure of von Neumann's inequality and the nonexistence of commuting unitary dilations for three or more commuting contractions.

Linear time-invariant one-dimensional systems without energy balance condition (1–8) are the basic object of study in system theory, starting with the work of Kalman; see, for example, [Kailath 1980; Bart et al. 1979]. There is a similar nonconservative analogue of the notion of vessel; it has been worked out by Kravitsky [1983]; see also [Livšic et al. 1995, Chapter 8; Vinnikov 1994; Ball and Vinnikov 1996]. In particular, we may use these nonconservative vessels to study meromorphic matrix functions on a compact Riemann surface via their realizations as the (normalized) joint transfer function of a vessel; see below in Section 2.

Another interesting generalization is a time-varying analogue of the notion of vessel that was considered by Gauchman [1983b; 1983a] (in a very general setting of Hilbert bundles on differentiable manifolds) and recently by Livšic [1996]; for simplicity we restrict ourselves again to the conservative two-dimensional case. We consider a linear time-varying conservative two-dimensional system exactly as in (1–20)–(1–22), except that $A_1 = A_1(t_1, t_2)$, $A_2 = A_2(t_1, t_2)$, $\Phi = \Phi(t_1, t_2)$, $\sigma_1 = \sigma_1(t_1, t_2)$ and $\sigma_2 = \sigma_2(t_1, t_2)$ are functions of $t = (t_1, t_2)$; we still assume that $\sigma_1(t)$ and $\sigma_2(t)$ are self-adjoint and the colligation conditions (1–9) hold (for all t). The condition for the system to be compatible for identically zero input and arbitrary initial state becomes the so-called zero-curvature condition:

$$\frac{\partial A_1}{\partial t_2} - \frac{\partial A_2}{\partial t_1} + i[A_1, A_2] = 0. \tag{1–39}$$

Repeating the derivation of (1–24) and (1–27) (taking into account various partial derivatives of system operators coming in) we see that we obtain again linear compatibility PDEs (but with variable coefficients) (1–26) and (1–28) at the input and at the output respectively if we assume that we have a *time-varying* *zero-curvature two-operator vessel*

$$\mathcal{V} = (A_1(t), A_2(t), H, \Phi(t), E, \sigma_1(t), \sigma_2(t), \gamma(t), \tilde{\gamma}(t)),$$

where $\gamma(t), \tilde{\gamma}(t) : E \to E$ are (bounded) operators such that

$$\sigma_1 \Phi A_2^* - \sigma_2 \Phi A_1^* - i(\sigma_1 \frac{\partial \Phi}{\partial t_2} - \sigma_2 \frac{\partial \Phi}{\partial t_1}) = \gamma \Phi, \tag{1–40}$$

$$\sigma_1 \Phi A_2 - \sigma_2 \Phi A_1 - i(\sigma_1 \frac{\partial \Phi}{\partial t_2} - \sigma_2 \frac{\partial \Phi}{\partial t_1}) = \tilde{\gamma} \Phi, \tag{1–41}$$

$$\tilde{\gamma} - \gamma = i\,(\sigma_1 \Phi \Phi^* \sigma_2 - \sigma_2 \Phi \Phi^* \sigma_1), \tag{1–42}$$

$$\frac{1}{i}\,(\gamma - \gamma^*) = \frac{\partial \sigma_2}{\partial t_1} - \frac{\partial \sigma_1}{\partial t_2}. \tag{1–43}$$

Note that $\gamma(t)$ and $\tilde{\gamma}(t)$ are generally not self-adjoint. An analogue of Theorem 1.3 holds for time-varying zero-curvature vessels, so that we may construct zero-curvature vessels by coupling elementary objects—zero-curvature vessels with one-dimensional state space. A more detailed study of the resulting "coupling chains", both discrete and continuous, remains to be done. Another basic problem is to study the input-output map of the vessel, which goes from the solution space of the input compatibility PDE to the solution space of the output compatibility PDE, and to describe the class of all input-output maps.

An especially important situation is when all the operators depend on only one of the two variables, let us say on t_1. In this case it seems reasonable to consider a perturbation problem, i.e., our time-varying vessel is either compactly supported or rapidly decaying perturbation of a usual time-invariant vessel.

2. The Joint Characteristic Function

The fundamental interplay between the spectral theory of a single nonselfadjoint operator in a Hilbert space and function theory is based on the notion of the *characteristic function* of an operator, more precisely of an operator colligation $\mathcal{C} = (A, H, \Phi, E, \sigma)$, defined by

$$S(\lambda) = I - i\,\Phi(A - \lambda I)^{-1} \Phi^* \sigma. \tag{2–1}$$

See, for example, [Brodskiĭ and Livšic 1958; Brodskiĭ 1969; Sz.-Nagy and Foiaş 1967; de Branges and Rovnyak 1966b; 1966a]. It is an analytic function of $\lambda \in \mathbb{C}$ for λ outside the spectrum of A, whose values are operators on E—or, since we are assuming $\dim E = M < \infty$, matrices. Equation (2–1) has the following consequences:

(1) $S(\lambda)$ is analytic in a neigbourhood of $\lambda = \infty$, and $S(\infty) = I$.

(2) $S(\lambda)$ is meromorphic on $\mathbb{C} \setminus \mathbb{R}$, and satisfies in its domain of analyticity the following metric properties with respect to the self-adjoint operator σ:

$$S^*(\lambda)\sigma S(\lambda) \geq \sigma \quad \text{when } \operatorname{Im}\lambda > 0, \tag{2–2}$$

$$S^*(\lambda)\sigma S(\lambda) \leq \sigma \quad \text{when } \operatorname{Im}\lambda < 0, \tag{2–3}$$

$$S^*(\lambda)\sigma S(\lambda) = \sigma \quad \text{when } \operatorname{Im}\lambda = 0. \tag{2–4}$$

Conversely, given E and σ with $\det\sigma \neq 0$, any function satisfying (1) and (2) is the characteristic function of some colligation with external part (E, σ). Furtermore, the characteristic function determines the correspoding colligation uniquely, up to the unitary equivalence on the principal subspace.

We mention two basic facts relating multiplicative properties of the characteristic function to the spectral properties of the operator A in the colligation.

- The set of singularities of $S(\lambda)$ (i.e., the set of points in the complex plane to a neighborhood of which $S(\lambda)$ cannot be continued analytically) coincides with the spectrum of A restricted to the principal subspace of the colligation.
- If $\mathcal{C} = \mathcal{C}' \vee \mathcal{C}''$ and $S'(\lambda)$, $S''(\lambda)$ are the characteristic functions of the colligations $\mathcal{C}'$, $\mathcal{C}''$ respectively, then $S(\lambda) = S''(\lambda)S'(\lambda)$.

It follows from the second fact that the canonical factorization of $S(\lambda)$ (the Riesz–Nevanlinna–Smirnov factorization for $\dim E = 1$, when $S(\lambda)$ is just a bounded analytic function in the lower or the upper half-plane, and the Potapov [1955] factorization for $\dim E > 1$) is related to the reduction of the operator A to a triangular form; more generally, factorizations of $S(\lambda)$ are related to invariant subspaces of A.

System-theoretically, the characteristic function of the colligation is the *transfer function* of the corresponding system (1–6)–(1–7). There are many equivalent ways to define the transfer function of a linear time-invariant system. The simplest one for our purposes is to assume that the input, the state and the output of the system are waves with the same frequency λ: $u(t) = u_0 e^{it\lambda}$, $f(t) = f_0 e^{it\lambda}$, $v(t) = v_0 e^{it\lambda}$, where $u_0, v_0 \in E$, $f_0 \in H$. Substitution into the system equations (1–6)–(1–7) shows that

$$v_0 = S(\lambda)u_0; \tag{2–5}$$

that is, the transfer function (the characteristic function) maps the input amplitude to the output amplitude.

We now consider a commutative vessel $\mathcal{V} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$ with the discriminant polynomial

$$p(\lambda_1, \lambda_2) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma) = \det(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma}),$$

and the (affine) discriminant curve C_0 with the equation $p(\lambda_1, \lambda_2) = 0$. We first define the *complete characteristic function* of the vessel by

$$W(\xi_1, \xi_2, z) = I - i\,\Phi(\xi_1 A_1 + \xi_2 A_2 - zI)^{-1}\Phi^*(\xi_1\sigma_1 + \xi_2\sigma_2). \tag{2–6}$$

It is an analytic function of $\xi_1, \xi_2, z \in \mathbb{C}$ for z outside the spectrum of $\xi_1 A_1 + \xi_2 A_2$. Since it consists essentially of usual characteristic functions of single-operator colligations obtained from $\mathcal{V}$ by averaging in all possible directions, it is not hard to show that the complete characteristic function determines the corresponding vessel uniquely, up to the unitary equivalence on the principal subspace. We also have appropriate metric properties (where $\xi_1, \xi_2 \in \mathbb{R}$):

$$W^*(\xi_1, \xi_2, z)(\xi_1\sigma_1 + \xi_2\sigma_2)W(\xi_1, \xi_2, z) \geq \xi_1\sigma_1 + \xi_2\sigma_2 \quad \text{when } \operatorname{Im} z > 0, \tag{2–7}$$

$$W^*(\xi_1, \xi_2, z)(\xi_1\sigma_1 + \xi_2\sigma_2)W(\xi_1, \xi_2, z) \leq \xi_1\sigma_1 + \xi_2\sigma_2 \quad \text{when } \operatorname{Im} z < 0, \tag{2–8}$$

$$W^*(\xi_1, \xi_2, z)(\xi_1\sigma_1 + \xi_2\sigma_2)W(\xi_1, \xi_2, z) = \xi_1\sigma_1 + \xi_2\sigma_2 \quad \text{when } \operatorname{Im} z = 0. \tag{2–9}$$

But since $W(\xi_1, \xi_2, z)$ is a function of two independent complex variables (two, because of the homogeneity), it does not admit a good factorization theory to relate to the spectral theory of the pair of operators A_1, A_2.

However it turns out that the complete characteristic function fits perfectly into the algebro-geometrical framework associated to the vessel, given by the discriminant polynomial $p(\lambda_1, \lambda_2)$ and its two determinantal representations. For each point $\lambda = (\lambda_1, \lambda_2) \in C_0$ we may define two nontrivial subspaces of the external space E:

$$\mathcal{E}(\lambda) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma), \tag{2–10}$$

$$\tilde{\mathcal{E}}(\lambda) = \ker(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma}). \tag{2–11}$$

THEOREM 2.1 [Livšic 1986b]. *For any point $\lambda = (\lambda_1, \lambda_2)$ on C_0 and for arbitrary complex numbers ξ_1, ξ_2 (such that $\xi_1\lambda_1 + \xi_2\lambda_2$ is outside the spectrum of $\xi_1 A_1 + \xi_2 A_2$), $W(\xi_1, \xi_2, \xi_1\lambda_1 + \xi_2\lambda_2)$ maps $\mathcal{E}(\lambda)$ into $\tilde{\mathcal{E}}(\lambda)$, and the restriction $W(\xi_1, \xi_2, \xi_1\lambda_1 + \xi_2\lambda_2)|\mathcal{E}(\lambda)$ is independent of ξ_1, ξ_2.*

This theorem allows us to define the *joint characteristic function* of the vessel by restricting the complete characteristic function to the discriminant curve and to the fibres of the "input family of subspaces" (2–10):

$$S(\lambda) = W(\xi_1, \xi_2, \xi_1\lambda_1 + \xi_2\lambda_2)|\mathcal{E}(\lambda) : \mathcal{E}(\lambda) \longrightarrow \tilde{\mathcal{E}}(\lambda), \tag{2–12}$$

where $\lambda = (\lambda_1, \lambda_2) \in C_0$ and ξ_1, ξ_2 are free complex parameters such that $\xi_1\lambda_1 + \xi_2\lambda_2$ is outside the spectrum of $\xi_1 A_1 + \xi_2 A_2$.

To clarify the definition of the joint characteristic function we shall interpret it as the *joint transfer function* of the corresponding commutative two-dimensional system (1–20)–(1–22) together with the compatibility PDEs (1–26) and (1–28) at the input and at the output. We assume as before that the input, the state and the output of the system are (planar) waves with the same (double) frequency $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$:

$$u(t_1, t_2) = u_0 e^{it_1\lambda_1 + it_2\lambda_2},$$

$$f(t_1, t_2) = f_0 e^{it_1\lambda_1 + it_2\lambda_2},$$

$$v(t_1, t_2) = v_0 e^{it_1\lambda_1 + it_2\lambda_2},$$

where $u_0, v_0 \in E,\ f_0 \in H$. The input compatibility PDE (1–26) yields

$$(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma)u_0 = 0; \tag{2–13}$$

hence $\lambda \in C_0$ and $u_0 \in \mathcal{E}(\lambda)$. The output compatibility PDE (1–28) yields

$$(\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma})v_0 = 0, \tag{2–14}$$

hence again $\lambda \in C_0$ (*this explains the equality of determinants in Theorem* 1.2) and $v_0 \in \tilde{\mathcal{E}}(\lambda)$. Substituting into the system equations, multiplying (1–20) and (1–21) by free complex parameters ξ_1 and ξ_2 respectively and adding, we obtain

$$v_0 = S(\lambda)u_0. \tag{2–15}$$

Hence as before the joint transfer function (the joint characteristic function) maps the input amplitude at a given (double) frequency to the output amplitude, except that because of the compatibility PDEs the double frequency is restricted to lie on the (affine) discriminant curve and the input and output amplitudes must lie in the fibres of the input and the output families of subspaces (2–10) and (2–11) respectively at this double frequency.

Unlike the complete characteristic function, which depends on two independent complex variables, the joint characteristic function depends on a point on a one-dimensional complex variety, namely the discriminant curve. We would like to claim that no information is lost by passing to the joint characteristic function. We make two geometric assumptions. The first one is mainly for the simplicity of exposition. We assume that the discriminant polynomial $p(\lambda_1, \lambda_2)$ has only one, possibly multiple, irreducible factor (and is of degree $M = \dim E$, so that there are no factors hidden at infinity). Thus $p(\lambda_1, \lambda_2) = \bigl(f(\lambda_1, \lambda_2)\bigr)^r$ for some $r \geq 1$, where $f(\lambda_1, \lambda_2) = 0$ is the irreducible affine equation of a real irreducible projective plane curve C—the projective closure of the affine discriminant curve C_0—of degree m, where $M = mr$.

The second assumption is deeper. We assume that for all smooth points μ on C, we have $\dim \mathcal{E}(\mu) = \dim \tilde{\mathcal{E}}(\mu) = r$; if $\mu \in C$ is a singular point of multiplicity s, we assume $\dim \mathcal{E}(\mu) = \dim \tilde{\mathcal{E}}(\mu) = rs$. In general we have only inequalities:

$$1 \leq \dim \mathcal{E}(\mu) \leq r, \quad 1 \leq \dim \tilde{\mathcal{E}}(\mu) \leq r$$

at smooth points, and

$$1 \leq \dim \mathcal{E}(\mu) \leq rs, \quad 1 \leq \dim \tilde{\mathcal{E}}(\mu) \leq rs$$

at a singular point of multiplicity s. We refer to this second assumption as the *maximality* of the input and the output determinantal representations of $p(\lambda_1, \lambda_2)$. Note that it holds automatically if $r = 1$, i.e., if $p(\lambda_1, \lambda_2)$ is irreducible, and C is smooth.

It follows from the maximality that the subspaces $\mathcal{E}(\mu)$, $\tilde{\mathcal{E}}(\mu)$ for different points μ on C (including, of course, the points at infinity) fit together to form two complex holomorphic rank r vector bundles $\mathcal{E}$, $\tilde{\mathcal{E}}$ on a compact Riemann surface X which is the desingularization (normalization) of C; here $X = C$ when C is smooth, and when C is singular X is obtained from C by resolving the singularities (see [Fulton 1969] or [Griffiths 1989], for example). Note that, since C is a real curve, X is a *real* Riemann surface, that is, a Riemann surface equipped with an anti-holomorphic involution (the complex conjugation on C). The joint characteristic function $S : \mathcal{E} \to \tilde{\mathcal{E}}$ is (after the natural extension to the points of C at infinity) simply a bundle mapping, holomorphic outside the joint spectrum of A_1, A_2. It is meromorphic on $X \setminus X_{\mathbb{R}}$, where $X_{\mathbb{R}}$ is the set of real points of X (fixed points of the anti-holomorphic involution). The following basic fact was established by Livšic in the dissipative case (i.e., when $\xi_1\sigma_1 + \xi_2\sigma_2 > 0$ for some ξ_1, ξ_2) and by Vinnikov in general.

THEOREM 2.2 [Vinnikov 1992; Ball and Vinnikov 1996]. *The joint characteristic function of a vessel* (*having maximal input and output determinantal representations*) *determines uniquely the complete characteristic function.*

PROOF. Since C is a plane curve of degree m, for $(\xi_1, \xi_2, z) \in \mathbb{C}^3$ generic the straight line $\xi_1\lambda_1 + \xi_2\lambda_2 = z$ intersects the curve C in m distinct (affine) points $\lambda^1, \dots, \lambda^m$ (that are all smooth points of C). These points correspond to the m distinct eigenvalues of the (one variable) matrix pencil obtained by restricting $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ to the given line. The corresponding eigenspaces are just $\mathcal{E}(\lambda^1), \dots, \mathcal{E}(\lambda^m)$; by the maximality assumption each one of them has dimension r, so that the sum of their dimensions equals mr and coincides with the dimension M of the ambient space E. Thus we have a (nonorthogonal) direct sum decomposition

$$\mathcal{E}(\lambda^1) \dot{+} \cdots \dot{+} \mathcal{E}(\lambda^m) = E. \tag{2–16}$$

Let $P(\xi_1, \xi_2, \lambda^i)$ be the corresponding projections of E onto $\mathcal{E}(\lambda^i)$, so that

$$P(\xi_1, \xi_2, \lambda^1) + \cdots + P(\xi_1, \xi_2, \lambda^m) = I. \tag{2–17}$$

Since $W(\xi_1, \xi_2, z)|_{\mathcal{E}(\lambda^i)} = S(\lambda^i)$ for all i by the definition of the joint characteristic function, we obtain from (2–17) an explicit formula, called the *restoration formula*

$$W(\xi_1, \xi_2, z) = \sum_{i=1}^{m} S(\lambda^i) P(\xi_1, \xi_2, \lambda^i) \tag{2–18}$$

for the complete characteristic function in terms of the joint (on an open dense subset of the domain of analyticity of $W(\xi_1, \xi_2, z)$). □

The next question is how to express the metric properties (2–7)–(2–9) in terms of the joint characteristic function. To this end we introduce an (indefinite) scalar product on the fibres of the input bundle $\mathcal{E}$ over nonreal (affine) points, by setting

$$[u, v]^{\mathcal{E}}_{\lambda} = i\,\frac{v^*(\xi_1\sigma_1 + \xi_2\sigma_2)u}{\xi_1(\lambda_1 - \bar{\lambda}_1) + \xi_2(\lambda_2 - \bar{\lambda}_2)} \quad \text{for } u, v \in \mathcal{E}(\lambda), \tag{2–19}$$

and similarly for the output bundle $\tilde{\mathcal{E}}$; here $\lambda = (\lambda_1, \lambda_2) \in C_0$, $\bar{\lambda} \neq \lambda$, and ξ_1, ξ_2 are free parameters—the value of (2–19) turns out to be independent of ξ_1, ξ_2. Note that this metric on the bundle generalizes the Poincaré metric on the upper half-plane. There is also a version of (2–19) at the real points, taking the limit and renormalizing it to be finite, namely

$$[u, v]^{\mathcal{E}}_{\lambda} = \frac{v^*(\xi_1\sigma_1 + \xi_2\sigma_2)u}{\xi_1\, d\lambda_1(\lambda) + \xi_2\, d\lambda_2(\lambda)} \quad \text{for } u, v \in \mathcal{E}(\lambda); \tag{2–20}$$

here $\lambda = (\lambda_1, \lambda_2) \in C_0$, $\bar{\lambda} = \lambda$, and ξ_1, ξ_2 are free parameters—the value of (2–20) turns out again to be independent of ξ_1, ξ_2. More generally, we introduce a hermitian pairing between the fibres of the bundle over nonconjugate (affine) points

$$[u, v]^{\mathcal{E}}_{\lambda^1,\lambda^2} = i\, \frac{v^*(\xi_1\sigma_1 + \xi_2\sigma_2)u}{\xi_1(\lambda^1_1 - \bar{\lambda}^2_1) + \xi_2(\lambda^1_2 - \bar{\lambda}^2_2)} \quad \text{for } u \in \mathcal{E}(\lambda^1),\ v \in \mathcal{E}(\lambda^2) \tag{2–21}$$

(with $\lambda^1 = (\lambda^1_1, \lambda^1_2), \lambda^2 = (\lambda^2_1, \lambda^2_2) \in C_0$, $\bar{\lambda}^2 \neq \lambda^1$), and over conjugate (affine) points

$$[u, v]^{\mathcal{E}}_{\lambda,\bar{\lambda}} = \frac{v^*(\xi_1\sigma_1 + \xi_2\sigma_2)u}{\xi_1\, d\lambda_1(\lambda) + \xi_2\, d\lambda_2(\lambda)} \quad \text{for } u \in \mathcal{E}(\lambda),\ v \in \mathcal{E}(\bar{\lambda}) \tag{2–22}$$

(with $\lambda = (\lambda_1, \lambda_2) \in C_0$). Then it can be shown, using the restoration formula (2–18), that *the properties* (2–7)–(2–9) *are equivalent to the following metric properties of the joint characteristic function in its domain of analyticity*:

$$[S(\lambda)u, S(\mu)v]^{\tilde{\mathcal{E}}}_{\lambda,\mu} \geq [u, v]^{\mathcal{E}}_{\lambda,\mu} \quad \text{for } u \in \mathcal{E}(\lambda),\ v \in \mathcal{E}(\mu), \tag{2–23}$$

$$[S(\lambda)u, S(\bar{\lambda})v]^{\tilde{\mathcal{E}}}_{\lambda,\bar{\lambda}} = [u, v]^{\mathcal{E}}_{\lambda,\bar{\lambda}} \quad \text{for } u \in \mathcal{E}(\lambda),\ v \in \mathcal{E}(\bar{\lambda}). \tag{2–24}$$

See [Vinnikov 1992] and, for details, [Livšic et al. 1995, Chapter 10]. The inequality in (2–23) means as usual that the expression appearing on the left-hand side is a positive definite kernel, i.e., for any N points $\lambda^1, \dots, \lambda^N$ on C_0 in the domain of analyticity of $S(\lambda)$ ($\lambda^i \neq \bar{\lambda}^j$) and any $u_i \in \mathcal{E}(\lambda^i)$ we have

$$\big([S(\lambda^i)u_i, S(\lambda^j)u_j]^{\tilde{\mathcal{E}}}_{\lambda^i,\lambda^j} - [u_i, u_j]^{\mathcal{E}}_{\lambda^i,\lambda^j}\big)_{i,j=1,\dots,N} \geq 0.$$

In particular, the bundle map S is expansive at nonreal points and isometric at real points with respect to the scalar product (2–19)–(2–20) on the input and the output bundles.

The joint characteristic function is not quite the end of the quest for a proper generalization of the usual characteristic function, since the kernel bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ and the scalar product (2–19)–(2–20) are hard to deal with analytically. However, it follows from the theory of determinantal representations of plane algebraic curves [Vinnikov 1989; 1993; Ball and Vinnikov 1996] that these bundles are isomorphic (up to an inessential twist) to certain vector bundles of the form $V_\chi \otimes \Delta$, where V_χ is the flat vector bundle corresponding to a representation $\chi : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ of the fundamental group of the Riemann surface X, and

Δ is a line bundle of half-order differentials on X (a square root of the canonical bundle). Sections of $V_\chi \otimes \Delta$ are thus multiplicative $\mathbb{C}^r$-valued half-order differentials on X; here a half-order differential is an expression locally of the form $f(t)\sqrt{dt}$ where t is a local parameter on X, and "multiplicative" means that our vector-valued half-order differential picks up a multiplier (factor of automorphy) $\chi(R)$ when we go around a closed loop R on X. We see then that we obtain from the joint characteristic function of the vessel a so-called *normalized joint characteristic function*, which is a mapping of flat vector bundles on X, i.e., a multiplicative $r \times r$ matrix function on X, with appropriate multipliers on the left and on the right. We proceed, following [Vinnikov 1992], to describe the scalar case $r = 1$ (line bundles); in this case the results are the most complete since we may use the classical theory of Jacobian varieties and theta functions (see also [Livšic et al. 1995, Chapters 10–11] for a detailed exposition, using elliptic functions only, of the simplest nontrivial case when the discriminant curve C is a smooth cubic, $m = 3$).

We let g be the genus (the "number of handles") of the compact Riemann surface X; when C is a smooth curve, the genus is given in terms of the degree of C by the formula $g = (m-1)(m-2)/2$. We choose a canonical homology basis on X consisting of the A-cycles $A_1, \dots, A_g$ and the B-cycles $B_1, \dots, B_g$ (and satisfying certain symmetry requirements with respect to the anti-holomorphic involution on X: see [Vinnikov 1993]). We can then construct a basis $\omega_1, \dots, \omega_g$ for the space of holomorphic differentials on X which is normalized with repect to our homology basis: $\int_{A_j} \omega_i = \delta_{ij}$; the so-called period matrix $\Omega = \left(\int_{B_j} \omega_i\right) \in \mathbb{C}^{g \times g}$; the period lattice $\Lambda = \mathbb{Z}^g + \Omega\mathbb{Z}^g \subset \mathbb{C}^g$ (this is the lattice in $\mathbb{C}^g$ formed by integrals of the column with entries $\omega_1, \dots, \omega_g$ over all possible closed loops on X); the Jacobian variety $J(X) = \mathbb{C}^g/\Lambda$, and the associated Riemann's theta function $\theta(w) = \theta(w; \Omega)$, $w \in \mathbb{C}^g$. See [Mumford 1983], for example, for all these classical notions. For $\zeta \in J(X)$ we let L_ζ be the flat line bundle with multipliers of absolute value 1 corresponding to ζ: we write $\zeta = b + \Omega a$ where $a, b \in \mathbb{R}^g$ have coordinates a_j, b_j, and the multipliers over the cycles A_j and B_j are given by $e^{-2\pi i a_j}$ and $e^{2\pi i b_j}$ respectively, for $j = 1, \dots, g$. We let Δ be the line bundle of half-order differentials corresponding to $-\kappa$, where $\kappa \in J(X)$ is the so-called Riemann's constant. Then $\mathcal{E}$ is isomorphic to the kernel bundle associated to a maximal self-adjoint determinantal representation of the irreducible defining polynomial $f(\lambda_1, \lambda_2)$ of C if and only if

$$\mathcal{E} \otimes \mathcal{O}(m-2)(-D) \cong L_\zeta \otimes \Delta, \tag{2–25}$$

where D is the divisor of singularities of C on X ($D = 0$ when C is smooth) and $\zeta \in J(X)$ satisfies

$$\theta(\zeta) \neq 0 \tag{2–26}$$

(this is equivalent, by Riemann's Theorem, to the fact that the line bundle $L_\zeta \otimes \Delta$ has no global holomorphic sections, and is a necessary and sufficient condition for $\mathcal{E}$ to be isomorphic to the kernel bundle associated to some—not necessarily self-adjoint—determinantal representation of $f(\lambda_1,\lambda_2)$), and

$$\zeta+\bar{\zeta}=\bar{\kappa}-\kappa \tag{2–27}$$

(this condition ensures the self-adjointness). The solution set of (2–26)–(2–27) in the g-dimensional complex torus $J(X)$ is a finite disjoint union of punctured g-dimensional real tori, the punctures coming from the zeroes of the theta function. In concrete terms, the isomorphism (2–25) means that there exists a nowhere zero section $u^\times(p)$ of $\mathcal{E}$ on X, whose entries are multiplicative meromorphic half-order differentials with multipliers corresponding to $-\zeta$ and with simple poles (at most) at the points of C at infinity; the isomorphism is then given by

$$y(p)\mapsto \frac{1}{\omega(p)}y(p)u^\times(p), \tag{2–28}$$

where $y(p)$ is a holomorphic multiplicative half-order differential on an open subset U of X with multipliers corresponding to ζ (a holomorphic section of $L_\zeta\otimes\Delta$ on U), and

$$\omega=\frac{d\lambda_1}{\partial f/\partial\lambda_2}=-\frac{d\lambda_2}{\partial f/\partial\lambda_1}$$

is a fixed meromorphic differential on X with zeroes of order $m-3$ at infinity and poles on the divisor of singularities D; note that the right-hand side of (2–28) is a section of $\mathcal{E}$ on U whose entries are meromorphic functions with poles of order $m-2$ (at most) at the points of C at infinity and vanishing on D (a holomorphic section of $\mathcal{E}\otimes\mathcal{O}(m-2)(-D)$ on U), as required. We call $u^\times(p)$ a normalized section of $\mathcal{E}$; it is determined uniquely up to a nonzero constant factor.

It follows that, if the input and the output line bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$ of the vessel $\mathcal{V}$ correspond as in (2–25) to the points ζ and $\tilde{\zeta}$ in $J(X)$ respectively, then the joint characteristic function $S:\mathcal{E}\to\tilde{\mathcal{E}}$ yields, under the corresponding isomorphisms, a scalar multiplicative function T on X with multipliers corresponding to $\tilde{\zeta}-\zeta$, called the *normalized joint characteristic function*. $T(p)$ is holomorphic and nonzero in a neighborhood of the points of C at infinity, and is meromorphic on $X\setminus X_{\mathbb{R}}$. In terms of normalized sections $u^\times(p)$ and $\widetilde{u}^\times(p)$ of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively we have

$$S(p)u^\times(p)=T(p)\widetilde{u}^\times(p). \tag{2–29}$$

Now—and this is the main point—the scalar product (2–19)–(2–20) on the line bundles $\mathcal{E}$ and $\tilde{\mathcal{E}}$, and more generally the pairing (2–21)–(2–22), can be expressed analytically in terms of theta functions. Explicitly, we have (after adjusting $u^\times(p)$ by an appropriate constant factor)

$$[u^\times(p),u^\times(q)]^{\mathcal{E}}_{p,q}=\varepsilon\,\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(p-\bar{q})}{i\,\theta\begin{bmatrix}a\\b\end{bmatrix}(0)\,E(p,\bar{q})}\quad\text{when } p\neq\bar{q}, \tag{2–30}$$

$$[u^\times(p), u^\times(p)]^{\mathcal{E}}_{p,\bar{p}} = \varepsilon, \tag{2–31}$$

and similarly for the output line bundle $\tilde{\mathcal{E}}$, the notation being this: $\varepsilon = \pm 1$ is the so-called sign of the input determinantal representation, which distinguishes between two self-adjoint determinantal representations that differ by a factor of -1 and hence have the same kernel bundle (it turns out that the output determinantal representation has the same sign ε);

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(w), \quad \text{for } w \in \mathbb{C}^g,$$

is the theta function with characteristics $a, b \in \mathbb{R}^g$ corresponding to ζ in $J(X)$, i.e., $\zeta = b + \Omega a$ (whenever we write a point on the Riemann surface X in the argument of a theta function, we mean the image of the point in $J(X)$ under the Abel–Jacobi map—more precisely, some lifting of the image to $\mathbb{C}^g$); and finally, $E(\,\cdot\,,\cdot\,)$ is the *prime form* on X, whose main property is that $E(p, s) = 0$ if and only if $p = s$ (this is a generalization to a compact Riemann surface of higher genus of the difference between two numbers in the complex plane). For more on the prime form, see [Mumford 1984; Fay 1973].

It follows from (2–30)–(2–31) that the metric properties (2–23)–(2–24) become, in terms of T

$$\varepsilon\, T(p)\overline{T(q)}\; \frac{\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(p - \bar{q})}{i\,\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(0)\, E(p, \bar{q})} - \varepsilon\; \frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(p - \bar{q})}{i\,\theta\begin{bmatrix} a \\ b \end{bmatrix}(0)\, E(p, \bar{q})} \geq 0, \tag{2–32}$$

$$T(p)\overline{T(\bar{p})} = 1 \tag{2–33}$$

in the domain of analyticity of $T(p)$ on X. Here $a, b \in \mathbb{R}^g$ and $\tilde{a}, \tilde{b} \in \mathbb{R}^g$ are the characteristics corresponding to ζ and $\tilde{\zeta}$ in $J(X)$ respectively, i.e., $\zeta = b + \Omega a$ and $\tilde{\zeta} = \tilde{b} + \Omega\tilde{a}$. The inequality in (2–32) means as in (2–23) that the expression appearing on the left-hand side is a positive definite kernel.

We can give now a complete analytic description of the class of normalized joint characteristic functions of vessels with an irreducible discriminant polynomial (and maximal input and output determinantal representations).

Theorem 2.3. *A multiplicative function $T(p)$ on X with multipliers corresponding to $\tilde{\zeta} - \zeta$ is the normalized joint characteristic function of a vessel with discriminant polynomial $f(\lambda_1, \lambda_2)$ and having maximal input and output determinantal representations of sign ε and corresponding to points ζ and $\tilde{\zeta}$ in $J(X)$ respectively if and only if $T(p)$ is holomorphic and nonzero in a neighborhood of the points of C at infinity*, *is meromorphic on $X \setminus X_{\mathbb{R}}$, and satisfies* (2–33) *and* (2–32).

It is worthwhile to mention that the corresponding point in $J(X)$ and the sign determine a maximal self-adjoint determinantal representation of $f(\lambda_1, \lambda_2)$ up to equivalence, where the equivalence relation is defined by multiplying a self-adjoint determinantal representation on the right and on the left by a (constant) invertible operator on the external space E and by its adjoint respectively. In Theorem 2.3 we may choose arbitrarily the input determinantal representation of the vessel within the equivalence class corresponding to $\zeta \in J(X)$ and ε, and then the output determinantal representation is uniquely (and explicitly) determined (within the equivalence class corresponding to $\tilde{\zeta} \in J(X)$ and ε) by the given normalized joint characteristic function $T(p)$ (actually, by the values of $T(p)$ at the points of C at infinity). This fact is of fundamental importance in the construction of triangular models; see [Livšic et al. 1995, Chapter 12].

Finally, it can be also shown that *the set of singularities of the normalized joint characteristic function coincides with the joint spectrum of the operators A_1 and A_2 in the vessel restricted to the principal subspace.*

It is natural to try to generalize all these results to the case when the discriminant polynomial has a multiple irreducible factor and the normalized joint characteristic function is a multiplicative matrix function on a compact real Riemann surface. An appropriate tool for such a generalization seems to be the notion of the *Cauchy kernel* $K(V_\chi; p, s)$ for a flat vector bundle V_χ of rank r on a compact Riemann surface X, where it is assumed that $V_\chi \otimes \Delta$ has no global holomorphic sections. The Cauchy kernel is defined as the unique meromorphic section of $\pi_1^* V_\chi \otimes \pi_2^* V_\chi^\vee \otimes \pi_1^* \Delta \otimes \pi_2^* \Delta$ on $X \times X$, where π_1 and π_2 denote the projections of $X \times X$ onto the first and the second factor respectively and $F^\vee$ denotes the dual of a vector bundle F, holomorphic except for a simple pole with residue I_r along the diagonal $p = s$. This notion was introduced in [Ball and Vinnikov 1996]; similar kernels were also considered in [Fay 1992]. In the scalar case $r = 1$, when $V_\chi = L_\zeta$ is the unitary flat line bundle corresponding to a point ζ in $J(X)$ with characteristics $a, b \in \mathbb{R}^g$ as before, we have

$$K(V_\chi; p, s) = \frac{\theta\begin{bmatrix} a \\ b \end{bmatrix}(s - p)}{\theta\begin{bmatrix} a \\ b \end{bmatrix}(0)\, E(s, p)}. \tag{2–34}$$

There are so far no similar explicit formulas for $r > 1$, but the Cauchy kernels themselves seem to provide the basic building blocks for the theory. For instance, Fay's trisecant identity—the fundamental identity satisfied by theta functions on a compact Riemann surface—was generalized in [Ball and Vinnikov] to vector bundles of higher rank in terms of the Cauchy kernels. Using the Cauchy kernels it should be possible to generalize Theorem 2.3 and the functional models of Section 3 below to the case of a matrix valued normalized joint characteristic function.

In the usual one-dimensional case, the notion of the transfer function is very important also for nonconservative systems, especially when the state space is finite dimensional, so that the transfer function is a rational matrix function;

in particular, realization of a rational matrix function as the transfer function is a basic tool in the study of various factorization and interpolation problems; see, for example, [Bart et al. 1979; Ball et al. 1990]. It is possible to introduce the joint transfer function and (assuming the maximality of determinantal representations) the normalized joint transfer function also for nonconservative vessels; see [Vinnikov 1994; Ball and Vinnikov 1996]. In the latter reference a nonconservative analog of Theorem 2.3 for multiplicative meromorphic $r \times r$ matrix functions on a compact Riemann surface was obtained, i.e., realization as the normalized joint transfer function of a (nonconservative) vessel with a finite-dimensional state space. The proof, at least for the case of simple poles only, is linear-algebraic in the spirit of [Bart et al. 1979], constructing the vessel explicitly from the poles and the residues of the meromorphic matrix function. Realizations are then used in [Ball and Vinnikov 1996] to solve completely, for the case of simple zeroes and poles, the problem of reconstructing a multiplicative meromorphic matrix function from its zero-pole data (including directional information), i.e., to solve the "homogenous interpolation problem" of [Ball et al. 1990] on a compact Riemann surface of a higher genus.

3. Semiexpansive Functions and Functional Models

Let X be a compact real Riemann surface, let $\zeta, \tilde{\zeta} \in J(X)$ satisfy (2–26) and (2–27), and let $T(p)$ be a multiplicative function on X with multipliers corresponding to $\tilde{\zeta} - \zeta$ that is meromorphic on $X \setminus X_{\mathbb{R}}$; we call $T(p)$ a *semiexpansive*, or more specifically $(\zeta, \tilde{\zeta})$-*expansive*, function if it satisfies (2–33) and (2–32) with $\varepsilon = +1$ (respectively semicontractive or $(\zeta, \tilde{\zeta})$-contractive for $\varepsilon = -1$). Theorem 2.3 suggests that the class of semiexpansive (or semicontractive) functions should be a proper generalization to the case of a compact real Riemann surface of a higher genus of the class of expansive (or contractive) functions on the upper half-plane, or on the unit disk. We first list some basic properties.

We assume the set $X_{\mathbb{R}}$ of real points of X is nonempty; it follows that $X_{\mathbb{R}}$ is a disjoint union of $k > 0$ topological circles $X_0, \dots, X_{k-1}$. It turns out—see [Vinnikov 1993]—that the real tori comprising the solution set of (2–27) can be naturally indexed as T_ν, where $\nu = (\nu_1, \dots, \nu_{k-1}) \in \{0,1\}^{k-1}$. Now, there can be two different situations: either $X_{\mathbb{R}}$ disconnects X, necessarily into two connected components interchanged by the anti-holomorphic involution, X_+ (the "interior" with $\partial X_+ = X_{\mathbb{R}}$ relative to a chosen orientation of $X_{\mathbb{R}}$) and X_- (the "exterior" with $\partial X_- = -X_{\mathbb{R}}$)—the *dividing case*, or $X \setminus X_{\mathbb{R}}$ remains connected—the *nondividing case*. Note that in the dividing case X is simply the double of a finite bordered Riemann surface X_+.

It is easy to see that ζ and $\tilde{\zeta}$ belong to the same real torus T_ν. Assume that X is dividing and that $\zeta, \tilde{\zeta} \in T_0$. Then it turns out that $T(p)$ is $(\zeta, \tilde{\zeta})$-expansive if and only if $|T(p)| \geq 1$ for $p \in X_+$ (equivalently, by (2–33), $|T(p)| \leq 1$ for $p \in X_-$), i.e., $T(p)$ is simply an expansive multiplicative function on X_+. This follows in a standard way from the fact that

$$-\frac{1}{2\pi}\,\frac{\theta\begin{bmatrix}a\\b\end{bmatrix}(\bar{q}-p)}{i\,\theta\begin{bmatrix}a\\b\end{bmatrix}(0)\,E(\bar{q},p)}$$

is (in the variable p) the reproducing kernel for the Hardy space $H^2(L_\zeta \otimes \Delta, X_+)$ of holomorphic sections of $L_\zeta \otimes \Delta$ on X_+ with the norm $\|y\| = \int_{X_\mathbb{R}} y\bar{y}$. Note that since y is a section of $L_\zeta \otimes \Delta$, $y\bar{y}$ is locally of the form $f(t)|dt|$ where t is a local parameter on X, so the above integral makes sense; in fact, Hardy spaces of half-order differentials on a finite bordered Riemann surface, being invariantly defined without any additional choices, turn out to be more convenient to handle than more traditional Hardy spaces of functions, which require a choice of some measure on the boundary; see [Alpay and Vinnikov b].

It is possible to give an operator-theoretic criterion for the above situation, i.e., when for a vessel $\mathcal{V}$ with irreducible discriminant polynomial and maximal input and output determinantal representations, X (the compact real Riemann surface which is the desingularization of the discriminant curve) is dividing, $\zeta, \tilde{\zeta} \in T_0$ and so the normalized joint characteristic function is simply expansive (or contractive if $\varepsilon = -1$) on X_+. This happens (assuming the mapping $\Phi : H \to E$ in the vessel is surjective) *if and only if there exists a real rational function* $r(\lambda_1, \lambda_2)$ *of two variables such that the operator* $r(A_1, A_2)$ *is defined and dissipative (i.e., has nonnegative imaginary part).* The proof uses the functional model and the description (see [Vinnikov 1993]) of definite self-adjoint determinantal representations of a real plane curve; the real rational function r is defined by $z(p) = r(\lambda_1(p), \lambda_2(p))$, where $\lambda_1(p)$ and $\lambda_2(p)$ are the affine coordinate functions on the discriminant curve and $z(p)$ is a meromorphic function on X mapping X_+ onto the upper half-plane. The existence of such functions was established in [Ahlfors 1950].

In general, if X is dividing and $\zeta, \tilde{\zeta} \in T_\nu$, it turns out that the nontangential boundary value (from the left) on $X_\mathbb{R}$ of a $(\zeta, \tilde{\zeta})$-expansive function $T(p)$ (which exists almost everywhere) satisfies $|T(p)| \geq 1$ if $\nu_j = 0$ and $|T(p)| \leq 1$ if $\nu_j = 1$ for $p \in X_j$ ($j = 0, \ldots, k-1$, we set $\nu_0 = 1$). Assuming that $T(p)$ has no zeroes in X_+ and $|T(p)|$ is bounded away from zero on $X_\mathbb{R}$, it follows that multiplication by $T(p)$ defines a contraction from $H^2_\nu(L_\zeta \otimes \Delta, X_-)$ to $H^2_\nu(L_{\tilde{\zeta}} \otimes \Delta, X_-)$, where $H^2_\nu(\cdot, X_-)$ is the Hardy space of holomorphic sections of an appropriate bundle on X_- with an indefinite inner product $[y,y]_\nu = \sum_{j=0}^{k-1} (-1)^{\nu_j} \int_{X_j} y\bar{y}$. This indefinite inner product space is actually a Kreĭn space, that is, an orthogonal direct sum of a Hilbert space and an anti-Hilbert space [Alpay and Vinnikov b]; this fact is entirely nonobvious even in the simplest case when X_+ is an annulus (and X is a torus). The reproducing kernel for this space is given by the same formula in terms of theta functions as in the Hilbert space case ($\nu = 0$) above.

We discuss briefly the canonical factorization theorem for semiexpansive functions, generalizing the Riesz–Nevanlinna–Smirnov factorization for expansive functions on the upper half-plane; see [Vinnikov 1992], and for an exposition in the case of genus 1 also [Livšic et al. 1995, Chapter 12]. Every semiexpansive function on X can be factored into the Blaschke product constructed from its poles on $X \setminus X_{\mathbb{R}}$, and the singular inner function and outer function constructed from the singular and absolutely continuous parts respectively (with respect to the local Lebesgue measure on $X_{\mathbb{R}}$) of a uniquely determined finite positive Borel measure on $X_{\mathbb{R}}$; furthermore, $\tilde{\zeta}$ can be computed from ζ, the poles and the measure. When X is dividing and $\zeta \in T_0$, so that we have an expansive function on X_+, the measure on $X_{\mathbb{R}}$ is arbitrary and the poles p_i are arbitrary points in X_+ satisfying an appropriate Blaschke condition ($\sum_{i=1}^{\infty}(\bar{p}_i - p_i)$ converges in $J(X)$); this is a fairly straightforward generalization of the usual canonical factorization to multiply connected domains and finite bordered Riemann surfaces; see [Voichick and Zalcman 1965; Hasumi 1966]. In general, the poles and the measure must satisfy certain explicit admissibility conditions with respect to the given ζ having to do with possible zeroes of the theta function; for example, we must have $\theta\bigl(\zeta + \sum_{i=1}^{\infty}(\bar{p}_i - p_i)\bigr) \neq 0$. The canonical factorization of the normalized joint characteristic function corresponds to the construction of the triangular model for the pair of operators A_1, A_2 in the vessel; and the admissibility conditions for the poles and the measure are precisely the solvability conditions for the systems of nonlinear difference and differential equations for self-adjoint determinantal representations that arise in the construction of the triangular model and were mentionned in Section 1.

We now turn to the description of functional models for commutative two-operator vessels with irreducible discriminant polynomial (and maximal input and output determinantal representations); see [Alpay and Vinnikov 1994; a]. For the single-operator case see [Sz.-Nagy and Foiaş 1967; de Branges and Rovnyak 1966a; 1966b], as well as the more recent surveys [Nikolskii and Vasyunin 1986; Ball and Cohen 1991]. We shall describe an analog of the de Branges–Rovnyak functional model; at least in the case when X is dividing and $\zeta, \tilde{\zeta} \in T_0$, an analog of Sz.-Nagy–Foiaş functional model can be constructed as well. It would be very interesting to further investigate, in this case, the geometry of the minimal joint unitary dilation of the corresponding commuting continuous semigroups of contractions, and also to find an analogue of the "coordinate free" functional model of Nikolskii and Vasyunin [1986; 1998].

For a given $(\zeta, \tilde{\zeta})$-expansive function $T(p)$ on X we let the corresponding model space $H(T)$ be the reproducing kernel Hilbert space with reproducing kernel

$$K_T(p,q) = \frac{\theta\begin{bmatrix}\tilde{a}\\ \tilde{b}\end{bmatrix}(\bar{q}-p)}{i\,\theta\begin{bmatrix}\tilde{a}\\ \tilde{b}\end{bmatrix}(0)\,E(\bar{q},p)} - T(p)\overline{T(q)}\,\frac{\theta\begin{bmatrix}a\\ b\end{bmatrix}(\bar{q}-p)}{i\,\theta\begin{bmatrix}a\\ b\end{bmatrix}(0)\,E(\bar{q},p)}. \tag{3–1}$$

By the definition of semiexpansive functions $K_T(p,q)$ is a positive definite kernel (here, as in (2–32), the characteristics $a, b \in \mathbb{R}^g$ and $\tilde{a}, \tilde{b} \in \mathbb{R}^g$ correspond to ζ and $\tilde{\zeta}$ in $J(X)$ respectively); therefore $H(T)$ exists and its elements are sections of $L_{\tilde{\zeta}} \otimes \Delta$ holomorphic on the domain of analyticity of $T(p)$. Assume that X is dividing and $\zeta, \tilde{\zeta} \in T_0$. If in addition we assume that $T(p)$ is $(\zeta, \tilde{\zeta})$-inner, i.e., the nontangential boundary values satisfy $|T(p)| = 1$ on $X_{\mathbb{R}}$ almost everywhere, then it follows in a standard way that

$$H(T) = H^2(L_{\tilde{\zeta}} \otimes \Delta, X_-) \ominus T H^2(L_\zeta \otimes \Delta, X_-). \tag{3–2}$$

(The equality sign here is a bit sloppy since the elements of the right-hand side space are defined only on X_-, while the elements of $H(T)$ are defined on X_+ as well, except for the poles of $T(p)$; we mean a natural isomorphism given by the restriction of an element of $H(T)$ to X_-.) If we don't assume that $T(p)$ is $(\zeta, \tilde{\zeta})$-inner, then $H(T)$ is the generalized orthogonal complement, in the sense of de Branges, of $TH^2(L_\zeta \otimes \Delta, X_-)$ in $H^2(L_{\tilde{\zeta}} \otimes \Delta, X_-)$. If X is dividing and $\zeta, \tilde{\zeta} \in T_\nu$ we have similar formulas using the Kreĭn spaces $H^2_\nu(L_\zeta \otimes \Delta, X_-)$ and $H^2_\nu(L_{\tilde{\zeta}} \otimes \Delta, X_-)$ instead (assuming that $T(p)$ has no zeroes in X_+ and $|T(p)|$ is bounded away from zero on $X_{\mathbb{R}}$).

We proceed to define the model operators. Let $z(p)$ be a meromorphic function on X whose poles are contained in the domain of analyticity of $T(p)$. For any section y of $L_{\tilde{\zeta}} \otimes \Delta$ which is holomorphic in a neighborhood of the poles of $z(p)$ we define

$$(M^z y)(p) = z(p) y(p) - \sum_{i=1}^n c_i y(p_i) \, \frac{\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(p_i - p)}{\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(0) \, E(p_i, p)}. \tag{3–3}$$

Here n is the degree of the meromorphic function $z(p)$, $p_1, \dots, p_n$ are the poles of $z(p)$—assumed to be all simple for the ease of notation, and c_i is the residue of $z(p)$ at p_i (in terms of some local parameter—since the other two factors in each term in the sum on the right-hand side of (3–3) are half-order differentials, the product is well-defined independently of the choice of local parameter). It follows that $M^z y$ is again holomorphic in a neighborhood of $p_1, \dots, p_n$. Actually, M^z is a bounded linear operator on $H(T)$, and for two meromorphic functions $z(p)$ and $w(p)$ the operators M^z and M^w commute. It is worthwhile to write down the resolvent $R^z_\alpha = (M^z - \alpha I)^{-1}$ of the operator M^z:

$$(R^z_\alpha y)(p) = \frac{y(p)}{z(p) - \alpha} - \sum_{i=1}^n \frac{1}{dz(p_i(\alpha))} \, y(p_i(\alpha)) \, \frac{\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(p_i(\alpha) - p)}{\theta\begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix}(0) \, E(p_i(\alpha), p)}, \tag{3–4}$$

where n is as before the degree of $z(p)$ and $p_1(\alpha), \dots, p_n(\alpha)$ are the points on X with $z(p) = \alpha$ (assumed to be all distinct, and to lie in the domain of analyticity of $T(p)$). Note that this is a natural generalization of the usual difference quotients transformation to a compact Riemann surface represented as a (ramified) covering of the Riemann sphere by means of the meromorphic function $z(p)$.

We now pick a pair of real meromorphic functions $\lambda_1(p)$, $\lambda_2(p)$ on X (i.e., meromorphic functions satisfying $\overline{\lambda_k(\bar{p})} = \lambda_k(p)$ for $k = 1, 2$) that generate the whole field of meromorphic functions on X. By standard results in the theory of compact Riemann surfaces, $\lambda_1(p)$ and $\lambda_2(p)$ satisfy an irreducible polynomial equation $f(\lambda_1(p), \lambda_2(p)) = 0$ of some degree m (with real coefficients) and X is the desingularization of the real irreducible projective plane curve C with the irreducible affine equation $f(\lambda_1, \lambda_2) = 0$. We assume that $T(p)$ is holomorphic and invertible at the poles of λ_1 and λ_2 on X, which are the points of C at infinity. Then M^{λ_1} and M^{λ_2} are commuting bounded linear operators on $H(T)$. Furthermore:

THEOREM 3.1. *$\mathcal{V}_T = (M^{\lambda_1}, M^{\lambda_2}, H(T), \Phi, \mathbb{C}^m, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$ is a commutative two-operator vessel with discriminant polynomial $f(\lambda_1, \lambda_2)$ and normalized joint characteristic function $T(p)$. Here $\Phi : H(T) \to \mathbb{C}^m$ is the evaluation at the poles of λ_1 and λ_2 (assuming all the poles to be simple—for a pole of order h we have to evaluate the derivatives up to order $h-1$, with respect to some local parameter, as well), and $\sigma_1, \sigma_2, \gamma, \tilde{\gamma}$ are given by certain explicit formulas in terms of theta functions with characteristics a, b and $\tilde{a}, \tilde{b}$ corresponding to ζ and $\tilde{\zeta}$ in $J(X)$ respectively, so that $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \gamma$ and $\lambda_1\sigma_2 - \lambda_2\sigma_1 + \tilde{\gamma}$ are maximal determinantal representations of $f(\lambda_1, \lambda_2)$ corresponding to ζ and $\tilde{\zeta}$ (and having sign $+1$).*

It can be shown that the vessel $\mathcal{V}_T$ is irreducible (or minimal), i.e., the principal subspace coincides with all of the inner space $H(T)$.

We call $\mathcal{V}_T$ the *model vessel* corresponding to the semiexpansive function $T(p)$. To justify this name we have to show that any commutative two-operator vessel $\mathcal{V} = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma, \tilde{\gamma})$ with discriminant polynomial $f(\lambda_1, \lambda_2)$ and maximal input and output determinantal representations corresponding to ζ and $\tilde{\zeta}$ in $J(X)$ (and having sign $+1$, say) is unitarily equivalent, on its principal subspace, to the model vessel corresponding to its normalized joint characteristic function (up to an automorphism of the external space E). The mapping from the inner space H of the given vessel $\mathcal{V}$ to the model space $H(T)$ is given explicitly by

$$h \mapsto \frac{\xi_1 d\lambda_1(p) + \xi_2 d\lambda_2(p)}{\omega(p)} P(\xi_1, \xi_2, p)\, \Phi\big(\xi_1 A_1 + \xi_2 A_2 - \xi_1\lambda_1(p) - \xi_2\lambda_2(p)\big)^{-1} h. \tag{3–5}$$

Here $h \in H$, $p \in X$, and ξ_1, ξ_2 are free parameters — the right-hand side of (3–5) turns out to be independent of ξ_1, ξ_2. $P(\xi_1, \xi_2, p)$ is the projection of E onto the fibre $\tilde{\mathcal{E}}(p)$ of the output bundle at p "in the direction" ξ_1, ξ_2, appearing in the restoration formula (2–18), and $\omega(p)$ is a meromorphic differential with zeroes of order $m-3$ at the points of C at infinity and poles on the divisor of singularities D as in (2–28). The right-hand side of (3–5) is a section of $\tilde{\mathcal{E}}$ with poles of order $m-2$ at the points of C at infinity and vanishing on D, and the isomorphism (2–28) gives us a section of $L_{\tilde{\zeta}} \otimes \Delta$ holomorphic outside the joint spectrum of A_1, A_2. We may arrive at the functional model by starting with the mapping (3–5) (restricted to the principal subspace), imposing the range norm on the image and then verifying that we actually obtain the reproducing kernel Hilbert space with reproducing kernel $K_T(p,q)$ of (3–1). The mapping (3–5) has a system-theoretic significance, since it is (at least in the stable dissipative case) the output of the two-dimensional system (1–20)–(1–22) with identically zero input and initial state $f(0,0) = h$ after taking a suitably defined "Laplace transform along the discriminant curve".

We make two final remarks on functional models. First, it may be checked that *the mapping* $z \mapsto M^z$ *defines a homomorphism from the algebra of meromorphic functions* $z(p)$ *on* X*, whose poles are contained in the domain of analyticity of* $T(p)$*, to the algebra of bounded linear operators on* $H(T)$. Thus when we construct the functional model for a given vessel $\mathcal{V}$, we obtain model operators not only for A_1 and A_2, but for all the operators in the algebra of rational functions in A_1, A_2.

Second, it is possible to characterize the spaces of the form $H(T)$ for a $(\zeta, \tilde{\zeta})$-expansive T as *reproducing kernel Hilbert spaces whose elements are meromorphic sections of* $L_{\tilde{\zeta}} \otimes \Delta$ *on* $X \setminus X_{\mathbb{R}}$*, which are invariant under a pair of operators of the form* $R_{\alpha_1}^{\lambda_1}$ *and* $R_{\alpha_2}^{\lambda_2}$ *as in* (3–4) *and such that a certain identity, generalizing de Branges identity for difference quotients, holds.* (Here $\tilde{\zeta}$ is fixed, while ζ may be arbitrary.) In particular, this yields a generalization of Beurling's Theorem on invariant subspaces of H^2 on the unit disk to multiply connected domains and finite bordered Riemann surfaces, proved by Sarason [1965], Voichick [1964] and Hasumi [1966].

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Victor Vinnikov
Department of Theoretical Mathematics
Weizmann Institute of Science
Rehovot 76100
Israel
vinnikov@wisdom.weizmann.ac.il