Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

# On the Stability of the Volume Radius

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Abstract. The volume radius of a given $n$-dimensional body is the radius of a euclidean ball having the same volume as this body. We prove that the volume radius of a given convex symmetric $n$-dimensional body with diameter at most $\sqrt{n}$ is almost equal to the volume radius of a body obtained by the intersection of this body with $n$ other bodies whose polars are bounded by 1 mean width.

In the last decade, interest in the problem of bounds for volumes of convex bodies was renewed mainly because of its applications to Banach Space Geometry and related topics. At the end of the 80's sharp bounds for volume radius of convex polytopes with given distance between antipodal faces were found independently by several authors: Carl and Pajor [1], Bourgain, Lindenstrauss and Milman [2], Gluskin [3]. Closely related results were obtained by Vaaler [4], Dilworth and Szarek [5] and Bárány and Furedi [6]. See also Ball and Pajor [7] where, following Kashin's conjecture, the problem was considered as a limiting case of a series of Vaaler-type results. Moreover in [3] it was observed that the volume radius of a unit cube has a certain stability property with respect to cutting the cube by a sequence of bands (see Proposition 1 below for the exact formulation). Some of Kashin's ideas enabled us to use this property for an alternative proof of Spencer's theorem [8] on a lacunary analogue of the Rudin–Shapiro polynomials (see [3]). Later Kashin [9] used the same approach for finite dimensional analogues of Menshov's correction theorem.

Here we continue to study this property. We show that it holds not only for cubes but also for a wide class of bodies. It is then observed that the condition on the width of the bands which appeared in [3] is very close to Talagrand's [10] description of bounded Gaussian processes. This observation permits us to restate the result in an invariant form which is more convenient for application. Moreover the result of [3] is extended to the intersection of a given body with a sequence of cylinders. This is made possible by a suitable generalization of the Khatri–Sidak theorem [11, 12].

Supported in part by the United States – Israel Binational Science Foundation.

We denote by $|x|$ the standard Euclidean norm of $x \in \mathbb{R}^n$. The unit ball of $\mathbb{R}^n$ is denoted by $D_n$, where

$$D_n = \{x \in \mathbb{R}^n : |x| \le 1\}.$$

We say that a convex body $V \subset \mathbb{R}^n$ is absolutely convex if $V = -V$. For such a body and $x \in \mathbb{R}^n$ one defines $\|x\|_V$ by setting

$$\|x\|_V = \inf\{\lambda \in \mathbb{R}_+ : x \in \lambda V\}.$$

A linear operator $S : \mathbb{R}^n \to \mathbb{R}^k$ is called a partial isometry if $|Sx| = |x|$ for any $x$ orthogonal to $\ker S$. For such an $S$ and positive $r$ we denote $W(S,r) = \{x \in \mathbb{R}^n : Sx \in rD_k\}$. Note that for $k = 1$ a partial isometry is just some norm one linear functional and the corresponding set $W(S,r)$ is a band of width $2r$. As usual, vol or $\mathrm{vol}_n$ is the standard Lebesgue measure on $\mathbb{R}^n$. The canonical Gaussian measure on $\mathbb{R}^n$ is denoted by $\gamma_n$; it is a probability measure with density $(2\pi)^{-n/2}\exp(-|x|^2/2)$.

As usual, we denote by $\chi_V$ the indicator function of the set $V$, such that $\chi_V(x) = 1$ if $x \in V$ and $\chi_V(x) = 0$ if $x \notin V$.

THEOREM. *For any $\varepsilon > 0$ there exists a positive constant $C = C(\varepsilon) < \infty$ such that the following assertion holds. For a given $n$, let $K \subset \mathbb{R}^n$ be an absolutely convex body such that $K \subset \sqrt{n} D_n$. Then, for any $n$ absolutely convex bodies $V_1, \dots, V_n \subset \mathbb{R}^n$ satisfying*

$$\int_{\mathbb{R}^n} \|x\|_{V_i}\, d\gamma_n(x) \le 1 \quad \textit{for } i = 1, 2, \dots, n,$$

*the following inequality holds*:

$$(1-\varepsilon) \le \left( \frac{\mathrm{vol}\big(K \cap C(V_1 \cap V_2 \cap \cdots \cap V_n)\big)}{\mathrm{vol}\, K} \right)^{1/n} \le 1.$$

Talagrand [10, Theorem 2] proved that for any $n$ and any absolutely convex body $V \subset \mathbb{R}^n$ satisfying

$$\int \|x\|_V\, d\gamma_n(x) \le 1$$

there exists a sequence of norm one linear functionals $f_i \in (\mathbb{R}^n)^*$ possessing the property that

$$\bigcap_{i=1}^{\infty} W\big(f_i, \sqrt{\log(2+i)}\big) \subset C_T V,$$

where $C_T < \infty$ is a universal constant. By Talagrand's result, the theorem is equivalent to the following proposition:

PROPOSITION 1. *For a given $n$ let $K \subset \mathbb{R}^n$ be an absolutely convex body satisfying $K \subset \sqrt{n} D_n$. Then, for any sequence of norm one linear functionals $f_i \in (\mathbb{R}^n)^*$, $i = 1, 2, \ldots$, with some constant $C = C(\varepsilon)$ depending on $\varepsilon$ only, we have*

$$(1-\varepsilon) \leq \left( \frac{\operatorname{vol}(K \cap \left(\bigcap_{i=1}^{\infty} W(f_i, r_i)\right))}{\operatorname{vol} K} \right)^{1/n} \leq 1,$$

*where $r_i \geq C\sqrt{\log(2+i/n)}$.*

PROOF. It is clear that, for any body $V \subset \lambda D_n$,

$$1 \leq \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{\operatorname{vol} V}{\gamma_n(V)} \leq e^{\lambda^2/2}.$$

In particular, for $K$ as in the Proposition, with $C_1 = C_1(\varepsilon)$,

$$C_1\sqrt{2\pi}\left(\gamma_n\left(\frac{1}{C_1}K\right)\right)^{1/n} \geq \sqrt{1-\varepsilon}\,(\operatorname{vol} K)^{1/n}.$$

On the other hand,

$$\left(\operatorname{vol}\left(K \cap \left(\bigcap_{i=1}^{\infty} W(f_i, r_i)\right)\right)\right)^{1//n} \geq C_1\sqrt{2\pi}\gamma_n\left(\frac{1}{C_1}K \cap \frac{1}{C_1}\left(\bigcap_{i=1}^{\infty} W(f_i, r_i)\right)\right).$$

By the Khatri–Sidak theorem,

$$\gamma_n\left(\frac{1}{C_1}K \cap \frac{1}{C_1}\left(\bigcap_{i=1}^{\infty} W(f_i, r_i)\right)\right) \geq \gamma_n\left(\frac{1}{C_1}K\right)\prod_{i=1}^{\infty}\gamma_n(W(f_i, r_i/C_1)).$$

The elementary bound $\gamma_n(W(f,r)) = \gamma_1((-r,r)) \geq 1 - e^{-r^2/2}$ implies that

$$\left(\prod_{i=1}^{\infty}\gamma_n(W(f_i, r_i/C_1))\right)^{1/n} \geq \sqrt{1-\varepsilon} \tag{1}$$

for $r_i = C\sqrt{\log(2+i/n)}$ with $C = C(\varepsilon)$, and the proposition follows. □

REMARK 1. The particular case of the cube instead of the general convex body $K$ was considered in [3]. In the first version of the paper the proof of Proposition 1 followed the same scheme as that of [3]. A significant simplification of the proof was found by the referee. I wish to express my gratitude to him for his suggestion to publish his proof here.

REMARK 2. We say that a body $V$ satisfies the positive correlation property (PCP) if for any absolutely convex body $W$ and for any positive constant $\lambda$ one has $\gamma_n(\lambda V \cap W) \geq \gamma_n(\lambda V)\,\gamma_n(W)$. The proof of Proposition 1 shows the following fact:

*For any $\varepsilon > 0$ there exists a constant $C > 0$ such that for any absolutely convex body $K \subset \sqrt{n} D_n$ and for any sequence of bodies $V_j$ satisfying the PCP one has*

$$1 - \varepsilon \leq \left( \frac{\operatorname{vol}(K \cap \left(\bigcap_{j=1}^{\infty} V_j\right))}{\operatorname{vol} K} \right)^{1/n} \leq 1,$$

*providing* $\left(\prod_{j=1}^{\infty} \gamma_n(V_j/C)\right) \geq (1 - \varepsilon/2)^n$.

The Gaussian Correlation Conjecture states that any absolutely convex body satisfies the PCP. In these terms the Khatri–Sidak theorem states that any band satisfies the PCP. A slight modification of its proof leads to this result:

LEMMA. *Let $V \subset \mathbb{R}^n$ be an absolutely convex body and $S : \mathbb{R}^n \to \mathbb{R}^k$ be a partial isometry of rank $S = k$. Then, for any positive $r$, we have*

$$\gamma_n(V \cap W(S, r)) \geq \gamma_n(V)\gamma_n(W(S, r)) = \gamma_n(V)\gamma_k(rD_k).$$

In other words, any cylinder satisfies the PCP. Certainly one can use the lemma to obtain analogues of Proposition 1 for the case of the intersection with a sequence of cylinders. We omit the precise statement, which is rather complicated. For the reader's convenience, we outline a proof of the lemma.

Let us fix some partial isometry $S$ from $\mathbb{R}^n$ onto $\mathbb{R}^k$. It is well known that its conjugate $S^*$ is a partial isometry from $\mathbb{R}^k$ to $\mathbb{R}^n$, which is right inverse to $S$, and $Q = Id_{\mathbb{R}^n} - S^* S$ is an orthogonal projection on $\ker S$. For $\alpha \in [0, \pi/2]$ let the operator $T(\alpha) : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ be defined by

$$T(\alpha)(x, y) = (Qx + \cos\alpha S^* Sx - \sin\alpha S^* y,\ \sin\alpha Sx + \cos\alpha y)$$

for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^k = \mathbb{R}^{n+k}$. For a given absolutely convex $V_1 \subset \mathbb{R}^n$ and $V_2 \subset \mathbb{R}^k$ we denote

$$\widetilde{V}_1(\alpha) = T(\alpha)(V_1 \times \mathbb{R}^k) \quad \text{and} \quad \widetilde{V}_2(\alpha) = T(\alpha)(\mathbb{R}^n \times V_2).$$

Consider the function

$$f_{V_1,V_2}(\alpha) = \operatorname{vol}\{\widetilde{V}_1(\alpha) \cap \widetilde{V}_2(0) \cap D_{n+k}\},$$

or equivalently

$$f_{V_1,V_2}(\alpha) = \int_{\widetilde{V}_2(0) \cap D_{n+k}} \chi_{\widetilde{V}_1(0)}(T^{-1}(\alpha)z)\, dz.$$

It is easy to see that $f_{V_1,V_2}$ is absolutely continuous and one has the following equality for a.e. $\alpha \in [0, \pi/2]$:

$$\frac{d}{d\alpha} f_{V_1,V_2}(\alpha) = \int_{\partial(\widetilde{V}_2(0) \cap D_{n+k})} \chi_{\widetilde{V}_1(0)}(T^{-1}(\alpha)z)\, \langle \widetilde{n}(z), Jz \rangle \, d\widetilde{\mu}(z),$$

where $\widetilde{\mu}$ is the Lebesgue measure on $\partial(\widetilde{V}_2(0) \cap D_{n+k})$, $\widetilde{n}$ is outer normal to $\partial(\widetilde{V}_2(0) \cap D_{n+k})$ and operator $J$ is given by $J = T(\alpha)\frac{d}{d\alpha}T^{-1}(\alpha)$. Straightforward computation shows that $J(x,y) = (S^*y, -Sx)$ for $(x,y) \in \mathbb{R}^n \times \mathbb{R}^k$. Since $\langle z, Jz\rangle = 0$ for any $z \in \mathbb{R}^{n+k}$ one can rewrite the previous equality as

$$\frac{d}{d\alpha} f_{V_1,V_2} = -\int_{\partial V_2} d\mu(y) \int_{W_y} \langle n(y), Sx\rangle \, dx, \tag{2}$$

where $\mu$ is the Lebesgue measure on $\partial V_2$, $n(y)$ is outer normal to $\partial V_2$ at point $y$ and

$$W_y = \{x \in \sqrt{(1-|y|^2)_+}\; D_n : Qx + \cos\alpha S^* Sx \in V_1 - \sin\alpha S^* y\} \ .$$

Proposition 2. *Let $p : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ be some nonincreasing positive function and $\nu$ be an absolutely continuous measure on $\mathbb{R}^{n+k}$ with density $p(|x|)$. Then, for any absolutely convex body $V_1 \subset \mathbb{R}^n$ and any positive $r$ the function*

$$h(\alpha) = \nu\{\widetilde{V}_1(\alpha) \cap r(\mathbb{R}^n \times D_k)\}$$

*is nondecreasing on* $[0, \pi/2]$.

Sketch of the proof. It is clear that without loss of generality, we can consider the case $p = \chi_{[0,1]}$ only. In this case $h(\alpha) = f_{V, rD_k}(\alpha)$. By Lemma 2 from [3] one has

$$\int_{W_y} \langle S^* y, x\rangle \, dx \le 0 \ . \tag{3}$$

Taking into account that $n(y) = r^{-1}y$ for $y \in \partial(rD_k) = rS^{k-1}$ we get Proposition 2 from (2) and (3). □

The inequality $h(0) \le h(\pi/2)$ with $p(t) = (2\pi)^{-n/2} e^{-t^2/2}$ proves the lemma. □

Note. When this paper was almost complete, we learned from S. Bobkov that Schechtman, Schlumprecht and Zinn [13] had found a very short proof of the Khatri–Sidak theorem, based on the Borel–Prékopa–Leindler characterization of log-concave measures. A simple modification of the method of [13] leads to an alternative proof of the lemma.

Remark 3. It is fairly simple to see that the conditions of Proposition 1 on $r_i$ are optimal for $i \gg n$; see [3, 14]. For $i < n$ this is not so. In fact the proof of Proposition 1 shows that it holds under better conditions on $r_i$, $i < n$ than those stated. To see this one has to take into account that we only need to estimate the product in (1) and to use for example the inequality $\gamma_1((-r, r)) \ge 2(2\pi)^{-1/2} r e^{-r^2/2}$.

The same remark holds in the case of the cylinders' intersection.

## Acknowledgements

Part of this work was done while the author was visiting at the Institute for Advanced Study, Princeton and MSRI, Berkeley. I wish to thank both Institutions for their hospitality.

I also wish to express my gratitude to Professors V. Milman, G. Schechtman, B. Shiff and N. Tomczak–Jaegerman for their interest in this work, to S. Bobkov for bringing my attention to the paper [13] and to the referee for his valuable remarks.

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