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Localization Technique on the Sphere and the Gromov–Milman Theorem on the Concentration Phenomenon on Uniformly Convex Sphere

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Abstract. We give a simpler proof of the Gromov–Milman theorem on concentration phenomenon on uniformly convex sphere. We also outline Rohlin's theory of measurable partitions used in the proof.

The purpose of this note is to present a localization technique for the sphere S^n on an example of the Gromov–Milman theorem [Gr-M] about the concentration phenomenon on uniformly convex spheres. This result was obtained in [Gr-M] in a some more general setting. Our approach follows the same general reasoning, but is simpler and more direct than the original approach. We also outline Rohlin's theory of measurable partitions, which is used in the proof. Note that the terminology of "localization" was introduced for $\mathbb{R}^n$ by L. Lovász and M. Simonovits [L-S1, L-S2]. [Gr-M] did not use such terminology and also did not put the scheme of localization explicitly.

Note. K. Ball has informed us recently that he, jointly with R. Villa, found an extremely short proof of the Gromov–Milman theorem for uniformly convex sphere as an application of the Prekopa–Leindler inequality (see, e.g., [P]).

1. Related Definitions and Formulation of the Gromov–Milman Theorem

Definition 1.1. Let us say that a finite dimensional normed space $X = (\mathbb{R}^{n+1}, \|\cdot\|)$ has modulus of convexity at least $\delta(\varepsilon) > 0$ for $\varepsilon > 0$, if for all vectors $x, y \in X$ such that $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$ we have $\|\frac{x+y}{2}\| \leq 1 - \delta(\varepsilon)$.

We may assume $\delta(\varepsilon)$ to be a monotone increasing function of positive ε. Denote by $K(X) := \{x \in X : \|x\| \leq 1\}$ the unit ball of X and by $S(X) := \{x \in X : \|x\| = 1\}$ the unit sphere of X.

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For any subset $A \subset S(X)$, let us denote $\hat{A} := \bigcup_{0\le t\le 1} t\cdot A$. Now define a probability measure $\hat{\mu}$ on $S(X)$ induced by the standard Lebesgue measure vol_{n+1} on $\mathbb{R}^{n+1}$: for any Borel subset $A \subset S(X)$, let

$$\hat{\mu}(A) := \mathrm{vol}_{n+1}(\hat{A})/\,\mathrm{vol}_{n+1} K(X).$$

We will prove the following theorem, due to Gromov and Milman [Gr-M].

Theorem 1.1. *Let $\delta(\varepsilon)$ be the modulus of convexity of the normed $(n+1)$-dimensional space X and let $\hat{\mu}$ be the probability measure on $S(X)$ as above. Then, for every Borel set $A \subset S(X)$ such that $\hat{\mu}(A) \ge \frac{1}{2}$, and every $\varepsilon > 0$,*

$$\hat{\mu}(A_\varepsilon) \ge 1 - \exp(-a(\varepsilon)n),$$

where $A_\varepsilon := \{x \in S(X) : \mathrm{dist}(x, A) \le \varepsilon\}$, $\mathrm{dist}(x, A) := \inf_{y\in A} \|x - y\|$, $a(\varepsilon) := \delta((\varepsilon/8) - \theta_n)$, where θ_n is such that $\delta(\theta_n) = 1 - (1/2)^{1/(n-1)} \approx \frac{\log 2}{n-1}$.

2. Rohlin's Theory

Following [Gr-M], we will use some results of Rohlin's theory [R]. Let (M, Ω_ν, ν) be a complete measure space, i.e. M is a set, Ω_ν is a σ-algebra of subsets of M, and ν is a complete probability measure on Ω_ν.

Let ζ be some partition of M into pairwise disjoint subsets, whose union is equal to M.

Definition 2.1. A partition ζ of M is called measurable, if there exists a countable family $\Sigma = \{S_\alpha\}_{\alpha=1}^\infty$ of measurable subsets of M such that each element $C \in \zeta$ has the form $C = \bigcap_{\alpha=1}^\infty R_\alpha$, where for all α either $R_\alpha = S_\alpha$ or $R_\alpha = \bar{S}_\alpha$, where $\bar{S}_\alpha$ denotes the complement of S_α.

Obviously, each element of a measurable partition is measurable.

Denote by H_ζ the canonical homomorphism from M onto the factor set M/ζ. Then M/ζ turns out to be a complete measure space, if we introduce a measure ν_ζ by setting a subset $X \subset M/\zeta$ to be measurable in M/ζ iff $H_\zeta^{-1}(X)$ is measurable in M and $\nu_\zeta(X) := \nu(H_\zeta^{-1}(X))$.

We will need the following theorem due to Rohlin:

Theorem 2.2 [R] *Let M be a metric separable complete space, ν be a complete Borel probability measure on M and ζ be a measurable partition of M generated by a countable family $\Sigma = \{S_\alpha\}_{\alpha=1}^\infty$ (in the sense of Definition 2.1). Then there exists a canonical family of complete Borel probability measures $\{\nu_C\}_{C\in M/\zeta}$ on M satisfying these conditions:*

(1) *For ν_ζ-a.e. element $C \in M/\zeta$, ν_C is concentrated on $C \subset M$, i.e. $\nu_C(C) = 1$ (here we denote both the element C of M/ζ and its preimage $H_\zeta^{-1}(C)$ in M by the same letter C).*
(2) *For every ν-measurable subset $A \subset M$, $\nu_C(A)$ is a ν_ζ-measurable function of $C \in M/\zeta$ and*

(3) $\nu(A) = \int_{M/\zeta} \nu_C(A \cap C)\, d\nu_\zeta(C)$.
(4) *The canonical family* $\{\nu_C\}$ *is unique, i.e. if* $\{\nu'_C\}$ *satisfies* (1)–(3), *then* $\nu_C = \nu'_C$ *for* ν_ζ*-a.e.* C.
(5) *Furthermore, the family* Σ', *which is an image of* Σ *under* H_ζ, *generates the* σ*-algebra of* ν_ζ*-measurable subsets of* M/ζ.

Corollary 2.3. *Let* M, ν, ζ *be as in Theorem* 2.2. *Let* $f \in L_1(M, \nu)$ *be an integrable function.*

Then the integral $\int_M f\, d\nu_C = \int_C f\, d\nu_C$ *is a* ν_ζ*- integrable function of* $C \in M/\zeta$ *and*

$$\int_M f\, d\nu = \int_{M/\zeta} \left(\int_C f\, d\nu_C \right) d\nu_\zeta(C).$$

Proof (standard). This corollary is obvious for the step functions. In general, we may assume $f \geq 0$.

For $k, j \in \mathbb{N} \cup \{0\}$, define

$$A_{kj} := \left\{ x \in M : \frac{j}{2^k} \leq f(x) < \min \left(\frac{j+1}{2^k}\, ,\, k \right) \right\}$$

(obviously, $A_{kj} = \varnothing$ for $j \geq k2^k$) and

$$f_k := \sum_{j=0}^{\infty} \frac{j}{2^k} \chi_{A_{kj}},$$

where $\chi_{A_{kj}}$ are characteristic functions of A_{kj}. Clearly, f_k are step functions, $0 \leq f_k(x) \leq f(x)$ for every $x \in M$, the sequence $\{f_k(x)\}_{k \in \mathbb{N}}$ is nondecreasing, and $f_k \longrightarrow f$ everywhere on M and in $L_1(M, \nu)$. For f_k we have:

$$\int_{M/\zeta} \left(\int_C f_k\, d\nu_C \right) d\nu_\zeta(C) = \int_M f_k\, d\nu \leq \int_M f\, d\nu.$$

Set $\phi_k(C) = \int_C f_k\, d\nu_C$. It is well defined for ν_ζ-a.e. $C \in M/\zeta$. Clearly, $\{\phi_k(C)\}$ is nondecreasing and $\sup_k \int_{M/\zeta} \phi_k(C)\, d\nu_\zeta(C) \leq const < \infty$.

Hence by B. Levy's theorem $\{\phi_k\}$ converges ν_ζ-a.e. and in $L_1(M/\zeta, \nu_\zeta)$ to some function $\phi(C) \in L_1(M/\zeta, \nu_\zeta)$, and $\phi_k(C) \leq \phi(C)$. Then for ν_ζ-a.e. C,

$$\int_C f\, d\nu_C = \lim_{k \to \infty} \int_C f_k\, d\nu_C = \phi(C)$$

again, by B. Levy's theorem applied to the measure ν_C.

Thus we obtain

$$\int_{M/\zeta} \left(\int_C f\, d\nu_C \right) d\nu_\zeta = \int_{M/\zeta} \phi(C)\, d\nu_\zeta = \lim_{k \to \infty} \int_{M/\zeta} \left(\int_C f_k\, d\nu_C \right) d\nu_\zeta$$
$$= \lim_{k \to \infty} \int_M f_k\, d\nu = \int_M f\, d\nu. \qquad \square$$

If the partition ζ is generated by the family $\Sigma = \{S_\alpha\}_{\alpha=1}^{\infty}$, denote by $\mathcal{F}_N$ a finite (σ -) algebra of sets generated by $\{S_\alpha\}_{\alpha=1}^{N}$, and let $\bar{\mathcal{F}}_N$ be its image in M/ζ under H_ζ. So $\bar{\mathcal{F}}_1 \subset \bar{\mathcal{F}}_2 \subset \cdots \subset \bar{\mathcal{F}}_N \subset \cdots$. Let $\bar{\mathcal{F}}_\infty$ be the minimal complete σ-algebra containing $\bigcup_{n=1}^{\infty} \bar{\mathcal{F}}_N$. By Theorem 2.2 (5), $\bar{\mathcal{F}}_\infty$ coincides with the σ-algebra of ν_ζ-measurable subsets of M/ζ.

For every element $C \in M/\zeta$ and every $N \in \mathbb{N}$, denote by $\bar{\Phi}_N(C)$ the unique minimal element of $\bar{\mathcal{F}}_N$, which contains C (clearly, $\bar{\Phi}_N(C) = H_\zeta(\bigcap_{\alpha=1}^{N} R_\alpha)$, where $R_\alpha = S_\alpha$ or $\bar{S}_\alpha$). Denote its preimage in M by $\Phi_N(C)$.

Corollary 2.4. *Let M, ν, ζ, f be as in Corollary 2.3. Then, for ν_ζ-a.e. $C \in M/\zeta$,*

$$\int_C f\, d\nu_C = \lim_{N\to\infty} \frac{1}{\nu(\Phi_N(C))} \int_{\Phi_N(C)} f\, d\nu.$$

Proof. The function $\phi(C) = \int_C f\, d\nu_C$ is $\bar{\mathcal{F}}_\infty$-measurable by Corollary 2.3. Then, by the classical P. Levy martingale convergence theorem (see, e.g., [L-Sh]),

$$\phi \overset{\nu_\zeta-a.e.}{=} \lim_{N\to\infty} \mathbb{E}\,(\phi \,|\, \bar{\mathcal{F}}_N).$$

But

$$\mathbb{E}\,(\phi \,|\, \bar{\mathcal{F}}_N)(C) = \frac{1}{\nu_\zeta(\bar{\Phi}_N(C))} \int_{\bar{\Phi}_N(C)} \phi(C_1)\, d\nu_\zeta(C_1).$$

By the definition of ν_ζ, $\nu_\zeta(\bar{\Phi}_N(C)) = \nu(\Phi_N(C))$. Using Corollary 2.3, we easily check that

$$\int_{\bar{\Phi}_N(C)} \phi(C_1)\, d\nu_\zeta(C_1) = \int_{\Phi_N(C)} f\, d\nu.$$

So $\mathbb{E}\,(\phi \,|\, \bar{\mathcal{F}}_N)(C) = \frac{1}{\nu(\Phi_N(C))} \int_{\Phi_N(C)} f\, d\nu$ and the corollary is proved. $\square$

3. Convex Restrictions of Measures

Let K be a convex bounded (not necessarily compact) subset of $\mathbb{R}^N$.

Definition 3.1. A function $\gamma : K \longrightarrow \mathbb{R}_+$ is called α-concave ($\alpha > 0$), if $\gamma^{1/\alpha}$ is concave.

Assume that $K \subset \mathbb{R}^k \subset \mathbb{R}^N$ and $\dim K = k$. Let μ be a nonnegative Borel measure on $\mathbb{R}^N$, which is absolutely continuous with respect to the standard Lebesgue measure m_N, and let $g := \frac{d\mu}{dm_N}$.

Definition 3.2. A measure ν on K is called a convex restriction of the measure μ, if there exists an $(n-k)$-concave function γ on K such that $d\nu = g\cdot\gamma\cdot dm_k$, where m_k is the Lebesgue measure on $\mathbb{R}^k$.

Remark. Our definition of the convex restriction of measures is different from that given in [Gr-M], but both definitions are equivalent.

LEMMA 3.3. *Assume that* $K_2 \subset K_1 \subset \mathbb{R}^N$ *and* $\dim K_i = k_i$. *Let a measure* ν_1 *on* K_1 *be a convex restriction of a measure* μ. *Let a measure* ν_2 *on* K_2 *be a convex restriction of* ν_1.

Then ν_2 *is a convex restriction of* μ.

PROOF. If $d\mu = g\, dm_N$, then $d\nu_1 = g\, \gamma_1\, dm_{k_1}$ and $d\nu_2 = g\, \gamma_1\, \gamma_2\, dm_{k_2}$, where γ_1 is an $(N-k_1)$-concave function on K_1, γ_2 is a (k_1-k_2)-concave on K_2. Set $\alpha = N - k_1$ and $\beta = k_1 - k_2$.

It is sufficient to show that $\gamma_1 \cdot \gamma_2$ is an $(\alpha+\beta)$-concave on K_2 [Gr-M, Appendix, Lemma 1]. Indeed, using the Hölder inequality with $p = (\alpha+\beta)/\alpha$ and $q = (\alpha+\beta)/\beta$, we obtain for every $x, y \in K_2$ and every $0 < \theta < 1$,

$$\begin{aligned}
&\theta \cdot [\gamma_1(x)\, \gamma_2(x)]^{1/(\alpha+\beta)} + (1-\theta) \cdot [\gamma_1(y)\, \gamma_2(y)]^{1/(\alpha+\beta)} \\
&\quad \le [\theta \cdot \gamma_1(x)^{1/\alpha} + (1-\theta) \cdot \gamma_1(y)^{1/\alpha}]^{\alpha/(\alpha+\beta)} \cdot [\theta \cdot \gamma_2(x)^{1/\beta} + (1-\theta) \cdot \gamma_2(y)^{1/\beta}]^{\beta/(\alpha+\beta)} \\
&\quad \le \gamma_1(\theta\, x + (1-\theta)\, y)^{1/(\alpha+\beta)} \cdot \gamma_2(\theta\, x + (1-\theta)\, y)^{1/(\alpha+\beta)}. \qquad \square
\end{aligned}$$

Later we will need the following result:

LEMMA 3.4. *Let a measure* μ *on* $\mathbb{R}^N$ *is such that* $d\mu = f \cdot dm_N$, *where* f *is continuous*, $f > 0$ m_N*-a.e., and suppose we are given a decreasing sequence of convex compact sets* $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ *of full dimension* N. *Let* $K := \bigcap_{n=1}^{\infty} K_n$, $k := \dim K$. *Define a sequence of probability measures* $\{\lambda_n\}$ *such that for any Borel subset* $A \subset \mathbb{R}^N$ $\lambda_n(A) := \frac{\mu(A \cap K_n)}{\mu(K_n)}$ *(note that our assumptions imply that* $\mu(K_n) \neq 0$*).*

Then one can choose a subsequence $\{n_l\}$ *such that* $\{\lambda_{n_l}\}$ *converges weakly to a measure concentrated on* K, *which is a convex restriction of* μ.

PROOF. Let E be the affine hull of K, and put $k = \dim E$.

Consider new convex sets

$$\tilde{K}_n := \left\{(x, y) \in E \oplus E^{\perp} : (x, \mathrm{vol}(K_n)^{1/(N-k)} \cdot y) \in K_n\right\}.$$

By the Cavalieri principle, $\mathrm{vol}(\tilde{K}_n) = 1$. Replace $\tilde{K}_n$ by its $(N-k)$-dimensional Schwarz symmetrization K'_n with respect to E. Then K'_n are also convex compact bodies, $\mathrm{vol}(K'_n) = 1$ and $K'_n \supset K$. This and their rotation invariance imply easily that K'_n are uniformly bounded. Hence by the Blaschke selection theorem one can choose a subsequence $\{n_l\}$ such that K'_{n_l} converges to some convex compact set M with respect to the Hausdorff metric. Obviously, M is also invariant with respect to rotations around E, $\mathrm{vol}(M) = 1$, $M \supset K$, and $M \cap E = K$, because $\bigcap_l K_{n_l} = K$.

Consider a function γ on K:

$$\gamma(x) = \mathrm{vol}_{N-k}\left(M \cap (x + E^{\perp})\right).$$

Then γ is $(N-k)$-concave by Brunn's theorem. We will show that for every continuous function u on $\mathbb{R}^N$

$$\frac{1}{\mu(K_{n_l})}\int_{K_{n_l}} u(x)\, f(x) dm_N(x) \longrightarrow \int_K u(x)\, f(x)\, \gamma(x)\, dm_k(x).$$

This will prove the lemma.

Denote $v := u\cdot f$ and consider a function $v'(x) := v(Pr_E\, x)$, where Pr_E is the orthogonal projection onto E. Since $v \equiv v'$ on K, for any $\varepsilon > 0$ there exists an open neighborhood U of K such that $|v - v'| < \varepsilon$ on U. But $K_n \subset U$ for large n, hence

$$\frac{1}{\mu(K_n)}\int_{K_n} (v - v')\, dm_N \longrightarrow 0,\ n \longrightarrow \infty.$$

By the Fubini theorem,

$$\frac{1}{\mu(K_{n_l})}\int_{K_{n_l}} v'(x)\, dm_N(x) = \int_{K'_{n_l}} v'(x)\, dm_N(x) \longrightarrow$$

$$\int_M v'(x) dm_N(x) = \int_K v'(x)\, \gamma(x) dm_k(x) = \int_K u(x)\, f(x)\, \gamma(x)\, dm_k(x). \quad \square$$

4. Convex Partitions

Assume that $M \subset \mathbb{R}^N$ is a convex compact body, $\dim M = N \geq 3$, $M \ni 0$. Let μ be a probability measure on M, which is absolutely continuous with respect to the Lebesgue measure m_N, $d\mu = f dm_N$, where f is continuous and $f > 0\, m_N$-a.e. on M.

Fix A_1, A_2 disjoint closed subsets of ∂M such that $\hat{A}_i := \bigcup_{0\leq t\leq 1} t\cdot A_i$, $i = 1, 2$ have nonzero measure μ. Set $\lambda := \frac{\mu(\hat{A}_1)}{\mu(\hat{A}_2)}$.

Using the idea of [Gr-M], we will construct a measurable (cf. Definition 2.1) partition ζ of the convex set M satisfying the following properties (in the notation of Section 2):

(4.1) Every element $C \in \zeta$ of this partition is a convex subset of M and has the form $C = \bigcup_{0\leq t\leq 1} t\cdot(C \cap \partial M)$.
(4.2) ν_C is the convex restriction of μ to C for ν_ζ-a.e. $C \in \zeta$.
(4.3) $\nu_C(\hat{A}_1) = \lambda\, \nu_C(\hat{A}_2)$ for ν_ζ-a.e. $C \in \zeta$.
(4.4) Moreover, if the measure μ is homogeneous of degree $\alpha > 0$, i.e. for every Borel subset $T \subset M$ and every $t \in [0, 1]$ $\mu(t\cdot T) = t^\alpha \cdot \mu(T)$, then ν_C is also homogeneous of degree α for ν_ζ-a.e. $C \in \zeta$.

The construction of such partition uses the Borsuk–Ulam theorem.

Let S^{N-1} be the Euclidean sphere in $\mathbb{R}^N$. For $x \in S^{N-1}$, denote $H_x^+ := \{y \in \mathbb{R}^N : (y, x) \geq 0\}$ the closed half-space. So $H_x^- := \mathbb{R}^N - H_x^+$ is an open half-space. Then $M^+ := M \cap H_x^+$ and $M^- := M \cap H_x^-$ are convex sets.

It will be more convenient to consider M^- as a compact set. Namely, replace M by a new set, where the hyperplane $H_x = \{y \mid \langle y, x\rangle = 0\}$ is considered as a "double" set, that is, one copy of it belongs to M^+ and another to M^- (this is similar to the situation where, if we consider the dyadic points of the unit interval as "double" points, we obtain the Cantor set). In the steps that follow, each hyperplane we construct will be considered as "double". This will not change M and its factor set by the partition constructed below, since these spaces are Lebesgue spaces in the sense of [R].

Consider a map $\phi : S^{N-1} \longrightarrow \mathbb{R}^2$ such that

$$\phi(x) = \big(\mu(\hat{A}_1 \cap H_x^+),\, \mu(\hat{A}_2 \cap H_x^+)\big).$$

Since ϕ is continuous and $N \geq 3$, we can apply the Borsuk–Ulam theorem and find $x \in S^{N-1}$ such that $\mu(\hat{A}_i \cap H_x^{\pm}) = \frac{1}{2}\mu(\hat{A}_i)$, for $i = 1, 2$. Now apply the same argument to M^+ and M^- separately, replacing A_i by $A_i^+ := A_i \cap H_x^+ \subset M^+$ and setting $A_i^- := A_i \cap H_x^- \subset M^-$ correspondingly. So after the second use of the Borsuk–Ulam theorem we obtain a partition of M into four disjoint convex subsets M^{++}, M^{+-}, M^{-+}, M^{--}. By construction $\mu(\hat{A}_1 \cap M^{++}) = \lambda\, \mu(\hat{A}_2 \cap M^{++})$, and this holds for all the other elements of the partition.

Repeating this procedure infinitely, we obtain a partition ζ of M, which is obviously measurable and satisfies (4.1) by construction. The property (4.2) follows immediately from Corollary 2.4 and Lemma 3.4. Corollary 2.4 implies also (4.3).

In order to prove (4.4), recall that the Borel σ-algebra of subsets of $\mathbb{R}^N$ is generated by a countable number of sets $\{T_j\}_{j=1}^{\infty}$. Since for ν_ζ-a.e. C ν_C is the convex restriction of μ, ν_C is absolutely continuous with respect to the Lebesgue measure on C; hence it is sufficient to check (4.4) only for $t \in \mathbb{Q}$. So we have to prove (4.4) for fixed T and t. And this again follows from Corollary 2.4.

By Theorem 2.1, $\mu(\hat{A}_1) = \int_{M/\zeta} \nu_C(\hat{A}_1)\, d\nu_\zeta(C)$. Hence we can choose C such that $\nu_C(\hat{A}_1 \cap C) = \nu_C(\hat{A}_1) \geq \mu(\hat{A}_1)$, and C satisfies (4.1)-(4.4). Let us show that $\dim C < N$. Indeed, $C = \bigcap_{k=1}^{\infty} V_k$, where V_k denotes the unique element of the partition of M constructed on the k-th step as above, which contains C. All V_k are convex, hence if $\dim C = N$, then $\dim V_k = N$. By Corollary 2.4 and the construction,

$$\nu_C(\hat{A}_1) = \lim_{k\to\infty} \frac{1}{\mu(V_k)}\, \mu(\hat{A}_1 \cap V_k) = \lim_{k\to\infty} \frac{1}{\mu(V_k)} \cdot \frac{1}{2^k}\, \mu(\hat{A}_1).$$

Since we have assumed that $\frac{d\mu}{dm_N} > 0\, m_N$-a.e., $\mu(V_k) \geq \mu(C) > 0$. So the right hand limit is equal to 0, contradicting the choice of C.

Let us fix such a C and denote it by M_1. Denote also ν_C by μ_1. Now we come back to the situation where $M = K(X)$ is the unit ball of $X = (\mathbb{R}^{n+1}, \|\cdot\|)$, μ is the normalized Lebesgue measure on M, and A_1, $A_2 \subset \partial M = S(X)$ are compact and disjoint. Thus μ_1 is a convex restriction of the Lebesgue measure, and it satisfies (4.4) with $\alpha = n + 1$.

Since $\mu_1(\hat{A}_1 \cap M_1) = \lambda\, \mu_1(\hat{A}_2 \cap M_1) > 0$ $\big($recall that $\lambda = \frac{\mu(\hat{A}_1)}{\mu(\hat{A}_2)} = \frac{m_{n+1}(\hat{A}_1)}{m_{n+1}(\hat{A}_2)}\big)$, we have $\dim M_1 \geq 2$. Obviously, M_1 is convex and compact.

If $\dim M_1 \geq 3$, the use of the Borsuk–Ulam theorem is possible and by the same procedure we construct a convex compact subset $M_2 \subset M_1$ and a convex restriction μ_2 of the measure μ_1 satisfying (4.1)-(4.4) with $\alpha = n+1$ and $\lambda\, \mu_2(\hat{A}_2 \cap M_2) = \mu_2(\hat{A}_1 \cap M_2) \geq \mu_1(\hat{A}_1 \cap M_1) \geq \mu(\hat{A}_1) = m_{n+1}(\hat{A}_1)$.

By Lemma 3.3, μ_2 is a convex restriction of m_{n+1}. Repeating this argument, after at most $n-1$ steps we obtain a 2-dimensional convex compact set $N \subset M$ and a measure ν on N such that:

(4.5) $N = \bigcup_{0 \leq t \leq 1} t \cdot (N \cap S(X))$ and N is contained in some half-plane (by construction).

(4.6) There exists an $(n-1)$-concave function γ on N such that $d\nu = \gamma\, dm_2$ (where m_2 is the Lebesgue measure on $\mathbb{R}^2$).

(4.7) $\lambda\, \nu(\hat{A}_2 \cap N) = \nu(\hat{A}_1 \cap N) \geq m_{n+1}(\hat{A}_1)$ $(= \hat{\mu}(A_1))$, where $\lambda = \frac{m_{n+1}(\hat{A}_1)}{m_{n+1}(\hat{A}_2)}$ as above.

(4.8) ν is homogeneous of degree $n+1$, i.e. for every Borel subset $T \subset \mathbb{R}^2$ and every $t \in [0,1]$,
$$\nu(t \cdot T) = t^{n+1} \cdot \nu(T).$$

Note that (4.6) and (4.8) immediately imply

(4.9) γ is homogeneous of degree $n-1$, i.e. $\gamma(t \cdot x) = t^{n-1}\, \gamma(x)$ for every $x \in N$, $t \in [0,1]$.

Clearly, by (4.5) $N \cap S(X)$ is a spherical segment. Denote it by $I = [a,\, b]$. Since the Banach–Mazur distance between any 2-dimensional normed space and the Euclidean ball is at most $\sqrt{2}$, we can find a Euclidean norm $|\cdot|$ on *span* N such that

$$\frac{1}{\sqrt{2}}\, |x| \leq \|x\| \leq |x|, \;\; \forall x \in span\, N. \tag{4.10}$$

For every two points $x,\, y \in I$, denote by $\rho(x,\, y)$ the length of the segment $[x,\, y] \subset I$ with respect to $\|\cdot\|$, i.e. if $[x,\, y]$ is parameterized by some parameter $\tau \in [0,1]$, then
$$\rho(x,\, y) := \sup_{0 \leq \tau_1 < \cdots < \tau_k \leq 1} \sum_{j=1}^{k-1} \|\tau_{j+1} - \tau_j\|.$$
Similarly, denote by $d(x,\, y)$ the length of $[x,\, y]$ with respect to $|\cdot|$.

By a result of [S],
$$\|x-y\| \leq \rho(x,\, y) \leq 2\, \|x-y\|.$$
Thus we obtain
$$|x-y| \leq d(x,\, y) \leq \sqrt{2}\rho(x,\, y) \leq 2\sqrt{2}\, \|x-y\| \leq 2\sqrt{2}\, |x-y|. \tag{4.11}$$

On I we have a measure $\hat\nu$ such that $\hat\nu(A) := \nu(\hat A)$, where $A \subset I$ is any Borel subset and $\hat A = \bigcup_{0\le t\le 1} t\cdot A$. Then $d\hat\nu = f_\nu\, dt$, where dt is an element of the Euclidean length and f_ν is a continuous function. By (4.6) and (4.9),

$$\nu(\hat A) = \int_{\hat A} \gamma\, dm_2 = \frac{1}{n+1}\int_A \gamma(t)\, dt.$$

So $f_\nu = \frac{1}{n+1}\gamma$. The rest of the paper closely follows [Gr-M].

For $x,\, y \in I$, (4.6) implies

$$\gamma^{1/(n-1)}\Big(\frac{x+y}{2}\Big) \ge \frac{\gamma^{1/(n-1)}(x) + \gamma^{1/(n-1)}(y)}{2}. \tag{4.12}$$

Set $z = \frac{x+y}{2}/\|\frac{x+y}{2}\| \in [x,\, y]$. By (4.9) and the inequality

$$\|\frac{x+y}{2}\| \le 1 - \delta(\|x-y\|),$$

we have

$$\gamma\Big(\frac{x+y}{2}\Big) \le (1-\delta(\|x-y\|))^{n-1}\ \gamma(z). \tag{4.13}$$

It easily follows from the inequality (4.11) that for some absolute constant $\alpha \in \big(0,\, \frac12\big)$,

$$d(z,\, x) \ge \alpha\, d(x,\, y), \ \text{ and } d(z,\, y) \ge \alpha\, d(x,\, y). \tag{4.14}$$

Let us parameterize the segment $I = [a,\, b]$ by the Euclidean length of the segment $[a,\, x]$, namely if x corresponds to t_1, it means $d(x,\, a) = t_1$. Let y corresponds to $t_2 > t_1$, then $d(x,\, y) = t_2 - t_1$. Clearly, (4.12)–(4.14) imply

$$\begin{aligned} &\frac{f_\nu^{1/(n-1)}(t_1) + f_\nu^{1/(n-1)}(t_2)}{2} \\ &\qquad \le (1-\delta(\|x-y\|)) \cdot \max_{z\in[t_1+\alpha\,(t_2-t_1),\, t_2-\alpha\,(t_2-t_1)]} f_\nu(z)^{1/(n-1)}. \end{aligned} \tag{4.15}$$

Then easily f_ν has no local minima and at most one local maximum inside I (this local maximum must be global). Denote the global maximum of f_ν by $t_0 \in [0,\, l]$ (where $l = d(a,\, b)$). Then obviously f_ν increases on $[0,\, t_0]$ and decreases on $[t_0,\, l]$.

For any $t \in [0,\, t_0]$ and any θ such that $0 \le t-\theta < t \le t_0$, (4.15) and the monotonicity of f_ν on $[0,\, t_0]$ imply

$$f_\nu(t-\theta) \le (1-\delta(\|x-y\|))^{n-1}\ f_\nu(t),$$

where x corresponds to $t-\theta$, and y corresponds to t. But by (4.11) $\|x-y\| \ge \frac{1}{2\sqrt2}\, d(x,\, y) = \frac{\theta}{2\sqrt2}$, and we obtain

$$f_\nu(t-\theta) \le \Big(1 - \delta\Big(\frac{\theta}{2\sqrt2}\Big)\Big)^{n-1}\ f_\nu(t). \tag{4.16}$$

Similarly, if $t_0 \le t < t+\theta \le l$, then

$$f_\nu(t+\theta) \le \left(1-\delta\Big(\frac{\theta}{\sqrt{2}}\Big)\right)^{n-1} f_\nu(t). \tag{4.17}$$

Hence, for $t_0 - 2\,\theta \ge 0$,

$$\begin{aligned}\hat\nu([0,\,t_0-2\,\theta]) &= \int_0^{t_0-2\,\theta} f_\nu(t)\,dt \le \left(1-\delta\Big(\frac{\theta}{2\sqrt{2}}\Big)\right)^{n-1}\int_\theta^{t_0-\theta} f_\nu(t)\,dt\\ &\le \left(1-\delta\Big(\frac{\theta}{2\sqrt{2}}\Big)\right)^{n-1}\left(\hat\nu([0,\,t_0-2\,\theta])+\hat\nu([t_0-2\,\theta, t_0-\theta])\right).\end{aligned}$$

Thus

$$\hat\nu([0,\,t_0-2\,\theta]) \le \frac{\left(1-\delta(\frac{\theta}{2\sqrt{2}})\right)^{n-1}}{1-\left(1-\delta(\frac{\theta}{2\sqrt{2}})\right)^{n-1}}\,\hat\nu([t_0-2\,\theta,\,t_0]). \tag{4.18}$$

In the same way, for $t_0 + 2\,\theta \le l$ we have

$$\hat\nu([t_0+2\,\theta,\,l]) \le \frac{\left(1-\delta(\frac{\theta}{2\sqrt{2}})\right)^{n-1}}{1-\left(1-\delta(\frac{\theta}{2\sqrt{2}})\right)^{n-1}}\,\hat\nu([t_0,\,t_0+2\theta]). \tag{4.19}$$

Adding (4.18) and (4.19) and using $\hat\nu([0,\,l]) = 1$, we obtain:

Lemma 4.20. $\hat\nu(I - [t_0-2\,\theta,\,t_0+2\,\theta]) \le \left(1-\delta(\frac{\theta}{2\sqrt{2}})\right)^{n-1} \approx e^{-\delta(\frac{\theta}{2\sqrt{2}}\,(n-1))}$.

5. Proof of Theorem 1.1

(We repeat the argument of [Gr-M].)

Let $A \subset S(X)$, $\hat\mu(A) \ge \frac{1}{2}$ (the measure $\hat\mu$ was defined in Section 1). Fix $\varepsilon \in (0,1)$. Set $A_1 := A$, $A_2 := S(X) - A_\varepsilon$. Hence we can find a compact convex 2-dimensional set N with a probability measure ν satisfying (4.5)–(4.9). Let c be the point on I with the maximal density of $\hat\nu$.

If θ_n is such that $\delta(\theta_n) = 1-(\frac{1}{2})^{1/(n-1)} \approx \frac{\log 2}{n-1}$, then $\hat\nu\{x \in I : d(x,c) \le 4\sqrt{2}\,\theta_n\} \ge \frac{1}{2}$. By (4.7), $\hat\nu(A_1 \cap I) \ge \frac{1}{2}$; hence there exists $x' \in A_1 \cap I$ such that $\|x'-c\| \le d(x',c) \le 4\sqrt{2}\theta_n$. Now let us take θ such that $\varepsilon = 4\sqrt{2}\,(\theta+\theta_n)$. For an ε - neighborhood of $\{x'\}$ (with respect to the original norm $\|\cdot\|$), we have $\{x'\}_\varepsilon \supset \{c\}_{4\sqrt{2}\,\theta}$ and $\{x'\}_\varepsilon \cap A_2 = \varnothing$. Therefore, again by Lemma 4.20 and (4.11)

$$\begin{aligned}\hat\nu(A_2\cap I) &\le \hat\nu(I - \{x : d(x,c) \le 4\sqrt{2}\theta\}) \le (1-\delta(\theta))^{n-1}\\ &\approx \exp(-\delta(\theta)\,(n-1)) = \exp\left(-\delta\Big(\frac{\varepsilon}{4\sqrt{2}}-\theta_n\Big)\,(n-1)\right).\end{aligned}$$

By (4.7),

$$\mu(A_2) = \frac{\hat{\nu}(A_2 \cap I)}{\hat{\nu}(A_1 \cap I)}\, \mu(A_1) \le \hat{\nu}(A_2 \cap I) \le \exp\left(-\delta\Big(\frac{\varepsilon}{4\sqrt{2}} - \theta_n\Big)(n-1)\right). \quad \square$$

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References

[Gr-M] Gromov, M.; Milman, V. D.; Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces. Compositio Mathematica **62** (1987), 263–282.

[L-S1] Lovász, L.; Simonovits M.; Mixing rate of Markov chaines, an isoperimetric inequality, and computing the volume. Proc. 31st Ann. Symp. on Found. of Comput. Sci., IEEE Computer Soc., 1990, 346–355.

[L-S2] Lovász, L.; Simonovits M.; Random walks in a convex body and an improved volume algorithm. Random Structures and Algorithms **4** (1993), 359–412.

[L-Sh] Liptser, R. S.; Shiryayev, A. N.; Statistics of Random Processes I, General Theory; Springer, New York, Heidelberg, Berlin, 1977.

[P] Pisier, G.; The Volume of Convex Bodies and Banach Space Geometry. Cambridge University Press, Cambridge 1989.

[R] Rohlin, V. A.; On the fundamental ideas of measure theory; Mat. Sbornik, N.S. 25 (67), (1949), 107–150 (Russian). Translation in Amer. Math. Soc. Translations **71**, 55 pp. (1952).

[S] Schäffer, J. J.; Inner diameter, perimeter, and girth of spheres. Math. Ann. **173** (1967), 59–82.

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