Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

Geometric Inequalities in Option Pricing

CHRISTER BORELL

Abstract. This paper discusses various geometric inequalities in option pricing assuming that the underlying stock prices are governed by a joint geometric Brownian motion. In particular, inequalities of isoperimetric type are proved for different classes of derivative securities. Moreover, the paper discusses the option on the minimum of several assets and, among other things, proves a log-concavity property of its price.

1. Introduction

The purpose of this paper is to prove various geometric inequalities in option pricing using familiar inequalities of the Brunn-Minkowski type in Gauss space.

To begin with, recall that a European (American) call [put] option is defined as the right to buy [sell] one share of stock at a specified price on (or before) a specified date. The specified price is referred to as the exercise price and the terminal date of the contract is called the expiration date or maturity date. In fact, already the early paper [20] by Merton treats a variety of convexity properties of puts and calls, sometimes without any distributional assumptions on the underlying stock prices. Here, however, it will always be assumed that the price process $X(t) = (X_1(t), \ldots, X_m(t)),\ t \geq 0$, of the underlying risky assets $\mathcal{X}_1, \ldots, \mathcal{X}_m$ is governed by a so called joint geometric Brownian motion. Furthermore, all options will be of European type and so, from now on, option will always mean option of European type.

Now suppose $f : \mathbb{R}_+^m \to [0, +\infty[$ is a continuous function such that

$$f(x) \leq A\left(1 + \sum_{i=1}^{m} |x_i|\right)^a \quad \text{for } x = (x_1, ..., x_m) \in \mathbb{R}_+^m,$$

1991 *Mathematics Subject Classification.* primary 52A40, 90A09; secondary 60G15, 60H05.

Key words and phrases. put, call, derivative security, isoperimetric inequality, log-concave, Gaussian random variable.

Research supported by the Swedish Natural Science Research Council.

for appropriate constants $a, A \geq 0$, and suppose a certain derivative security $\mathcal{U}_f^T$ pays $f(X(T))$ at the maturity time T. Here f is termed a payoff function. If t is a time point prior to T, set $\tau = T - t$, and denote by $u_f(\tau, X(t))$ the (theoretic) price of $\mathcal{U}_f^T$ at time t. If $u_f(\tau, x) = u_f(\tau, x_1, \ldots, x_m)$ is positive and $i \in \{1, \ldots, m\}$ is fixed, the quantity

$$\psi_f^i(\tau, x) = \frac{x_i}{u_f(\tau, x)} \frac{\partial u_f(\tau, x)}{\partial x_i}$$

is called the elasticity of the price $u_f(\tau, x)$ relative to the price x_i. The quantities $\psi_f^1(\tau, x), \ldots, \psi_f^m(\tau, x)$ enter quite naturally in option pricing in connection with so called hedging against the contingent claim $\mathcal{U}_f^T$. Actually, we will below occasionally consider a slightly larger class of payoff functions than stated here.

Now let the function $f(x),\ x \in \mathbb{R}_+^m$, be a log-concave function of the log-price vector $\ln x = (\ln x_1, \ldots, \ln x_m)$. In Section 3, we prove, among other things, that the function $\tau^{m/2} u_f(\tau, x)$ is a log-concave function of $(\tau, \ln x)$. In particular, if f is not identically equal to zero, then for any fixed $i \in \{1, \ldots, m\}$ and $\tau > 0$ the elasticity function $\psi_f^i(\tau, x)$ is a non-increasing function of x_i when the other prices $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ are held fixed. Note that these results apply to the payoff function

$$f(x) = \min_{i=1,\ldots,m} x_i \tag{1}$$

which is of interest in connection with the cheapest to deliver option. The derivative security corresponding to the payoff function in (1) is sometimes referred to as the quality option (see e.g. Boyle [11]).

The main concern in this paper is to prove certain inequalities of isoperimetric type. More explicitly, consider the same risky assets as above and suppose $a > 0$ is given. We shall write $f \in \mathcal{C}_a$ if $f : \mathbb{R}_+^m \to [0, +\infty[$ is a locally Lipschitz continuous function such that

$$\sum_{i=1}^m x_i \left| \frac{\partial f}{\partial x_i} \right| \leq a + f(x) \text{ a.e.}$$

with respect to Lebesgue measure in $\mathbb{R}^m$. The class $\mathcal{C}_a$ is convex and contains the zero payoff function. Moreover, the class $\mathcal{C}_a$ contains the payoff functions of all puts and calls on the $\mathcal{X}_i, i = 1, \ldots, m$, with exercise prices less than or equal to a. In addition, if $f, g \in \mathcal{C}_a$, then $\max(f, g) \in \mathcal{C}_a$ and $\min(f, g) \in \mathcal{C}_a$. In particular, the function in (1) as well as the function

$$f(x) = \max_{i=1,\ldots,m} x_i \tag{2}$$

belong to the class $\mathcal{C}_a$ for all $a > 0$.

In Section 4 we discuss, among other things, the Monte Carlo method for computing the option price $u_f(\tau, x)$ when f is as in (2). Let $\mathcal{X}_m$ be the most volatile asset of the risky assets $\mathcal{X}_1, \ldots, \mathcal{X}_m$ and let σ_m be the volatility of $\mathcal{X}_m$.

The (crude) Monte Carlo method then gives us a certain unbiased estimator $\overline{Z}_N$ of the option price $u_f(\tau, x)$ and we prove that

$$\mathbb{P}\left[\left|\frac{\overline{Z}_N - u_f(\tau,x)}{u_f(\tau,x)}\right| \geq \varepsilon\right] \leq \frac{e^{\sigma_m^2\tau}-1}{\varepsilon^2 N}, \quad \varepsilon > 0. \tag{3}$$

Note that the right-hand side of (3) is independent of the option price $u_f(\tau, x)$.

In Section 4 we also prove the following property of the class $\mathcal{C}_a$ for fixed $a > 0$. Suppose v is the expected exercise value of a call on $\mathcal{X}_m$ with the maturity date T and exercise price a. Then, amongst all derivative securities $\mathcal{U}_f^T$ with $f \in \mathcal{C}_a$ and with the expected payoff v at time T, the payoff at time T has maximal variance for the call on $\mathcal{X}_m$ with the exercise price a.

Finally, in Section 5 we discuss inequalities of isoperimetric type for other classes of payoff functions than those considered above.

2. Notation and Basic Results

Throughout this paper $\mathcal{X}_i, i = 1, \ldots, m$, stand for m risky assets with a joint price process $X(t) = (X_1(t), \ldots, X_m(t)),\ t \geq 0$, governed by an m-dimensional geometric Brownian motion. Stated more explicitly, there are linearly independent unit vectors $c_i,\ i = 1, \ldots, m$, in $\mathbb{R}^n$ and a normalized Brownian motion $(W(t))$ in $\mathbb{R}^n$ such that

$$\frac{dX_i(t)}{X_i(t)} = (\mu_i + \sigma_i^2/2)dt + \sigma_i dW_i(t), \quad i = 1, \ldots, m$$

for suitable $\mu_1, \ldots, \mu_m \in \mathbb{R}$ and $\sigma_1 > 0, \ldots, \sigma_m > 0$, where

$$W_i(t) = \langle c_i, W(t)\rangle, \quad i = 1, \ldots, m.$$

Here, $\langle\,\cdot\,,\cdot\,\rangle = \langle\,\cdot\,,\cdot\,\rangle_{\mathbb{R}^n}$ denotes the standard scalar product in $\mathbb{R}^n$.

In what follows, $t < T$ and we set

$$M_{\sigma_i}^{W_i}(\tau) = e^{-\frac{\sigma_i^2}{2}\tau + \sigma_i W_i(\tau)} \quad \text{for } i = 1, \ldots, m$$

and

$$M_\sigma^W(\tau) = (M_{\sigma_1}^{W_1}(\tau), \ldots, M_{\sigma_m}^{W_m}(\tau)),$$

where $\tau = T - t$. Moreover, if $\xi = (\xi_1, \ldots, \xi_m)$, $\eta = (\eta_1, \ldots, \eta_m) \in \mathbb{R}^m$, we will make frequent use of the following notation:

$$|\xi| = (|\xi_1|, \ldots, |\xi_m|)$$

$$\|\xi\|_1 = \sum_1^m |\xi_i|$$

$$\|\xi\|_2 = \sqrt{\langle \xi, \xi\rangle_{\mathbb{R}^m}}$$

$$\|\xi\|_\infty = \max_{i=1,\ldots,m} |\xi_i|$$

$$e^{\xi} = (e^{\xi_1}, \dots, e^{\xi_m})$$

$$\ln e^{\xi} = \xi$$

and

$$\xi\eta = (\xi_1\eta_1, \dots, \xi_m\eta_m).$$

Now consider a derivative security $\mathcal{U}_f^T$ with the payoff $f(X(T))$ at time T. Below, for technical reasons, it will be assumed that $f : \mathbb{R}_+^m \to \mathbb{R}$ is a continuous function such that

$$\mathbb{E}[|f(xM_\sigma^W(\tau))|^p] < +\infty$$

for all $x \in \mathbb{R}_+^m$, $\tau > 0$, and $p > 0$ and a function f satisfying these assumptions will be called a payoff function. If r denotes the risk-free interest rate and if $u_f(\tau, X(t))$ denotes the value of the derivative security $\mathcal{U}_f^T$ at time $t \in [0, T[$, we have

$$u_f(\tau, x) = \mathbb{E}[e^{-r\tau} f(xe^{r\tau} M_\sigma^W(\tau))]. \tag{4}$$

A proof this equation is given e.g. in Duffie's book [14] or in the basic paper [16] by Harrison and Pliska. If f is a payoff function it is readily seen that $u_f(\tau, x)$ is a payoff function as a function of x for fixed τ. We now define $u_f(\tau, x)$ for all $\tau > 0$ by the equation (4) and set $(S_\tau f)(x) = u_f(\tau, x)$ if $\tau > 0$. Then the family $(S_\tau)_{\tau > 0}$ becomes a semi-group, the so called option semi-group of the underlying risky assets $\mathcal{X}_1, \dots, \mathcal{X}_m$.

Throughout this paper, if $\xi \in \mathbb{R}$, we let $\xi^+ = \max(0, \xi)$ and $\xi^- = (-\xi)^+$. Moreover, given $a > 0$ and $i \in \{1, \dots, m\}$, let

$$c_{a,i}(x) = (x_i - a)^+$$

and

$$p_{a,i}(x) = (x_i - a)^- = (a - x_i)^+.$$

Here the derivative security $\mathcal{U}_{c_{a,i}}^T$ is called a call on $\mathcal{X}_i$ with exercise price a and maturity time T and the derivative security $\mathcal{U}_{p_{a,i}}^T$ is called a put on $\mathcal{X}_i$ with exercise price a and maturity time T. From (4) we have the following famous formula by Black and Scholes, viz.

$$u_{c_{a,i}}(x, \tau) = x\Phi\left(\frac{\ln \frac{x_i}{a} + \left(r + \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}}\right) - ae^{-r\tau}\Phi\left(\frac{\ln \frac{x_i}{a} + \left(r - \frac{\sigma_i^2}{2}\right)\tau}{\sigma_i\sqrt{\tau}}\right)$$

where

$$\Phi(\xi) = \int_{-\infty}^{\xi} e^{-\frac{\eta^2}{2}} \frac{d\eta}{\sqrt{2\pi}}$$

is the distribution function of a real-valued Gaussian random variable with unit variance and expectation zero. Moreover, by the put-call parity relation we have

$$u_{p_{a,i}}(\tau, x) = ae^{-r\tau} + u_{c_{a,i}}(\tau, x) - x_i.$$

In the following $\mathcal{X}_0$ denotes a bond with the value $X_0(t) = e^{-r\tau}$ at time t. Furthermore, we set

$$\phi_f^i(\tau, x) = \frac{\partial u_f}{\partial x_i}(\tau, x)\,, \quad i = 1, \dots, m \tag{5}$$

and

$$\phi_f^0(\tau, x) = e^{r\tau}(u_f(\tau, x) - \sum_1^m x_i \phi_f^i(\tau, x)).$$

A portfolio consisting of $\phi_f^j(T-s, X(s))$ units of $\mathcal{X}_j$ for all $j = 0, \dots, m$ at any time $s \in [t, T[$ has the value $u_f(\tau, X(t))$ at time t and

$$f(X(T)) = u_f(\tau, X(t)) + \sum_{j=0}^m \int_t^T \phi_f^j(T-s, X_j(s))\, dX_j(s).$$

This so called self-financing trading strategy in the $\mathcal{X}_j$, $j = 0, \dots, m$, is basic to the theory of option pricing and much more details may be found in [14] and [16]. The portfolio $\phi_f = (\phi_f^0, \phi_f^1, \dots, \phi_f^m)$ is often called a hedge against the contingent claim $\mathcal{U}_f^T$. If $u_f(\tau, x)$ is positive for all x, the corresponding relative portfolio $\psi_f = (\psi_f^0, \psi_f^1, \dots, \psi_f^m)$ is defined by

$$\psi_f^i(\tau, x) = x_i \phi_f^i(\tau, x)/u_f(\tau, x)\,, \quad i = 1, \dots, m$$

and

$$\psi_f^0 = 1 - \sum_{i=1}^m \psi_f^i\,.$$

Given $i \in \{1, \dots, m\}$ the quantity $\psi_f^i(\tau, x)$ is called the elasticity of the price $u_f(\tau, x)$ relative to the price x_i.

A payoff function f is said to be homogeneous if

$$f(\alpha x) = \alpha f(x), \quad \alpha > 0,\ x \in \mathbb{R}_+^m$$

and for such functions, $\phi_f^0 = 0$ and u_f is independent of r. Typical examples of homogeneous payoff functions are

$$f_{\min}(x) = \min_{i=1,\dots,m} x_i$$

and

$$f_{\max}(x) = \max_{i=1,\dots,m} x_i.$$

Finally, for future reference recall that a real-valued random variable is said to have a $N(0;1)$-distribution if its distribution function equals Φ.

3. Derivative Securities with Log-Concave Payoff Functions

Recall that a non-negative function h defined on a convex subset D of a vector space is log-concave if

$$h(\theta\xi + (1-\theta)\eta) \geq h(\xi)^{\theta} h(\eta)^{1-\theta} \tag{6}$$

for all $\xi, \eta \in D$ and all $\theta \in \,]0,1[$. If the inequality in (6) is reversed, then h is said to be log-convex. It is well known and simple to prove that the class of all log-convex functions on a convex set is closed under addition. However, the class of all log-concave functions defined on a convex set containing more than one point is not closed under addition.

The options on the minimum and maximum of several assets have been treated by Stulz [22], Johnson [18], Boyle and Tse [12] and others. The important cheapest to deliver option involves the consideration of options on the minimum of several assets i.e., the so called quality option (for more details see e.g. Boyle [11]). In fact, options on the minimum and maximum of two assets already appear implicitly in the Margrabe paper [19], which considers the option to exchange one asset for another. We will comment more on the Margrabe option below. Note that the payoff function $f_{\min}(x)$ is a log-concave function of the asset price vector $x = (x_1, \ldots, x_m)$ as well as of the asset log-price vector $\ln x = (\ln x_1, \ldots, \ln x_m)$. Moreover, the payoff function $f_{\max}(x)$ is a log-convex function of the asset log-price vector $\ln x$. If a payoff function f is concave (convex), then the security price $u_f(\tau, x)$ is a concave (convex) function of x for fixed τ as is readily seen from equation (4) (cf. [20]). If the payoff function $f(x)$ is a log-convex function of the log-price vector $\ln x$ and f is not identically equal to zero, then it follows from the equation (4) that the option price $u_f(\tau, x)$ is a log-convex and positive function of $\ln x$ for fixed τ. In particular, for any fixed $i = 1, \ldots, m$, the function

$$x_i \to \psi_f^i(\tau, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) \tag{7}$$

is non-decreasing since

$$\psi_f^i(\tau, x) = \frac{\partial \ln u_f(\tau, e^{\xi})}{\partial \xi_i}, \quad x = e^{\xi}. \tag{8}$$

The main purpose of this section is to prove that the function in (7) is non-increasing if the payoff function $f(x)$ is a log-concave function of the log-price vector $\ln x$ and f is not identically equal to zero. To this end we will make use of a very nice property of log-concave functions, first proved in a general setting by Prékopa [21] and which reads as follows:

If the function $f(\xi, \eta_1, \ldots, \eta_n)$ is a log-concave function of $(\xi, \eta_1, \ldots, \eta_n) \in D \times \mathbb{R}^n$, where D is convex, then the integral

$$\int_{\mathbb{R}^n} f(\xi, \eta_1, \ldots, \eta_n)\, d\eta_1 \ldots d\eta_n$$

is a log-concave function of $\xi \in D$.

Below, this result will be referred to as Prékopa's theorem. Note that the Davidovič, Korenbljum, and Hacet early paper [13] treats an important special case of Prékopa's theorem.

THEOREM 3.1. (a) *If the payoff function* $f(x)$ *is a log-concave function of the log-price vector* $\ln x$, *then the function* $\tau^{m/2}u_f(\tau, x)$ *is a log-concave function of* $(\tau, \ln x)$. *In particular, if* f *is not identically equal to zero, then for any* $i = 1, \ldots, m$ *and* $\tau > 0$, *the function in* (7) *is non-increasing.*
(b) *If the payoff function* $f(x)$ *is homogeneous and a log-concave function of the log-price vector* $\ln x$, *then the function* $\tau^{(m-1)/2}u_f(\tau, x)$ *is a log-concave function of* $(\tau, \ln x)$.

To prove Theorem 3.1, we need the following result:

LEMMA 3.1. *If* $g : \mathbb{R}^m_+ \to \mathbb{R}$ *is a homogeneous payoff function and* $m \geq 2$, *then*

$$\mathbb{E}[g(M^W_\sigma(\tau))] = \mathbb{E}[g(M^{W^*_1}_{\sigma^*_1}(\tau), \ldots, M^{W^*_{m-1}}_{\sigma^*_{m-1}}(\tau), 1)]$$

where

$$\sigma^*_i = \sqrt{\sigma_i^2 - 2\langle c_i, c_j\rangle \sigma_i\sigma_m + \sigma_m^2}$$

and

$$W^*_i = (\sigma_i W_i - \sigma_m W_m)/\sigma^*_i$$

for $i = 1, \ldots, m-1$.

In the special case $m = 2$, Lemma 3.1 is implicit in [20] (with a proof different from the one below).

PROOF. We have that

$$\mathbb{E}[g(M^W_\sigma(\tau))] = \mathbb{E}[g(e^{a_1+\sigma^*_1 W^*_1(\tau)}, \ldots, e^{a_{m-1}+\sigma^*_{m-1}W^*_{m-1}(\tau)}, 1)e^{a_m+\sigma_m W_m(\tau)}]$$

for appropriate constants $a_1, \ldots, a_m$ independent of g. By conditioning on $W^*(\tau) = (W^*_1(\tau), \ldots, W^*_{m-1}(\tau))$ the right-hand side equals

$$\mathbb{E}[g(e^{a_1+\sigma^*_1 W^*_1(\tau)}, \ldots, e^{a_{m-1}+\sigma^*_{m-1}W^*_{m-1}(\tau)}, 1)e^{a'_m+\langle b', W^*(\tau)\rangle}]$$

for appropriate $a'_m \in \mathbb{R}$ and $b' \in \mathbb{R}^{m-1}$. Therefore, by translating the probability law of $W^*(\tau)$, we get

$$\mathbb{E}[g(M^W_\sigma(\tau))] = \mathbb{E}[g(M^{W^*_1}_{\sigma^*_1}(\tau), \ldots, M^{W^*_{m-1}}_{\sigma^*_{m-1}}(\tau), 1)e^{a+\langle b, W^*(\tau)\rangle}] \tag{9}$$

for suitable $a \in \mathbb{R}$ and $b \in \mathbb{R}^{m-1}$. Now let C denote the covariance matrix of $W^*(1)$ and let $e_1, \ldots, e_{m-1}$ be the standard basis in $\mathbb{R}^{m-1}$. Then by choosing $g(x) = x_i$ for $i = 1, \ldots, m$, we have

$$\begin{cases} \langle b + \sigma^*_i e_i, C(b + \sigma^*_i e_i)\rangle + 2a - {\sigma^*_i}^2 = 0 \quad \text{for } i = 1, \ldots, m-1, \\ \langle b, Cb\rangle + 2a = 0. \end{cases}$$

From this we get $\langle e_i, Cb\rangle = 0$ for $i = 1, \dots, m-1$, and it follows that $b = 0$ since C is invertible. Hence also $a = 0$. In view of (9), Lemma 3.1 is thereby completely proved. $\square$

Proof of Theorem 3.1. We first prove Part (b). However, for the sake of simplicity we restrict ourselves to the special case $m = 2$; the general case is proved in a similar way. To prove Part (b) with $m = 2$ note that Lemma 3.1 yields

$$u_f(\tau, x) = \int_{\mathbb{R}} f(x_1 e^{-\frac{\sigma_1^{*2}}{2}\tau + \sigma_1^* \sqrt{\tau}\zeta}, x_2) e^{-\frac{\zeta^2}{2}} \frac{d\zeta}{\sqrt{2\pi}}$$

and hence

$$\sqrt{\tau} u_f(\tau, x) = \int_{\mathbb{R}} f(x_1 e^{-\frac{\sigma_1^{*2}}{2}\tau + \sigma_1^* \zeta}, x_2) e^{-\frac{\zeta^2}{2\tau}} \frac{d\zeta}{\sqrt{2\pi}}.$$

We next introduce the new vector variable $\xi = \ln x$ and have

$$\sqrt{\tau} u_f(\tau, x) = \int_{\mathbb{R}} g(\tau, \xi, \zeta)\, d\zeta$$

where

$$g(\tau, \xi, \zeta) = f(e^{\xi_1 - \frac{\sigma_1^{*2}}{2}\tau + \sigma_1^* \zeta}, e^{\xi_2}) \frac{e^{-\frac{\zeta^2}{2\tau}}}{\sqrt{2\pi}}.$$

Since the function

$$\frac{\zeta^2}{\tau}, \quad \zeta \in \mathbb{R}, \quad \tau > 0$$

is convex we conclude that the function $g(\tau, \xi, \zeta)$ is a log-concave function of (τ, ξ, ζ). The Prékopa theorem now implies that the integral

$$\int_{\mathbb{R}} g(\tau, \xi, \zeta)\, d\zeta$$

is a log-concave function of (τ, ξ). This proves Part (b) of Theorem 3.1. The first statement in Part (a) of Theorem 3.1 is proved in a similar way as Part (b) of Theorem 3.1. Moreover, the last statement in Part (a) of Theorem 3.1 now follows from (8). This concludes our proof of Theorem 3.1. $\square$

Example 3.1. Set $f_0(x) = \min(x_1, x_2)$ and suppose $\alpha \in]0, +\infty[$. Then, in view of Theorem 3.1, the function $\tau^\alpha u_{f_0}(\tau, x)$ is a log-concave function of $(\tau, \ln x)$ if $\alpha \geq \frac{1}{2}$. We now claim that the condition $\alpha \geq \frac{1}{2}$ is necessary for this conclusion. To see this first note the equation

$$u_{f_0}(\tau, x) = x_1 \Phi\left(\frac{\ln\frac{x_2}{x_1} - \frac{\sigma_1^{*2}\tau}{2}}{\sigma_1^*\sqrt{\tau}}\right) + x_2 \Phi\left(\frac{\ln\frac{x_1}{x_2} - \frac{\sigma_1^{*2}\tau}{2}}{\sigma_1^*\sqrt{\tau}}\right) \tag{10}$$

which is implicit in the Margrabe paper [19] (here σ_1^* is as in Lemma 3.1 with $m = 2$). In fact, Margrabe determines u_{f_1} when $f_1(x) = \max(0, x_2 - x_1)$ and, since $u_{f_0}(\tau, x) = x_2 - u_{f_1}(\tau, x)$, equation (10) is an immediate consequence of his paper. A direct derivation of (10) is also simple using Lemma 3.1. To see this, set $a = \sigma_1^* \sqrt{\tau}$ and let G be a real-valued centered Gaussian random variable with unit variance. Now using Lemma 3.1 it follows that

$$u_{f_0}(\tau, x) = \mathbb{E}[\min(x_1 e^{-\frac{a^2}{2}+aG}, x_2)]$$

so that

$$u_{f_0}(\tau, x) = x_1 \mathbb{E}\left[e^{-\frac{a^2}{2}+aG}; G \leq \frac{1}{a}\left(\ln\frac{x_2}{x_1} + \frac{a^2}{2}\right)\right] + x_2 \mathbb{P}\left[G > \frac{1}{a}\left(\ln\frac{x_2}{x_1} + \frac{a^2}{2}\right)\right].$$

By applying the translation formula of Gaussian measures, we get

$$u_{f_0}(\tau, x) = x_1 \mathbb{P}\left[G \leq \frac{1}{a}\left(\ln\frac{x_2}{x_1} - \frac{a^2}{2}\right)\right] + x_2 \mathbb{P}\left[-G < \frac{1}{a}\left(\ln\frac{x_1}{x_2} - \frac{a^2}{2}\right)\right]$$

and (10) follows. In particular, we have

$$u_{f_0}(\tau, (x_1, x_1)) = 2x_1 \Phi(-\frac{\sigma_1^*}{2}\sqrt{\tau}).$$

Now set

$$g(\tau) = \alpha \ln\tau + \ln(\Phi(-\sqrt{\tau})), \quad \tau > 0.$$

The claim above follows if we prove that g is not concave for any $\alpha \in \left]0, \frac{1}{2}\right[$. To this end, set $\varphi = \Phi'$ so that

$$g'(\tau) = \frac{\alpha}{\tau} - \frac{\varphi(-\sqrt{\tau})}{2\sqrt{\tau}\Phi(-\sqrt{\tau})}.$$

The function g' is non-increasing if and only if the function

$$h(s) = \frac{\alpha}{s^2} - \frac{\varphi(s)}{2s\Phi(-s)}, \quad s > 0$$

is non-increasing. Now

$$h'(s) = -\frac{2\alpha}{s^3} + \frac{\varphi(s)}{2\Phi(-s)}\left(1 + \frac{1}{s^2} - \frac{\varphi(s)}{s\Phi(-s)}\right)$$

and, by using the Laplace-Feller inequality (see e.g. Tong [24]),

$$\Phi(-s) = \varphi(s)\left(\frac{1}{s} - \frac{1}{s^3} + \frac{3}{s^5} + O\left(\frac{1}{s^7}\right)\right), \quad \text{as } s \to +\infty.$$

From this

$$\frac{\varphi(s)}{\Phi(-s)} = \frac{s}{1 - \left(\frac{1}{s^2} - \frac{3}{s^4} + O(\frac{1}{s^6})\right)}, \quad \text{as } s \to +\infty,$$

and we get

$$h'(s) = -\frac{2\alpha}{s^3} + \frac{\varphi(s)}{2\Phi(-s)}\left[1 + \frac{1}{s^2} - \left(1 + \frac{1}{s^2} - \frac{3}{s^4} + \left(\frac{1}{s^2} - \frac{3}{s^4}\right)^2 + O\left(\frac{1}{s^6}\right)\right)\right],$$

as $s \to +\infty$. Thus

$$h'(s) = -\frac{2\alpha}{s^3} + \frac{\varphi(s)}{2\Phi(-s)}\frac{2}{s^4}\left(1 + O\left(\frac{1}{s^2}\right)\right), \quad \text{as } s \to +\infty,$$

and, finally,

$$h'(s) = -\frac{2\alpha}{s^3} + \frac{1}{1+O\left(\frac{1}{s^2}\right)}\frac{1}{s^3}\left(1+O\left(\frac{1}{s^2}\right)\right), \quad \text{as } s \to +\infty.$$

Therefore, if $h'(s) \leq 0$ for all $s > 0$, then necessarily $\alpha \geq \frac{1}{2}$. This proves that the function g is not concave for any $\alpha \in \left]0, \frac{1}{2}\right[$ and, hence, that the function $\tau^\alpha u_{f_0}(\tau, x)$ is not a log-concave function of $(\tau, \ln x)$ for any $\alpha \in \left]0, \frac{1}{2}\right[$. □

Theorem 3.2. *If $f_0 = f_{\min}$, then the functions $\tau^{(m-1)/2}\phi^i_{f_0}(\tau, x)$, $i = 1, \ldots, m$, are log-concave functions of* $(\tau, \ln x)$.

Proof. Using (5) with $f = f_0$, we have from Lemma 3.1 that

$$\phi^m_{f_0}(\tau, x) = \mathbb{E}[h(x_1 M^{W^*_1}_{\sigma^*_1}(\tau), \ldots, x_{m-1} M^{W^*_{m-1}}_{\sigma^*_{m-1}}(\tau), x_m)]$$

where

$$h(x) = \begin{cases} 1 & \text{if } x_m < f_0(x_1, \ldots, x_{m-1}, 1), \\ 0 & \text{if } x_m \geq f_0(x_1, \ldots, x_{m-1}, 1). \end{cases}$$

Since h is a log-concave function of $\ln x$, as in the proof of Theorem 3.1, the Prékopa theorem implies that the function $\tau^{(m-1)/2}\phi^m_{f_0}(\tau, x)$ is a log-concave function of $(\tau, \ln x)$. In a similar way we conclude that the functions

$$\tau^{(m-1)/2}\phi^i_{f_0}(\tau, x), \quad i = 1, \ldots, m-1,$$

are log-concave functions of $(\tau, \ln x)$. This completes the proof of Theorem 3.2. □

4. Extremal Properties of Calls

In this section we are going to prove an inequality of the so called Berwald's type (cf. [3]) for a certain class of option prices. To begin with we therefore review the Berwald inequality as well as some other closely related results due to the author [6], [8].

A real-valued function ψ is said to be convex with respect to another real-valued function φ if there exists a convex continuous function κ such that $\psi = \kappa \circ \varphi$. We shall write $\psi \in \mathcal{V}_0(\varphi)$ if the function $\psi : [0, +\infty[\to \mathbb{R}$ is convex with respect to the non-decreasing continuous function $\varphi : [0, +\infty[\to \mathbb{R}$.

Now let K be a convex body in $\mathbb{R}^n$ with volume $|K|$ and suppose $f : K \to]0, +\infty[$ is a given concave function. Moreover, suppose $\psi \in \mathcal{V}_0(\varphi)$ and

$$\frac{1}{|K|}\int_K \varphi(f(x))\,dx = n\int_0^1 \varphi(\xi t)(1-t)^{n-1}\,dt$$

where ξ is a suitable positive number. Under these premises Berwald [3] proves that

$$\frac{1}{|K|}\int_K \psi(f(x))\,dx \leq n\int_0^1 \psi(\xi t)(1-t)^{n-1}\,dt.$$

In [6] the same inequality is established for so called dome functions on K (i.e. functions on K which are possible to represent as the supremum of a suitable family of uniformly bounded and positive concave functions on K). Clearly, the Berwald inequality then also remains true for all functions on K which are equimeasurable with appropriate dome functions on K, a class of functions, which is optimal in connection with the Berwald inequality [6]. All these results depend on the standard Brunn-Minkowski inequality for volume measure in $\mathbb{R}^n$. In our paper [8] we proved an inequality of the Berwald type for certain sublinear functions using the so called isoperimetric inequality in Gauss space. Here, again, we will apply the isoperimetric inequality in Gauss space but this time to a class of functions different from the one in [8].

Throughout the remaining part of this paper we assume that $G = (G_1, \dots, G_n)$ is the standard Gaussian random vector in $\mathbb{R}^n$ with stochastically independent and $N(0;1)$-distributed components. The isoperimetric inequality for the random vector $G = (G_1, \dots, G_n)$, independently discovered by Sudakov and Tsyrelson [23] and the author [7], reads as follows:

If $A \subseteq \mathbb{R}^n$ is a Borel set and $\mathbb{P}[G \in A] = \mathbb{P}[G_n \leq \alpha]$ for an appropriate $\alpha \in [-\infty, +\infty]$, then $\mathbb{P}[G \in A + \bar{B}(0;\varepsilon)] \geq \mathbb{P}[G_n \leq \alpha + \varepsilon]$ for $\varepsilon > 0$, where $\bar{B}(0;\varepsilon) = \{\xi \in \mathbb{R}^n; \|\xi\|_2 \leq \varepsilon\}$.

For new proofs of the isoperimetric inequality in Gauss space, see Bakry and Ledoux [1] and Bobkov [5]. Before we apply isoperimetry in Gauss space to option pricing we have to discuss some properties of so called Lipschitz functions.

A real-valued function g defined on an open subset V of $\mathbb{R}^m$ belongs to the Lipschitz class $\text{Lip}_\infty(V;C)$, if $C > 0$ and

$$|g(\xi) - g(\eta)| \leq C\|\xi - \eta\|_\infty, \quad \xi, \eta \in V.$$

By a theorem of Rademacher (see e.g. Federer [15]), any function g of Lipschitz class $\text{Lip}_\infty(V;C)$ is differentiable a.e. with respect to Lebesgue measure and

$$\|\nabla g(\xi)\|_1 \leq C \text{ a.e.}$$

Furthermore, if $0 < C_0 \leq C$ and

$$\|\nabla g(\xi)\|_1 \leq C_0 \text{ a.e.}$$

then $g \in \text{Lip}_\infty(V;C_0)$. Given an open set $U \subseteq \mathbb{R}^m$, we will write $g \in \text{Lip}_{\text{loc}}(U)$, if to any relatively compact open subset V of U, the restriction of g to V belongs to the class $\text{Lip}_\infty(V;C)$ for an appropriate $C > 0$.

A function $f \in \text{Lip}_{\text{loc}}(\mathbb{R}^m_+)$ is said to belong to the class $\mathcal{C}$ if $f > 0$, that is, $f(x) > 0, x \in \mathbb{R}^m_+$, and

$$\langle x, |\nabla f(x)|\rangle \leq f(x) \text{ a.e.} \tag{11}$$

Given $a > 0$, we define

$$\mathcal{C}_a = (\mathcal{C} - a)^+.$$

Stated more explicitly, a function f belongs to the class $\mathcal{C}_a$ if and only if $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m_+)$, f is non-negative, and

$$\langle x, |\nabla f(x)|\rangle \leq a + f(x) \ \text{ a.e.}$$

THEOREM 4.1. *Suppose* $f : \mathbb{R}^m_+ \to [0, +\infty[$ *and let* $a > 0$. *Then* $f \in \mathcal{C}$ *if and only if* $f > 0$ *and*

$$f(xe^{\xi}) \leq f(x)e^{\|\xi\|_\infty}, \quad x \in \mathbb{R}^m_+,\ \xi \in \mathbb{R}^m. \tag{12}$$

Moreover, $f \in \mathcal{C}_a$ *if and only if*

$$a + f(xe^{\xi}) \leq (a + f(x))e^{\|\xi\|_\infty}, \quad x \in \mathbb{R}^m_+,\ \xi \in \mathbb{R}^m.$$

In particular, *any* $f \in \mathcal{C}_a$ *is a payoff function.*

PROOF. Suppose first that $f > 0$ and set

$$g(\xi) = \ln f(e^{\xi}), \quad \xi \in \mathbb{R}^m.$$

Clearly, the inequality (12) just means that $g \in \mathrm{Lip}_\infty(\mathbb{R}^m; 1)$.

Now let $f \in \mathcal{C}$. Then $g \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m)$ and

$$\nabla g(\xi) = \frac{e^{\xi}\nabla f(e^{\xi})}{f(e^{\xi})} \ \text{ a.e.} \tag{13}$$

Moreover, $\|\nabla g(\xi)\|_1 \leq 1$, a.e. and, hence, $g \in \mathrm{Lip}_\infty(\mathbb{R}^m; 1)$.

Conversely, suppose $g \in \mathrm{Lip}_\infty(\mathbb{R}^m; 1)$. Then $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^m_+)$ and (13) holds. Accordingly, the inequality (11) must be true. Summing up, we have proved that $f \in \mathcal{C}$ if and only if (12) is true. The remaining part of Theorem 4.1 is now obvious from the very definition of the class $\mathcal{C}_a$. This concludes our proof of Theorem 4.1. □

In general, the following properties are immediate consequences of either Theorem 4.1 or the very definition of the class $\mathcal{C}_a$:

- **(a)** $\mathcal{C}_a$ is convex.
- **(b)** $\mathcal{C}_a \subseteq \mathcal{C}_b$ if $a \leq b$.
- **(c)** $c \in \mathcal{C}_a$ if $c \geq 0$.
- **(d)** $\lambda\mathcal{C}_a = \mathcal{C}_{\lambda a},\ \lambda > 0$.
- **(e)** $\theta\mathcal{C}_a \subseteq \mathcal{C}_a,\ 0 < \theta < 1$.
- **(f)** $\mathcal{C}_a + \mathcal{C}_b \subseteq \mathcal{C}_{a+b}$.
- **(g)** $f, g \in \mathcal{C}_a \Rightarrow \max(f, g) \in \mathcal{C}_a$.
- **(h)** $f, g \in \mathcal{C}_a \Rightarrow \min(f, g) \in \mathcal{C}_a$.
- **(i)** If T is an n by n permutation matrix or an n by n diagonal matrix with positive entries, then $f(x) \in \mathcal{C}_a \Rightarrow f(Tx) \in \mathcal{C}_a$.
- **(j)** For any $i = 1, \ldots, m$, $c_{b,i} \in \mathcal{C}_a$ if and only if $b \leq a$.
- **(k)** For any $i = 1, \ldots, m$, $\lambda c_{a,i} \notin \mathcal{C}_a$ if $\lambda > 1$.
- **(l)** For any $i = 1, \ldots, m$, $p_{b,i} \in \mathcal{C}_a$ if and only if $b \leq a$.
- **(m)** For any $i = 1, \ldots, m$, $\lambda p_{a,i} \notin \mathcal{C}_a$ if $\lambda > 1$.

(n) $f \in \mathcal{C}_a$ if f is non-negative and concave.
(o) $f \in \mathcal{C}_a \Rightarrow e^{r\tau} S_\tau f \in \mathcal{C}_a$.

Here, for the sake of completeness, we indicate a proof of Property **(n)**. To begin with it is well known that a concave function f on $\mathbb{R}_+^m$ belongs to the class $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+^m)$ and that the convex set $\{(x,t) \in \mathbb{R}_+^m \times \mathbb{R}; t \le f(x)\}$ has a hyperplane of support at each point $(x, f(x)), x \in \mathbb{R}_+^m$ (see e.g. Hörmander [17]). Moreover, by the Rademacher theorem referred to above, there exists a set $D \subseteq \mathbb{R}_+^m$ such that f is differentiable at each point of D and such that the complement of D in $\mathbb{R}_+^m$ is a null set. Accordingly,

$$f(y) \le f(x) + \langle \nabla f(x), y - x\rangle, \quad x \in D,\ y \in \mathbb{R}_+^m.$$

Thus, given $x \in D$, we have $\langle \nabla f(x), x\rangle \le f(x)$ as f is non-negative. But here $\nabla f(x) \ge 0$ since the function $h(s) = f(x + sy)$, $s \ge 0$, is non-decreasing for all $x, y \in \mathbb{R}_+^m$. This proves (11) so that $f \in \mathcal{C}_a$ for every $a > 0$.

Throughout the remaining part of this paper we assume that

$$\max_{i=1,\ldots,m} \sigma_i = \sigma_m.$$

Now given $a > 0$ and a continuous function $f : \mathbb{R}_+^m \to [0, +\infty[$, set for fixed $\tau > 0$,

$$g = g_\tau = \frac{1}{\sigma_m \sqrt{\tau}} \ln(1 + f/a). \tag{14}$$

We shall say that the function f belongs to the class $\mathcal{C}_{a,m}$ if the function

$$\Phi^{-1}(\mathbb{P}[g_\tau(x_1 e^{\sigma_1\sqrt{\tau}\langle c_1, G\rangle}, \ldots, x_m e^{\sigma_m\sqrt{\tau}\langle c_m, G\rangle}) \le s]) - s, \quad s > 0$$

is non-decreasing for every $x \in \mathbb{R}_+^m$ and every $\tau > 0$. It is readily seen that any $f \in \mathcal{C}_{a,m}$ is a payoff function. The class $\mathcal{C}_{a,m}$ turns out to be optimal in connection with a certain isoperimetric inequality we prove below. However, before stating this result we want to prove

THEOREM 4.2. *For any $a > 0$,*

$$\mathcal{C}_a \subseteq \mathcal{C}_{a,m}.$$

PROOF. Suppose $f \in \mathcal{C}_a$ and let g be as in (14), where $\tau > 0$ is fixed. We now claim that

$$g(ze^\xi) \le g(z) + \frac{\|\xi\|_\infty}{\sigma_m\sqrt{\tau}}$$

if $z \in \mathbb{R}_+^m$ and $\xi \in \mathbb{R}^m$. But

$$g(ze^\xi) = \frac{1}{\sigma_m\sqrt{\tau}} \ln((a + f(ze^\xi))/a)$$

and since $f \in \mathcal{C}_a$, Theorem 4.1 yields

$$g(ze^\xi) \le \frac{1}{\sigma_m\sqrt{\tau}} \ln((a + f(z))e^{\|\xi\|_\infty}/a)$$

and the claim above follows at once.

To complete the proof of Theorem 4.2 we represent the standard Gaussian random vector G in $\mathbb{R}^n$ as the identity mapping in $\mathbb{R}^n$ and put for any fixed $x \in \mathbb{R}^n_+$ and $s > 0$,

$$A(s) = [g(x_1 e^{\sigma_1 \sqrt{\tau}\langle c_1, G\rangle}, \ldots, x_m e^{\sigma_m \sqrt{\tau}\langle c_m, G\rangle}) \leq s].$$

Then, if $\varepsilon > 0$,

$$A(s) + \bar{B}(0;\varepsilon) \subseteq A(s+\varepsilon)$$

and the isoperimetric inequality for G gives

$$\Phi^{-1}(\mathbb{P}[A(s+\varepsilon)]) \geq \Phi^{-1}(\mathbb{P}[A(s)]) + \varepsilon.$$

Since $x \in \mathbb{R}^m_+$ and $\tau > 0$ are arbitrary, $f \in \mathcal{C}_{a,m}$ and Theorem 4.2 is proved. $\square$

In what follows we shall write $\psi \in \mathcal{V}(\varphi)$ if $\psi \in \mathcal{V}_0(\varphi)$ and

$$\overline{\lim}_{s\to+\infty} s^{-p}(|\varphi(s)| + |\psi(s)|) < +\infty$$

for an appropriate $p > 0$.

Theorem 4.3. *Suppose $\psi \in \mathcal{V}(\varphi)$. Then, if $f \in \mathcal{C}_{a,m}$ and*

$$u_{\varphi\circ f}(\tau, x) = u_{\varphi\circ c_{a,m}}(\tau, y)$$

where $x, y \in \mathbb{R}^m_+$ and $\tau > 0$ are fixed,

$$u_{\psi\circ f}(\tau, x) \leq u_{\psi\circ c_{a,m}}(\tau, y).$$

Proof. In the proof, without loss of generality, we assume that $\varphi(0) = \psi(0) = 0$. We have

$$c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)) = (y_m e^{(r-\sigma_m^2/2)\tau + \sigma_m W_m(\tau)} - a)^+$$

and hence

$$c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)) = a(e^{a_m + \sigma_m W_m(\tau)} - 1)^+$$

for a suitable constant a_m. Setting $B_m = W_m(\tau)/\sqrt{\tau}$, we get

$$c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)) = a(e^{\sigma_m\sqrt{\tau}(B_m - b_m)} - 1)^+$$

for a suitable constant b_m. Thus

$$c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)) = a(e^{\sigma_m\sqrt{\tau}(B_m - b_m)^+} - 1).$$

Now define

$$j(s) = a(e^{\sigma_m\sqrt{\tau}s} - 1), \quad s \geq 0$$

and set $\varphi_0 = \varphi(j)$ so that

$$\varphi(c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)) = \varphi_0((B_m - b_m)^+)$$

and

$$\mathbb{E}[\varphi(c_{a,m}(ye^{r\tau} M^W_\sigma(\tau)))] = \int_0^{+\infty} \mathbb{P}[(B_m - b_m)^+ > s]\, d\varphi_0(s)$$

since $\varphi_0(0) = 0$.

In the next step we introduce the function $g = g_\tau$ by the equation (14) and have $f = j(g)$ and $\varphi(f) = \varphi_0(g)$. Thus

$$\varphi(f(xe^{r\tau} M_\sigma^W(\tau))) = \varphi_0(g(xe^{r\tau} M_\sigma^W(\tau)))$$

and

$$\mathbb{E}[\varphi(f(xe^{r\tau} M_\sigma^W(\tau)))] = \int_0^{+\infty} \mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) > s]\, d\varphi_0(s).$$

Further, we define

$$h(s) = \mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) \leq s], \quad s \geq 0$$

and have

$$h(s) = \mathbb{P}[g(z_1 e^{\sigma_1 \sqrt{\tau} \langle c_1, G\rangle}, \dots, z_m e^{\sigma_m \sqrt{\tau}\langle c_m, G\rangle}) \leq s], \quad s \geq 0$$

for appropriate $z_1, \dots, z_m \in \mathbb{R}_+$.

Now suppose $s_0 \geq 0$ and

$$h(s_0) \geq \mathbb{P}[(B_m - b_m)^+ \leq s_0]. \tag{15}$$

We then have

$$h(s_0 + \varepsilon) \geq \mathbb{P}[(B_m - b_m)^+ \leq s_0 + \varepsilon], \quad \varepsilon > 0$$

because $f \in \mathcal{C}_{a,m}$ and B_m is a $N(0;1)$-distributed random variable. To complete the proof of Theorem 4.3, first set $\psi_0 = \psi(j)$ so that

$$\mathbb{E}[\psi(c_{a,m}(ye^{r\tau} M_\sigma^W(\tau)))] = \int_0^{+\infty} \mathbb{P}[(B_m - b_m)^+ > s]\, d\psi_0(s)$$

and

$$\mathbb{E}[\psi(f(xe^{r\tau} M_\sigma^W(\tau)))] = \int_0^{+\infty} \mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) > s]\, d\psi_0(s)$$

since $\psi_0(0) = 0$. Moreover, let $d\psi_0 = \lambda d\varphi_0$, where the function λ is non-decreasing, and let s_* denote the infimum over all $s_0 \geq 0$ such that (15) holds. Here, by convention, the infimum over the empty set equals $+\infty$. The extreme cases $s_* = 0$ and $s_* = +\infty$ are simple and so we concentrate on the case $0 < s_* < +\infty$. Then, for any $S \in \,]s_*, +\infty[$,

$$\begin{aligned}
&\int_0^S \mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) > s]\, d\psi_0(s) - \int_0^S \mathbb{P}[(B_m - b_m)^+ > s]\, d\psi_0(s) \\
&\quad = \int_0^{s_*} (\mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) > s] - \mathbb{P}[(B_m - b_m)^+ > s])\lambda(s)\, d\varphi_0(s) \\
&\qquad + \int_{s_*}^S (\mathbb{P}[g(xe^{r\tau} M_\sigma^W(\tau)) > s] - \mathbb{P}[(B_m - b_m)^+ > s])\lambda(s)\, d\varphi_0(s).
\end{aligned}$$

Here the right-hand side does not exceed

$$\lambda(s_*)\int_0^{s_*}(\mathbb{P}[g(xe^{r\tau}M_\sigma^W(\tau))>s]-\mathbb{P}[(B_m-b_m)^+>s])\,d\varphi_0(s)$$
$$+\lambda(s_*)\int_{s_*}^{S}(\mathbb{P}[g(xe^{r\tau}M_\sigma^W(\tau))>s]-\mathbb{P}[(B_m-b_m)^+>s])\,d\varphi_0(s),$$

which is equal to

$$\lambda(s_*)\int_0^{S}(\mathbb{P}[g(xe^{r\tau}M_\sigma^W(\tau))>s]-\mathbb{P}[(B_m-b_m)^+>s])\,d\varphi_0(s).$$

By letting S tend to plus infinity it is immediate that

$$\int_0^{+\infty}\mathbb{P}[g(xe^{r\tau}M_\sigma^W(\tau))>s]\,d\psi_0(s)\leq\int_0^{+\infty}\mathbb{P}[(B_m-b_m)^+>s]\,d\psi_0(s)$$

and Theorem 4.3 follows at once. □

If X is a non-negative random variable with positive expectation, we set

$$D_{\mathrm{rel}}[X]=\frac{\sqrt{\mathbb{E}[X^2]-(\mathbb{E}[X])^2}}{\mathbb{E}[X]}.$$

Moreover, if f is a payoff function, we use the notation

$$Z(\tau,x;f)=e^{-r\tau}f(xe^{r\tau}M_\sigma^W(\tau)).$$

Note that

$$u_f(\tau,x)=\mathbb{E}[Z(\tau,x;f)]$$

by (4).

Corollary 4.1. *Suppose $x,y\in\mathbb{R}_+^m$. If $f\in\mathcal{C}_{a,m}$ and*

$$u_f(\tau,x)\geq u_{c_{a,m}}(\tau,y) \tag{16}$$

then

$$D_{\mathrm{rel}}[Z(\tau,x;f)]\leq D_{\mathrm{rel}}[Z(\tau,y;c_{a,m})].$$

Proof. If there is equality in (16) the conclusion in Corollary 4.1 follows from Theorem 4.3. To prove the general case it therefore suffices to show that the function

$$F(y;a)=\frac{\mathbb{E}[Z^2(\tau,y,c_{a,m})]}{(\mathbb{E}[Z(\tau,y,c_{a,m})])^2}$$

is a non-increasing function of y_m. To this end, first choose $0<b<a$ and note that $\theta c_{b,m}\in\mathcal{C}_a$ for all $0<\theta\leq 1$. Since $c_{b,m}\geq c_{a,m}$ there is a $\theta\in]0,1]$ such that

$$u_{\theta c_{b,m}}(\tau,y)=u_{c_{a,m}}(\tau,y).$$

Accordingly, in view of Theorem 4.3,

$$u_{(\theta c_{b,m})^2}(\tau,y)\leq u_{c^2_{a,m}}(\tau,y)$$

and hence

$$F(y;b) \leq F(y;a).$$

Now since

$$F\left(\frac{a}{b}y;a\right) = F(y;b)$$

we are done. This completes our proof of Corollary 4.1. □

Example 4.1. The use of the Monte Carlo method in computing option prices goes back to Boyle [10]. It is especially attractive for options depending on several assets (see, for example, Barraquand [2]). To estimate the price function $u_f(\tau, x)$ in this way, let $Z_1, \ldots, Z_N$ be stochastically independent copies of $Z(\tau, x; f)$. Then the arithmetic mean

$$\bar{Z}_N = \frac{1}{N}\sum_1^N Z_j$$

is an unbiased estimator of $u_f(\tau, x)$. The variance of $\bar{Z}_N$ equals $1/N$ times the variance of $Z(\tau, x; f)$ and, assuming $u_f(\tau, x) > 0$, the Chebychev inequality yields

$$\mathbb{P}\left[\left|\frac{\bar{Z}_N - u_f(\tau, x)}{u_f(\tau, x)}\right| \geq \varepsilon\right] \leq \frac{1}{\varepsilon^2 N}(D_{\mathrm{rel}}[Z(\tau, x; f)])^2,\ \varepsilon > 0.$$

Therefore, it is interesting to have an explicit upper bound of $D_{\mathrm{rel}}[Z(\tau, x; f)]$.

As an example, consider the special case $f = f_{\max}$. If $f = f_{\max}$, clearly $f \in \mathcal{C}_a$ for all $a > 0$. Now, if $a > 0$, (16) is true with $y = x$ and Corollary 4.1 yields

$$D_{\mathrm{rel}}[Z(\tau, x; f_{\max})] \leq \lim_{a \to 0^+} D_{\mathrm{rel}}[Z(\tau, x; c_{a,m})].$$

Thus

$$D_{\mathrm{rel}}[Z(\tau, x; f_{\max})] \leq \sqrt{e^{\sigma_m^2 \tau} - 1}.$$

A completely different approximate method for computing the value of the option on the maximum (or minimum) of several assets is treated by Boyle and Tse [12]. □

The next theorem shows that the class $\mathcal{C}_{a,m}$ in Theorem 4.3 is the best possible. Indeed, we have

Theorem 4.4. *Let f be a payoff function in $\mathbb{R}_+^m$ and suppose $a > 0$. Furthermore, suppose*

$$u_{\psi \circ f}(\tau, x) \leq u_{\psi \circ c_{a,m}}(\tau, y)$$

for all $x, y \in \mathbb{R}_+^m$, all $\tau > 0$, and all ψ and bounded φ such that $\psi \in \mathcal{V}(\varphi)$ and

$$u_{\varphi \circ f}(\tau, x) = u_{\varphi \circ c_{a,m}}(\tau, y).$$

Then $f \in \mathcal{C}_{a,m}$.

Proof. To begin with, given $0 < b < c$, define $\varphi_\alpha(s) = (s-\alpha)^+$, $\alpha > 0$, and $\varphi_{b,c}(s) = \varphi_b(s) - \varphi_c(s)$ so that

$$\varphi_{b,c}(s) = \min((s-b)^+, c-b).$$

First we assume that $x \in \mathbb{R}_+^m$ and $\tau > 0$ are fixed and

$$0 < \mathbb{E}[\varphi_{b,c}(f(xe^{r\tau} M_\sigma^W(\tau)))] < c - b. \tag{17}$$

Now let $\theta > 0$ be such that

$$\mathbb{E}[\varphi_{b,c}(f(xe^{r\tau} M_\sigma^W(\tau)))] = \mathbb{E}[\varphi_{b,c}(c_{a,m}(\theta xe^{r\tau} M_\sigma^W(\tau)))]. \tag{18}$$

We next choose $h > 0$ so small that $b < b+h < c$ and define $\kappa(s) = -\min(s,h)$. Then $-\varphi_{b,b+h} = \kappa \circ \varphi_{b,c}$ and thus

$$\mathbb{E}[\varphi_{b,b+h}(f(xe^{r\tau} M_\sigma^W(\tau)))] \geq \mathbb{E}[\varphi_{b,b+h}(c_{a,m}(\theta xe^{r\tau} M_\sigma^W(\tau)))].$$

In the following, if $A \subseteq \mathbb{R}$, the function χ_A defined on $\mathbb{R}$ equals one in A and zero off A. Using this notation,

$$h\chi_{]b,+\infty[}(s) \geq \varphi_{b,b+h} \tag{19}$$

and hence

$$h\mathbb{P}[f(xe^{r\tau} M_\sigma^W(\tau)) > b] \geq \mathbb{E}[\varphi_{b,b+h}(c_{a,m}(xe^{r\tau} M_\sigma^W(\tau)))].$$

Thus

$$\begin{aligned} &h\mathbb{P}[f(xe^{r\tau} M_\sigma^W(\tau)) > b] \\ &\quad \geq \mathbb{E}[((\theta x_m e^{r\tau} M_{\sigma_m}^{W_m}(\tau) - a)^+ - b)^+] - \mathbb{E}[((\theta x_m e^{r\tau} M_{\sigma_m}^{W_m}(\tau) - a)^+ - b - h)^+]. \end{aligned}$$

From this we get

$$\mathbb{P}[f(xe^{r\tau} M_\sigma^W(\tau)) > b] \geq -\frac{d}{db}\mathbb{E}[((\theta x_m e^{r\tau} M_{\sigma_m}^{W_m}(\tau) - a)^+ - b)^+]$$

where the right-hand side equals

$$\mathbb{P}[(\theta x_m e^{r\tau} M_{\sigma_m}^{W_m}(\tau) - a)^+ - b > 0] = \mathbb{P}[\theta x_m e^{r\tau} M_{\sigma_m}^{W_m}(\tau) - a - b > 0]$$

and, accordingly,

$$\mathbb{P}[f(xe^{r\tau} M_\sigma^W(\tau)) > b] \geq \Phi\left(\frac{1}{\sigma_m\sqrt{\tau}} \ln \frac{\theta x_m e^{(r-\sigma_m^2/2)\tau}}{a+b}\right). \tag{20}$$

Now suppose $b < c-h < c$ and observe that $\varphi_{c-h,c} = \varphi_{c-h-b} \circ \varphi_{b,c}$. Remembering (18), we have

$$\mathbb{E}[\varphi_{c-h,c}(f(xe^{r\tau} M_\sigma^W(\tau)))] \leq \mathbb{E}[\varphi_{c-h,c}(c_{a,m}(\theta xe^{r\tau} M_\sigma^W(\tau)))].$$

Furthermore, since

$$h\chi_{]c,+\infty[}(s) \leq \varphi_{c-h,c}(s) \tag{21}$$

it follows that

$$h\mathbb{P}[f(xe^{r\tau}M_\sigma^W(\tau)) > c] \leq \mathbb{E}[\varphi_{c-h,c}(c_{a,m}(xe^{r\tau}M_\sigma^W(\tau)))]$$

and as above we conclude that

$$\mathbb{P}[f(xe^{r\tau}M_\sigma^W(\tau)) > c] \leq \Phi\left(\frac{1}{\sigma_m\sqrt{\tau}}\ln\frac{\theta x_m e^{(r-\sigma_m^2/2)\tau}}{a+c}\right). \qquad (22)$$

Comparing (20) and (22) it follows that

$$(a+b)e^{\sigma_m\sqrt{\tau}\Phi^{-1}(\mathbb{P}[f(xe^{r\tau}M_\sigma^W(\tau))>b])} \geq (a+c)e^{\sigma_m\sqrt{\tau}\Phi^{-1}(\mathbb{P}[f(xe^{r\tau}M_\sigma^W(\tau))>c])}.$$

Clearly, this inequality also holds if (17) is violated and we conclude that the function

$$\sigma_m\sqrt{\tau}\Phi^{-1}(\mathbb{P}[f(xe^{r\tau}M_\sigma^W(\tau)) \leq s]) - \ln(a+s), \quad s \geq 0$$

is non-decreasing. Since this is true for all $x \in \mathbb{R}_+^m$ and all $\tau > 0$ we conclude that the function f belongs to the class $\mathcal{C}_{a,m}$, which completes our proof of Theorem 4.4. □

Suppose now that f is a general payoff function. The expectation at time t of the value of the derivative security $\mathcal{U}_f^T$ at the maturity date T equals $v_f(\tau, X(t); 0)$, where

$$v_f(\tau, x; 0) = \mathbb{E}[f(xe^{\mu\tau+\sigma W(\tau)})].$$

Here we employ the vector notation $\mu = (\mu_1, \ldots, \mu_m), \sigma = (\sigma_1, \ldots, \sigma_m)$, and $W(\tau) = (W_1(\tau), \ldots, W_m(\tau))$. Thus

$$v_f(\tau, x; 0) = e^{r\tau}u_f(\tau, xe^{(\mu-r+\sigma^2/2)\tau}).$$

Now, suppose $t < t_* \leq T$ and set $\tau_* = T - t_*$. If $\tau_* > 0$, the expectation at time t of the value of $\mathcal{U}_f^T$ at time t_* equals $v_f(\tau, X(t); \tau_*)$, where

$$v_f(\tau, x; \tau_*) = \mathbb{E}[u_f(\tau_*, xe^{\mu(t_*-t)+\sigma W^0(t_*-t)})]$$

and where W^0 is a stochastically independent copy of W. Hence

$$v_f(\tau, x; \tau_*) = \mathbb{E}[e^{-r\tau_*}f(xe^{\mu(t_*-t)+\sigma W^0(t_*-t)}e^{r\tau_*}M_\sigma^W(\tau_*))].$$

Since $t_* - t = \tau - \tau_*$, we get

$$v_f(\tau, x; \tau_*) = \mathbb{E}[e^{-r\tau_*}f(xe^{(\mu+\sigma^2/2)(\tau-\tau_*)}e^{r\tau_*}M_\sigma^W(\tau))].$$

Thus

$$v_f(\tau, x; \tau_*) = e^{r(\tau-\tau_*)}u_f(\tau, xe^{(\mu-r+\sigma^2/2)(\tau-\tau_*)}).$$

Alternatively, it is simple to derive the same formula using the semi-group property of the family $(S_\tau)_{\tau>0}$.

Theorem 4.3 thus has the following consequence:

Corollary 4.2. *Let $\psi \in \mathcal{V}(\varphi)$. Then, if $f \in \mathcal{C}_{a,m}$ and*

$$v_{\varphi \circ f}(\tau, x; \tau_*) = v_{\varphi \circ c_{a,m}}(\tau, y; \tau_*)$$

where $x, y \in \mathbb{R}_+^m$ and $\tau \geq \tau_* \geq 0$ are fixed,

$$v_{\psi \circ f}(\tau, x; \tau_*) \leq v_{\psi \circ c_{a,m}}(\tau, y; \tau_*).$$

5. Extremal Properties of Puts

Given $a > 0$, we define

$$\mathcal{P}_a = (a - \mathcal{C})^+.$$

Stated more explicitly, a function $f \in \mathcal{P}_a$ if and only if $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}_+^m)$, $0 \leq f < a$ and

$$\langle x, |\nabla f(x)| \rangle + f(x) \leq a, \quad \text{a.e.}$$

In view of Theorem 4.1 we now have

Theorem 5.1. *Suppose $a > 0$ and let $f : \mathbb{R}_+^m \to [0, a[$. Then $f \in \mathcal{P}_a$ if and only if*

$$\frac{1}{a - f(xe^{\xi})} \leq \frac{e^{\|\xi\|_\infty}}{a - f(x)}, \quad x \in \mathbb{R}_+^m, \ \xi \in \mathbb{R}_+^m.$$

In general, the following properties are immediate consequences of either Theorem 5.1 or the very definition of the class $\mathcal{P}_a$:

- **(a)** $\mathcal{P}_a$ is convex.
- **(b)** $\mathcal{P}_a \subseteq \mathcal{P}_b$ if $a \leq b$.
- **(c)** $c \in \mathcal{P}_a$ if $0 \leq c < a$.
- **(d)** $\lambda \mathcal{P}_a = \mathcal{P}_{\lambda a}, \ \lambda > 0$.
- **(e)** $\theta \mathcal{P}_a \subseteq \mathcal{P}_a, \ 0 < \theta < 1$.
- **(f)** $\mathcal{P}_a + \mathcal{P}_b \subseteq \mathcal{P}_{a+b}$.
- **(g)** $f, g \in \mathcal{P}_a \Rightarrow \max(f, g) \in \mathcal{P}_a$.
- **(h)** $f, g \in \mathcal{P}_a \Rightarrow \min(f, g) \in \mathcal{P}_a$.
- **(i)** If T is an n by n permutation matrix or an n by n diagonal matrix with positive entries, then $f(x) \in \mathcal{P}_a \Rightarrow f(Tx) \in \mathcal{P}_a$.
- **(j)** For any $i = 1, \ldots, m$, $p_{b,i} \in \mathcal{P}_a$ if and only if $b \leq a$.
- **(k)** For any $i = 1, \ldots, m$, $\lambda p_{a,i} \notin \mathcal{P}_a$ if $\lambda > 1$.
- **(l)** $f \in \mathcal{P}_a$ if $0 \leq f < a$ is convex.
- **(m)** $f \in \mathcal{P}_a \Rightarrow e^{r\tau} S_\tau f \in \mathcal{P}_a$.

We are now going to introduce slightly larger classes of payoff functions than the classes $\mathcal{P}_a$, $a > 0$. To this end, let $a > 0$ be given and suppose $f : \mathbb{R}_+^m \to [0, a[$ is a continuous function and set for fixed $\tau > 0$,

$$g = g_\tau = -\frac{1}{\sigma_m \sqrt{\tau}} \ln(1 - f/a). \tag{23}$$

We shall say that the the function f belongs to the class $\mathcal{P}_{a,m}$ if the function

$$\Phi^{-1}(\mathbb{P}[g_\tau(x_1 e^{\sigma_1\sqrt{\tau}\langle c_1,G\rangle},\dots,x_m e^{\sigma_m\sqrt{\tau}\langle c_m,G\rangle})\leq s])-s,\quad s>0$$

is non-decreasing for every $x\in\mathbb{R}^m_+$ and $\tau>0$. Here, again, $G=(G_1,\dots,G_n)$ denotes the standard Gaussian random vector in $\mathbb{R}^n$ with stochastically independent $N(0;1)$-distributed components.

We now set, for any $f\in\mathcal{P}_{a,m}$,

$$I_a(f)=\frac{af}{a-f}$$

and have

$$(1-f/a)(1+I_a(f)/a)=1.$$

THEOREM 5.2. (a) *The map I_a is a bijection of $\mathcal{P}_{a,m}$ onto $\mathcal{C}_{a,m}$.*
(b) *The restriction map of I_a to $\mathcal{P}_a$ is a bijection of $\mathcal{P}_a$ onto $\mathcal{C}_a$.*

PROOF. Part (a) follows at once from the equations (14) and (23). Moreover Part (b) is an immediate consequence of Theorems 4.1 and 5.1. □

THEOREM 5.3. *Suppose $\psi\in\mathcal{V}(\varphi)$. Then, if $f\in\mathcal{P}_{a,m}$ and*

$$u_{\varphi\circ f}(\tau,x)=u_{\varphi\circ p_{a,m}}(\tau,y)$$

where $x,y\in\mathbb{R}^m_+$ and $\tau>0$ are fixed,

$$u_{\psi\circ f}(\tau,x)\leq u_{\psi\circ p_{a,m}}(\tau,y).$$

PROOF. Set $f^*=I_a(f)$ and $p^*_{a,m}=I_a(p_{a,m})$. Then

$$f=\frac{af^*}{a+f^*}$$

and

$$p_{a,m}=\frac{ap^*_{a,m}}{a+p^*_{a,m}}.$$

Moreover, we define

$$\varphi^*(s)=\varphi\Big(\frac{as}{a+s}\Big),\quad s\geq 0.$$

Then

$$u_{\varphi\circ f}(\tau,x)=u_{\varphi^*\circ f^*}(\tau,x)$$

and

$$u_{\varphi\circ p_{a,m}}(\tau,y)=u_{\varphi^*\circ p^*_{a,m}}(\tau,y).$$

From the definition of the map I_a it follows that

$$p^*_{a,m}(v)=\Big(\frac{a^2}{v_m}-a\Big)^+,\quad v\in\mathbb{R}^m_+$$

and using (4) we conclude that

$$u_{\varphi^*\circ p^*_{a,m}}(\tau,y)=u_{\varphi^*\circ c_{a,m}}(\tau,z)$$

where $z \in \mathbb{R}_+^m$ and

$$z_m = \frac{a^2 e^{-2r\tau + \sigma_m^2 \tau}}{y_m}.$$

The result is now an immediate consequence of Theorem 4.3. This completes our proof of Theorem 5.3. □

THEOREM 5.4. *Suppose $a > 0$ and let $f : \mathbb{R}_+^m \to [0, a[$ be a payoff function. Furthermore, suppose*

$$u_{\psi\circ f}(\tau, x) \le u_{\psi\circ p_{a,m}}(\tau, y)$$

for all $x, y \in \mathbb{R}_+^m$, all $\tau > 0$, and all ψ and bounded φ such that $\psi \in \mathcal{V}(\varphi)$ and

$$u_{\varphi\circ f}(\tau, x) = u_{\varphi\circ p_{a,m}}(\tau, y).$$

Then $f \in \mathcal{P}_{a,m}$.

PROOF. By exploiting the map I_a as in the proof of Theorem 5.3 the result follows at once from Theorem 4.4. □

The next result follows from Theorem 5.3 in the same way as Corollary 4.2 follows from Theorem 4.3.

COROLLARY 5.1. *Let $\psi \in \mathcal{V}(\varphi)$. Then, if $f \in \mathcal{P}_{a,m}$ and*

$$v_{\varphi\circ f}(\tau, x; \tau_*) = v_{\varphi\circ p_{a,m}}(\tau, y; \tau_*),$$

where $x, y \in \mathbb{R}_+^m$ and $\tau > \tau_* \ge 0$ are fixed,

$$v_{\psi\circ f}(\tau, x; \tau_*) \le v_{\psi\circ p_{a,m}}(\tau, y; \tau_*).$$

References

[1] Bakry, D. and Ledoux, M. Lévy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator. Invent. math. **123** (1995), 259–281.

[2] Barraquand, J. Numerical valuation of high dimensional multivariate European securities. Paris Research Laboratory, Report No. 26 (1993).

[3] Berwald, L. Verallgemeinerungen eines Mittelwertsatzes von J. Favard für positive konkave Funktionen. Acta Math. **79** (1947), 16–37.

[4] Black, F. and Scholes, M. The pricing of options and corporate liabilities. Journal of Political Economy **72** (1973), 164–175.

[5] Bobkov, S. An isoperimetric inequality on the discrete cube and an elementary proof of the isoperimetric inequality in Gauss space. Ann. Prob. **25** (1997), 206–214.

[6] Borell, Ch. Integral inequalities for generalized concave or convex functions. J. Math. Anal. Appl. **43** (1973), 419–440.

[7] Borell, Ch. The Brunn-Minkowski inequality in Gauss space. Invent. math. **30** (1975), 207–216.

[8] Borell, Ch. Gaussian Radon measures on locally convex spaces. Math. Scand. **38** (1976), 265–284.

[9] Borell, Ch. Geometric properties of some familiar diffusions in $\mathbb{R}^n$. Ann. Probability **21** (1993), 482–489.

[10] Boyle P. P. Options: A Monte Carlo approach. Journal of Financial Economics **4** (1977), 323–338.

[11] Boyle P. P. The quality option and timing options in future contracts. The Journal of Finance **44** (1989), 101–113.

[12] Boyle, P. P. and Tse, Y. K. An algorithm for computing values of options on the maximum or minimum of several assets. Journal of Finance and Quantitative Analysis **25** (1990), 215–227.

[13] Davidovič, Ju. S., Korenbljum, B. I., Hacet, T. A property of logarithmically concave functions. Soviet Math. Dokl. **10** (1969), 477–480.

[14] Duffie, D. Dynamic Asset Pricing Theory. Princeton University Press, Princeton (1992).

[15] Federer, H. Geometric Measure Theory. Springer, Berlin and Heidelberg (1969).

[16] Harrison, J. M. and Pliska S. R. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Processes and their Applications **11** (1981), 215–216.

[17] Hörmander, L. Notions of Convexity. Birkhäuser, Boston, Basel, and Berlin (1994).

[18] Johnson, H. Options on the maximum or minimum of several assets. Journal of Financial and Quantitative Analysis **22** (1987), 277–283.

[19] Margrabe, W. The value of an option to exchange one asset for another. Journal of Finance **33** (1978), 177–186.

[20] Merton, R. The theory of rational option pricing. Bell Journal of Economics and Management Science **4** (1973), 141–183.

[21] Prékopa, A. On logarithmic concave measures and functions. Acta Sci. Math. (Szeged) **34** (1973), 335–343.

[22] Stulz, R. Options on the minimum or the maximum of two risky assets: Analysis and Applications. Journal of Financial Economics **10** (1982), 161–185.

[23] Sudakov, V. N. and Tsyrelson, B. S. Extremal properties of half-spaces for spherically invariant measures. J. Sov. Math. **9** (1978), 9–18. Translated from Zap. Nauchn. Sem. Leninggrad. Otdel. Mat. Inst. Steklov **41** (1974), 14–24.

[24] Tong, Y. L. The Multivariate Normal Distribution. Springer, Berlin and New York (1990).

Christer Borell
Chalmers University of Technology
Department of Mathematics
CTH&GU
S-412 96 Göteborg
Sweden
borell@math.chalmers.se