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Some Function-Theoretic Issues in Feedback Stabilisation

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ABSTRACT. This article aims to present for a mathematical audience some interesting function theory elaborated over recent years by control engineers in connection with the problem of robustly stabilising an imperfectly modelled physical device.

The theory of feedback extends over a broad field, from rarefied differential geometry to down-to-earth nuts-and-bolts engineering. Function theory enters into the study of a class of problems of great practical importance, those relating to linear time-invariant systems. It is still the case that the great majority of engineering devices are modelled by such systems, and function theory remains an important strand in recent engineering researches on the stabilisation of uncertain systems. The connection with H^∞ and Hankel operators is widely known by now, but the extent to which engineers have developed the mathematical theory along novel lines deserves publicity. Challenges of an engineering nature have given rise to some beautiful ideas and results in function theory, and the purpose of this expository article is to present some notions arising from studies of robust stabilisation which deserve the attention of mathematicians. These notions relate to certain spaces of functions on the real line or subsets of the complex plane and to sundry metrics on these spaces which measure closeness of functions from the point of view of stabilisability. Many engineers have contributed to these developments, notably M. Vidyasagar, T. T. Georgiou, M. C. Smith, K. Glover, D. C. McFarlane, and G. Vinnicombe. Virtually everything in this paper is from [Vidyasagar 1984; Georgiou and Smith 1990; 1993; McFarlane and Glover 1990; Vinnicombe 1993; Curtain and Zwart 1995]. A useful textbook covering the elements of H^∞-control is [Doyle et al. 1992]. However, these sources assume

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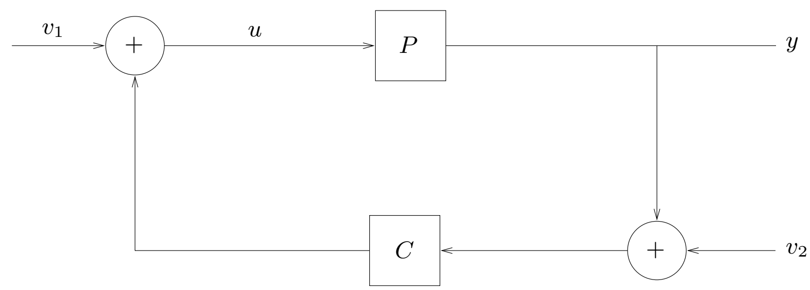
knowledge of control theory, and my hope here is to provide function-theorists with a way into some fine and relevant mathematics in places they would not normally look. I am grateful to Keith Glover and Malcolm Smith for helpful comments.

1. Robust Stabilisation

Consider this standard feedback configuration:



Here, as usual, P is the *plant*—that is, a function representing the object to be controlled—and C is the controller. We work in the frequency domain, so u and y represent the Laplace transforms of input and output signals to the plant. Subject to zero initial conditions, u and y are related by $y = Pu$, where u, y, and P are functions of the frequency variable s. We describe P as the *transfer function* of the system. The signal v_2 can be thought of as noise in the sensors that measure y. If the plant is modelled by a system of linear constant-coefficient differential equations then P will be a rational function, but if the model contains delays then P will have some exponential terms. Although much of the engineering literature concentrates on rational P there is a substantial body of work addressing the question of the most appropriate spaces of functions for the analysis of stablisation of general (not necessarily rational) plants. We shall return to this issue below, but to begin with let us think of rational plants P, and let us suppose that P is unstable, i.e., has at least one pole in the closed right half plane $\{s \in \mathbb{C} : \operatorname{Re} s \geq 0\}$. We wish to stabilise P with a rational controller C. In the feedback loop of Figure 1 we have

$$y = Pu = P(v_1 + Cy + Cv_2)$$

and so

$$y = (1 - PC)^{-1}Pv_1 + (1 - PC)^{-1}PCv_2$$

Thus for the loop to be stable we need the rational functions $(1 - PC)^{-1}P$ and $(1 - PC)^{-1}PC$ to be analytic in the closed right half-plane. For true peace of mind, though, we need somewhat more, since if there is instability in any of the connections of the loop then that connection will be liable to burn out.

Note that

$$u = v_1 + Cy + Cv_2 = v_1 + CPu + Cv_2$$

and so

$$u = (1 - CP)^{-1}v_1 + (1 - CP)^{-1}Cv_2.$$

Thus

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} (1-PC)^{-1}P & (1-PC)^{-1}PC \\ (1-CP)^{-1} & (1-CP)^{-1}C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} P \\ 1 \end{bmatrix} (1-CP)^{-1} \begin{bmatrix} 1 & C \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

We shall say that the system (P, C) is *stable* or that C *stabilises* P if

$$H(P,C) \stackrel{\text{def}}{=} \begin{bmatrix} P \\ 1 \end{bmatrix} (1-CP)^{-1} \begin{bmatrix} 1 & C \end{bmatrix} \tag{1–1}$$

is bounded and analytic in the closed right half-plane (this property is sometimes also called *internal stability*). The simplest form of the *stabilisation problem* is:

Given a plant P, find a controller C such that (P, C) is stable.

Classical control is full of recipes for constructing such Cs and simultaneously achieving various desirable performance characteristics [Rohrs et al. 1993]. This simple formulation, however, leaves out of account one of the most important aspects of control system design: the fact that models of plants are only approximate. Control engineers have long had ways of addressing this difficulty, but the development of a theory that meets it head-on is relatively recent. "H^∞ control" began around 1980 and is based on the notion that instead of finding C to stabilise a single plant P one should be looking for a C that stabilises not only the chosen model P but also all sufficiently close plants. The idea is natural: one constructs a model P_0 of a physical device by making several idealisations, simplifications, approximations and estimates. This P_0 is called the "nominal plant". One posits that there is a "true plant" P that is close to P_0, and it is P that one really wants to stabilise. Since P is unknown one should try to find a C that simultaneously stabilises as large a neighbourhood of P_0 as possible. To put this approach into effect for any given application one must decide

(1) What is the appropriate space of functions P?

(2) What is the appropriate notion of closeness?

The most natural interpretation of the second question is that one should give a metric on the chosen space of transfer functions; then one could try to stabilise the ball of greatest possible radius about P_0. However a weaker version is also of interest: what is the appropriate topology on the space of transfer functions [Vidyasagar 1984; Zames and El-Sakkary 1980]? Answering these questions requires significant mathematical as well as engineering considerations.

Corresponding to any answer to questions (1) and (2), then, we have a version of the *robust stabilisation problem*:

Given a nominal plant P_0 find the largest possible neighbourhood U of P_0 such that there is a controller C that stabilises every element of U.

In the case that closeness of plants is measurable by a metric then it is natural to seek the neighbourhood U that is the open ball of greatest possible radius.

The simplest version of the theory arises from choosing the space of rational functions without purely imaginary poles and the metric to be the L^∞ norm on the imaginary axis. This norm is physically well motivated, since it is roughly speaking the square root of the maximum ratio of the energies of output to input signals. Thus two plants are close in the L^∞ norm if, for the same input of unit energy they produce uniformly close outputs. This form of the robust stabilisation problem has an elegant solution in which an important ingredient is some classical analysis: Nevanlinna-Pick theory or Nehari's theorem. An account can be found in [Francis 1987, Chapter 6], and a simplified one in [Young 1988, Chapter 14]. The solution has, however, a slightly undesirable feature. Suppose we are given a strictly proper rational plant P_0 (a rational function is *strictly proper* if it vanishes at infinity). For $\varepsilon > 0$ we denote by $V(P_0, \varepsilon)$ the set of strictly proper rational plants P, analytic on the imaginary axis, such that $\|P - P_0\|_\infty < \varepsilon$ *and P has the same number of poles as P_0 in the right half-plane*. The theory tells us that the largest value of ε for which all members of $V(P_0, \varepsilon)$ can be simultaneously stabilised by a single controller is the reciprocal of the norm of a certain Hankel operator. This is an elegant robust stabilisation result, but it is not precisely the answer to the problem as posed. The restriction on the number of right-half-plane poles of the perturbed plant P is a requirement of the method of solution rather than a natural engineering assumption: the nominal plant P_0 and the nearby "true plant" P may perfectly easily have different numbers of poles in the right half-plane. Indeed, there is no reason to rule out plants with poles *on* the imaginary axis. These considerations led engineers to seek alternative approaches even within the framework of rational functions. One that has been particularly successful is a different representation of rational functions that allows us to use the L^∞ norm even in the presence of poles on the imaginary axis.

2. Graphs and Metrics

Stable plants have transfer functions that are bounded and analytic in the closed right half-plane, and so the H^∞ norm immediately gives a natural metric on the set of stable plants. The analysis of robust stabilization demands a metric for *unstable* plants, and it is less clear how that should be defined. An approach that has been successful is to focus not so much on the concrete rational transfer function but rather on the operator from inputs to outputs, and more particularly on its graph. One could after all argue that this is a more fundamental entity than the transfer function. This approach leads to a single natural topology on rational plants, the *graph topology*, which seems to have gained acceptance among engineers as the appropriate one for the robust stabilisation problem. However, there are several different metrics that give rise to this topology, or what is the same thing, several inequivalent ways of representing modelling uncertainty. Each corresponds to a version of the robust stabilisation problem; some have elegant solutions, some are as yet unsolved. The operator-theoretic view also has the merit of giving a lead towards generalisation of the theory to non-rational and even nonlinear plants. I particularly recommend [Georgiou and Smith 1993] as enjoyable reading for operator-theorists, since the paper gives a concise and elegant account of the basic notions of stabilisation from a congenial perspective.

For control applications one has to study plants with several inputs and outputs, so that transfer functions are matrix-valued. This makes problems significantly more complex and more interesting, but for the present purpose it is enough to consider plants with a single input and output, so that transfer functions will be scalar-valued. Consider a rational plant P. It determines a possibly unbounded closed linear operator M_P on the Hardy space H^2 of the right half-plane: $M_P u = Pu$ for $u \in H^2$ such that $Pu \in H^2$. Note that the domain of M_P,

$$\mathcal{D}(M_P) \stackrel{\text{def}}{=} \{u \in H^2 : Pu \in H^2\},$$

may be the whole of H^2 (if $P \in H^\infty$), a proper dense subspace (e.g., $P(s) = s^{-1}$), a proper closed subspace (e.g., $P(s) = (s-1)^{-1}$), or a subspace that is neither closed nor dense (e.g., $P(s) = s^{-1}(s-1)^{-1}$). Define the *graph of* P, denoted by $\mathcal{G}_P$, to be the graph of M_P:

$$\mathcal{G}_P = \left\{ \begin{bmatrix} u \\ Pu \end{bmatrix} : u \in \mathcal{D}(M_P) \right\}.$$

$\mathcal{G}_P$ is a closed subspace of $H^2 \oplus H^2$, and moreover it is invariant under multiplication by e^{-as} for all $a > 0$. By the Lax-Beurling theorem there is an inner $2 \times r$ function θ_P such that $\mathcal{G}_P = \theta_P H^2(\mathbb{C}^r)$ for some $r \in \mathbb{N}$. Obviously $r \leq 2$, and in fact we must have $r = 1$. For suppose $\theta_P = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ with M, N of type $1 \times r$ and let $Mx = 0$ for some $x \in H^2(\mathbb{C}^r)$. Then

$$\begin{bmatrix} 0 \\ Nx \end{bmatrix} = \begin{bmatrix} Mx \\ Nx \end{bmatrix} = \theta_P x \in \mathcal{G}_P,$$

and hence $Nx = 0$. Thus $\theta_P x = 0$, and since θ_P is inner, $x = 0$. That is, $Mx = 0$ implies $x = 0$, from which it follows that $r = 1$. Thus θ_P is a 2×1 inner function, which is called a *graph symbol* for P or a *normalised coprime factor representation* for P. The point of the latter terminology is that if $\theta_P = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ then $P = NM^{-1}$ is an expression of P as a ratio of two stable rational plants that are coprime elements of the algebra $\mathcal{S}$ of stable rational functions ($\mathcal{S} = \mathbb{C}(s) \cap H^\infty$ where $\mathbb{C}(s)$ denotes the field of rational functions in the variable s over $\mathbb{C}$), and that are normalised in the sense that $|N|^2 + |M|^2 = 1$ on the imaginary axis. Consider for example the unstable plant $P(s) = (s-1)/(s-2)$. For any choice of a in the left half plane

$$P(s) = \frac{s-1}{s-a} \Big/ \frac{s-2}{s-a}$$

is an expression of P as a ratio of coprime elements of $\mathcal{S}$. A little calculation determines the unique a for which normalisation is achieved, and in fact

$$\theta_P(s) = \frac{1}{\sqrt{2}s + \sqrt{5}} \begin{bmatrix} s-1 \\ s-2 \end{bmatrix}.$$

Of course θ_P is only unique up to multiplication by complex numbers of unit modulus.

The correspondences

$$\text{Plants } P \longleftrightarrow \text{Closed subspaces } \mathcal{G}_P \text{ of } H^2(\mathbb{C}^2) \longleftrightarrow 2 \times 1 \text{ inner functions } \theta_P$$

provide a useful conceptual and analytic framework for the study of robust stabilisation. Topologies and metrics on the family of closed subspaces of a Hilbert space induce the corresponding objects on the set of plants and the inner functions θ_P provide a tool for detailed analysis and computation.

The simplest metric on the set of closed subspaces of a Hilbert space is the *gap metric*. This is the metric induced by the operator norm via the identification of a closed subspace with the corresponding orthogonal projection. That is, if Π_K denotes the orthogonal projection operator in $\mathcal{L}(H)$ with range K then the gap between closed subspaces K_1, K_2 of H is

$$\operatorname{gap}(K_1, K_2) = \|\Pi_{K_1} - \Pi_{K_2}\|$$

Accordingly we may define a metric on rational plants by

$$\operatorname{gap}(P_1, P_2) = \operatorname{gap}(\mathcal{G}_{P_1}, \mathcal{G}_{P_2})$$

This metric was introduced in the present context in [Zames and El-Sakkary 1980] and argued to be an appropriate measure of closeness. Computing the gap is a (generalised) Nevanlinna-Pick interpolation problem [Georgiou 1988]:

$$\operatorname{gap}(P_1, P_2) = \max\{\delta_1(P_1, P_2),\, \delta_1(P_2, P_1)\}$$

where

$$\delta_1(P_1, P_2) = \inf_{Q \in H^\infty} \|\theta_{P_1} - \theta_{P_2} Q\|_{H^\infty}.$$

A variant is the *graph metric*

$$\operatorname{graph}(P_1, P_2) = \max\{\delta_2(P_1, P_2), \delta_2(P_2, P_1)\}$$

where

$$\delta_2(P_1, P_2) = \inf_{\|Q\|_{H^\infty} \le 1} \|\theta_{P_1} - \theta_{P_2} Q\|_{H^\infty}.$$

This metric was proposed by Vidyasagar [1984], who gives examples and motivation. The graph and gap metrics generate the same topology on the space of rational functions. Vidyasagar calls this the *graph topology*, and advances convincing arguments for the thesis that it is the appropriate one for robust stabilisation. One is that the graph topology is the weak topology generated by the functions $H(\cdot, C)$ on $\mathbb{C}(s)$. That is, for any $P_0 \in \mathbb{C}(s)$, a neighbourhood sub-base of P_0 in the graph topology is furnished by the sets of the form

$$\left\{P \in \mathbb{C}(s) : H(P,C) \in \mathcal{S} \text{ and } \left\|H(P,C) - H(P_0,C)\right\|_{H^\infty} < \varepsilon\right\}$$

where ε ranges over the positive reals and C ranges over the rational functions that stabilise P_0 (here the notation $H(P,C) \in \mathcal{S}$ means that each entry of the 2×2 matrix function $H(P,C)$ belongs to $\mathcal{S}$). One could express this characterisation by saying that the graph topology is the coarsest topology for which stabilisation by any controller is a robust property. The control community has evidently accepted that the graph topology is indeed the relevant one for discussion of the robust-stabilisation problem, but there are several rival metrics that induce this topology and it is perhaps not yet finally resolved which of them is best suited to engineering applications. There is a very strong candidate, due to G. Vinnicombe [1993], which is comparatively easy to compute and admits some sharp robustness results; we shall discuss this metric in the next section. By way of preparation we need two further notions. The first is that of the L^2*-gap metric* δ_{L^2}. This is just like the gap metric, except that we identify a plant P with the operator it induces on L^2 (of the imaginary axis) rather than H^2. More precisely, we define the possibly unbounded operator L_P on L^2 by

$$L_P u = Pu \quad \text{for } u \in L^2 \text{ such that } Pu \in L^2,$$

and we define $\delta_{L^2}(P_1, P_2)$ to be the gap between the closed subspaces of $L^2 \oplus L^2$ that are the graphs of L_{P_1} and L_{P_2}. The L^2-gap metric is much easier to compute than the gap metric—in fact, we have [Georgiou and Smith 1990]

$$\delta_{L^2}(P_1, P_2) = \left\|(1 + P_2 P_2^*)^{-1/2}(P_2 - P_1)(1 + P_1^* P_1)^{-1/2}\right\|_{L^\infty}. \tag{2–1}$$

To see this note that if a rational plant P has graph symbol $\theta \stackrel{\text{def}}{=} \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right]$ then $P = NM^{-1}$ and the L^2-graph of P is θL^2. Since θ is inner, the orthogonal projection on θL^2 is the multiplication operator $L_{\theta\theta^*}$ on L^2, and the projection on its orthogonal complement is $L_{1-\theta\theta^*}$, which equals $L_{\bar{\theta}_c \theta_c^T}$ where θ_c is the "complementary inner function",

$$\theta_c = \begin{bmatrix} -N \\ M \end{bmatrix},$$

so that $[\,\theta \quad \bar{\theta}_c\,]$ is unitary-valued on the imaginary axis. The *directed gap*

$$\vec{\delta}(\theta_1 L^2, \theta_2 L^2)$$

between the closed subspaces $\theta_1 L^2$ and $\theta_2 L^2$ of L^2 is defined to be the norm of the orthogonal projection from $\theta_1 L^2$ to the orthogonal complement of $\theta_2 L^2$.

Hence, if θ_j is the symbol of P_j,

$$\vec{\delta}(\theta_1 L^2, \theta_2 L^2) = \|L_{\bar{\theta}_{2c}\theta_{2c}^T} L_{\theta_1\theta_1^*}\| = \|L_{\bar{\theta}_{2c}\theta_{2c}^T\theta_1\theta_1^*}\| = \|\bar{\theta}_{2c}\theta_{2c}^T\theta_1\theta_1^*\|_\infty = \|\theta_{2c}^T\theta_1\|_\infty,$$

the last equation because θ_1 and $\bar{\theta}_{2c}$ are isometries. Now

$$\theta_{2c}^T\theta_1 = [\,-N_2 \quad M_2\,]\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} = M_2N_1 - N_2M_1 = M_2(P_1 - P_2)M_1,$$

so that

$$\vec{\delta}(\theta_1 L^2, \theta_2 L^2) = \big\|M_2(P_1 - P_2)M_1\big\|_\infty.$$

On the imaginary axis we have

$$1 = M^*M + N^*N = M^*(1 + P^*P)M$$

and hence

$$|M| = (1 + P^*P)^{-1/2}.$$

Thus

$$\vec{\delta}(\theta_1 L^2, \theta_2 L^2) = \big\|(1 + P_2^*P_2)^{-1/2}(P_1 - P_2)(1 + P_1^*P_1)^{-1/2}\big\|_\infty.$$

The gap between $\theta_1 L^2$ and $\theta_2 L^2$ is the maximum of the two directed gaps

$$\vec{\delta}(\theta_1 L^2, \theta_2 L^2) \quad \text{and} \quad \vec{\delta}(\theta_2 L^2, \theta_1 L^2),$$

and the formula (2–1) for $\delta_{L^2}(P_1, P_2)$ follows. A modification of this proof works for matrix-valued functions.

However, δ_{L^2} does *not* induce the graph topology—stabilisation is not a robust property with respect to this metric. Indeed, if P is a stable plant then it is stabilised by the controller $C = 0$, and yet every δ_{L^2}-neighbourhood of P contains unstable plants. It would appear that δ_{L^2} is no use for the study of feedback systems, which is a pity given that it is so manageable. Vinnicombe's bright idea was to rescue this metric by introducing a winding number constraint.

An important notion for the study of robustness of stabilisation is the *stability margin*. If a controller C stabilises a plant P it may do so with more or less to spare, and various quantitative measures of this notion are in use. A natural one is the stability margin $b_{P,C}$:

$$b_{P,C} = \begin{cases} \big\|H(P,C)\big\|_{H^\infty}^{-1} & \text{if } (P,C) \text{ is stable,} \\ 0 & \text{otherwise.} \end{cases}$$

That this quantity truly deserves the name of stability margin is shown by the following fact [Georgiou and Smith 1990].

THEOREM 2.1. *Let C be a controller that stabilises a plant P and let $\beta > 0$. Then C stabilises the closed ball of radius β about C (with respect to the gap metric) if and only if $b_{P,C} > \beta$.*

Thus robust stabilisation of P in the gap metric is equivalent to finding a stabilising controller C for which $b_{P,C}$ is large, or in other words, $\|H(P,C)\|_{H^\infty}$ is small. This turns out to be a Nehari problem. There are numerous further striking results about this range of questions, of which we mention only two.

- For sufficiently small radii, the gap-metric ball of radius β centred at P coincides with the set of functions NM^{-1} with $\left\|\theta_P - \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]\right\|_{H^\infty} < \beta$. Thus closeness in the gap metric is equivalent to smallness of perturbations of the "numerator" and "denominator" in θ_P [Georgiou and Smith 1990, Theorem 4].
- The maximum stability margin $\sup_C b_{P,C}$ is equal to $\left(1 - \|H_{\theta_P^*}\|^2\right)^{1/2}$, where $H_{\theta_P^*}$ is the Hankel operator with symbol θ_P^* (see [McFarlane and Glover 1990, Theorem 4.3], or [Georgiou and Smith 1990, Theorem 2]).

It is worth mentioning that $b_{P,C}$ has an interpretation in terms of the geometry of $H^2 \oplus H^2$. For a stable system (P,C) let $T_{P,C} \in \mathcal{L}(H^2 \oplus H^2)$ be the operation of multiplication by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} H(P,C) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ P \end{bmatrix} (1 - CP)^{-1} \left[1 \quad -C \right].$$

It is immediate that $T_{P,C}$ is idempotent. Its range is $\mathcal{G}_P$ and its kernel is

$$\tilde{\mathcal{G}}_C \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} Cv \\ v \end{bmatrix} : v,\, Cv \in H^2 \right\}.$$

That is, $T_{P,C}$ is the projection on $\mathcal{G}_P$ along $\tilde{\mathcal{G}}_C$. Clearly $b_{P,C} = \|T_{P,C}\|^{-1}$.

3. Duality and Vinnicombe's Metric

Theorem 2.1 suggests a notion of duality between stability margins and metrics on plants. Let us say that a metric δ is *dual* to the stability margin $b_{P,C}$ when the following holds:

C stabilises the closed δ–ball of radius $\beta > 0$ about P if and only if $b_{P,C} > \beta$.

Another way of expressing this notion is to say that δ is dual to $b_{P,C}$ if, for any stable pair (P,C), the supremum of the radii of the δ-balls about P that are stabilised by C is $b_{P,C}$.

One might incline to think that Theorem 2.1 would be the last word on metrics dual to $b_{P,C}$ and that one needs to look no further than the gap metric. However, Vinnicombe argues that we can do better in two ways by the use of his metric δ_V, which is also dual to $b_{P,C}$. Firstly δ_V is simpler to calculate *and* to analyse than the gap metric. Secondly, δ_V is *smaller* than the gap metric, so that it enables us to establish a larger "uncertainty ball" of plants about a nominal plant P_0 that are all stabilised by a single controller. Indeed, δ_V is the best possible metric in this sense: it is the smallest metric that is dual to $b_{P,C}$. It enjoys a property converse to that of duality to $b_{P,C}$. For small enough $\beta > 0$ the closed δ_V ball of radius β about P is the largest set of linear time–invariant plants that can be guaranteed to be stabilised by a controller C if all we know about C is that $b_{P,C} > \beta$.

For a rational function f without zeros on the imaginary axis $i\mathbb{R}$ we define the *winding number of f on the imaginary axis* (denoted by $\operatorname{wno}(f)$) to be the winding number of f about 0 along the anticlockwise-oriented contour

$$\{Re^{i\theta} : -\pi/2 \le \theta \le \pi/2\} \cup \{iy : R \ge y \ge -R\}$$

indented round any pole of f on $i\mathbb{R}$, with R chosen so that all zeros and poles of f in the closed right half-plane lie inside the contour.

DEFINITION 3.1. For rational plants P_1 and P_2, we have

$$\delta_V(P_1, P_2) = \begin{cases} \delta_{L^2}(P_1, P_2) & \text{if } \theta^*_{P_2}\theta_{P_1} \text{ has no zero on } i\mathbb{R} \text{ and } \operatorname{wno}(\theta^*_{P_2}\theta_{P_1}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Here θ^*_P denotes the unique rational function satisfying $\theta^*_P(iy) = \theta_P(iy)^*$ for all $y \in \mathbb{R}$ (so that $\theta^*_P(s) = \theta_P(-\bar{s})^*$).

Thus, in view of (2–1), we have

$$\delta_V(P_1, P_2) = \begin{cases} \left\| (1+P_2P_2^*)^{-1/2}(P_2-P_1)(1+P_1^*P_1)^{-1/2} \right\|_\infty & \text{if } \operatorname{wno}(\theta^*_{P_2}\theta_{P_1}) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Vinnicombe shows that δ_V is a metric—indeed, he proves the stronger assertion that $\sin^{-1}\delta_V$ is a metric. The proof of the triangle inequality requires some calculation. He also shows that δ_V induces the graph topology on $\mathbb{C}(s)$. (An exercise for the reader: show that the metric $\sin^{-1}\delta_V$ is dual to the stability margin $\sin^{-1} b_{P,C}$.) The chief virtue of δ_V is its tight duality relationship with the stability margin $b_{P,C}$, which is in fact even further–reaching than is stated above. The metric δ_V gives precise information as to how adversely a perturbation of P can affect the stability margin $b_{P,C}$, as witness the following result [Vinnicombe 1993, Theorems 4.2 and 4.5].

THEOREM 3.2. *Let (P_0, C_0) be a rational stable pair and let $0 < \beta \le \alpha < \sup_C b_{P_0,C}$. Then these two conditions are equivalent*:

(i) $b_{P_0,C_0} > \alpha$.
(ii) $b_{P,C_0} > \sin(\sin^{-1}\alpha - \sin^{-1}\beta)$ *for all P in the δ_V-ball of radius β about P_0.*

The next two conditions are also equivalent:

(iii) $b_{P,C} > \sin(\sin^{-1}\alpha - \sin^{-1}\beta)$ *for all C satisfying $b_{P_0,C} > \alpha$.*
(iv) $\delta_V(P_0, P) \le \beta$.

Hence a perturbation of P_0 of $(\sin^{-1}\delta_V)$-magnitude ε reduces the stability margin $\sin^{-1} b_{\cdot,C_0}$ by at most ε, and moreover this estimate is sharp.

The foregoing results about rational functions are so elegant that one can hardly resist the temptation to try to generalise them to non-rational functions.

But to what class of functions? Much attention has been devoted in the engineering literature to the identification of a class that is wide enough to encompass all functions of physical interest and yet enjoys the structural properties that allow analysis of the robust stabilisation problem. Not every measurable function on the imaginary axis can be stabilised: there exist P such that $H(P, C) \notin H^\infty$ for every function C. A necessary and sufficient condition that P be stabilisable is that its graph $\mathcal{G}_P$ be closed and have a left-invertible symbol [Georgiou and Smith 1993, Proposition 3]. However there are also conditions of an algebraic character that are desirable. In the rational case one often uses the fact that the ring $\mathcal{S}$ of stable rational functions has the *Bézout property*; that is, if f and g are coprime elements of $\mathcal{S}$, there exist a and $b \in \mathcal{S}$ such that $af + bg = 1$. This property fails in many natural classes of functions.

Since stabilisation is defined by the condition $H(P, C) \in H^\infty$ a plausible class of functions to analyse is the field of fractions of the integral domain H^∞. However this class does not have all the properties needed for the generalisation of the strong results obtained for rational functions. In particular, it does not have the Bézout property. It is in any case unnecessarily large. It includes functions that can hardly be held to represent any physical system, such as $e^{1/s}$. Georgiou and Smith [1992] obtained good results for plants belonging to the field of fractions of the algebra of functions analytic in the open right half-plane and continuous on its closure (in the Riemann sphere). However, they observe that an example of S. Treil shows that plants in this class do not necessarily have normalised coprime factor representations in the class. The same authors, in collaboration with C. Foiaş, have extended their geometric approach to time-varying systems [Foiaş et al. 1993]—but that ceases to be function theory.

A class that has been much studied in the present context is the *Callier–Desoer class* $\hat{\mathcal{B}}(\beta)$ for suitable $\beta \in \mathbb{R}$. A concise and accessible account of the properties of this class and the reasons for introducing it is given in [Curtain and Zwart 1995, Sec 7.1]. It is defined as follows. Denote by $\mathcal{A}(\beta)$ the space of measures μ on $[0, \infty)$ having trivial singular part with respect to Lebesgue measure and satisfying

$$\int_{[0,\infty)} e^{-\beta t} |\mu| \, dt < \infty.$$

Let $\hat{\mathcal{A}}(\beta)$ be the space of Laplace-Stieltjes transforms of elements of $\mathcal{A}(\beta)$. Let

$$\hat{\mathcal{A}}_-(\beta) = \bigcup_{\alpha < \beta} \hat{\mathcal{A}}(\alpha);$$

thus $\hat{\mathcal{A}}_-(\beta)$ is an algebra of functions analytic on the closed half-plane $\{\operatorname{Re} s \geq \beta\}$ under pointwise operations. Let $\hat{\mathcal{A}}_\infty(\beta)$ be the subset of $\hat{\mathcal{A}}_-(\beta)$ consisting of those members that are bounded away from 0 at infinity. Then $\hat{\mathcal{B}}(\beta)$ is defined to be the field of fractions of $\hat{\mathcal{A}}_-(\beta)$ by the multiplicative subset $\hat{\mathcal{A}}_\infty(\beta)$.

The class $\hat{\mathcal{B}}(\beta)$ has numerous good properties. It is a commutative algebra with identity. If $P \in \hat{\mathcal{B}}(\beta)$ then P is meromorphic in $\{\operatorname{Re} s \geq \alpha\}$ for some $\alpha < \beta$. It is not true that $\hat{\mathcal{B}}(\beta)$ has the Bézout property, because one can easily find coprime elements that both tend to zero along the same sequence of points of $\{\operatorname{Re} s \geq \beta\}$, whence the ideal they generate cannot contain 1. (Caveat: in the engineering literature two elements of an algebra are often *defined* to be coprime if and only if the ideal they generate is the whole algebra; in this article I have stuck to the usual mathematical definition, according to which two elements are coprime if and only if their only common factors are units of the algebra.)

There is, however, a good replacement for the Bézout property. Any $P \in \hat{\mathcal{B}}(\beta)$ has a factorisation $P = NM^{-1}$ where $N \in \hat{\mathcal{A}}_-(\beta)$, $M \in \hat{\mathcal{A}}_\infty(\beta)$ and the ideal of $\hat{\mathcal{B}}(\beta)$ generated by N and M is the whole of $\hat{\mathcal{B}}(\beta)$. This fact permits the extension to $\hat{\mathcal{B}}(\beta)$ of numerous techniques from the rational case.

Another approach to identifying a suitable class of functions is to start from a state space description of a system, or evolution equation, and see what functions arise as the corresponding transfer functions. Some very subtle questions arise in this way. G. Weiss [1994] has characterised the transfer functions of "regular systems", a class that probably includes all linear time-invariant state-space systems of practical interest; they are analytic in a half-plane $\{\operatorname{Re} s > \beta\}$ for some $\beta > 0$ and have a limit as $s \to \infty$ along the real axis. It will be interesting if this line of investigation leads eventually to the same conclusions as the function-theoretic viewpoint of Georgiou and Smith. So far, though, the description of the ideal holomorphic space for the analysis of robust stabilisation awaits the final word.

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