Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

A Note on Gowers' Dichotomy Theorem

BERNARD MAUREY

Abstract. We present a direct proof, slightly different from the original, for an important special case of Gowers' general dichotomy result: If X is an arbitrary infinite dimensional Banach space, either X has a subspace with unconditional basis, or X contains a hereditarily indecomposable subspace.

The first example of dichotomy related to the topic discussed in this note is the classical combinatorial result of Ramsey: for every set A of pairs of integers, there exists an infinite subset M of $\mathbb{N}$ such that, either every pair $\{m_1, m_2\}$ from M is in A, or no pair from M is in the set A. There exist various generalizations to "infinite Ramsey theorems" for sets of finite or infinite sequences of integers, beginning with the result of Nash-Williams [NW]: for any set A of finite increasing sequences of integers, there exists an infinite subset M of $\mathbb{N}$ such that either no finite sequence from M is in A, or every infinite increasing sequence from M has some initial segment in A (although it does not look so at the first glance, notice that the result is symmetric in A and A^c, the complementary set of A; for further developments, see also [GP], [E]). The first naive attempt to generalize this result to a vector space setting would be to ask the following question: given a normed space X with a basis, and a set A of finite sequences of blocks in X (i.e., finite sequences of vectors $(x_1, \dots, x_k)$ where $x_1, \dots, x_k \in X$ are successive linear combinations from the given basis), does there exist a vector subspace Y of X spanned by a block basis, such that either every infinite sequence of blocks from Y has some initial segment in A, or no finite sequence of blocks from Y belongs to A, up to some obviously necessary perturbation involving the norm of X. It turns out that the answer to this question is negative, as a consequence of the existence of *distortable spaces*, like Tsirelson's space [T]. A correct vector generalization requires a more delicate statement, which in particular is not symmetric in A and A^c. Gowers' dichotomy theorem is such a result; in its first form [G1], this theorem is about sets of finite sequences of blocks in a normed space, and it was later extended in [G2] to analytic sets of infinite sequences of blocks. We will not state these general results here, in particular we will not describe the very interesting "vector game" that seems necessary for expressing Gowers' theorem. The first striking application of this result (probably the one for which the combinatorial result was proved) is an application to the *unconditional basic sequence problem*. This problem asks whether it is possible to find in a given Banach space X an infinite unconditional basic sequence (x_n), or, in equivalent terms, an infinite dimensional subspace Y of X with an unconditional basis. The answer is negative for some spaces X, as was shown in [GM1]. Furthermore, the example in [GM1] has a property which seems rather extreme in the direction opposite to an unconditional behaviour: this space X is *Hereditarily Indecomposable*, or H.I. for short. This means that no vector subspace of X is the topological direct sum of two infinite dimensional subspaces (and of course X is infinite dimensional).

It was natural to investigate more closely the connection between the failure of the unconditional basic sequence property and the H.I. property. Was it just accidental if the first example of a space not containing any infinite unconditional sequence was actually a H.I. space? Gowers' result completely clarifies the situation.

Theorem 1 (Special case of Gowers' dichotomy theorem). *Let X be an arbitrary infinite dimensional Banach space. Either X has a subspace with unconditional basis, or X contains a H.I. subspace.*

Let us mention that there exist non trivial examples of non H.I. spaces not containing any infinite unconditional basic sequence (see [GM2]; on the other hand, trivial examples of this situation are simply obtained by considering spaces of the form $X \oplus X$, with an H.I. space X). Recall that Theorem 1 above, together with the results by Komorowski and Tomczak [KT] gave a positive solution to the *homogeneous Banach space problem*, which appeared in Banach's book [B] sixty years before: if a Banach space X is isomorphic to all its infinite dimensional closed subspaces, then X is isomorphic to the Hilbert space ℓ_2.

The purpose of this note is to present a variant for a direct proof of this important special case of Gowers' general dichotomy result. It is of course not essentially different from the original argument in [G1], and the attentive reader will easily detect several steps here that are very similar to some parts of [G1], for example our Lemma 2 below and its Corollary. Our main intention is to give a more geometric exposition. We shall try to gather all the easy geometric information that we need before embarking for the central part of the argument, which is the combinatorial part.

We begin with some notation and definitions. For any normed space X we denote by $S(X)$ the unit sphere of X. The notation Y, Z, or U, V will be used for infinite dimensional vector subspaces of X, and E, F, G for finite dimensional subspaces of X. Given a real number $C \geq 1$, a finite or infinite sequence (e_n) of non zero vectors in a normed space X is called C-*unconditional* if

$$\left\|\sum \varepsilon_i a_i e_i\right\| \le C \left\|\sum a_i e_i\right\|$$

for any sequence of signs $\varepsilon_i = \pm 1$ and any finitely supported sequence (a_i) of scalars. An infinite sequence (e_n) of non zero vectors is called *unconditional* if it is C-unconditional for some C. It is usual to normalize the sequence (e_n) by the condition $\|e_n\| = 1$ for each integer n, but this is unimportant here.

An infinite dimensional normed space X is called *Hereditarily Indecomposable* (in short H.I.) if for any infinite dimensional vector subspaces Y and Z of X,

$$\inf\{\|y - z\| : y \in S(Y),\, z \in S(Z)\} = 0. \tag{1}$$

It is easy to check that this property is equivalent to the fact that no subspace of X is the topological direct sum of two infinite dimensional subspaces Y and Z. Property (1) says that the *angle* between any two infinite dimensional subspaces of X is equal to 0. This notion of angle will be discussed with more details below.

In order to compare easily the H.I. property and the unconditionality property, we rephrase unconditionality in terms of angle of subspaces. Saying that (e_n) is C-unconditional is of course equivalent to saying that

$$\left\|\sum a_i e_i\right\| \le C \left\|\sum \varepsilon_i a_i e_i\right\|$$

for all signs (ε_i) and all scalars (a_i) (we just moved the signs to the other side). For any finite subset K of the set of indices, let E_K denote the linear span of $(e_k)_{k\in K}$. Consider a linear combination $\sum \varepsilon_i a_i e_i$, let $I = \{i : \varepsilon_i = 1\}$ and $J = \{i : \varepsilon_i = -1\}$. Letting $x = \sum_{i\in I} a_i e_i \in E_I$ and $y = \sum_{i\in J} a_i e_i \in E_J$ we may restate the above inequality as

$$\|x + y\| \le C\, \|x - y\|$$

for all $x \in E_I$ and $y \in E_J$, whenever I and J are disjoint. This is again an angle property. There are however several ways for measuring the angle between two subspaces, and we want to introduce two of them. For any L, M finite or infinite dimensional subspaces of X, we denote by $a(L, M)$ the measure of the angle between L and M given by

$$a(L, M) = \inf\{\|x - y\| : x \in S(L), y \in S(M)\}.$$

This expression is symmetric, decreasing in L and M, and (Lipschitz-) continuous for the metric $\delta(L, M)$ given by the Hausdorff distance between the unit spheres $S(L)$ and $S(M)$,

$$\delta(L, M) = \max\big\{\sup\{d(x, S(M)) : x \in S(L)\},\, \sup\{d(y, S(L)) : y \in S(M)\}\big\}.$$

An equivalent expression for the angle is

$$b(L, M) = \inf\big\{\inf\{d(x, M) : x \in S(L)\},\, \inf\{d(y, L) : y \in S(M)\}\big\}.$$

It is clear that $b(L,M) \le a(L,M)$; in the other direction we have $a(L,M) \le 2b(L,M)$. To see this, let $b > b(L,M)$, and assume for example that $b(L,M) = \inf\{d(x,M) : x \in S(L)\}$. Let $x \in S(L)$ and $u \in M$ be such that $\|x-u\| < b$, hence $1-b < \|u\| < 1+b$. Letting $u' = u/\|u\|$, we have $\|u-u'\| = \big|\|u\|-1\big| < b$, and $d(x, S(M)) \le \|x-u'\| < 2b$.

According to the above discussion, we see that a sequence $(e_n)_{n\in\mathbb{N}}$ of non zero vectors in X is unconditional iff there exists $\beta > 0$ such that

$$b(\mathrm{span}\{e_n : n \in I\}, \mathrm{span}\{e_n : n \in J\}) \ge \beta \tag{2}$$

whenever I and J are finite disjoint subsets of $\mathbb{N}$. The relations between β and the unconditional constant of the sequence (e_n) are as follows: given $\beta > 0$ with the above property, the sequence (e_n) is C-unconditional with $C \le 2/\beta$. Conversely, if the sequence (e_n) is C-unconditional, then (2) is true with $\beta \ge 2/(C+1)$.

Let us check these two facts. Suppose first that (2) is true for some $\beta > 0$. If I and J are disjoint, and $x \in E_I$, $y \in E_J$, we see that

$$\|x+y\| \le \frac{2}{\beta}\,\|x-y\|,$$

proving that (e_n) is $2/\beta$-unconditional. Indeed, suppose that $\|x\| = 1 \ge \|y\|$; we know that $\|x-y\| \ge b(E_I, E_J) \ge \beta$ and $\|x+y\| \le 2$, and the inequality above follows by homogeneity. Conversely, if (e_n) is C-unconditional, the projection $P_I : \sum a_i e_i \to \sum_{i\in I} a_i e_i$ on E_I has norm $\le (C+1)/2$ for any subset I of the set of indices, and this implies that $\|x\| \le \frac{C+1}{2}\|x-y\|$, hence we may choose $\beta = 2/(C+1)$.

The following easy technical Lemma will be used in the proof of Theorem 2 below.

Lemma 1. *Assume that E, E' are finite dimensional subspaces of X, M any subspace of X and Z an infinite dimensional subspace of X. We have*

$$\sup_{U\subset Z} a(E'+U,\, M) \le \sup_{U\subset Z} a(E+U,\, M) + 2\delta(E', E),$$

where the supremum above runs over all infinite dimensional subspaces U of Z.

Proof. Let

$$s > \sup_{V\subset Z} a(E+V,\, M), \quad \delta = \delta(E', E),$$

$t > 1$ and let U be any infinite dimensional subspace of Z. By a standard argument, we may find an infinite dimensional subspace $U' \subset U$ such that $t\|e+u'\| \ge \|e\|$ for every $e \in E$ and $u' \in U'$ (we intersect U with the kernels of a finite set of functionals forming a t^{-1}-norming set for E). By assumption we have $a(E+U',\, M) < s$, hence we can find $e + u' \in S(E+U')$ and $y \in S(M)$ such that $\|(e+u') - y\| < s$. We know then that $\|e\| \le t$, thus there exists

$e' \in E'$ such that $\|e' - e\| \le t\delta$. Now $1 - t\delta \le \|e' + u'\| \le 1 + t\delta$ and we can find $x \in S(E' + U')$ such that $\|x - (e' + u')\| \le t\delta$. Finally,

$$a(E' + U,\, M) \le a(E' + U',\, M) \le \|x - y\| < s + 2t\delta,$$

ending the proof. □

So far we did not say if our normed spaces are real or complex, and everything above applies to both cases. In the complex case however, it is customary to define the *complex unconditional constant* by replacing in the definition above the signs $\varepsilon_i = \pm 1$ by arbitrary complex numbers of modulus one. This makes no essential difference, because a sequence of vectors in a complex normed space is complex-unconditional iff it is real-unconditional, except that the complex unconditional constant may differ from the real constant by some factor (less than 3 say). We shall therefore work with the real definition of the unconditional constant.

We introduce the intermediate notion of a $HI(\varepsilon)$ space. Given $\varepsilon > 0$, an infinite dimensional normed space X will be called a $HI(\varepsilon)$ *space* if for every infinite dimensional subspaces Y and Z of X we have

$$a(Y, Z) \le \varepsilon.$$

Obviously, a normed space X is H.I. iff it is $HI(\varepsilon)$ for every $\varepsilon > 0$.

THEOREM 2. *Let X be an infinite dimensional normed space. For each $\varepsilon > 0$, either X contains an infinite sequence with unconditional constant $\le 4/\varepsilon$, or X contains a $HI(\varepsilon)$ subspace Z.*

Of course, when X does not contain any infinite sequence with unconditional constant $\le 4/\varepsilon$, this implies that *every* infinite dimensional subspace Y of X contains a $HI(\varepsilon)$ subspace. Theorem 2 implies Theorem 1 by a simple diagonalization procedure that already appears in [G1]: assume that X does not contain any infinite unconditional sequence; by Theorem 2, every subspace Y of X contains for each $\varepsilon > 0$ a subspace Z which is $HI(\varepsilon)$. Taking successively $\varepsilon = 2^{-n}$, we construct a decreasing sequence (Z_n), where Z_n is a $HI(2^{-n})$ subspace of X. Let Z be a subspace obtained from the sequence (Z_n) by the diagonal procedure. For every n, this space Z is contained in Z_n up to finitely many dimensions, therefore Z is $HI(\varepsilon)$ for every $\varepsilon > 0$, so Z is H.I.

PROOF OF THEOREM 2. We may clearly restrict our attention to separable spaces X. Let (E, F) be a couple of finite dimensional subspaces of X and let Z be an infinite dimensional subspace of X. We set

$$A(E, F, Z) = \sup_{U,V \subset Z} a(E + U,\, F + V),$$

where the supremum is taken over all infinite dimensional subspaces U and V of Z. It follows from Lemma 1 that $A(E', F', Z) \le A(E, F, Z) + 2\delta(E', E) +$

$2\delta(F', F)$ for all finite dimensional subspaces E' and F'. We will keep $\varepsilon > 0$ fixed throughout the proof.

We introduce a convenient terminology, inspired by [GP]. We say that the couple (E, F) *accepts* the subspace Z if

$$A(E, F, Z) < \varepsilon.$$

This perhaps unnatural strict inequality is necessary for approximation reasons. Indeed, we get from Lemma 1 that when (E, F) accepts a subspace Z of X, then (E', F') also accepts Z provided $\delta(E', E)$ and $\delta(F', F)$ are small enough. When (E, F) accepts Z, we know that $a(E + U,\ F + V) < \varepsilon$ for all infinite dimensional subspaces U and V of Z, and except for the small technicality just mentioned, this is exactly the idea that the reader should keep in mind. Before going any further, let us notice that when the couple $(\{0\}, \{0\})$ accepts a subspace Z, then Z is $HI(\varepsilon)$ (actually, Z is then $HI(\varepsilon')$ for some $\varepsilon' < \varepsilon$). Acceptance is clearly symmetric: (F, E) accepts Z iff (E, F) accepts Z. If (E, F) accepts Z, it also accepts every $Z' \subset Z$ (obvious) and every $Z + G$, when $\dim G < +\infty$; this last fact is easy: given two infinite dimensional subspaces U, V of $Z + G$, we may consider the two infinite dimensional subspaces $U' = U \cap Z$ and $V' = V \cap Z$ of Z; since (E, F) accepts Z, we have

$$a(E + U,\ F + V) \le a(E + U',\ F + V') \le A(E, F, Z) < \varepsilon.$$

Notice that what we just did was proving the equality $A(E, F, Z) = A(E,\ F,\ Z + G)$, which is one of the main ingredients for the proof: we are dealing here with a function of Z that does not depend upon changing finitely many dimensions.

We say that a couple $\tau = (E, F)$ *rejects* Z if no subspace $Z' \subset Z$ is accepted by τ. Rejection is also symmetric, and saying that (E, F) rejects some subspace Z (or simply: does not accept Z) implies that

$$a(E, F) \ge \varepsilon$$

because $a(E, F) \ge a(E + U,\ F + V)$ for all U, V, hence $a(E, F) \ge A(E, F, Z)$ for every Z. This yields $b(E, F) \ge \frac{1}{2} a(E, F) \ge \varepsilon/2$ and will be used in connection with the property (2) for $\beta = \varepsilon/2$, in order to produce an upper bound $4/\varepsilon$ for the unconditional constant. This notion of rejection will therefore be the tool for constructing inductively subspaces with an angle bounded away from 0; the strength of the rejection hypothesis will allow the induction to run. Observe that when a couple τ rejects a subspace Z, it is clearly true by definition that τ rejects every subspace Z' of Z, and τ also rejects "supspaces" of Z of the form $Z + G$, when G is finite dimensional (otherwise, τ would accept some $Z' \subset Z + G$, hence also accept $Z'' = Z' \cap Z$, contradicting the fact that τ rejects Z); combining the above observations, we see that when τ accepts or rejects Z, the same is true for every Z' such that $Z' \subset Z + G$, when G is any finite dimensional subspace of X. This simple remark is the basis for our first step. Since X was assumed separable, we may select a countable family $\mathcal{E}$, dense in the set of finite dimensional subspaces of X (for the Hausdorff metric of spheres).

Claim 1. *There exists an infinite dimensional subspace Z_0 of X such that for every couple (E, F) with $E, F \in \mathcal{E}$ and every rational α in $(0, \varepsilon)$, either $A(E, F, Z_0) < \alpha$ or, for every infinite dimensional subspace Z' of Z_0, we have $A(E, F, Z') \geq \alpha$.*

Proof. We use a very usual diagonal argument. Let $(\sigma_n)_{n\geq 1}$, with $\sigma_n = (E_n, F_n, \alpha_n)$ be a listing of all triples (E, F, α) such that $E, F \in \mathcal{E}$ and α is a rational number in $(0, \varepsilon)$. We construct a decreasing sequence $(X_n)_{n\geq 0}$ of subspaces of X in the following way: $X_0 = X$, and if $A(E_{n+1}, F_{n+1}, Z') \geq \alpha_{n+1}$ for every subspace Z' of X_n, we simply let $X_{n+1} = X_n$. Otherwise, there exists a subspace of X_n, which we call X_{n+1}, such that $A(E_{n+1}, F_{n+1}, X_{n+1}) < \alpha_{n+1}$. We consider then a diagonal infinite dimensional subspace Z_0 which is the linear span of a sequence $(z_n)_{n\geq 1}$ built by picking inductively z_{n+1} in X_{n+1} and not in the linear span of $z_1, z_2, \ldots, z_n$. For each integer $n \geq 1$, we see that $Z_0 \subset X_n + G_n$ for some finite dimensional subspace G_n, and either

$$A(E_n, F_n, Z_0) \leq A(E_n,\, F_n,\, X_n + G_n) = A(E_n, F_n, X_n) < \alpha_n,$$

or for every subspace Z' of Z_0, $A(E_n, F_n, Z') = A(E_n,\, F_n,\, Z' \cap X_n) \geq \alpha_n$. □

By an easy approximation argument, we can state a version of Claim 1 above that will apply to any couple τ, and not only to those from the dense subset $\mathcal{E}$. Let (E, F) be an arbitrary couple. If (E, F) does not reject Z_0, it accepts some $Z' \subset Z_0$ and we may choose a rational α in $(0, \varepsilon)$ such that $A(E, F, Z') < \alpha$; let β be rational and $0 < \beta < (\varepsilon - \alpha)/8$; let $E', F' \in \mathcal{E}$ be such that $\delta(E', E) < \beta$ and $\delta(F', F) < \beta$. This implies by Lemma 1 that $A(E', F', Z') < \alpha + 4\beta < \varepsilon$. But then by Claim 1 it follows that $A(E', F', Z_0) < \alpha + 4\beta$; by approximation again (E, F) accepts Z_0. Finally:

Claim 2. *For each couple (E, F) of finite dimensional subspaces of X, either (E, F) rejects Z_0 or (E, F) accepts Z_0.*

From now on the whole construction will be performed inside our "stabilizing" subspace Z_0. Here is where the dichotomy really starts. There are two possibilities: either the couple $(\{0\}, \{0\})$ accepts Z_0, or it rejects. As was mentioned before, saying that $(\{0\}, \{0\})$ accepts Z_0 implies that Z_0 is $HI(\varepsilon)$. Suppose now that $(\{0\}, \{0\})$ rejects Z_0; we will find in Z_0 a sequence $(e_k)_{k\geq 1}$ with unconditional constant $C \leq 4/\varepsilon$. This will be done in the following manner: we will choose the sequence (e_k) of non zero vectors in such a way that for each $n \geq 1$ and for all disjoint sets $I, J \subset \{1, \ldots, n\}$, the couple (E_I, E_J) rejects Z_0 (as before, we denote by E_K the linear span of $\{e_k : k \in K\}$). The next Lemma and its Corollary give the tool for constructing the next vector e_{n+1} of our unconditional sequence, when $e_1, \ldots, e_n$ are already selected.

Lemma 2. *If (E, F) rejects Z_0, then for every infinite dimensional subspace Z' of Z_0 there exists a further infinite dimensional subspace $U' \subset Z'$ such that for every finite dimensional subspace E' of U', the couple $(E + E',\, F)$ rejects Z_0.*

Proof. Otherwise, there exists $Z' \subset Z_0$ such that, for every subspace $U' \subset Z'$, there exists $E' \subset U'$ such that $(E + E',\, F)$ does not reject Z_0. We know that $(E + E',\, F)$ accepts Z_0 by Claim 2; for every subspace $V' \subset Z'$ we have, since $E + U' = E + E' + U'$,

$$a(E + U',\, F + V') = a(E + E' + U',\, F + V') \leq A(E + E',\, F,\, Z_0) < \varepsilon,$$

which implies that (E, F) accepts Z', therefore (E, F) accepts Z_0 by Claim 2, contrary to the initial hypothesis. □

Corollary 1. *Suppose that $(E_\alpha, F_\alpha)_{\alpha \in A}$ is a finite family of couples, and that (E_α, F_α) rejects Z_0 for each $\alpha \in A$. For every infinite dimensional subspace Z'' of Z_0 there exists a further infinite dimensional subspace $U'' \subset Z''$ such that for every finite dimensional subspace E' of U'', the couple $(E_\alpha + E',\, F_\alpha)$ rejects Z_0 for each $\alpha \in A$.*

Proof. We set $A = \{\alpha_1, \ldots, \alpha_p\}$. Let $Z'' = Z'_0$ be a subspace of Z_0. By Lemma 2, there exists $U' = Z'_1 \subset Z'_0$ such that for every $E' \subset Z'_1$, $(E_{\alpha_1} + E',\, F_{\alpha_1})$ rejects Z_0. We apply again Lemma 2, this time to the couple $(E_{\alpha_2}, F_{\alpha_2})$, with $Z' = Z'_1$, and so on until $U'' = Z'_p \subset Z'_{p-1} \subset \ldots \subset Z'$ is reached. □

The Corollary will be applied in the following weakened form; the notation $[z]$ stands for the line $\mathbb{R}z$ or $\mathbb{C}z$ generated by a non zero vector z:

Suppose that $(E_\alpha, F_\alpha)_{\alpha \in A}$ is a finite family of couples, and that (E_α, F_α) rejects Z_0 for each $\alpha \in A$. For every infinite dimensional subspace Z' of Z_0 there exists a non zero vector $z \in Z'$ such that the couple $(E_\alpha + [z],\, F_\alpha)$ rejects Z_0 for each $\alpha \in A$.

Let us finish the proof of the Theorem. Recall that E_K denotes the linear span of $(e_k)_{k \in K}$. Assuming that $(\{0\}, \{0\})$ rejects Z_0, we build by induction a sequence $(e_k)_{k=1}^{\infty}$ of non zero vectors in Z_0, such that for every integer $n \geq 1$ and every partition (I, J) of $\{1, \ldots, n\}$, the couple (E_I, E_J) rejects Z_0. Assuming that $e_1, \ldots, e_n$ are already selected, let us call *partition of length n* any couple of the form (E_I, E_J), for some partition (I, J) of $\{1, \ldots, n\}$. Consider the finite list A_n of all partitions (E_α, F_α) of length n, where $\alpha \in A_n$. Our induction hypothesis is that for every $\alpha \in A_n$, the couple (E_α, F_α) rejects Z_0; let Z' be an infinite dimensional subspace of Z_0 such that $Z' \cap \operatorname{span}\{e_1, \ldots, e_n\} = \{0\}$. By the Corollary, we can find a non zero vector $z \in Z'$ such that for every $\alpha \in A_n$, the couple $(E_\alpha + [z],\, F_\alpha)$ rejects Z_0; observe that (F_α, E_α) also belongs to the list, hence $(F_\alpha + [z],\, E_\alpha)$ also rejects Z_0. We choose now $e_{n+1} = z$. It is clear that with this choice, (E_I, E_J) rejects Z_0 for every partition of length $n+1$. This implies that the infinite sequence $(e_k)_{k \geq 1}$ satisfies property (2) with the constant $\beta = \varepsilon/2$, thus this sequence is $4/\varepsilon$-unconditional. □

Remark. It is possible to obtain directly a H.I. space, without passing through the intermediate stage of $HI(\varepsilon)$ spaces, by replacing the study of couples by that of triples (E, F, ε), for ε varying. The first version of this paper was indeed written in this way, but the referee said (and was probably right about it) that the earlier version in [M] gave a clearer view of the combinatorics, by dealing first with a countable situation (a countable vector space over $\mathbb{Q}$) and treating the boring approximation afterwards. This version is a sort of midpoint between the two, which perhaps only adds the disadvantages of both…

References

[B] S. Banach, Théorie des opérations linéaires, Warszawa, 1932.

[E] E. Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163–165.

[GP] F. Galvin and K. Prikry, *Borel sets and Ramsey's theorem*, J. Symbolic Logic **38** (1973), 193–198.

[G1] W. T. Gowers, *A new dichotomy for Banach spaces*, Geom. Funct. Anal. **6** (1996), 1083–1093.

[G2] W. T. Gowers, *Analytic sets and games in Banach spaces*, preprint IHES M/94/42.

[G3] W. T. Gowers, *Recent results in the theory of infinite dimensional Banach spaces*, Proceedings ICM 1994, Birkhäuser 1995.

[GM1] W. T. Gowers and B. Maurey, *The unconditional basic sequence problem*, Jour. Amer. Math. Soc. **6** (1993), 851–874.

[GM2] W. T. Gowers and B. Maurey, *Banach spaces with small spaces of operators*, Math. Ann. **307** (1997), 543–568.

[KT] R. Komorowski and N. Tomczak-Jaegermann, *Banach spaces without local unconditional structure*, Israel J. Math. **89** (1995), 205–226.

[M] B. Maurey, *Quelques progrès dans la compréhension de la dimension infinie*, Journée Annuelle 1994, Société Mathématique de France.

[NW] C. St J. A. Nash-Williams, *On well quasi-ordering transfinite sequences*, Proc. Cam. Phil. Soc. **61** (1965), 33–39.

[T] B. S. Tsirelson, *Not every Banach space contains ℓ_p or c_0*, Funct. Anal. Appl. **8** (1974), 139–141.

Bernard Maurey
Equipe d'Analyse et Mathématiques Appliquées
Université de Marne la Vallée
5 boulevard Descartes, Champs sur Marne
77454 Marne la Vallée Cedex 2
France
maurey@math.univ-mlv.fr