Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

The Extension of the Finite-Dimensional Version of Krivine's Theorem to Quasi-Normed Spaces

ALEXANDER E. LITVAK

Abstract. In 1980 D. Amir and V. D. Milman gave a quantitative finite-dimensional version of Krivine's theorem. We extend their version of the Krivine's theorem to the quasi-convex setting and provide a quantitative version for p-convex norms.

Recently, a number of results of the Local Theory have been extended to the quasi-normed spaces. There are several works [Kal1, Kal2, D, GL, KT, GK, BBP1, BBP2, M2] where such results as Dvoretzky–Rogers lemma [DvR], Dvoretzky theorem [Dv1, Dv2], Milman's subspace-quotient theorem [M1], Krivine's theorem [Kr], Pisier's abstract version of Grotendick's theorem [P1, P2], Gluskin's theorem on Minkowski compactum [G], Milman's reverse Brunn–Minkowski inequality [M3], and Milman's isomorphic regularization theorem [M4] are extended to quasi-normed spaces after they were established for normed spaces. It is somewhat surprising since the first proofs of these facts substantially used convexity and duality.

In [AM2] D. Amir and V. D. Milman proved the local version of Krivine's theorem (see also [Gow], [MS]). They studied quantitative estimates appearing in this theorem. We extend their result to the q- and quasi-normed spaces.

Recall that a quasi-norm on a real vector space X is a map $\|\cdot\| : X \to \mathbb{R}^+$ satisfying these conditions:

(1) $\|x\| > 0$ for all $x \neq 0$.
(2) $\|tx\| = |t|\|x\|$ for all $t \in \mathbb{R}$ and $x \in X$.
(3) There exists $C \geq 1$ such that $\|x + y\| \leq C(\|x\| + \|y\|)$ for all $x, y \in X$.

If (3) is substituted by

(3a) $\|x + y\|^q \leq \|x\|^q + \|y\|^q$ for all $x, y \in X$, for some fixed $q \in (0, 1]$,

This research was supported by Grant No. 92–00285 from United States–Israel Binational Science Foundation (BSF).

then $\|\cdot\|$ is called a q-norm on X. Note that 1-norm is the usual norm. It is obvious that every q-norm is a quasi-norm with $C = 2^{1/q-1}$. However, not every quasi-norm is q-norm for some q. Moreover, it is even not necessary continuous. It can be shown by the following simple example. Let f be a positive function on the Euclidean sphere S^{n-1} defined by

$$f(x) = \begin{cases} |x| & \text{for } x \in A, \\ 2|x| & \text{otherwise.} \end{cases}$$

Here A is a subset of S^{n-1} such that both A and $S^{n-1} \setminus A$ are dense in S^{n-1}. Denote $\|x\| = |x| f(x/|x|)$. Because f is not continuous it is clear that $\|\cdot\|$ is not q-norm for any q though it is the quasi-norm.

The next lemma is the Aoki–Rolewicz Theorem ([KPR, R]; see also [K, p. 47]).

Lemma 1. *Let $\|\cdot\|$ be a quasi-norm with the constant C in the quasi-triangle inequality. Then there exists a q-norm $\|\cdot\|$ for which*

$$\|x\|_q \leq \|x\| \leq 2C\|x\|_q$$

with q satisfying $2^{1/q-1} = C$. This q-norm can be defined as follows

$$\|x\|_q = \inf\left\{ \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} : n > 0,\ x = \sum_{i=1}^{n} x_i \right\}.$$

We refer to [KPR] for further properties of the quasi- and q-norms.

Theorem 1. *Let $\{e_i\}_1^n$ be a unit vector basis in $\mathbb{R}^n$, $\|\cdot\|_p$ be a l_p-norm on $\mathbb{R}^n$, i.e., $\|\sum_{i=1}^n a_i e_i\|_p = \left(\sum_i |a_i|^p\right)^{1/p}$, for $0 < p < \infty$. Let $\|\cdot\|$ be a q-norm on $\mathbb{R}^n$ such that*

$$C_1^{-1}\|x\|_p \leq \|x\| \leq C_2\|x\|_p \tag{1}$$

for every $x \in \mathbb{R}^n$. Then for every $\varepsilon > 0$ and $C = C_1C_2$ there exists a block sequence $u_1, u_2, \dots, u_m$ of $e_1, e_2, \dots, e_n$ which satisfies

$$(1-\varepsilon)\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{m} a_i u_i\right\| \leq (1+\varepsilon)\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p} \tag{2}$$

for all $a_1, a_2, \dots, a_m$ and $m \geq C(\varepsilon, p, q)\,(n/\log n)^{\nu}$, where

$$\nu = \frac{\alpha\varepsilon_0}{\varepsilon_0 + p + \alpha\varepsilon_0} \quad \textit{for} \quad p < 1 \qquad \textit{and} \qquad \nu = \frac{\varepsilon_0}{2\varepsilon_0 + 1} \quad \textit{for} \quad p \geq 1;$$

$$\alpha = \min\{p, q\}, \quad \varepsilon_0 = \left(\frac{q\varepsilon/2}{1 + C^q 12^{q/p}} \right)^{p/q}.$$

Remark 1. If $p \geq 1$ in this theorem, then we have the well-known finite-dimensional version of Krivine's theorem with some modifications concerning change of the usual norm to the q-norm. In this case for small enough q we get $\varepsilon_0 \approx (q\varepsilon/4)^{p/q}$ and $\nu \approx \varepsilon_0$.

The case $p < 1$ is more interesting. We get an extension of the finite-dimensional version of Krivine's theorem. To provide an intuition for the behavior of the constant in the theorem we point out that for small enough p and q with $p = q$ we can take $\varepsilon_0 \approx q\varepsilon/30$ and $\nu \approx \varepsilon_0$.

REMARK 2. By Lemma 1 in the case of quasi-norm with the constant C_0 the inequality (2) is substituted with

$$(1-\varepsilon)\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p} \leq \left\|\sum_{i=1}^{m} a_i u_i\right\| \leq 2(1+\varepsilon)C_0\left(\sum_{i=1}^{m} |a_i|^p\right)^{1/p}.$$

Due to the example above, we can not remove the constant C_0 in this inequality.

The proof of the theorem consists of two lemmas.

LEMMA 2. *For every $\eta > 0$ there exists a constant $C(\eta) > 0$ such that if $\|\cdot\|$ is a q-norm on $\mathbb{R}^n$ satisfying* (1) *then there exists a block sequence $y_1, y_2, \ldots, y_k$ of $e_1, e_2, \ldots, e_n$ which is $(1+\eta)$-symmetric and $k \geq C(\eta, q, p)n/\log n$.*

LEMMA 3. *If $y_1, y_2, \ldots, y_k$ is a 1-symmetric sequence in a normed space satisfying*

$$C_1^{-1}\|a\|_p \leq \left\|\sum_{i=1}^{k} a_i y_i\right\| \leq C_2\|a\|_p$$

for all $a = (a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ then for every $\varepsilon > 0$ there exists a block sequence $u_1, u_2, \ldots, u_m$ of $y_1, y_2, \ldots, y_k$ such that

$$(1-\varepsilon)\|a\|_p \leq \left\|\sum_{i=1}^{m} a_i u_i\right\| \leq (1+\varepsilon)\|a\|_p$$

for all $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$, where $m \geq C(p,q)\varepsilon^{p/q}k^{\nu}$,

$$\nu = \frac{\alpha\varepsilon_0}{\varepsilon_0 + p + \alpha\varepsilon_0} \quad \text{for } p < 1 \quad \text{and} \quad \nu = \frac{\varepsilon_0}{2\varepsilon_0 + 1} \quad \text{for } p \geq 1,$$

$$\alpha = \min\{p, q\}, \quad \varepsilon_0 = \left(\frac{q\varepsilon}{1 + C^q 12^{q/p}}\right)^{p/q}.$$

At first, D. Amir and V. D. Milman ([AM2]; see also [MS]) proved Lemma 2 for $q = 1$, $p \geq 1$ with the estimate $k \geq C(\eta, q, p)n^{1/3}$. Their proof can be modified to obtain result for $0 < p < \infty$, $q \leq 1$. Afterwards, W. T. Gowers [Gow] showed that the estimate of k can be improved to $k \geq C(\eta, q, p)n/\ln n$. In fact, he gave two different, though similar, proofs for cases $p = 1$ and $p > 1$. The proof given for case $p = 1$ strongly used the convexity of the norm and the fact that p is equal to 1. However, the method used for $p > 1$ actually works for every $0 < p < \infty$ and even for q-norms. Let us recall the idea of W. T. Gowers. First we will introduce some definition.

Let Ω be the group $\{-1,1\}^n \times S_n$, where S_n is the permutation group. Let Ψ be the group $\{-1,1\}^k \times S_k$. For

$$b=\sum_{i=1}^{n} b_i e_i \in \mathbb{R}^n, \quad a=\sum_{i=1}^{k} a_i e_i \in \mathbb{R}^k, \quad (\varepsilon,\pi)\in\Omega, \quad (\eta,\sigma)\in\Psi$$

set

$$b_{\varepsilon\pi}=\sum_{i=1}^{n} \varepsilon_i b_i e_{\pi(i)}, \qquad a_{\eta\sigma}=\sum_{i=1}^{k} \eta_i a_i e_{\sigma(i)}.$$

Let $h\cdot k=n$. For $i\le k$, $j\le h$ put

$$e_{ij}=e_{(i-1)h+j}, \quad \varepsilon_{ij}=\varepsilon_{(i-1)h+j}, \quad \pi_{ij}=\pi((i-1)h+j).$$

Define an action of Ψ on Ω by

$$\Psi_{\eta\sigma}((\varepsilon,\pi))=(\varepsilon^1,\pi^1), \text{ where } \varepsilon^1_{ij}=\eta_i\varepsilon_{\sigma(i)j}, \quad \pi^1_{ij}=\pi_{\sigma(i)j}.$$

For any $(\varepsilon,\pi)\in\Omega$ define the operator

$$\Phi_{\varepsilon\pi}:\ \mathbb{R}^k\to\mathbb{R}^n \qquad \text{by} \qquad \Phi_{\varepsilon\pi}\left(\sum_{i=1}^{k} a_i e_i\right)=\sum_{i=1}^{k}\sum_{j=1}^{h}\varepsilon_{ij}a_i e_{\pi_{ij}}.$$

For every $a\in\mathbb{R}^k$ by M_a denote the median of $\Phi_{\varepsilon\pi}(a)$ taken over Ω. Finally, let $A=\{a\in l_p^k:\ \|a\|_p\le 1, a_1\ge a_2\ge\cdots\ge a_k\ge 0\}$.

The following claim, which W. T. Gowers proved for case $p>1$ and $q=1$, is the main step in the proof of Lemma 2.

Claim 1. *Let $\|\cdot\|$ be a q-norm on $\mathbb{R}^n$ satisfying $\|x\|_p\le\|x\|\le B\|x\|_p$. There is a constant $C_0=C(p,q,\delta,B)$ such that given $\lambda>0$ for every $a\in A$*

$$\mathbf{Prob}_\Omega\left\{\exists(\eta,\sigma)\ :\ \left|\|\Phi_{\varepsilon\pi}(a_{\eta\sigma})\|^q-M_a^q\right|^{1/q}>\frac{1}{2^{1/q}}\delta\|a\|_p h^{1/p}\right\}<1/N$$

with $k=C_0\frac{n}{\lambda\log n}$ and $N=k^\lambda$.

The proof of this claim can be equally well applied for all $0<p<\infty$ and $0<q\le 1$. The only change that we have to do is to replace the triangle inequality

$$\big|\|x\|-\|y\|\big|\le\|x-y\| \quad \text{by} \quad \big|\|x\|^q-\|y\|^q\big|^{1/q}\le\|x-y\|.$$

The following two claims are technical and can be proved using ideas of [Gow] with small changes, connected with replacing $p\ge 1$ by $p<1$ and the norm by q-norm.

Claim 2. *Let $0<p<\infty$ and $\delta>0$. There exist a constant λ, depending on p and δ only, such that for every integer k the set A contains a δ-net K of cardinality k^λ.*

Claim 3. *Let $\|\cdot\|$ be a q-norm on $\mathbb{R}^n$ satisfying $\|x\|_p \le \|x\| \le B\|x\|_p$. If there is $(\varepsilon,\pi) \in \Omega$ such that for every a in some δ-net K of A*

$$\left|\|\Phi_{\varepsilon\pi}(a_{\eta\sigma})\|^q - \|\Phi_{\varepsilon\pi}(a_{\eta_1\sigma_1})\|^q\right|^{1/q} \le \delta\|a\|_p h^{1/p}$$

for every $(\eta,\sigma),(\eta_1,\sigma_1) \in \Psi$ then the block basis

$$\{\Phi_{\varepsilon\pi}(e_i)\}_{i=1}^k$$

of $(\mathbb{R}^k,\|\cdot\|)$ is $(1+6\,(B\delta)^q)^{1/q}$-symmetric.

These three claims imply Lemma 2 in the standard way (see [Gow] for the details).

Proof of Lemma 3. Our method of proof is close to the method used in [AM1], but our notation follows that of [MS, chapter 10].

First, we will give the Krivine's construction of block basis. Let a and N be some integers which will be specified later. Let us introduce some set of numbers $\{\lambda_j\}_J$. We will say that set

$$\{B_{j,i}\}_{j\in J, i\in I}$$

(if card $I = 1$ then we have only one index j) is $\{\lambda_j\}_J$-set if

(1) $B_{j,i} \subset \{1,\dots,n\}$ for every $j \in J, i \in I$,
(2) $B_{j,i}$ are mutually disjoint,
(3) card $B_{j,i} = \lambda_j$ for every $j \in J,\ i \in I$.

Let us fix some $\{[\rho^j]\}$-set

$$\{A_{j,s}\}_{0\le j\le N-1, 1\le s\le m}$$

for $\rho = 1 + 1/a$.

For $0 \le j \le N-1$ and $1 \le s \le m$, define

$$Y_{j,s} = \sum_{i\in A_{j,s}} y_i$$

and

$$z_s = \sum_{j=0}^{N-1} \rho^{(N-j)/p} Y_{j,s}.$$

Clearly, $\|z_1\| = \|z_2\| = \cdots = \|z_m\|$. The integer m will be defined from

$$k \approx m \sum_{j=0}^{N-1} [\rho^{(N-j)/p}] \approx m\rho^N(\rho-1)^{-1} = ma\Big(\frac{a+1}{a}\Big)^N.$$

Finally, we define the block sequence $\{u_s\}_{s=1}^m$ by

$$u_s = z_s/\|z_s\|.$$

Now, as in [MS], we will establish the necessary estimates.

Fix $N, M \in \{T+1, T+2, \dots, m\}$ and $t_s \in \{0, \dots, T\}$ for $s \in \{1, \dots, m\}$ such that

$$\sum_{s=1}^{M} \rho^{-t_s} = 1 + \eta, \quad \text{with } |\eta| = 1.$$

Then

$$\sum_{s=1}^{M} \rho^{-t_s/p} z_s = \sum_{s=1}^{M} \sum_{j=0}^{N-1} \rho^{(N-j-t_s)/p} Y_{j,s}$$

$$= \sum_{i=0}^{N-1+T} \rho^{(N-i)/p} \sum_{\substack{s \le M,\ j \le N-1 \\ j+t_s = i}} \sum_{l \in A_{j,s}} y_l = \sum_{i=0}^{N-1+T} \rho^{(N-i)/p} \sum_{l \in B_i} y_l$$

for some $\{a_i\}$-set $\{B_i\}_{i=0}^{N-1+T}$, where

$$a_i = \sum_{\substack{s \le M,\ j \le N-1 \\ j+t_s = i}} [\rho^{i-t_s}], \quad \text{for } 0 \le i \le N-1+T.$$

Therefore, we can choose a vector z which has the same structure as z_s (i.e., $z = \sum_{j=0}^{N-1} \rho^{(N-j)/p} \sum_{i \in A_j} y_i$ for some $\{[\rho^j]\}$-set $\{A_j\}_{0 \le j \le N-1}$) such that the difference Δ is

$$\Delta = \sum_{s=1}^{M} \rho^{-t_s/p} z_s - z = \sum_{s=1}^{N-1} \rho^{(N-i)/p} \sum_{l \in C_i} y_l + \sum_{s=N}^{N-1+T} \rho^{(N-i)/p} \sum_{l \in C_i} y_l$$

for some $\{b_j\}$-set $\{C_j\}_{i=0}^{N-1+T}$, where

$$b_j = \begin{cases} |[\rho^j - a_j]| & \text{for } 0 \le j \le N-1, \\ a_j & \text{for } N \le j \le N-1+T. \end{cases}$$

Using techniques from [MS, pp. 66-67] we obtain

$$\|\Delta\| \le C_2 \rho^{N/p} \big(4T + N|\eta| + NM\rho^{-T}\big)^{1/p} \quad \text{and} \quad \|z\| \ge (1/C_1) \rho^{N/p} (N/2)^{1/p}.$$

Hence

$$\left| \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s \right\|^q - 1 \right| \le \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s - \frac{z}{\|z\|} \right\|^q$$

$$= \left(\frac{\|\Delta\|}{\|z\|} \right)^q \le (C_1 C_2)^q \left(\frac{8T}{N} + 2|\eta| + 2M\rho^{-T} \right)^{q/p}.$$

Thus

$$\left| \left\| \sum_{s=1}^{M} \rho^{-t_s/p} u_s \right\|^q - 1 \right| \le C^q (12\varepsilon_0)^{q/p},$$

provided $T \le N\varepsilon_0$, $|\eta| \le \varepsilon_0$, and $M\rho^{-T} \le m\rho^{-T} \le \varepsilon_0$, for some ε_0. Assume $T = [N\varepsilon_0]$.

Case 1: $p<1$.

Let $\sum_{s=1}^m |\alpha_s|^p = 1$ and $a_s = |\alpha_s|$. Let $\alpha = \min\{p,q\}$ and $\delta = \varepsilon_0^{1/p}/m^{1/\alpha}$. Take $\beta_s = \rho^{-t_s/p}$ or $\beta_s = 0,\ t_s \in \{0,1,\dots,T\}$ such that $|a_s-\beta_s| \le \delta$ for every s. It is possible if $\rho^{-T/p} \le \delta$ and $1-\rho^{-1/p} \le \delta$. Since $p \le 1$ it is enough to take a such that it satisfies following the inequalities

$$\left(\frac{a}{a+1}\right)^{[N\varepsilon_0]} \le \delta^p = \frac{\varepsilon_0}{m^{p/\alpha}} \quad \text{and} \quad \delta \ge \frac{1}{p(a+1)}.$$

Take $a = [1/(\delta p)] = \left[m^{1/\alpha}/(p\varepsilon_0^{1/p})\right]$. Thus $\delta \ge \frac{1}{p(a+1)}$,

$$\begin{aligned}\left|\sum \rho^{-t_s} - 1\right| &= \left|\sum \beta_s^p - 1\right| \le \left|\sum (a_s+\delta)^p - 1\right| \\ &\le \left|\sum (a_s^p + \delta^p) - 1\right| = \delta^p m \le \varepsilon_0\end{aligned}$$

and

$$\begin{aligned}\left|\left\|\sum_{s=1}^m \beta_s u_s\right\|^q - \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q\right| &\le \left\|\sum_{s=1}^m |\beta_s - a_s| u_s\right\|^q \\ &\le \delta^q \left\|\sum_{s=1}^m u_s\right\|^q \le \delta^q m \le \varepsilon_0^{q/p}.\end{aligned}$$

Hence

$$\left|\left\|\sum_{s=1}^m \alpha_s u_s\right\|^q - 1\right| \le \varepsilon_0^{q/p}(1 + C^q 12^{q/p}),$$

if $m^{p/\alpha} \le \varepsilon_0(\frac{1+a}{a})^{[N\varepsilon_0]}$ and $ma(\frac{1+a}{a})^N \le k$, when $a = \left[\frac{m^{1/\alpha}}{p\varepsilon_0^{1/p}}\right]$. Choose N such that $(\frac{a}{1+a})^{N\varepsilon_0}$ is of the order $\varepsilon_0/m^{p/\alpha}$. Then

$$m\ \frac{m^{1/\alpha}}{p\varepsilon_0^{1/p}}\ \left(\frac{m^{p/\alpha}}{\varepsilon_0}\right)^{1/\varepsilon_0} = \frac{m^{1+1/\alpha+p/(\alpha\varepsilon_0)}}{\varepsilon_0^{1/p} p \varepsilon_0^{1/\varepsilon_0}} \sim k.$$

Thus, since $1/\alpha \ge \max\{1/p, 1/q\}$,

$$m \sim \varepsilon_0\,(pk)^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}} \sim \varepsilon_0 k^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}}$$

and for $\varepsilon_1 = \varepsilon_0^{q/p}\left(1 + c^q 12^{q/p}\right)$

$$(1-\varepsilon_1)^{1/q}\,\|(\alpha_s)\|_p \le \left\|\sum \alpha_s u_s\right\| \le (1+\varepsilon_1)^{1/q}\,\|(\alpha_s)\|_p$$

holds. For ε_1 small enough ($\varepsilon_1 < 2^q - 1$) we obtain $1-\varepsilon_1/q \le (1-\varepsilon_1)^{1/q}$ and $1+2\varepsilon_1/q \ge (1+\varepsilon_1)^{1/q}$. Take $\varepsilon = 2\varepsilon_1/q$, then

$$\varepsilon_0 = \left(\frac{q\varepsilon/2}{1+C^q 12^{q/p}}\right)^{p/q}$$

and

$$m \ge C(p,q)\varepsilon^{p/q} k^{\frac{\alpha\varepsilon_0}{\varepsilon_0+p+\alpha\varepsilon_0}}.$$

Case 2: $p \geq 1$. We use the same idea. Let $\sum_{s=1}^m |\alpha_s|^p = 1$ and $a_s = |\alpha_s|$. Let $\delta = \varepsilon_0/(C^p m)$. Take $\beta_s = \rho^{-t_s/p}$ or $\beta_s = 0$, $t_s \in \{0, 1, \dots, T\}$ such that $|a_s^p - \beta_s^p| \leq \delta$ for every s. It is possible if $\rho^{-T} \leq \delta$ and $1 - \rho^{-1} \leq \delta$. These two conditions are met if

$$\left(\frac{a}{a+1}\right)^{[N\varepsilon_0]} \leq \delta = \frac{\varepsilon_0}{C^p m} \quad \text{and} \quad \delta \geq \frac{1}{a+1}.$$

Take $a = [1/\delta] = [C^p m/\varepsilon_0]$. Thus

$$\left|\sum \rho^{-t_s} - 1\right| = \left|\sum \beta_s^p - 1\right| \leq \left|\sum (a_s^p + \delta) - 1\right| = \delta m \leq \varepsilon_0.$$

Since

$$\left\|\sum_{s=1}^m u_s\right\| \leq C_1 C_2 \frac{\|\sum_{s=1}^m u_s\|_p}{\|z\|_p} \leq C_1 C_2 \left(\frac{m \sum \rho^{N-j}[\rho^j]}{\|z\|_p^p}\right)^{1/p} = C m^{1/p}$$

and

$$|\beta_s - a_s| \leq |\beta_s^p - a_s^p|^{1/p} \leq \delta^{1/p},$$

we obtain

$$\begin{aligned} \left|\left\|\sum_{s=1}^m \beta_s u_s\right\|^q - \left\|\sum_{s=1}^m \alpha_s u_s\right\|^q\right| \;&\leq\; \left\|\sum_{s=1}^m |\beta_s - a_s| u_s\right\|^q \\ &\leq\; \delta^{q/p} \left\|\sum_{s=1}^m u_s\right\|^q \leq \delta^{q/p} C^q m^{q/p} \leq \varepsilon_0^{q/p}. \end{aligned}$$

Hence

$$\left|\left\|\sum_{s=1}^m \alpha_s u_s\right\|^q - 1\right| \;\leq\; \varepsilon_0^{q/p}(1 + C^q 12^{q/p}),$$

if $m \leq \frac{\varepsilon_0}{C^p}(\frac{1+a}{a})^{[N\varepsilon_0]}$ and $ma(\frac{1+a}{a})^N \leq k$, when $a = [C^p m/\varepsilon_0]$. Choose N such that $(\frac{a}{1+a})^{N\varepsilon_0}$ is of the order $\varepsilon_0/(C^p m)$. Then

$$m \; \frac{C^p m}{\varepsilon_0} \; \left(\frac{C^p m}{\varepsilon_0}\right)^{1/\varepsilon_0} = \left(\frac{C^p}{\varepsilon_0}\right)^{1+1/\varepsilon_0} m^{2+1/\varepsilon_0} \sim k.$$

Thus

$$m \geq \frac{\varepsilon_0}{C^p} k^{\frac{\varepsilon_0}{2\varepsilon_0+1}}$$

and, for $\varepsilon_1 = \varepsilon_0^{q/p}\left(1 + C^q 12^{q/p}\right)$,

$$(1 - \varepsilon_1)^{1/q} \|(\alpha_s)\|_p \leq \left\|\sum \alpha_s u_s\right\| \leq (1 + \varepsilon_1)^{1/q} \|(\alpha_s)\|_p$$

holds. For ε_1 small enough ($\varepsilon_1 < 2^q - 1$) we obtain $1 - \varepsilon_1/q \leq (1 - \varepsilon_1)^{1/q}$ and $1 + 2\varepsilon_1/q \geq (1 + \varepsilon_1)^{1/q}$. Take $\varepsilon = 2\varepsilon_1/q$, then

$$\varepsilon_0 = \left(\frac{q\varepsilon/2}{1 + C^q 12^{q/p}}\right)^{p/q}$$

and

$$m \geq C(p,q)\varepsilon^{p/q} k^{\frac{\varepsilon_0}{2\varepsilon_0+1}}. \qquad \square$$

Acknowledgement

I thank Prof. V. Milman for his guidance and encouragement, and the referee for his helpful remarks.

References

[AM1] D. Amir and V. D. Milman, *Unconditional and symmetric sets in n-dimensional normed spaces*, Isr. J. Math. **37** (1980), 3–20.

[AM2] D. Amir and V. D. Milman, *A quantitative finite-dimensional Krivine theorem*, Isr. J. Math. **50** (1985), 1–12.

[BBP1] J. Bastero, J. Bernués, and A. Peña, *An extension of Milman's reverse Brunn–Minkowski inequality*, GAFA **5**:3 (1995), 572–581.

[BBP2] J. Bastero, J. Bernués, and A. Peña, *The theorem of Caratheodory and Gluskin for* $0 < p < 1$, Proc. Amer. Math. Soc. **123**:1 (1995), 141–144.

[D] S. J. Dilworth, *The dimension of Euclidean subspaces of quasi-normed spaces*, Math. Proc. Camb. Phil. Soc. **97** (1985), 311–320.

[Dv1] A. Dvoretzky, *A theorem on convex bodies and applications to Banach spaces*, Proc. Nat. Acad. Sci. USA **45** (1959), 223–226.

[Dv2] A. Dvoretzky, *Some results on convex bodies and Banach spaces*, Proc. Symp. Linear spaces, Jerusalem, 123–160, 1961.

[DvR] A. Dvoretzky and C. Rogers, *Absolute and unconditional convergence in normed linear spaces*, Proc. Nat. Acad. Sci. USA **36** (1950), 192–197.

[G] E. Gluskin, *The diameter of the Minkowski compactum is approximately equal to n*, Functional Anal. Appl. **15** (1981), 72–73.

[GK] Y. Gordon and N. J. Kalton, *Local structure theory for quasi-normed spaces*, Bull. Sci. Math. **118** (1994), 441–453.

[GL] Y. Gordon, D. R. Lewis, *Dvoretzky's theorem for quasi-normed spaces*, Illinois J. Math. **35** (1991), 250-259.

[Gow] W. T. Gowers, *Symmetric block bases in finite-dimensional normed spaces*, Isr. J. Math. **68** (1989), 193–219.

[Kal1] N. J. Kalton, *The convexity type of a quasi-Banach space*, unpublished note, 1977.

[Kal2] N. J. Kalton, *Convexity, type and the three space problem*, Studia Math. **69** (1981), 247–287.

[KPR] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-space sampler*, London Mathematical Society Lecture Note Series, 89, Cambridge University Press, Cambridge, 1984.

[KT] N. Kalton and Sik-Chung Tam, *Factorization theorems for quasi-normed spaces*, Houston J. Math. **19** (1993), 301-317.

[K] H. König, *Eigenvalue distribution of compact operators*, Birkhäuser, 1986.

[Kr] J. L. Krivine, *Sous-espaces de dimension finie des espaces de Banach réticulés*, Ann. Math. **104** (1976), 1–29.

[M1] V. D. Milman, *Almost Euclidean quotient spaces of subspaces of a finite dimensional normed space*, Proc. Amer. Math. Soc. **94**:3 (1985), 445–449.

[M2] V. Milman, *Isomorphic Euclidean regularization of quasi-norms in* $\mathbb{R}^n$, C. R. Acad. Sci. Paris **321** (1996), 879-884.

[M3] V. D. Milman, *Inégalité de Brunn–Minkowsky inverse et applications á la théorie locale des espaces normés*, C. R. Acad. Sci. Paris Sér. 1 **302** (1986), 25–28.

[M4] V. Milman, *Some applications of duality relations*, GAFA Seminar 1989–90, Lecture Notes in Math. 1469, Springer, 1991, 13–40.

[MS] V. Milman, G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. 1200, Springer, 1986.

[P1] G. Pisier, *Un théorème sur les opérateurs entre espaces de Banach qui se factorisent par un espace de Hilbert*, Ann. École Norm. Sup. **13** (1980), 23–43.

[P2] G. Pisier, *Factorization of linear operators and geometry of Banach spaces*, NSF-CBMS Regional Conference Series 60, AMS, Providence, 1986.

[R] S. Rolewicz, *Metric linear spaces*, Monografie Matematyczne 56, PWN, Warsaw, 1972.

Alexander E. Litvak
Department of Mathematics
Tel Aviv University
Raymond and Beverly Sackler Faculty of Exact Sciences
Ramat Aviv, Tel Aviv 69978
Israel
alexandr@math.tau.ac.il

Current address: Department of Mathematical Sciences, 632 CAB, University of Alberta, Edmonton, AB, Canada T6G 2G1 (alexandr@math.ualberta.ca)