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On the Constant in the Reverse Brunn–Minkowski Inequality for p-Convex Balls

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Abstract. This note is devoted to the study of the dependence on p of the constant in the reverse Brunn–Minkowski inequality for p-convex balls (that is, p-convex symmetric bodies). We will show that this constant is estimated as $c^{1/p} \leq C(p) \leq C^{ln(2/p)/p}$, for absolute constants $c > 1$ and $C > 1$.

Let $K \subset \mathbb{R}^n$ and $0 < p \leq 1$. K is called a p-convex set if for any $\lambda, \mu \in (0,1)$ such that $\lambda^p + \mu^p = 1$ and for any points $x, y \in K$ the point $\lambda x + \mu y$ belongs to K. We will call a p-convex compact centrally symmetric body a p-ball.

Recall that a p-norm on real vector space X is a map $\|\cdot\| : X \to \mathbb{R}^+$ satisfying these conditions:

(1) $\|x\| > 0$ for all $x \neq 0$.
(2) $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$ and $x \in X$.
(3) $\|x+y\|^p \leq \|x\|^p + \|y\|^p$ for all $x, y \in X$.

Note that the unit ball of p-normed space is a p-ball and, vice versa, the gauge of p-ball is a p-norm.

Recently, J. Bastero, J. Bernués, and A. Peña [BBP] extended the reverse Brunn–Minkowski inequality, which was discovered by V. Milman [M], to the class of p-convex balls. They proved the following result:

Theorem 0. *Let $0 < p \leq 1$. There exists a constant $C = C(p) \geq 1$ such that for all $n \geq 1$ and all p-balls $A_1, A_2 \subset \mathbb{R}^n$, there exists a linear operator $u : \mathbb{R}^n \to \mathbb{R}^n$ with $|\det(u)| = 1$ and*

$$|uA_1 + A_2|^{1/n} \leq C\big(|A_1|^{1/n} + |A_2|^{1/n}\big), \tag{1}$$

where $|A|$ denotes the volume of body A.

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Their proof yields an estimate $C(p) \leq C^{ln(2/p)/p^2}$.

We will obtain a much better estimate for $C(p)$:

Theorem 1. *There exist absolute constants $c > 1$ and $C > 1$ such that the constant $C(p)$ in* (1) *satisfies*

$$c^{1/p} \leq C(p) \leq C^{ln(2/p)/p}.$$

The proof of Theorem 0 [BBP] was based on an estimate of the entropy numbers (see also [Pi]). We use the same idea, but obtain the better dependence of the constant on p.

Let us recall the definitions of the Kolmogorov and entropy numbers. Let $U : X \to Y$ be an operator between two Banach spaces. Let $k > 0$ be an integer. The Kolmogorov numbers are defined by the following formula

$$d_k(U) = \inf \{\|Q_S U\| \mid S \subset Y, \ \ \dim S = k \ \},$$

where $Q_S : Y \to Y/S$ is a quotient map. For any subsets K_1, K_2 of Y denote by $N(K_1, K_2)$ the smallest number N such that there are N points $y_1, \dots, y_N$ in Y such that

$$K_1 \subset \bigcup_{i=1}^{N} (y_i + K_2).$$

Denote the unit ball of the space X (Y) by B_X (B_Y) and define the entropy numbers by

$$e_k(U) = \inf \left\{\varepsilon > 0 \mid N(UB_X, \varepsilon B_Y) \leq 2^{k-1}\right\}.$$

For p-convex balls $B_1, B_2 \subset \mathbb{R}^n$, with $0 < p \leq 1$, we will denote the identity operator from $(\mathbb{R}^n, \|\cdot\|_1)$ to $(\mathbb{R}^n, \|\cdot\|_2)$ by $B_1 \to B_2$, where $\|\cdot\|_i$ $(i = 1, 2)$ is the p-norm whose unit ball is B_i.

Theorem 2. *Given $\alpha > 1/p - 1/2$, there exists a constant $C = C(\alpha, p)$ such that, for any n and any p-convex ball $B \subset \mathbb{R}^n$, there exists an ellipsoid $D \subset \mathbb{R}^n$ such that, for every $1 \leq k \leq n$,*

$$\max\{d_k(D \to B), \ e_k(B \to D)\} \ \leq C(n/k)^{\alpha} \ .$$

Moreover, there is an absolute constant c such that

$$C(\alpha, p) \leq \left(\frac{2}{p}\right)^{c/p} \left(\frac{1}{1-\delta}\right)^{8/\delta} \quad \text{for } \alpha > \frac{3(1-p)}{2p}, \ \delta = \frac{3(1-p)}{2p\alpha}, \ p \leq \frac{1}{2} \tag{2}$$

and

$$C(\alpha, p) \leq \left(\frac{2}{p}\right)^{c/p^2} \left(\frac{1}{1-\varepsilon}\right)^{\frac{2}{\varepsilon p^2}} \quad \text{for } \alpha > \frac{1}{p} - \frac{1}{2}, \ \varepsilon = \frac{1/p - 1/2}{\alpha}. \tag{3}$$

Remark 1. In fact, in [BBP] Theorem 2 was proved with estimate (3). Using this result we prove estimate (2).

In the following $C(\alpha, p)$ will denote the best possible constant from Theorem 2.

The main point of the proof is the following lemma.

LEMMA 1. *Let $p, q, \theta \in (0,1)$ such that $1/q-1 = (1/p-1)(1-\theta)$ and $\gamma = \alpha(1-\theta)$. Then*

$$C(\alpha, p) \le 2^{1/p} 2^{1/(1-\theta)} (e/(1-\theta))^{\alpha} C_{p\theta}^{1/(1-\theta)} C(\gamma, q)^{1/(1-\theta)},$$

where

$$C_{p\theta} = \frac{\Gamma\big(1+(1-p)/p\big)}{\Gamma\big(1+\theta(1-p)/p\big)\Gamma\big(1+(1-\theta)(1-p)/p\big)}, \quad \Gamma \text{ is the gamma function.}$$

For the reader's convenience we postpone the proof of this lemma.

PROOF OF THEOREM 2. Take $q = 1/2,\ 1-\theta = p/(1-p)$. Then $C_{p\theta} = (1-p)/p$ and, consequently, by Lemma 1,

$$C(\alpha, p) \le c \left(\frac{e}{p}\right)^{\alpha} 2^{2/p} \left(\frac{1}{p}\right)^{1/p} C\left(\frac{\alpha p}{1-p}, \frac{1}{2}\right).$$

Inequality (3) implies

$$C\left(\frac{\alpha p}{1-p}, \frac{1}{2}\right) \le c \left(\frac{1}{1-\delta}\right)^{8/\delta}, \qquad \text{where } \delta = \frac{3(1-p)}{2p\alpha}.$$

Thus for $\alpha > 3(1-p)/(2p)$ and $p \le 1/2$ we obtain

$$C(\alpha, p) \le \left(\frac{2}{p}\right)^{c/p} \left(\frac{1}{1-\delta}\right)^{8/\delta}. \qquad \square$$

PROOF OF THEOREM 1. By B. Carl's theorem ([C], or see Theorem 5.2 of [Pi]) for any operator u between Banach spaces the following inequality holds

$$\sup_{k \le n} k^{\alpha} e_k(u) \le \rho_{\alpha} \sup_{k \le n} k^{\alpha} d_k(u).$$

One can check that Carl's proof works in the p-convex case also and gives

$$\rho_{\alpha} \le C^{1/p} (C\alpha)^{C\alpha}$$

for some absolute constant C. Let us fix $\alpha = 2/p$. Then, by Theorem 2, we have that for any p-convex body K there exists an ellipsoid D such that

$$\max\{e_n(D \to B),\ e_n(B \to D)\} \le C^{ln(2/p)/p}.$$

The standard argument [Pi] gives the upper estimate for C_p.

To show the lower bound we use the following example. Let B_p^n be a unit ball in the space l_p^n and B_2^n be a unit ball in the space l_2^n. Denote

$$A = \frac{|B_2^n|^{1/n}}{|B_p^n|^{1/n}} = \frac{\Gamma(3/2)\Gamma^{1/n}(1+n/p)}{\Gamma^{1/n}(1+n/2)\Gamma(1+1/p)} \ge C_0 \frac{n^{1/p-1/2}}{\sqrt{1/p}},$$

where C_0 is an absolute constant.

Consider a body

$$K = AB_p^n.$$

We are going to estimate from below

$$\frac{|UB_2^n + K|^{1/n}}{|UB_2^n|^{1/n} + |K|^{1/n}} = \frac{|UB_2^n + K|^{1/n}}{2|B_2^n|^{1/n}}$$

for an arbitrary operator $U : \mathbb{R}^n \to \mathbb{R}^n$ with $|\det U| = 1$.

To simplify the sum of bodies in the example let us use the Steiner symmetrization with respect to vectors from the canonical basis of $\mathbb{R}^n$ (see, e.g., [BLM], for precise definitions). Usually the Steiner symmetrization is defined for convex bodies, but if we take the unit ball of l_p^n and any coordinate vector then we have the similar situation. The following properties of the Steiner symmetrization are well-known (and can be directly checked):

(i) It preserves volume.
(ii) The symmetrization of sum of two bodies contains sum of symmetrizations of these bodies.
(iii) Given an ellipsoid UB_2^n, a consecutive application of the Steiner symmetrizations with respect to all vectors from the canonical basis results in the ellipsoid VB_2^n, where V is a diagonal operator (depending on U).

That means that in our example it is enough to consider a diagonal operator U with $|\det U| = 1$.

Let $b \in (0,1)$ and P_1 be the orthogonal projection on a coordinate subspace of dimension $n-1$. Then direct computations give for every $r > 0$

$$|UB_2^n + rB_p^n| \geq 2\int_0^{rb_p} |P_1 UB_2^n + brP_1 B_p^n|\, dx \geq 2rb_p |P_1 UB_2^n + brP_1 B_p^n|,$$

where $b_p = \big(p(1-b)\big)^{1/p}$. Since $P_1 K = AB_p^{n-1}$, by induction arguments one has

$$|UB_2^n + K| \geq \big(2Ab^{(k-1)/2} b_p\big)^k |P_k UB_2^n + b^k P_k K|,$$

where P_k is the orthogonal projection on an arbitrary $(n-k)$-dimensional coordinate subspace of $\mathbb{R}^n$. Choosing $b = \exp(-2/(kp))$, P_k such that $|P_k UB_2^n| \geq |B_2^{n-k}|$ and $k = [n/2]$ we get

$$\begin{aligned} C(p) &\geq \frac{|UB_2^n + K|^{1/n}}{2|B_2^n|^{1/n}} \geq \tfrac{1}{2}\big(2Ae^{-1/p}\,(2/k)^{1/p}\big)^{k/n} \left(\frac{|B_2^{n-k}|}{|B_2^n|}\right)^{1/n} \\ &\geq c_1 \sqrt{p^{1/2}\,(4/e)^{1/p}} \end{aligned}$$

for sufficiently large n and an absolute constant c_1. That gives the result for p small enough, namely, $p \leq c_2$, where c_2 is an absolute constant. For $p \in (c_2, 1]$ the result follows from the convex case. □

To prove Lemma 1 we will use the Lions–Peetre interpolation [BL, K] with parameters $(\theta, 1)$.

Let us recall some definitions.

Let X be a quasi-normed space with an equivalent quasi-norms $\|\cdot\|_0$ and $\|\cdot\|_1$. Let $X_i = (X, \|\cdot\|_i)$.

Define $K(t,x) = \inf\{\|x_0\|_0 + \ t\|x_1\|_1 \mid x = x_0 + x_1\}$ and

$$\|x\|_{\theta,1} = \theta(1-\theta)\int_0^{+\infty} \frac{K(t,x)}{t^{1+\theta}}dt,$$

for $\theta \in (0,1)$.

The interpolation space $(X_0, X_1)_{\theta,1}$ is the space $(X, \|\cdot\|_{\theta,1})$.

Claim 1. *Let $\|\cdot\|_0 = \|\cdot\|_1 = \|\cdot\|$ be p-norms on space X. Then*

$$\frac{1}{C_{p\theta}}\|x\| \le \|x\|_{\theta,1} \le \|x\|$$

for every $x \in X$, with $C_{p\theta}$ as in Lemma 1.

Proof. $\|x\|_{\theta,1} \le \|x\|$ since

$$\inf\left\{\|x_0\|_0 + \ t\left\|x_1\right\|_1 \ \mid x = x_0 + x_1\right\} \le \min(1,t)\left\|x\right\|$$

and

$$\|x\|_{\theta,1} = \theta(1-\theta)\int_0^{+\infty}\frac{K(t,x)}{t^{1+\theta}}dt \le \theta(1-\theta)\int_0^{+\infty}\frac{\min(1,t)}{t^{1+\theta}}\|x\|dt = \|x\|.$$

By p-convexity of the norm $\|\cdot\|$ for $a = \|y\|/\|x\| \le 1$ we have

$$\frac{\|y\| + t\|x-y\|}{\|x\|} \ge a + t(1-a^p)^{1/p} \ge \frac{t}{(1+t^s)^{1/s}}, \quad \text{where} \quad s = \frac{p}{1-p}\,.$$

Hence

$$K(t,x) = \inf\left\{\|x_0\|_0 + \ t\left\|x_1\right\|_1 \ \mid x = x_0 + x_1\right\} \ge \|x\|\,\frac{t}{(1+t^s)^{1/s}}$$

and

$$\begin{aligned}\frac{\|x\|_{\theta,1}}{\|x\|} &\ge \theta(1-\theta)\int_0^{+\infty}\frac{dt}{(1+t^s)^{1/s}t^\theta} = B\left(\frac{1-\theta}{s},\frac{\theta}{s}\right)\,\frac{\theta(1-s)}{s}\\ &= \frac{(\theta/s)\Gamma\big(\theta/s\big)\big((1-\theta)/s\big)\Gamma\big((1-\theta)/s\big)}{(1/s)\Gamma\big(1/s\big)} = \frac{1}{C_{p\theta}},\end{aligned}$$

where $B(x,y)$ is the beta function. This proves the claim. □

Claim 2. *Let $\|\cdot\|_0 = \|\cdot\|_1 = \|\cdot\|$ be norms on X. Then $\|x\|_{\theta,1} = \|x\|$ for every $x \in X$.*

Proof. In case of norm $K(t,x) = \min(1,t)\|x\|$. So, $\|x\|_{\theta,1} = \|x\|$. □

The next statement is standard (see [BL] or [K]).

CLAIM 3. *Let X_i, Y_i $(i = 0, 1)$ be quasi-normed spaces. Let $T : X_i \to Y_i$ $(i = 0, 1)$ be a linear operator. Then*

$$\|T : (X_0, X_1)_{\theta,1} \to (Y_0, Y_1)_{\theta,1}\| \le \|T : X_0 \to Y_0\|^{1-\theta} \|T : X_1 \to Y_1\|^{\theta}.$$

CLAIM 4. *Let X_i $(i = 0, 1)$ be quasi-normed spaces. Then for every $N \ge 1$,*

$$\big(l_1^N(X_0), l_1^N(X_1)\big)_{\theta,1} = l_1^N\big((X_0, X_1)_{\theta,1}\big)$$

with equal norms.

PROOF. The conclusion of this claim follows from the equality

$$K(t, x = (x_1, x_2, \dots, x_N), l_1^N(X_0), l_1^N(X_1)) = \sum_{i=1}^{N} K(t, x_i, X_0, X_1). \qquad \square$$

CLAIM 5. *Let X_i $(i = 0, 1)$ be quasi-normed spaces, Y be a p-normed space. Let $T : X_i$ $(i = 0, 1) \to Y$ be a linear operator. Then for every $k_0, k_1 \ge 1$*

$$d_{k_0+k_1-1}\big(T : (X_0, X_1)_{\theta,1} \to Y\big) \le C_{p\theta}\, d_{k_0}^{1-\theta}\big(T : X_0 \to Y\big) d_{k_1}^{\theta}\big(T : X_1 \to Y\big).$$

PROOF. As in the convex case [P], fix $\varepsilon > 0$. Consider a subspace $S_i \subset Y$ $(i = 0, 1)$ such that $\dim S_i < k_i$ and

$$\|Q_{S_i} T : X_i \to Y/S_i\| \le (1 + \varepsilon) d_{k_i}\big(T : X_i \to Y\big).$$

Let $S = span(S_0, S_1) \subset Y$. Then $\dim S < k_0 + k_1 - 1$ and

$$\|Q_S T : X_i \to Y/S\| \le \|Q_{S_i} T : X_i \to Y/S_i\|.$$

Note that quotient space of a p-normed space is again a p-normed one. Because of this, and by Claims 1 and 3,

$$\begin{aligned}
\|Q_S T : (X_0, X_1)_{\theta,1} \to Y/S\| &\le C_{p\theta} \|Q_S T : (X_0, X_1)_{\theta,1} \to (Y/S, Y/S)_{\theta,1}\| \\
&\le C_{p\theta} \|Q_S T : X_0 \to Y/S\|^{1-\theta} \|Q_S T : X_1 \to Y/S\|^{\theta} \\
&\le C_{p\theta} \|Q_{S_0} T : X_0 \to Y/S_0\|^{1-\theta} \|Q_{S_1} T : X_1 \to Y/S_1\|^{\theta} \\
&\le C_{p\theta} (1 + \varepsilon)^2 d_{k_0}\big(T : X_0 \to Y\big)^{1-\theta} d_{k_1}\big(T : X_1 \to Y\big)^{\theta}.
\end{aligned}$$

This completes the proof. $\square$

PROOF OF LEMMA 1.

Step 1. Let D be an optimal ellipsoid such that

$$d_k(D \to B) \le C(\alpha, p)(n/k)^{\alpha} \quad \text{and} \quad e_k(B \to D) \le C(\alpha, p)(n/k)^{\alpha}$$

for every $1 \le k \le n$.

Let $\lambda = C(\alpha, p)(n/k)^{\alpha}$.

Step 2. Now denote the body $(B,D)_{\theta,1}$ by B_θ. By Claim 5 (applied for $k_0=1$), for every $1\le k\le n$ we have

$$d_k(B_\theta\to B)\le C_{p\theta}\|B\to B\|^{1-\theta}(d_k(D\to B))^\theta\le C_{p\theta}\lambda^\theta.$$

It follows from the definition of entropy numbers that B is covered by 2^{k-1} translates of λD with centers in $\mathbb{R}^n$. Replacing λD with $2\lambda D$ we can choose these centers in B. Therefore there are 2^{k-1} points $x_i\in B$ $(1\le i\le 2^{k-1})$ such that

$$B\subset\bigcup_{i=1}^{2^{k-1}}(x_i+2\lambda D).$$

This means that for any $z\in B$ there is some $x_i\in B$ such that $\|z-x_i\|_D\le 2\lambda$. Also, by p-convexity, $\|z-x_i\|_B\le 2^{1/p}$. By taking the operator $u_x:\mathbb{R}\to X$, $u_xt=tx$ for some fixed x, and applying Claim 3 (or see [BL], [BS]) it is clear that

$$\|x\|_{B_\theta}\le\|x\|_B^{1-\theta}\|x\|_D^\theta.$$

Hence, for any $z\in B$ there exists $x_i\in B$ such that

$$\|z-x_i\|_{B_\theta}\le(2^{1/p})^{1-\theta}(2\lambda)^\theta,$$

that is,

$$e_k(B\to B_\theta)\le 2^{(1-\theta)/p}(2\lambda)^\theta.$$

Thus, we obtain

$$d_k(B_\theta\to B)\le C_{p\theta}\lambda^\theta\quad\text{and}\quad e_k(B\to B_\theta)\le 2^\theta 2^{(1-\theta)/p}\lambda^\theta$$

for every $1\le k\le n$.

LEMMA 2. *Let $B\subset\mathbb{R}^n$ be a p-convex ball and $D\subset\mathbb{R}^n$ be a convex body. Let $0<\theta<1$ and $B_\theta=(B,D)_{\theta,1}$. Then there exists a q-convex body B^q such that $B_\theta\subset B^q\subset 2^{1/q}B_\theta$, where $1/q-1=(1/p-1)(1-\theta)$.*

PROOF. Take the operator $U:l_1^2(\mathbb{R}^n)\to\mathbb{R}^n$ defined by $U((x,y))=x+y$. Since

$$\|x+y\|_B\le 2^{1/p-1}\left(\|x\|_B+\|y\|_B\right)\quad\text{and}\quad\|x+y\|_D\le\left(\|x\|_D+\|y\|_D\right)$$

and by Claims 3, 4 we have

$$\|x+y\|_{B_\theta}\le 2^{(1-\theta)(1/p-1)}\left(\|x\|_{B_\theta}+\|y\|_{B_\theta}\right).$$

But by the Aoki–Rolewicz theorem for every quasi-norm $\|\cdot\|$ with the constant C in the quasi-triangle inequality there exists a q-norm

$$\|\cdot\|_q=\inf\left\{\left(\sum_{i=1}^n\|x_i\|^q\right)^{1/q}\,\middle|\;n>0,\ x=\sum_{i=1}^n x_i\right\}$$

such that $\|x\|_q\le\|x\|\le 2C\|x\|_q$ with q satisfying $2^{1/q-1}=C$ ([KPR, R]; see also [K], p.47).

Thus, $B_\theta\subset B^q\subset 2^{1/q}B_\theta$, where B^q is a unit ball of q-norm $\|\cdot\|_q$. $\square$

Remark 2. Essentially, Lemma 2 goes back to Theorem 5.6.2 of [BL]. However, the particular case that we need is simpler and we are able to estimate the constant of equivalence.

Note that Lemma 2 can be easily extended to the more general case:

Lemma 2′. *Let $B_i \subset \mathbb{R}^n$ be a p_i-convex bodies for $i = 0, 1$ and $B_\theta = (B_0, B_1)_{\theta,1}$. Then there exists a q-convex body B^q such that $B_\theta \subset B^q \subset 2^{1/q} B_\theta$, where*

$$\frac{1}{q} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Remark 3. N. Kalton pointed out to us that the interpolation body $(B, D)_{\theta,1}$ between a p-convex B and an ellipsoid D is equivalent to some q-convex body for any $q \in (0, 1]$ satisfying

$$1/q - 1/2 > (1/p - 1/2)(1 - \theta).$$

To prove this result one have to use methods of [Kal] and [KT]. Certainly, with growing q the constant of equivalence becomes worse.

Step 3. By definition of $C(\alpha, p)$ for B^q from Lemma 2 and $\gamma = \alpha(1-\theta)$ there exists an ellipsoid D_1 such that for every $1 \le k \le n$

$$d_k(D_1 \to B^q) \le C(\gamma, q)(n/k)^\gamma \quad \text{and} \quad e_k(B^q \to D_1) \le C(\gamma, q)(n/k)^\gamma.$$

By the ideal property of the numbers d_k, e_k and because of the inclusion $B_\theta \subset B^q \subset 2^{1/q} B_\theta$, for every $1 \le k \le n$

$$d_k(D_1 \to B_\theta) \le 2^{1/q} C(\gamma, q)(n/k)^\gamma \quad \text{and} \quad e_k(B_\theta \to D_1) \le C(\gamma, q)(n/k)^\gamma.$$

Step 4. Let $a = 1 + [k(1-\theta)]$. Using multiplicative properties of the numbers d_k, e_k we get

$$\begin{aligned} d_k(D_1 \to B) &\le d_{k+1-a}(D_1 \to B_\theta) d_a(B_\theta \to B) \\ &\le C_{p\theta} \lambda^\theta 2^{1/q} C(\gamma, q)(n/k)^\gamma \left(\frac{1}{(1-\theta)^{1-\theta}\theta^\theta}\right)^\alpha \\ &\le C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} C_{p\theta} 2^{1/q} C(\gamma, q)(n/k)^\alpha \end{aligned}$$

and

$$\begin{aligned} e_k(B \to D_1) &\le e_{k+1-a}(B \to B_\theta) e_a(B_\theta \to D_1) \\ &\le 2^\theta 2^{(1-\theta)/p} \lambda^\theta C(\gamma, q)(n/k)^\gamma \left(\frac{1}{(1-\theta)^{1-\theta}\theta^\theta}\right)^\alpha \\ &\le C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} 2^\theta 2^{(1-\theta)/p} C(\gamma, q)(n/k)^\alpha. \end{aligned}$$

By the minimality of $C(\alpha, p)$ and since $1/q \le 1 + (1-\theta)/p$ we have

$$C(\alpha, p) \le C(\alpha, p)^\theta \left(\frac{e}{1-\theta}\right)^{\alpha(1-\theta)} C_{p\theta} 2^{1-\theta/p} 2 C(\gamma, q)(n/k)^\alpha.$$

That proves Lemma 1. □

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