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On a Generalization of the Busemann–Petty Problem

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Abstract. The generalized Busemann–Petty problem asks: If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and the volume of $K \cap H$ is smaller than the volume of $L \cap H$ for every i-dimensional subspace H, $1 < i < n$, does it follow that the volume of K is smaller than the volume of L? The hyperplane case $i = n-1$ is known as the Busemann–Petty problem. It has a negative answer when $n > 4$, and has a positive answer when $n = 3, 4$. This paper gives a negative answer to the generalized Busemann–Petty problem for $3 < i < n$ in the stronger sense that the integer i is not fixed. For the 2-dimensional case $i = 2$, it is proved that the problem has a positive answer when L is a ball and K is close to L.

1. Introduction

Denote by $\mathrm{vol}_i(\cdot)$ the i-dimensional Lebesgue measure, and denote by $G_{i,n}$ the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$. The *generalized Busemann–Petty problem* asks:

GBP. *If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, is there the implication*

$$\mathrm{vol}_i(K \cap \xi) \le \mathrm{vol}_i(L \cap \xi), \quad \forall \xi \in G_{i,n} \quad \Longrightarrow \quad \mathrm{vol}_n(K) \le \mathrm{vol}_n(L)? \tag{1.1}$$

The case of $i = 1$ is trivially true. The hyperplane case $i = n - 1$ is well-known as the *Busemann–Petty problem* (see [BP] and [Bu]). Many authors contributed to the solution of the Busemann–Petty problem (see [Ba] [Bo] [G1] [Gia] [Gie] [GR] [Ha] [Lu] [LR] [Pa] [Z1]). The problem has a negative answer when $n > 4$ (see [G1], [Pa] and [Z2]), and it has a positive answer when $n = 3, 4$ (see [G2] and [Z4]). The notion of *intersection body*, introduced by Lutwak [Lu], plays an

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important role in the solution of the Busemann–Petty problem. It relates to the positivity of the inverse spherical Radon transform.

Because of the special feature of the answer to the Busemann–Petty problem, it is interesting to consider the generalized Busemann–Petty problem. What are the dimensions of cross sections and ambient spaces so that the generalized Busemann–Petty problem has a positive or negative answer? By introducing the notion of *i-intersection body* and using techniques in functional analysis and Radon transforms on Grassmannians, it is proved in [Z3] that the answer to the generalized Busemann–Petty problem is equivalent to the existence of origin-symmetric convex bodies which are not i-intersection bodies. When $3 < i < n$, we give a negative answer to the problem. The argument shows that cylinders are not i-intersection bodies if $3 < i < n$. We also give a partial answer to the case of 2-dimensional sections. We remark that one of the results in [Z3] that no polytope is an i-intersection body is not correct.

It is shown in [Z3] that the generalized Busemann–Petty problem has a positive answer if K is an i-intersection body, in particular, if K is a ball in $\mathbb{R}^n$. However, when L is a ball, the generalized Busemann–Petty problem may still have a negative answer. For instance, Keith Ball observed that one can construct counterexamples by using the techniques in [Ba] and letting $K =$ the unit cube, $L =$ a ball of appropriate radius when n and i are sufficiently large. We prove that, when L is a ball and K is sufficiently close to L, the generalized Busemann–Petty problem of 2-dimensional sections has a positive answer. The result is contained in the following theorem.

Theorem 1.1. *Let K be a centered convex body and let B_n be the standard unit ball in $\mathbb{R}^n$. There exists $\delta_0 > 0$ which only depends on the dimension so that if* $\operatorname{dist}(K, B_n) < \delta_0$ *then*

$$\operatorname{vol}_2(K \cap \xi) \le \operatorname{vol}_2(B_n \cap \xi), \quad \forall \xi \in G_{2,n} \quad \Longrightarrow \quad \operatorname{vol}_n(K) \le \operatorname{vol}_n(B_n).$$

Let ω_n be the volume of B_n. By the homogeneity of the inequalities in the last implication, we obtain the following corollary.

Corollary 1.2. *Let K be a centered convex body in $\mathbb{R}^n$. There exists $\delta_0 > 0$ which only depends on the dimension so that if the distance of K to a ball is less than δ_0, then*

$$\operatorname{vol}_n(K)^{\frac{2}{n}} \le \frac{\omega_n^{\frac{2}{n}}}{\pi} \max_{\xi \in G_{2,n}} \operatorname{vol}_2(K \cap \xi). \tag{1.2}$$

Inequality (1.2) is proved for any centered convex bodies in $\mathbb{R}^3$ in [G3]. It might be still true for any centered convex bodies in all dimensions as well.

Note that, for the generalized Busemann–Petty problem, the dimension i of sections in the implication (1.1) is fixed. It is natural to ask what will happen if the dimension i of sections is not fixed but takes different values. We would like to thank V.D. Milman who brought our attention to this question. Our answer is contained in the following theorem.

THEOREM 1.3. *There exist centered convex bodies of revolution K and L so that, for all* $3 < i < n$,

$$\mathrm{vol}_i(K \cap \xi) < \mathrm{vol}_i(L \cap \xi), \quad \forall \xi \in G_{i,n},$$

but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

This result is best possible in the class of convex bodies of revolution. It is proved that, *if K is a centered convex body of revolution, then*

$$\mathrm{vol}_i(K \cap \xi) \le \mathrm{vol}_i(L \cap \xi), \quad \forall \xi \in G_{i,n}, \quad \Longrightarrow \quad \mathrm{vol}_n(K) \le \mathrm{vol}_n(L),$$

when $i = 2$, *or* 3. See [G1], [Z2] and [Z3].

The proofs of Theorems 1.1 and 1.3 use the tools of Radon transforms on Grassmannians. We give definitions and basic facts of the Radon transforms for later use.

Let $C_e(S^{n-1})$ be the space of continuous even functions on the unit sphere S^{n-1}, and denote by $C(G_{i,n})$ the space of continuous functions on $G_{i,n}$. The *Radon transform*, for $2 \le i \le n-1$,

$$\mathrm{R}_i : C_e(S^{n-1}) \longrightarrow C(G_{i,n})$$

is defined by

$$(\mathrm{R}_i f)(\xi) = \frac{1}{i\omega_i} \int_{u \in S^{n-1} \cap \xi} f(u)\, du, \quad \xi \in G_{i,n},\ f \in C_e(S^{n-1}),$$

where ω_i and du are the volume and the surface area element of the i-dimensional unit ball, respectively.

Let ρ_K be the *radial function* of a centered convex body K in $\mathbb{R}^n$ given by

$$\rho_K(u) = \max\{\lambda \ge 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

The Radon transform R_i is closely connected with the central sections of centered bodies by the following formula

$$(\mathrm{R}_i \rho_K^i)(\xi) = \frac{1}{\omega_i} \mathrm{vol}_i(K \cap \xi), \quad \xi \in G_{i,n}. \tag{1.3}$$

The *dual transform* R_i^t of $\mathbb{R}_i$ is the map $C(G_{i,n}) \to C_e(S^{n-1})$ given by

$$(\mathrm{R}_i^t g)(u) = \int_{u \in \xi \in G_{i,n}} g(\xi)\, d\xi, \quad u \in S^{n-1},\ g \in C(G_{i,n}).$$

We have the following duality [He, pp. 144 and 161]:

$$\langle \mathrm{R}_i f, g \rangle = \langle f, \mathrm{R}_i^t g \rangle, \quad f \in C_e(S^{n-1}),\ g \in C(G_{i,n}), \tag{1.4}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product of functions in homogeneous spaces.

2. Two-Dimensional Sections

In this section we give the proof of Theorem 1.1. One technical part of the proof is to approximate arbitrary convex bodies by smooth convex bodies quantitatively. We use convolutions on the rotation group $\mathrm{SO}(n)$ of $\mathbb{R}^n$.

Let G be a compact Lie group. Let $C(G)$ be the space of continuous functions on G with the uniform topology. For $f,\ g \in C(G)$, the *convolution* $f * g \in C(G)$ of f and g is defined by

$$(f * g)(u) = \int_{v\in G} f(uv^{-1})g(v)\,dv = \int_{v\in G} f(v)g(v^{-1}u)\,dv,$$

where dv is the invariant probability measure of G.

Associated with a convex body K is its *support function* h_K defined on S^{n-1} by

$$h_K(u) = \max\{\langle u, x\rangle : x \in K\}, \quad u \in S^{n-1},$$

where $\langle u, x\rangle$ is the usual inner product of u and x in $\mathbb{R}^n$. The *polar body* K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \le 1 \text{ for all } y \in K\}.$$

Its support function is given by

$$h_{K^*}(u) = \rho_K^{-1}(u), \quad u \in S^{n-1}.$$

If h_K is the support function of K, and f is a positive function on $\mathrm{SO}(n)$, then $f * h_K$ is the support function of another convex body. Moreover, the convolution preserves the symmetry of the convex body. For a proof of this fact and more details on convex bodies and convolutions, see [GZ].

Lemma 2.1. *Let G be a compact Lie group of dimension m. If f is Lipschitz continuous on G, then there exists $\delta_0 > 0$ which depends only on G and the Lipschitz constant of f, so that for any $\delta < \delta_0$ there exists C^∞ positive function ϕ_δ satisfying*

$$|\phi_\delta * f - f| < \delta, \quad \|\phi_\delta * f\|_{C^2} < \delta^{-m-3}.$$

Proof. Let B_δ be the geodesic ball of radius δ at the unit of G. Let ϕ be a C^∞ nonnegative function which is strictly positive inside $B_{\delta/2}$ but is zero outside B_δ. Let $\exp : T_eG \to G$ be the exponential map. Condiser the C^∞ function

$$\phi_\delta(x) = a_\delta^{-1}\phi\big(\exp(\delta^{-1}\exp^{-1}(x))\big),$$

where $a_\delta = \int_G \phi\big(\exp(\delta^{-1}\exp^{-1}(x))\big)\,dx$. When δ is small,

$$a_\delta \sim c\delta^m, \tag{2.1}$$

for some constant c.

Since f is Lipschitz continuous, we have

$$\begin{aligned}|\phi_\delta * f(x) - f(x)| &= \left|\int_G \phi_\delta(y) f(yx)\,dy - \int_G \phi_\delta(y) f(x)\,dy\right| \\ &\le \int_G \phi_\delta(y)|f(yx) - f(x)|\,dy \le c_1\delta. \qquad (2.2)\end{aligned}$$

From the following equalities,

$$\phi_\delta * f(x) = \int_G \phi_\delta(xy^{-1}) f(y)\,dy = a_\delta^{-1}\int_G \phi\big(\exp(\delta^{-1}\exp^{-1}(xy^{-1}))\big) f(y)\,dy,$$

the second order derivatives of $\phi_\delta * f$ yield a factor δ^{-2}. Therefore, when δ is small (depending on the upper bound of f), (2.1) gives

$$\|\phi_\delta * f\|_{C^2} \le c_2\delta^{-m-2}. \qquad (2.3)$$

Note that c_1 and c_2 only depend on G and the Lipschitz constant of f. From (2.2) and (2.3), the required inequalities follow immediately. □

LEMMA 2.2. *Let K be a centered convex body in $\mathbb{R}^n$. There exists $\delta_0 > 0$ which only depends on the dimension and the diameters of K and its polar body, so that for any $\delta < \delta_0$ there exists a centered convex body K_δ with C^∞ radial function ρ_{K_δ} so that*

$$|\rho_{K_\delta} - \rho_K| < \delta, \quad \|\rho_{K_\delta}\|_{C^2} < \delta^{-n^2}. \qquad (2.4)$$

PROOF. Consider the support function h_{K^*} of the polar body K^* of K. Since the sphere $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is a homogeneous space, the support function h_{K^*} on S^{n-1} can be viewed as a function on $\mathrm{SO}(n)$. From Lemma 2.1, for any $\delta < \delta_0$ there exists C^∞ function ϕ_δ so that

$$|\phi_\delta * h_{K^*} - h_{K^*}| < \delta, \quad \|\phi_\delta * h_{K^*}\|_{C^2} < \delta^{-m-3},$$

where $m = \dim \mathrm{SO}(n) = \frac{1}{2}(n^2 - n)$. The number δ_0 depends on the dimension and the Lipschitz constant of h_{K^*}. Since the Lipschitz constant of h_{K^*} depends only on the dimension and the diameter of K^*, the number δ_0 only depends on the dimension and the diameter of K^*.

Define a centered convex body K_δ by

$$\rho_{K_\delta}^{-1} = h_{K_\delta^*} = \phi_\delta * h_{K^*}.$$

Therefore, ρ_{K_δ} is C^∞ and satisfies the inequalities

$$|\rho_{K_\delta}^{-1} - \rho_K^{-1}| < \delta, \quad \|\rho_{K_\delta}^{-1}\|_{C^2} < \delta^{-m-3}.$$

By using the fact that the function ρ_{K_δ} and its first order derivative are bounded by a constant depending on the diameter of K, we conclude (2.4) from the last two inequalities. □

We will use the symbol $\lesssim$ which means that the expression on the left-hand side of the symbol is less than the expression on the right-hand side by a constant factor depending only on the dimension.

LEMMA 2.3. *Let F be a uniform Lipschitz function on S^{n-1}. If, for $0<\delta<1$,*

$$\|\mathrm{R}_2 F\|_2 < \delta, \tag{2.5}$$

then

$$\|F\|_\infty \lesssim \delta^{\frac{1}{n+3}}. \tag{2.6}$$

PROOF. Consider the spherical harmonic expansion of F, $F=\sum Y_k$. Then

$$\|\mathrm{R}_2 F\|_2 \sim \left(\sum k^{-1}\|Y_k\|_2^2\right)^{\frac{1}{2}}, \tag{2.7}$$

where $\sim$ means that the quantities on both sides of it are bounded by each other with constant factors depending only on the dimension. See [St].

For $0<r<1$, let

$$P_r F = \sum r^k Y_k.$$

Since the Lipschitz constant of F is uniformly bounded on S^{n-1}, we have

$$\|P_r F - F\|_\infty \lesssim (1-r)^{\frac{1}{2}}. \tag{2.8}$$

See [BL].

From (2.7) and (2.5), we obtain

$$\begin{aligned}\|P_r F\|_\infty &< \sum r^k \|Y_k\|_\infty \lesssim \sum r^k k^{\frac{n-1}{2}} \|Y_k\|_2 \\ &\lesssim \left(\sum r^k k^{\frac{n-1}{2}} k^{\frac{1}{2}}\right) \|\mathrm{R}_2 F\|_2 \lesssim (1-r)^{-\frac{n}{2}-1}\delta. \end{aligned} \tag{2.9}$$

Thus by (2.8), (2.9) and choosing $r = 1-\delta^{\frac{2}{n+3}}$, we have

$$\|F\|_\infty \lesssim (1-r)^{\frac{1}{2}} + (1-r)^{-\frac{n}{2}-1}\delta \le \delta^{\frac{1}{n+3}}.$$

This completes the proof. □

PROOF OF THEOREM 1.1. Consider the ratio

$$\bar{V}_n(K) = \frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(B_n)}.$$

Choose the invariant probability measures on the sphere S^{n-1} and on the Grassmannian $G_{2,n}$. From (1.4), we have

$$\bar{V}_n(K) = \int \rho_K^n = \int [\mathrm{R}_2(\mathrm{R}_2^t \mathrm{R}_2)^{-1}\rho_K^{n-2}](\mathrm{R}_2\rho_K^2). \tag{2.10}$$

The implication in Theorem 1.1 becomes

$$\bar{V}_2(K\cap\xi) \le 1, \quad \xi \in G_{2,n} \Longrightarrow \bar{V}_n(K) \le 1. \tag{2.11}$$

Assume that

$$\begin{cases} \bar{V}_n(K) > 1, \\ \bar{V}_2(K \cap \xi) \le 1, \quad \xi \in G_{2,n}, \end{cases} \tag{2.12}$$

and

$$\operatorname{dist}(K, B_n) = \delta_0 > 0. \tag{2.13}$$

From Lemma 2.2, there exists K_1 such that

$$|\rho_{K_1} - \rho_K| < \delta, \quad \|\rho_{K_1}\|_{C^2} < \delta^{-n^2}, \tag{2.14}$$

where $\delta = \delta_0^N$, and N is a constant to be chosen which only depends on the dimension.

Apply (2.10) to K_1. Then by (2.12) and (2.14) we have

$$\bar{V}_n(K) + o(\delta) = \int [\mathrm{R}_2(\mathrm{R}_2^t \mathrm{R}_2)^{-1} \rho_{K_1}^{n-2}](\mathrm{R}_2 \rho_{K_1}^2), \tag{2.15}$$

and

$$\begin{aligned} \int \mathrm{R}_2(\mathrm{R}_2^t \mathrm{R}_2)^{-1} \rho_{K_1}^{n-2} &= \int \rho_{K_1}^{n-2} \le \left(\int \rho_{K_1}^n \right)^{\frac{n-2}{n}} \\ &= \bar{V}_n(K_1)^{\frac{n-2}{n}} + o(\delta) \le \bar{V}_n(K) + o(\delta). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), we obtain

$$o(\delta) \le \int [\mathrm{R}_2(\mathrm{R}_2^t \mathrm{R}_2)^{-1} \rho_{K_1}^{n-2}](\mathrm{R}_2 \rho_{K_1}^2 - 1). \tag{2.17}$$

The assumption (2.13) gives

$$|\rho_K - 1| \le \delta_0, \tag{2.18}$$

$$|\rho_{K_1} - 1| \le 2\delta_0. \tag{2.19}$$

Hence

$$\|\nabla \rho_{K_1}\| \lesssim \delta_0^{\frac{1}{2}}. \tag{2.20}$$

A proof of the last inequality can be found in [Bo]. From (2.20), one has

$$\left\| \nabla \left(\rho_{K_1}^{n-2} - \int \rho_{K_1}^{n-2} \right) \right\| \lesssim \delta_0^{\frac{1}{2}}.$$

If E_{2-n} extends a function on S^{n-1} to a homogeneous function of degree $2-n$ in $\mathbb{R}^n$, and S restricts a function in $\mathbb{R}^n$ to S^{n-1}, then,

$$\left\| S(-\Delta_{\mathbb{R}^n})^{\frac{1}{2}} E_{2-n} \left(\rho_{K_1}^{n-2} - \int \rho_{K_1}^{n-2} \right) \right\|_{BMO} \lesssim \delta_0^{\frac{1}{2}}. \tag{2.21}$$

Use here the fact that L^∞-control on the tangential derivative yields BMO-control on the normal derivative.

By (2.14), the inequality (2.21) implies

$$\left\| S(-\Delta_{\mathbb{R}^n})^{\frac{1}{2}} E_{2-n} \left(\rho_{K_1}^{n-2} - \int \rho_{K_1}^{n-2} \right) \right\|_\infty \lesssim \delta_0^{\frac{1}{2}} \log(\delta^{-n^2}/\delta_0^{\frac{1}{2}}) < \delta_0^{\frac{1}{3}}.$$

The last inequality used the fact that the constant N only depends on the dimension. Since $S(-\Delta_{\mathbb{R}^n})^{\frac{1}{2}}E_{2-n} = (\mathrm{R}_2^t\mathrm{R}_2)^{-1}$ (see [St]), it follows that

$$\|(\mathrm{R}_2^t\mathrm{R}_2)^{-1}(\rho_{K_1}^{n-2} - \int \rho_{K_1}^{n-2})\|_\infty < \delta_0^{\frac{1}{3}}. \tag{2.22}$$

From (2.18) and (2.22), we have

$$\|(\mathrm{R}_2^t\mathrm{R}_2)^{-1}\rho_{K_1}^{n-2} - 1\|_\infty = \|(\mathrm{R}_2^t\mathrm{R}_2)^{-1}\rho_{K_1}^{n-2} - \int \rho_{K_1}^{n-2}\|_\infty + o(\delta_0) \lesssim \delta_0^{\frac{1}{3}}. \tag{2.23}$$

It follows in particular from (2.23) that

$$\frac{1}{2} < \mathrm{R}_2(\mathrm{R}_2^t\mathrm{R}_2)^{-1}\rho_{K_1}^{n-2} < 2. \tag{2.24}$$

Assumption (2.12) means that

$$\mathrm{R}_2\rho_K^2 \le 1. \tag{2.25}$$

It follows from (2.17), (2.24) and (2.25) that

$$\int [\mathrm{R}_2(\mathrm{R}_2^t\mathrm{R}_2)^{-1}\rho_{K_1}^{n-2}](1 - \mathrm{R}_2\rho_K^2) \le o(\delta), \tag{2.26}$$

$$\int |1 - \mathrm{R}_2\rho_K^2| \le o(\delta). \tag{2.27}$$

By Lemma 2.3, this yields for small δ_0

$$\delta_0 \sim \|1 - \rho_K^2\|_\infty < \delta_0^{\frac{N}{n+3}}, \tag{2.28}$$

Hence, for N large enough, a contradiction follows. Therefore, the assumption (2.12) is impossible, and the implication (2.11) is true. This completes the proof. □

3. High-Dimensional Sections

In this section we give a proof for Theorem 1.3. The following lemma gives the the Radon transforms of functions which are $\mathrm{SO}(n-1)$ invariant.

Lemma 3.1. *Let g be a continuous function on S^{n-1} which is* $\mathrm{SO}(n-1)$ *invariant. Then*

$$\mathrm{R}_i g(u) = \frac{c_1}{\cos\phi}\int_\phi^{\frac{\pi}{2}} g(v)\left(1 - \frac{\cos^2\psi}{\cos^2\phi}\right)^{\frac{i-3}{2}} \sin\psi\, d\psi \tag{3.1}$$

$$\int_{S^{n-1}} g\, dv = c_2\int_0^{\frac{\pi}{2}} g(v)\sin^{n-2}\psi\, d\psi, \tag{3.2}$$

where ϕ and ψ are the angles of the unit vectors u and v with the x_n-axis, respectively.

Proof. The proof of formula (3.1) is similar to that of Lemma 2.1 in [Z1]. See also Lemma 8 in [Z2]. (3.2) follows from the spherical coordinates. □

LEMMA 3.2. *There exist a convex body K and a C^∞ function g so that*

$$\mathrm{R}_i\big(\rho_K^{i-4} g\big) < 0, \quad \langle \rho_K^{n-4}, g\rangle > 0, \tag{3.3}$$

for all integers $3 < i < n$.

PROOF. Consider a convex body of revolution K. From (3.1) and (3.2), the inequalities in (3.3) become

$$\int_\phi^{\frac{\pi}{2}} \rho_K(\psi)^{i-4} g(\psi) \left(1 - \frac{\cos^2\psi}{\cos^2\phi}\right)^{\frac{i-3}{2}} \sin\psi \, d\psi < 0, \tag{3.4}$$

$$\int_0^{\frac{\pi}{2}} \rho_K^{n-4}(\psi) g(\psi) \sin^{n-2}\psi \, d\psi > 0. \tag{3.5}$$

We need to choose K and g so that the last two inequalities are satisfied.

Let K be a cylinder. Then

$$\rho_K(\psi) = \frac{1}{\sin\psi}, \quad \psi_1 \le \psi \le \frac{\pi}{2},$$

for some $\psi_1 > 0$. Choose g so that $g(\psi) = 0$ when $0 \le \psi \le \psi_1$. Then (3.4) and (3.5) can be written as

$$\int_\phi^{\frac{\pi}{2}} g(\psi) \left(1 - \frac{\cos^2\psi}{\cos^2\phi}\right)^{\frac{i-3}{2}} \sin^{5-i}\psi \, d\psi < 0, \tag{3.6}$$

$$\int_{\psi_1}^{\frac{\pi}{2}} g(\psi) \sin^2\psi \, d\psi > 0, \quad \psi_1 \le \phi \le \frac{\pi}{2}. \tag{3.7}$$

Let $\cos\psi = t$. Then (3.6) and (3.7) become

$$\int_0^x g(\psi(t)) \left(1 - \frac{t^2}{x^2}\right)^{\frac{i-3}{2}} (1-t^2)^{\frac{4-i}{2}} \, dt < 0 \tag{3.8}$$

and

$$\int_0^{x_1} g(\psi(t))(1-t^2)^{\frac{1}{2}} \, dt > 0, \quad 0 \le x \le x_1, \tag{3.9}$$

where $x_1 = \cos\psi_1$.

Let $f(t) = g(\psi(t))(1-t^2)^{\frac{1}{2}}$. We write (3.8) and (3.9) as

$$\int_0^x f(t) \left(\frac{x^2 - t^2}{1 - t^2}\right)^{\frac{i-3}{2}} dt < 0 \tag{3.10}$$

and

$$\int_0^{x_1} f(t) \, dt > 0, \quad 0 \le x \le x_1. \tag{3.11}$$

Let

$$g(t,x) = \left(\frac{x^2 - t^2}{1 - t^2}\right)^{\frac{i-3}{2}}.$$

When $i > 3$, $g(t,x)$ is strictly decreasing for $t \in [0,x]$. Choose $f(t)$ such that

$$\begin{aligned} f(t) < 0 &\quad \text{if } 0 < t < x_0, \\ f(t) > 0 &\quad \text{if } x_0 < t < x_1 \\ f(t) = 0 &\quad \text{if } x_1 < t < 1, \end{aligned}$$

and

$$\int_0^{x_1} f(t)\,dt = 0.$$

It follows that

$$\begin{aligned} \int_0^x f(t)g(t,x)\,dt &= \int_0^{x_0} f(t)g(t,x)\,dt + \int_{x_0}^x f(t)g(t,x)\,dt \\ &< \int_0^{x_0} f(t)g(x_0,x)\,dt + \int_{x_0}^x f(t)g(x_0,x)\,dt \\ &= g(x_0,x)\int_0^x f(t)\,dt \le 0. \end{aligned}$$

By a small perturbation of f, there is

$$\int_0^x f(t)g(t,x)\,dt < 0$$

and

$$\int_0^{x_1} f(t)\,dt > 0, \quad 0 < x < x_1.$$

Therefore, one can choose f so that (3.10) and (3.11) are true. This proves the lemma. $\square$

Proof of Theorem 1.3. By the Lemma 3.2, there is a C^∞ convex body K of positive curvature and a C^∞ function g on S^{n-1} so that for all $3 < i < n$,

$$\mathrm{R}_i\big(\rho_K^{i-4} g\big) < 0, \quad \langle \rho_K^{n-4}, g\rangle > 0.$$

Define a centered convex body of revolution K_ε by

$$\rho_{K_\varepsilon}^4 = \rho_K^4 + \varepsilon g,$$

for $\varepsilon > 0$ small. We have

$$V(K_\varepsilon) - V(K) = \frac{n}{4}\langle \rho_K^{n-4}, g\rangle\varepsilon + o(\varepsilon),$$

$$\mathrm{vol}_i(K_\varepsilon \cap \xi) - \mathrm{vol}_i(K \cap \xi) = \frac{i}{4}\mathrm{R}_i(\rho_K^{i-4} g)(\xi)\varepsilon + o(\varepsilon).$$

Therefore, when ε is small enough, we have $V(K_\varepsilon) > V(K)$ and

$$\mathrm{vol}_i(K_\varepsilon \cap \xi) < \mathrm{vol}_i(K \cap \xi), \quad \forall \xi \in G_{i,n},\ 3 < i < n. \qquad \square$$

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