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On the Equivalence Between Geometric and Arithmetic Means for Log-Concave Measures

RAFAŁ LATAŁA

Abstract. Let X be a random vector with log-concave distribution in some Banach space. We prove that $\|X\|_p \le C_p \|X\|_0$ for any $p > 0$, where $\|X\|_p = (E\|X\|^p)^{1/p}$, $\|X\|_0 = \exp E \ln \|X\|$ and C_p are constants depending only on p. We also derive some estimates of log-concave measures of small balls.

Introduction. Let X be a random vector with log-concave distribution (for precise definitions see below). It is known that for any measurable seminorm and $p, q > 0$ the inequality

$$\|X\|_p \le C_{p,q} \|X\|_q$$

holds with constants $C_{p,q}$ depending only on p and q (see [4], Appendix III). In this paper we show that the above constants can be made independent of q, which is equivalent to the inequality

$$\|X\|_p \le C_p \|X\|_0, \tag{1}$$

where $\|X\|_0$ is the geometric mean of $\|X\|$. In the particular case in which X is uniformly distributed on some convex compact set in R^n and the seminorm is given by some functional, inequality (1) was established by V. D. Milman and A. Pajor [3]. As a consequence of (1) we prove the result of Ullrich [6] concerning the equivalence of means for sums of independent Steinhaus random variables with vector coefficients, even though these random-variables are not log-concave (Corollary 2).

To prove (1) we derive some estimates of log-concave measures of small balls (Corollary 1), which are of independent interest. In the case of Gaussian random variables they were formulated and established in a weaker version in [5] and completelely proved in [2].

Definitions and Notation. Let E be a complete, separable, metric vector space endowed with its Borel $\sigma-$algebra $\mathcal{B}_E$.By μ we denote a log-concave probability measure on $(E, \mathcal{B}_E)$ (for some characterizations, properties and examples, see [1]) i.e. a probability measure with the property that for any Borel subsets A, B and all $0 < \lambda < 1$ we have

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}.$$

We say that a random vector X with values in E is log-concave if the distribution of X is log-concave.For a random vector X and a measurable seminorm $\|.\|$ on E (i.e. Borel measurable, nonnegative, subadditive and positively homogeneous function on E) we define

$$\|X\|_p = (E\|X\|^p)^{1/p} \text{ for } p > 0$$

and

$$\|X\|_0 = \lim_{p \to 0^+} \|X\|_p = \exp(E \ln \|X\|).$$

Let us begin with the following Lemma from [1].

Lemma 1. *For any convex, symmetric Borel set B and $k \geq 1$ we have*

$$\mu((kB)^c) \leq \mu(B)\left(\frac{1-\mu(B)}{\mu(B)}\right)^{(k+1)/2}.$$

Proof. The statement follows immediately from the log-concavity of μ and the inclusion

$$\frac{k-1}{k+1}B + \frac{2}{k+1}(kB)^c \subset B^c. \qquad \square$$

Lemma 2. *If B is a convex, symmetric Borel set, with $\mu(KB) \geq (1+\delta)\mu(B)$ for some $K > 1$ and $\delta > 0$ then*

$$\mu(tB) \leq Ct\mu(B) \textit{ for any } t \in (0,1),$$

where $C = C(K/\delta)$ is a constant depending only on K/δ.

Proof. Obviously it's enough to prove the result for $t = 1/2n$, $n = 1, 2, \ldots$. So let us fix n and define, for $u \geq 0$,

$$P_u = \{x : \|x\|_B \in (u - 1/2n, u + 1/2n)\},$$

where

$$\|x\|_B = \inf\{t > 0 : x \in tB\}.$$

By simple calculation $\lambda P_u + (1-\lambda)(2n)^{-1}B \subset P_{\lambda u}$, so

$$\mu(P_{\lambda u}) \geq \mu(P_u)^{\lambda}\mu((2n)^{-1}B)^{1-\lambda} \text{ for } \lambda \in (0,1). \tag{2}$$

From the assumptions it easily follows that there exists $u \geq 1$ such that $\mu(P_u) \geq \delta\mu(B)/Kn$. Let $\mu((2n)^{-1}B) = \kappa\mu(B)/n$. If $\kappa \leq 2\delta/K$ we are done, so we will assume that $\kappa \geq 2\delta/K$. Then by (2) it follows that $\mu(P_1) \geq \delta\mu(B)/Kn$. The sets $P_{(n-1)/n}, P_{(n-2)/n}, \ldots, P_{1/n}, (2n)^{-1}B$ are disjoint subsets of B, and hence

$$\mu(B) \geq \mu(P_{(n-1)/n}) + \cdots + \mu(P_{1/n}) + \mu((2n)^{-1}B).$$

Using our estimations of $\mu(P_1)$ and $\mu((2n)^{-1}B)$ we obtain by (2)

$$\mu(B) \geq n^{-1}\mu(B)((\delta/K)^{(n-1)/n}\kappa^{1/n} + \cdots + (\delta/K)^{1/n}\kappa^{(n-1)/n} + \kappa) =$$

$$= \frac{\kappa}{n}\mu(B)\frac{1-\delta/K\kappa}{1-(\delta/K\kappa)^{1/n}} \geq \frac{\kappa}{2n}\mu(B)\frac{1}{1-(\delta/K\kappa)^{1/n}}.$$

Therefore

$$\kappa \leq 2n(1-(\delta/K\kappa)^{1/n}) \leq 2\ln K\kappa/\delta,$$

so that $\kappa \leq C(K/\delta)$ and the lemma follows. □

Corollary 1. *For each $b < 1$ there exists a constant C_b such that for every log-concave probability measure μ and every measurable convex, symmetric set B with $\mu(B) \leq b$ we have*

$$\mu(tB) \leq C_b t\mu(B) \text{ for } t \in [0,1].$$

Proof. If $\mu(B) = 2/3$ then by Lemma 1 $\mu(3B) \geq 5/6 = (1+1/4)\mu(B)$, so by Lemma 2 for some constant $\tilde{C}_1$, $\mu(tB) \leq \tilde{C}_1 t\mu(B)$.

If $\mu(B) \in [1/3, 2/3]$ then obviously $\mu(tB) \leq 2\tilde{C}_1 t\mu(B)$.

If $\mu(B) < 1/3$, let K be such that $\mu(KB) = 2/3$. By the above case $\mu(B) \leq \tilde{C}_1 K^{-1}\mu(KB)$, and hence

$$K \leq 2\tilde{C}_1\left(\frac{\mu(KB)}{\mu(B)} - 1\right).$$

So Lemma 2 gives in this case that $\mu(tB) \leq \tilde{C}_2 t\mu(B)$ for some constant $\tilde{C}_2$.

Finally if $\mu(B) > 2/3$, but $\mu(B) \leq b < 1$ then by Lemma 1 for some $K_b < \infty$, $\mu(K_b^{-1}B) \leq 2/3$ and we can use the previous calculations. □

Theorem 1. *For any $p > 0$ there exists a universal constant C_p, depending only on p such that for any sequence $X_1, \ldots, X_n$ of independent log-concave random vectors and any measurable seminorm $\|.\|$ on E we have*

$$\left\|\sum_{i=1}^n X_i\right\|_p \leq C_p\left\|\sum_{i=1}^n X_i\right\|_0.$$

Proof. Since a convolution of log-concave measures is also log-concave (see [1]) we may and do assume that $n = 1$. Let

$$M = \inf\{t : P(\|X_1\| \geq t) \leq 2/3\}.$$

Then by Lemma 1 (used for $B = \{x \in E : \|x\| \leq M\}$) it follows easily that $\|X_1\|_p \leq a_p M$ for $p > 0$ and some constants a_p depending only on p. By similar reasoning Corollary 1 yields $\|X_1\|_0 \geq a_0 M$. □

Corollary 2. *Let E be a complex Banach space and $X_1, \ldots, X_n$ be a sequence of independent random variables uniformly distributed on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Then for any sequence of vectors $v_1, \ldots, v_n \in E$ and any $p > 0$ the following inequality holds*:

$$\left\|\sum v_k X_k\right\|_p \leq K_p \left\|\sum v_k X_k\right\|_0,$$

where K_p is a constant depending only on p.

Proof. It is enough to prove Corollary for $p \geq 1$. Let $Y_1, \ldots, Y_n$ be a sequence of independent random variables uniformly distributed on the unit disc $\{z : |z| \leq 1\}$. By Theorem 1 we have

$$\left\|\sum v_k Y_k\right\|_p \leq C_p \left\|\sum v_k Y_k\right\|_0. \tag{3}$$

But we may represent Y_k in the form $Y_k = R_k X_k$, where R_k are independent, identically distributed random variables on $[0, 1]$ (with an appropriate distribution), which are independent of X_k. Hence, by taking conditional expectation we obtain

$$\left\|\sum v_k Y_k\right\|_p \geq (ER_1) \left\|\sum v_k X_k\right\|_p. \tag{4}$$

Finally let us observe that for any $u, v \in E$ the function $f(z) = \ln \|u + zv\|$ is subharmonic on $\mathbb{C}$, so $g(r) = E \ln \|u + rvX_1\|$ is nondecreasing on $[0, \infty)$ and therefore

$$\left\|\sum v_k X_k\right\|_0 \geq \left\|\sum v_k Y_k\right\|_0. \tag{5}$$

The corollary follows from (3), (4) and (5). □

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Rafał Latała
Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warsaw
Poland
rlatala@mimuw.edu.pl