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Floating Body, Illumination Body, and Polytopal Approximation

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Abstract. Let K be a convex body in $\mathbb{R}^d$ and K_t its floating bodies. There is a polytope that satisfies $K_t \subset P_n \subset K$ and has at most n vertices, where

$$n \le e^{16d} \frac{\mathrm{vol}_d(K \setminus K_t)}{t\ \mathrm{vol}_d(B_2^d)}.$$

Let K^t be the illumination bodies of K and Q_n a polytope that contains K and has at most n $(d-1)$-dimensional faces. Then

$$\mathrm{vol}_d(K^t \setminus K) \le cd^4 \,\mathrm{vol}_d(Q_n \setminus K),$$

where

$$n \le \frac{c}{dt}\ \mathrm{vol}_d(K^t \setminus K).$$

1. Introduction

We investigate the approximation of a convex body K in $\mathbb{R}^d$ by a polytope. We measure the approximation by the symmetric difference metric. The symmetric difference metric between two convex bodies K and C is

$$d_S(C,K) = \mathrm{vol}_d((C \setminus K) \cup (K \setminus C)).$$

We study in particular two questions: How well can a convex body K be approximated by a polytope P_n that is contained in K and has at most n vertices and how well can K be approximated by a polytope Q_n that contains K and has at most n $(d-1)$-dimensional faces. Macbeath [Mac] showed that the Euclidean Ball B_2^d is an extremal case: The approximation for any other convex body is better. We have for the Euclidean ball

$$c_1\ d\,\mathrm{vol}_d(B_2^d) n^{-\frac{2}{d-1}} \le d_S(P_n, B_2^d) \le c_2\ d\,\mathrm{vol}_d(B_2^d) n^{-\frac{2}{d-1}}, \tag{1.1}$$

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provided that $n \geq (c_3\ d)^{(d-1)/2}$. The right hand inequality was first established by Bronshtein and Ivanov [BI] and Dudley [D_1,D_2]. Gordon, Meyer, and Reisner [GMR_1,GMR_2] gave a constructive proof for the same inequality. Müller [Mü] showed that random approximation gives the same estimate. Gordon, Reisner, and Schütt [GRS] established the left hand inequality. Gruber [Gr_2] obtained an asymptotic formula. If a convex body K in $\mathbb{R}^d$ has a C^2-boundary with everywhere positive curvature, then

$$\inf\{d_S(K,P_n) \mid P_n \subset K \text{ and } P_n \ \text{ has at most } n \text{ vertices}\}$$

is asymptotically the same as

$$\tfrac{1}{2}\mathrm{del}_{d-1}\left(\int_{\partial K}\kappa(x)^{\frac{1}{d+1}}\,d\mu(x)\right)^{\frac{d+1}{d-1}}\left(\frac{1}{n}\right)^{\frac{2}{d-1}},$$

where del_{d-1} is a constant that is connected with Delone triangulations. In this paper we are not concerned with asymptotic estimates, but with uniform.

$\mathrm{Int}(M)$ denotes the interior of a set M. $H(x,\xi)$ denotes the hyperplane that contains x and is orthogonal to ξ. $H^+(x,\xi)$ denotes the halfspace that contains the vector $x-\xi$, and $H^-(x,\xi)$ the halfspace containing $x+\xi$. $e_i, i=1,\dots,d$ denotes the unit vector basis in $\mathbb{R}^d$. $[A,B]$ is the convex hull of the sets A and B. The convex floating body K_t of a convex body K is the intersection of all halfspaces whose defining hyperplanes cut off a set of volume t from K.

The illumination body K^t of a convex body K is [W]

$$\{x\in\mathbb{R}^d \mid \mathrm{vol}_d([x,K]\setminus K)\leq t\,\}.$$

K^t is a convex body. It is enough to show this for polytopes. Let F_i denote the faces of a polytope P, ξ_i the outer normal and x_i an element of F_i. Then

$$\mathrm{vol}_d([x,P]\setminus P)=\frac{1}{d}\sum_{i=1}^{n}\max\{0,\langle\xi_i,x-x_i\rangle\}\,\mathrm{vol}_{d-1}(F_i).$$

The right-hand side is a convex function.

2. The Floating Body

THEOREM 2.1. *Let K be a convex body in $\mathbb{R}^d$. Then, for every t satisfying $0\leq t\leq \frac{1}{4}e^{-5}\,\mathrm{vol}_d(K)$, there exist $n\in\mathbb{N}$ with*

$$n\leq e^{16d}\ \frac{\mathrm{vol}_d(K\setminus K_t)}{t\ \mathrm{vol}_d(B_2^d)}$$

and a polytope P_n that has n vertices and such that

$$K_t\subset P_n\subset K.$$

We want to see what kind of asymptotic estimate we get for bodies with smooth boundary from Theorem 2.1. We have [SW]

$$\begin{aligned}\operatorname{vol}_d(K \setminus K_t) &\sim t^{\frac{2}{d+1}} \frac{1}{2} \left(\frac{d+1}{\operatorname{vol}_{d-1}(B_2^{d-1})} \right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu(x) \\ &\sim t^{\frac{2}{d+1}} d \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu(x).\end{aligned}$$

Since

$$n \sim d^{\frac{d}{2}} \frac{1}{t} \operatorname{vol}_d(K \setminus K_t),$$

we get

$$\begin{aligned}\operatorname{vol}_d(K \setminus K_t) &\sim d \left(d^{\frac{d}{2}} \frac{1}{n} \operatorname{vol}_d(K \setminus K_t) \right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu(x), \\ \operatorname{vol}_d(K \setminus K_t)^{\frac{d-1}{d+1}} &\sim d^2 n^{-\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu(x).\end{aligned}$$

Thus we get

$$\operatorname{vol}_d(K \setminus P_n) \leq \operatorname{vol}_d(K \setminus K_t) \sim d^2 n^{-\frac{2}{d-1}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}} \, d\mu(x) \right)^{\frac{d+1}{d-1}}.$$

When K is the Euclidean ball we get

$$\operatorname{vol}_d(B_2^d \setminus P_n) \leq c d^2 n^{-\frac{2}{d-1}} \operatorname{vol}_d(B_2^d),$$

where c is an absolute constant. If one compares this to the optimal result (1.1) one sees that there is an additional factor d.

The volume difference $\operatorname{vol}_d(P) - \operatorname{vol}_d(P_t)$ for a polytope P is of a much smaller order than for a convex body with smooth boundary. In fact, we have [S] that it is of the order $t\,|\ln t|^{d-1}$. In [S] this has been used to get estimates for approximation of convex bodies by polytopes.

The same result as in Theorem 2.1 holds if we fix the number of (d-1)-dimensional faces instead of the number of vertices. This follows from the same proof as for Theorem 2.1 and also from the economic cap covering for floating bodies [BL, Theorem 6]. I. Bárány showed us a proof for Theorem 2.1 using the economic cap covering. The constants are not as good as in Theorem 2.1.

The following lemmata are not new. They have usually been formulated for symmetric, convex bodies [B,H,MP]. Lemma 2.2 is due to Grünbaum [Grü].

Lemma 2.2. *Let K be a convex body in $\mathbb{R}^d$ and let $H(cg(K), \xi)$ be the hyperplane passing through the center of gravity $cg(K)$ of K and being orthogonal to ξ. Then we have, for all $\xi \in \partial B_2^d$:*

(i) $(1 - \frac{1}{d+1})^d \operatorname{vol}_d(K) \leq \operatorname{vol}_d(K \cap H^+(cg(K), \xi)) \leq (1 - (1 - \frac{1}{d+1})^d) \operatorname{vol}_d(K)$.

(ii) *For all hyperplanes H in $\mathbb{R}^d$ that are parallel to $H(cg(K),\xi)$,*

$$\Big(1-\frac{1}{d+1}\Big)^{d-1}\operatorname{vol}_{d-1}(K\cap H)\le \operatorname{vol}_{d-1}(K\cap H(cg(K),\xi)).$$

The sequence $(1-\frac{1}{d+1})^d$, $d=2,3,\dots$ is monotonely decreasing. Indeed, by Bernoulli's inequality we have $1-\frac{1}{d}\le(1-\frac{1}{d^2})^d$, or $\frac{d-1}{d}\le(\frac{d^2-1}{d^2})^d$. Therefore we get $(\frac{d}{d+1})^d\le(\frac{d-1}{d})^{d-1}$, which implies $(1-\frac{1}{d+1})^d\le(1-\frac{1}{d})^{d-1}$.

Therefore we get for the inequalities (i)

$$\frac{1}{e}\operatorname{vol}_d(K)\le\operatorname{vol}_d(K\cap H^+(cg(K),\xi))\le(1-\frac{1}{e})\operatorname{vol}_d(K). \tag{2.1}$$

By the preceding calculations, $(1+\frac{1}{d})^d$ is a monotonely increasing sequence. Thus $(1+\frac{1}{d})^{d-1}<e$. For (ii) we get

$$\operatorname{vol}_{d-1}(K\cap H)\le e\operatorname{vol}_{d-1}(K\cap H(cg(K),\xi)). \tag{2.2}$$

Proof. (i) We can reduce the inequality to the case that K is a cone with a Euclidean ball of dimension $d-1$ as base. To see this we perform a Schwarz symmetrization parallel to $H(cg(K),\xi)$ and denote the symmetrized body by $S(K)$. The Schwarz symmetrization replaces a section parallel to $H(cg(K),\xi)$ by a $(d-1)$-dimensional Euclidean sphere of the same $(d-1)$-dimensional volume. This does not change the volume of K and $K\cap H^+(cg(K),\xi)$ and the center of gravity $cg(K)$ is still an element of $H(cg(K),\xi)$. Now we consider the cone

$$[z,S(K)\cap H(cg(K),\xi)]$$

such that

$$\operatorname{vol}_d([z,S(K)\cap H(cg(K),\xi)])=\operatorname{vol}_d(K\cap H^-(cg(K),\xi))$$

and such that z lies on the axis of symmetry of $S(K)$ and in $H^-(cg(K),\xi)$. See Figure 1.

The set

$$\tilde K=(K\cap H^+(cg(K),\xi))\cup[z,S(K)\cap H(cg(K),\xi)]$$

is a convex set such that $\operatorname{vol}_d(K)=\operatorname{vol}_d(\tilde K)$ and such that the center of gravity $cg(\tilde K)$ of $\tilde K$ is contained in $[z,S(K)\cap H(cg(K),\xi)]$. Thus

$$\operatorname{vol}_d(\tilde K\cap H^+(cg(\tilde K),\xi))\ge\operatorname{vol}_d(\tilde K\cap H^+(cg(K),\xi))=\operatorname{vol}_d(K\cap H^+(cg(K),\xi)).$$

We apply a similar argument to the set $S(K)\cap H^+(cg(K),\xi)$ and show that we may assume that $S(K)$ is a cone with z as its vertex. Thus we may assume that

$$K=\big[(0,\dots,0,1),\ \big\{(x_1,\dots,x_{d-1},0)\mid \textstyle\sum_{i=1}^{d-1}|x_i|^2\le1\big\}\big]\quad\text{and}\quad\xi=(0,\dots,0,1).$$

Then

$$\operatorname{vol}_d(K)=\frac{1}{d}\operatorname{vol}_{d-1}(B_2^{d-1})$$

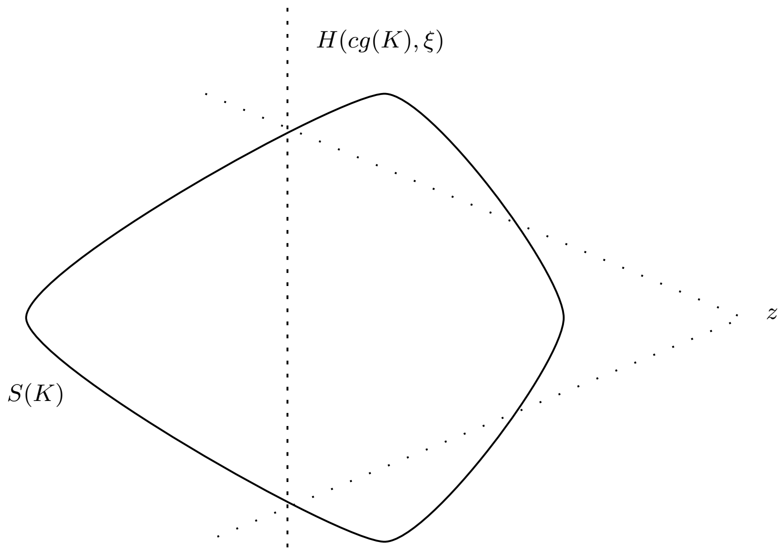

Figure 1.

and

$$\frac{1}{\mathrm{vol}_d(K)}\int_K x_d\,dx_d = d\int_0^1 t(1-t)^{d-1}\,dt = d\int_0^1 (1-s)s^{d-1}\,ds = \frac{1}{d+1}\,.$$

We obtain

$$\mathrm{vol}_d(K\cap H^-(cg(K),(0,\dots,0,1))) = \left(1-\frac{1}{d+1}\right)^d \mathrm{vol}_d(K).$$

(ii) Let H be a hyperplane parallel to $H(cg(K),\xi)$ and such that $\mathrm{vol}_{d-1}(K\cap H) > \mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))$. Otherwise there is nothing to prove. We apply a Schwarz symmetrization parallel to $H(cg(K),\xi)$ to K. The symmetrized body is denoted by $S(K)$. Let z be the element of the axis of symmetry of $S(K)$ such that

$$[z,S(K)\cap H]\cap H(cg(K),\xi) = S(K)\cap H(cg(K),\xi).$$

Since $\mathrm{vol}_{d-1}(K\cap H) > \mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))$ there is such a z. We may assume that $H^+(cg(K),\xi)$ is the half-space containing z. Then

$$[z,S(K)\cap H]\cap H^-(cg(K),\xi)\subset S(K)\cap H^-(cg(K),\xi),$$
$$[z,S(K)\cap H]\cap H^+(cg(K),\xi)\supset S(K)\cap H^+(cg(K),\xi).$$

Therefore

$$cg([z,S(K)\cap H])\in H^+(cg(K),\xi).$$

Therefore, if h_{cg} denotes the distance of z to $H(cg(K),\xi)$ and h the distance of z to H, we get as in the proof of (i) that

$$h_{cg} \geq h\Big(1-\frac{1}{d+1}\Big).$$

Thus we get

$$\begin{aligned}\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)) &= \mathrm{vol}_{d-1}(S(K)\cap H(cg(K),\xi))\\ &\geq (1-\frac{1}{d+1})^{d-1}\,\mathrm{vol}_{d-1}(S(K)\cap H)\\ &= (1-\frac{1}{d+1})^{d-1}\,\mathrm{vol}_{d-1}(K\cap H). \qquad \square\end{aligned}$$

LEMMA 2.3. *Let K be a convex body in $\mathbb{R}^d$ and let $\Theta(\xi)$ be the infimum of all positive numbers t such that*

$$\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)) \geq e\ \mathrm{vol}_{d-1}(K\cap H(cg(K)+t\xi,\xi)).$$

Then

$$\frac{1}{2e^3}\,\mathrm{vol}_d(K) \leq \Theta(\xi)\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)) \leq e\ \mathrm{vol}_d(K).$$

PROOF. The right hand inequality follows from Fubini's theorem and Brunn–Minkowski's theorem. Now we verify the left hand inequality. We consider first the case in which, for all t such that $t>\Theta(\xi)$,

$$K\cap H(cg(K)+t\xi,\xi)=\varnothing.$$

Then, by (2.1) and (2.2),

$$\begin{aligned}\frac{1}{e}\,\mathrm{vol}_d(K) &\leq \mathrm{vol}_d(K\cap H^+(cg(K),\xi))\\ &= \int_0^{\Theta(\xi)} \mathrm{vol}_{d-1}(K\cap H(cg(K)+t\xi,\xi))\,dt\\ &\leq e\ \Theta(\xi)\ \mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)).\end{aligned}$$

If, for some t such that $t>\Theta(\xi)$, we have $K\cap H(cg(K)+t\xi,\xi)\neq\varnothing$, then, by continuity,

$$\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)) = e\ \mathrm{vol}_{d-1}(K\cap H(cg(K)+\Theta(\xi)\xi,\xi)).$$

We perform a Schwarz symmetrization parallel to $H(cg(K),\xi)$. We consider the cone

$$[z, S(K)\cap H(cg(K),\xi)]$$

such that z is an element of the axis of symmetry of $S(K)$ and such that

$$[z,S(K)\cap H(cg(K),\xi)]\cap H(cg(K)+\Theta(\xi)\xi,\xi) = S(K)\cap H(cg(K)+\Theta(\xi)\xi,\xi).$$

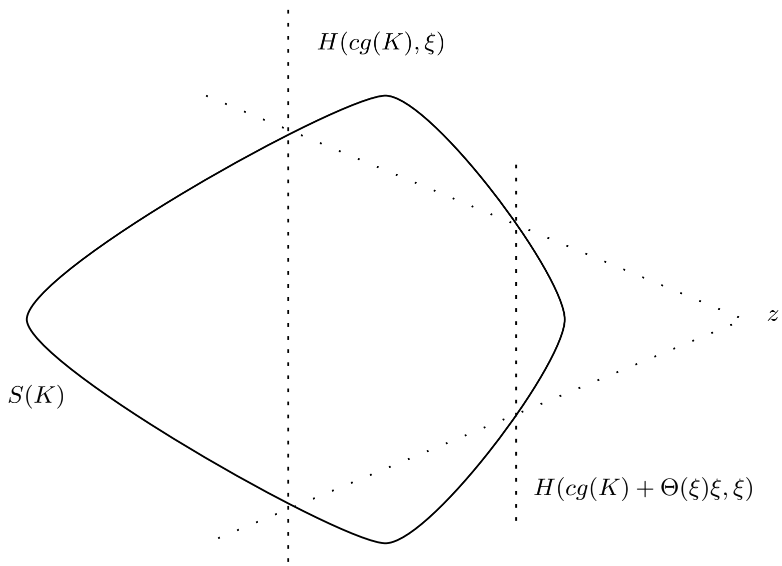

Figure 2.

Let $H^+(cg(K),\xi)$ and $H^+(cg(K)+\Theta(\xi)\xi,\xi)$ be the half-spaces that contain z. Then, by convexity,

$$[z, S(K) \cap H(cg(K),\xi)] \cap H^+(cg(K)+\Theta(\xi)\xi,\xi) \supset S(K) \cap H^+(cg(K)+\Theta(\xi)\xi,\xi). \quad (2.3)$$

We get by (2.1)

$$\begin{aligned}\frac{1}{e}\operatorname{vol}_d(K) \le \operatorname{vol}_d(K \cap H^+(cg(K),\xi)) \\ &= \operatorname{vol}_d(K \cap H^+(cg(K),\xi) \cap H^-(cg(K)+\Theta(\xi)\xi,\xi)) \\ &\quad + \operatorname{vol}_d(K \cap H^+(cg(K)+\Theta(\xi)\xi,\xi)) \\ &= \operatorname{vol}_d(S(K) \cap H^+(cg(K),\xi) \cap H^-(cg(K)+\Theta(\xi)\xi,\xi)) \\ &\quad + \operatorname{vol}_d(S(K) \cap H^+(cg(K)+\Theta(\xi)\xi,\xi)).\end{aligned}$$

By the hypothesis of the lemma we have, for all s with $0 \le s \le \Theta(\xi)$,

$$\operatorname{vol}_{d-1}(K \cap H(cg(K),\xi)) \le e \operatorname{vol}_{d-1}(K \cap H(cg(K)+s\xi,\xi)).$$

Using this and (2.2) we estimate the first summand. The second summand is estimated by using (2.3). Thus the above expression is not greater than

$$e^2 \operatorname{vol}_d([z, S(K) \cap H(cg(K),\xi)] \cap H^-(cg(K)+\Theta(\xi)\xi,\xi)) \\ + \operatorname{vol}_d([z, S(K) \cap H(cg(K),\xi)] \cap H^+(cg(K)+\Theta(\xi)\xi,\xi)).$$

This is the volume of a cone with the base $S(K)\cap H(cg(K),\xi)$. By an elementary computation for the volume of a cone we get that the latter expression is smaller than

$$2e^2 \operatorname{vol}_d([z, S(K)\cap H(cg(K),\xi)] \cap H^-(cg(K)+\Theta(\xi)\xi,\xi)).$$

Since in a cone the base has the greatest surface area, the above expression is smaller than

$$2e^2\Theta(\xi)\operatorname{vol}_{d-1}(K\cap H(cg(K),\xi)). \qquad \square$$

Lemma 2.4. *Let K be a convex body in $\mathbb{R}^d$. Then there is a linear transform T with $\det(T)=1$ so that, for all $\xi\in\partial B_2^d$,*

$$\int_{T(K)} |\langle x,\xi\rangle|^2\,dx = \frac{1}{d}\int_{T(K)}\sum_{i=1}^d |\langle x,e_i\rangle|^2\,dx.$$

We say that a convex body is in an isotropic position if the linear transform T in Lemma 2.4 can be chosen to be the identity. See [B,H].

Proof. We claim that there is a orthogonal transform U such that, for all $i,j=1,\dots,d$ with $i\neq j$,

$$\int_{U(K)} \langle x,e_i\rangle\langle x,e_j\rangle\,dx = 0.$$

Clearly, the matrix

$$\left(\int_K \langle x,e_i\rangle\langle x,e_j\rangle\,dx\right)_{i,j=1}^d$$

is symmetric. Therefore there is an orthogonal $d\times d$-matrix U so that

$$U\left(\int_K \langle x,e_i\rangle\langle x,e_j\rangle\,dx\right)_{i,j=1}^d U^t$$

is a diagonal matrix. We have

$$\begin{aligned} U\left(\int_K \langle x,e_i\rangle\langle x,e_j\rangle\,dx\right)_{i,j=1}^d U^t &= \left(\int_K \sum_{i,j=1}^d u_{l,i}\langle x,e_i\rangle\langle x,e_j\rangle u_{k,j}\,dx\right)_{l,k=1}^d \\ &= \left(\int_K \langle x,U^t(e_l)\rangle\langle x,U^t(e_k)\rangle\,dx\right)_{l,k=1}^d \\ &= \left(\int_{U(K)} \langle y,e_l\rangle\langle y,e_k\rangle\,dy\right)_{l,k=1}^d. \end{aligned}$$

So the latter matrix is a diagonal matrix. All the diagonal elements are strictly positive. This argument is repeated with a diagonal matrix so that the diagonal elements turn out to be equal. Therefore there is a matrix T with $\det T = 1$ such that

$$\int_{T(K)} \langle x, e_i\rangle\langle x, e_j\rangle\, dx = \begin{cases} 0 & \text{if } i \neq j, \\ \dfrac{1}{d}\displaystyle\int_{T(K)} \sum_{j=1}^{d} |\langle x, e_j\rangle|^2\, dx & \text{if } i = j. \end{cases}$$

From this the lemma follows. □

LEMMA 2.5. *Let K be a convex body in $\mathbb{R}^d$ that is in an isotropic position and whose center of gravity is at the origin. Then, for all $\xi \in \partial B_2^d$,*

$$\begin{aligned}\frac{1}{24e^{10}} \operatorname{vol}_d(K)^3 &\leq \operatorname{vol}_{d-1}(K \cap H(cg(K), \xi))^2 \frac{1}{d} \int_K \sum_{i=1}^{d} |\langle x, e_i\rangle|^2\, dx \\ &\leq 6\ e^3 \operatorname{vol}_d(K)^3.\end{aligned}$$

PROOF. By Lemma 2.4 we have, for all $\xi \in \partial B_2^d$,

$$\frac{1}{d} \int_K \sum_{i=1}^{d} |\langle x, e_i\rangle|^2\, dx = \int_K |\langle x, \xi\rangle|^2\, dx.$$

By Fubini's theorem, this equals

$$\int_{-\infty}^{\infty} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt \geq \int_0^{\Theta(\xi)} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt,$$

where $\Theta(\xi)$ is as defined in Lemma 2.3. By the definition of $\Theta(\xi)$ the above expression is greater than

$$\frac{1}{e} \operatorname{vol}_{d-1}(K \cap H(cg(K), \xi)) \int_0^{\Theta(\xi)} t^2\, dt \geq \frac{1}{3e} \Theta(\xi)^3 \operatorname{vol}_{d-1}(K \cap H(cg(K), \xi)).$$

By Lemma 2.3 this is greater than

$$\frac{1}{24e^{10}} \frac{\operatorname{vol}_d(K)^3}{\operatorname{vol}_{d-1}(K \cap H(cg(K), \xi))^2}\,.$$

Now we show the right hand inequality. By Lemma 2.4 we have

$$\begin{aligned}\frac{1}{d} \int_K \sum_{i=1}^{d} |\langle x, e_i\rangle|^2\, dx &= \int_K |\langle x, \xi\rangle|^2\, dx = \int_{-\infty}^{\infty} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt \\ &= \int_0^{\Theta(\xi)} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt + \int_{\Theta(\xi)}^{\infty} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt \\ &\quad + \int_{\Theta(-\xi)}^{0} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt + \int_{-\infty}^{\Theta(-\xi)} t^2\ \operatorname{vol}_{d-1}(K \cap H(t\xi, \xi))\, dt.\end{aligned}$$

By (2.2) this is not greater than

$$\frac{e}{3}\,\Theta(\xi)^3\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))+\int_{\Theta(\xi)}^{\infty} t^2\;\mathrm{vol}_{d-1}(K\cap H(t\xi,\xi))\,dt$$
$$+\frac{e}{3}\,\Theta(-\xi)^3\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))+\int_{-\infty}^{\Theta(-\xi)} t^2\;\mathrm{vol}_{d-1}(K\cap H(t\xi,\xi))\,dt.$$

The integrals can be estimated by

$$2\;\Theta(\xi)^3\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))\quad\text{and}\quad 2\;\Theta(-\xi)^3\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)),$$

respectively. We treat here only the case ξ; the case $-\xi$ is treated in the same way. If the integral equals 0, there is nothing to show. If the integral does not equal 0, we have

$$\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))=e\;\mathrm{vol}_{d-1}(K\cap H(cg(K)+\Theta(\xi)\xi,\xi)).$$

We consider the Schwarz symmetrization $S(K)$ of K with respect to the plane $H(cg(K),\xi)$. We consider the cone C that is generated by the Euclidean spheres $S(K)\cap H(cg(K),\xi)$ and $S(K)\cap H(cg(K)+\Theta(\xi)\xi,\xi)$. We

$$S(K)\cap H^+(cg(K)+\Theta(\xi)\xi,\xi)\subset C$$

and the height of C equals

$$\frac{\Theta(\xi)}{1-e^{-\frac{1}{d-1}}}.$$

Since $(1+\frac{1}{d-1})^{d-1}<e$, we have $1-e^{-\frac{1}{d-1}}>\frac{1}{d}$. Thus the height of the cone C is less than $d\;\Theta(\xi)$. Thus, for all t with $\Theta(\xi)\le t\le d\;\Theta(\xi)$,

$$\mathrm{vol}_{d-1}(K\cap H(cg(K)+t\xi,\xi))\le\left(1-\frac{t}{d\Theta(\xi)}\right)^{d-1}\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)).$$

Now we get

$$\begin{aligned}
&\int_{\Theta(\xi)}^{\infty} t^2\;\mathrm{vol}_{d-1}(K\cap H(t\xi,\xi))\,dt\\
&\qquad\le\int_{\Theta(\xi)}^{d\;\Theta(\xi)} t^2\;(1-\frac{t}{d\Theta(\xi)})^{d-1}\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))\,dt\\
&\qquad\le\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))(d\;\Theta(\xi))^3\int_0^1 s^2(1-s)^{d-1}\,ds\\
&\qquad=\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))(d\;\Theta(\xi))^3\frac{2}{d(d+1)(d+2)}\\
&\qquad\le 2\;\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))\Theta(\xi)^3.
\end{aligned}$$

Therefore

$$\frac{1}{d}\int_K\sum_{i=1}^d|\langle x,e_i\rangle|^2\,dx\le\left(\frac{e}{3}+2\right)(\Theta(\xi)^3+\Theta(-\xi)^3)\,\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi)).$$

Now we apply Lemma 2.3 and get

$$2(\frac{e}{3}+2)e^3\frac{\mathrm{vol}_d(K)^3}{\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))^2}. \qquad \square$$

LEMMA 2.6. *Let K be a convex body in $\mathbb{R}^d$ such that the origin is an element of K. Then*

$$\frac{1}{d}\int_K\sum_{i=1}^d|\langle x,e_i\rangle|^2\,dx \ge \frac{d^{\frac{2}{d}}}{d+2}\ \mathrm{vol}_{d-1}(\partial B_2^d)^{-\frac{2}{d}}\,\mathrm{vol}_d(K)^{\frac{d+2}{d}}.$$

PROOF. Let $r(\xi)$ be the distance of the origin to the boundary of K in direction ξ. By passing to spherical coordinates we get

$$\frac{1}{d}\int_K\sum_{i=1}^d|\langle x,e_i\rangle|^2\,dx=\frac{1}{d}\int_{\partial B_2^d}\int_0^{r(\xi)}\rho^{d+1}\,d\rho\,d\xi=\frac{1}{d(d+2)}\int_{\partial B_2^d}r(\xi)^{d+2}\,d\xi$$

By Hölder's inequality, this expression is greater than

$$\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{d(d+2)}\left(\frac{1}{\mathrm{vol}_{d-1}(\partial B_2^d)}\int_{\partial B_2^d}r(\xi)^d\,d\xi\right)^{\frac{d+2}{d}}$$

$$=\frac{d^{\frac{2}{d}}}{d+2}\,\mathrm{vol}_{d-1}(\partial B_2^d)^{-\frac{2}{d}}\,\mathrm{vol}_d(K)^{\frac{d+2}{d}}. \qquad \square$$

The following lemma can be found in [MP]. It is formulated there for the case of symmetric convex bodies.

LEMMA 2.7. *Let K be a convex body in $\mathbb{R}^d$ such that the origin coincides with the center of gravity of K and such that K is in an isotropic position. Then*

$$B_2^d(cg(K),\frac{1}{24e^5\sqrt{\pi}}\,\mathrm{vol}_d(K)^{\frac{1}{d}})\subset K_{\frac{1}{4e^4}\,\mathrm{vol}_d(K)}.$$

An affine transform can put a convex body into this position.

PROOF. As in Lemma 2.3, let $\Theta(\xi)$ be the infimum of all numbers t such that

$$\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))\ge e\ \mathrm{vol}_{d-1}(K\cap H(cg(K)+t\xi,\xi)).$$

By Lemma 2.3,

$$\Theta(\xi)\ge\frac{1}{2e^3}\frac{\mathrm{vol}_d(K)}{\mathrm{vol}_{d-1}(K\cap H(cg(K),\xi))}.$$

By Lemma 2.5 we get

$$\Theta(\xi)\ge\frac{1}{2e^3\sqrt{6}e^{\frac{3}{2}}}\left(\frac{1}{\mathrm{vol}_d(K)}\frac{1}{d}\int_K\sum_{i=1}^d|\langle x,e_i\rangle|^2\,dx\right)^{\frac{1}{2}}.$$

We have

$$\mathrm{vol}_d(B_2^d)=\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\le\frac{\pi^{(d-1)/2}(2e)^{\frac{d}{2}}}{d^{\frac{d+1}{2}}},$$

and thus

$$\mathrm{vol}_d(B_2^d)^{\frac{1}{d}} \leq \sqrt{\frac{2\pi e}{d}}.$$

Therefore, by Lemma 2.6,

$$\Theta(\xi) \geq \frac{1}{2e^3\sqrt{6}e^{\frac{3}{2}}} \frac{d^{\frac{1}{d}}}{\sqrt{d+2}} \left(\frac{\mathrm{vol}_d(K)}{\mathrm{vol}_{d-1}(\partial B_2^d)}\right)^{\frac{1}{d}} \geq \frac{1}{12e^5\sqrt{\pi}} \ \mathrm{vol}_d(K)^{\frac{1}{d}}.$$

On the other hand,

$$\mathrm{vol}_d(K \cap H^-(cg(K) + \tfrac{1}{2}\Theta(\xi)\xi, \xi)) \geq \int_{\frac{1}{2}\Theta(\xi)}^{\Theta(\xi)} \mathrm{vol}_{d-1}(K \cap H(cg(K) + t\xi, \xi))\, dt,$$

where $H^-(cg(K) + \frac{1}{2}\Theta(\xi)\xi, \xi)$ is the half-space not containing the origin. By the definition of $\Theta(\xi)$ this expression is greater than

$$\frac{\Theta(\xi)}{2e} \, \mathrm{vol}_{d-1}(K \cap H(cg(K), \xi)).$$

By Lemma 2.3 we get that this is greater than

$$\frac{1}{4e^4} \, \mathrm{vol}_d(K).$$

Therefore, every hyperplane that has distance

$$\frac{1}{24e^5\sqrt{\pi}} \, \mathrm{vol}_d(K)^{\frac{1}{d}}$$

from the center of gravity cuts off a set of volume greater than $\frac{1}{4e^4} \, \mathrm{vol}_d(K)$. □

Proof of Theorem 2.1. We are choosing the vertices $x_1, \dots, x_n \in \partial K$ of the polytope P_n. $N(x_k)$ denotes the normal to ∂K at x_k. x_1 is chosen arbitrarily. Having chosen $x_1, \dots, x_{k-1}$ we choose x_k such that

$$\{x_1, \dots, x_{k-1}\} \cap \mathrm{Int}(K \cap H^-(x_k - \Delta_k N(x_k), N(x_k)) = \varnothing,$$

where Δ_k is determined by

$$\mathrm{vol}_d(K \cap H^-(x_k - \Delta_k N(x_k), N(x_k))) = t.$$

If the normal at x_k is not unique it suffices that just one of the normals satisfies the condition. It could be that the hyperplane $H(x_k - \Delta_k N(x_k), N(x_k))$ is not tangential to the floating body K_t, but this does not affect the computation. We claim that this process terminates for some n with

$$n \leq e^{16d} \frac{\mathrm{vol}_d(K \setminus K_t)}{t \ \mathrm{vol}_d(B_2^d)}. \tag{2.4}$$

This claim proves the theorem: If we cannot choose another x_{n+1}, then there is no cap of volume t that does not contain an element of the polytope $P_n = [x_1, \dots, x_n]$. By the theorem of Hahn–Banach we get $K_t \subset P_n$. We show now

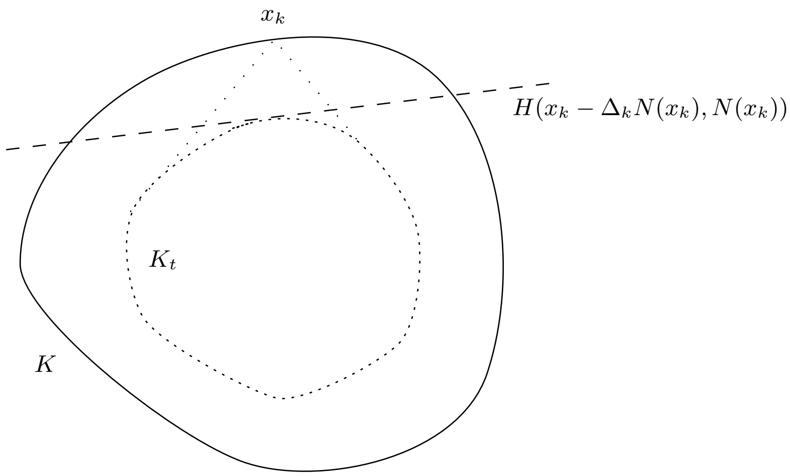

Figure 3.

the claim. We assume that we manage to choose points $x_1, \dots, x_n$ where n is to big that (2.4) does not hold. We put

$$S_n = K \cap H^-(x_n - \Delta_n N(x_n), N(x_n)) \tag{2.5}$$

and

$$S_k = K \cap \left(\bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^-(x_k - \Delta_k N(x_k), N(x_k))$$

for $k = 1, \dots, n-1$. For $k \neq l$, we have

$$\mathrm{vol}_d(S_k \cap S_l) = 0.$$

Let $k < l < n$. Then

$$\begin{aligned} S_k \cap S_l = K \cap &\left(\bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^-(x_k - \Delta_k N(x_k), N(x_k)) \\ &\cap K \cap \left(\bigcap_{i=l+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^-(x_l - \Delta_l N(x_l), N(x_l)) \\ &\subset H^+(x_l - \Delta_l N(x_l), N(x_l)) \cap H^-(x_l - \Delta_l N(x_l), N(x_l)) \\ &= H(x_l - \Delta_l N(x_l), N(x_l)). \end{aligned}$$

Thus we have

$$\mathrm{vol}_d(S_k \cap S_l) \leq \mathrm{vol}_d(H(x_l - \Delta_l N(x_l), N(x_l))) = 0. \tag{2.6}$$

The case $k < l = n$ is shown in the same way. We have, for $k = 1, \dots, n-1$,

$$\begin{aligned} S_k &= K \cap \left(\bigcap_{i=k+1}^{n} H^+(x_i - \Delta_i N(x_i), N(x_i)) \right) \cap H^-(x_k - \Delta_k N(x_k), N(x_k)) \\ &\supset [x_k, K_t] \cap H^-(x_k - \Delta_k N(x_k), N(x_k)) \\ &\supset [x_k, (K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))_t] \cap H^-(x_k - \Delta_k N(x_k), N(x_k)), \end{aligned}$$

where $\tilde{\Delta}_k$ is determined by

$$\mathrm{vol}_d(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) = 4e^4 t.$$

By Lemma 2.7 there is an ellipsoid $\mathcal{E}$ contained in

$$(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))_t$$

whose center is $cg(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))$ and that has volume

$$\mathrm{vol}_d(\mathcal{E}) = \frac{4e^4}{(24e^5\sqrt{\pi})^d}\ t\ \mathrm{vol}_d(B_2^d)$$

Since $(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k)))_t$ is contained in K_t, $\mathcal{E}$ is contained in K_t. Thus

$$S_k \supset [x_k, \mathcal{E}] \cap H^-(x_k - \Delta_k N(x_k), N(x_k)).$$

We claim now that $[x_k, \mathcal{E}] \cap H^-(x_k - \Delta_k N(x_k), N(x_k))$ contains an ellipsoid $\tilde{\mathcal{E}}$ such that

$$\mathrm{vol}_d(\tilde{\mathcal{E}}) = \frac{4e^4}{(24e^5\sqrt{\pi})^d} \frac{1}{(4e^5)^d}\ t\ \mathrm{vol}_d(B_2^d),$$

and consequently

$$\mathrm{vol}_d(S_k) \geq \frac{4e^4}{(24e^5\sqrt{\pi})^d} \frac{1}{(4e^5)^d}\ t\ \mathrm{vol}_d(B_2^d) = \frac{4e^4}{(96e^{10}\sqrt{\pi})^d}\ t\ \mathrm{vol}_d(B_2^d). \tag{2.7}$$

For this we have to see that $\tilde{\Delta}_k \leq 4e^5\ \Delta_k$. By the assumption $t \leq \frac{1}{4}e^{-5}\,\mathrm{vol}_d(K)$ we get

$$\mathrm{vol}_d(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) \leq \frac{1}{e}\ \mathrm{vol}_d(K).$$

Therefore, by (2.1), $cg(K) \in H^+(x_k - \tilde{\Delta}_k N(x_k), N(x_k))$. We consider two cases. If

$$\mathrm{vol}_{d-1}(K \cap H(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) \leq \mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k))),$$

the theorem of Brunn–Minkowski implies that, for all s in the range $\Delta_k \leq s \leq \tilde{\Delta}_k$, we have

$$\begin{aligned} \mathrm{vol}_{d-1}(K \cap H(cg(K), N(x_k))) &\leq \mathrm{vol}_{d-1}(K \cap H(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) \\ &\leq \mathrm{vol}_{d-1}(K \cap H(x_k - sN(x_k), N(x_k))). \end{aligned} \tag{2.8}$$

We get, by (2.2),

$$\Delta_k \geq \frac{t}{e \ \mathrm{vol}_{d-1}(K \cap H(cg(K), N(x_k)))}\,.$$

By (2.8),

$$\begin{aligned}&(\tilde{\Delta}_k - \Delta_k)\,\mathrm{vol}_{d-1}(K \cap H(cg(K), N(x_k)))\\ &\quad \leq \mathrm{vol}_d(K \cap H^-(x_k - \tilde{\Delta}_k N(x_k), N(x_k))) - \mathrm{vol}_d(K \cap H^-(x_k - \Delta_k N(x_k), N(x_k))).\end{aligned}$$

This implies

$$\tilde{\Delta}_k - \Delta_k \leq \frac{(4e^4-1)t}{\mathrm{vol}_{d-1}(K \cap H(cg(K), N(x_k)))}.$$

Therefore

$$\tilde{\Delta}_k \leq \frac{(4e^4-1)t}{\mathrm{vol}_{d-1}(K \cap H(cg(K), N(x_k)))} + \Delta_k \leq 4e^5 \ \Delta_k.$$

If

$$\mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k))) \leq \mathrm{vol}_{d-1}(K \cap H(x_k - \tilde{\Delta}_k N(x_k), N(x_k))),$$

the theorem of Brunn–Minkowski implies that, for all u in the range $0 \leq u \leq \Delta_k$, and all s in the range $\Delta_k \leq s \leq \tilde{\Delta}_k$, we have

$$\begin{aligned}\mathrm{vol}_{d-1}(K \cap H(x_k - uN(x_k), N(x_k))) &\leq \mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k)))\\ &\leq \mathrm{vol}_{d-1}(K \cap H(x_k - sN(x_k), N(x_k))).\end{aligned}$$

We get

$$\Delta_k \geq \frac{t}{\mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k)))}$$

and

$$\tilde{\Delta}_k - \Delta_k \leq \frac{(4e^4-1)t}{\mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k)))}.$$

Therefore

$$\tilde{\Delta}_k \leq \frac{(4e^4-1)t}{\mathrm{vol}_{d-1}(K \cap H(x_k - \Delta_k N(x_k), N(x_k)))} + \Delta_k \leq 4e^4 \Delta_k.$$

We have verified (2.7). From (2.6) and (2.7) we get

$$\mathrm{vol}_d(K \setminus K_t) \geq \mathrm{vol}_d(\bigcup_{k=1}^{n} S_k) = \sum_{k=1}^{n} \mathrm{vol}_d(S_k) \geq n \frac{4e^4}{(96e^{10}\sqrt{\pi})^d} \ t \ \mathrm{vol}_d(B_2^d).$$

Thus we get the desired equation (2.4):

$$\mathrm{vol}_d(K \setminus K_t) \geq e^{-16d} n \ t \ \mathrm{vol}_d(B_2^d). \qquad \square$$

3. The Illumination Body

THEOREM 3.1. *Let K be a convex body in $\mathbb{R}^d$ such that*

$$\frac{1}{c_1} B_2^d \subset K \subset c_2 B_2^d.$$

Let $0 \leq t \leq (5c_1c_2)^{-d-1} \operatorname{vol}_d(K)$ and let $n \in \mathbb{N}$ be such that

$$(\frac{128}{7}\pi)^{(d-1)/2} \leq n \leq \frac{1}{32\ edt} \operatorname{vol}_d(K^t \setminus K).$$

Then we have, for every polytope P_n that contains K and has at most n $(d-1)$-dimensional faces,

$$\operatorname{vol}_d(K^t \setminus K) \leq 10^7\ d^2 (c_1c_2)^{2+\frac{1}{d-1}} \operatorname{vol}_d(P_n \setminus K).$$

We want to see what this result means for bodies with a smooth boundary. We have the asymptotic formula [W]

$$\lim_{t \to 0} \frac{\operatorname{vol}_d(K^t) - \operatorname{vol}_d(K)}{t^{\frac{2}{d+1}}} = \frac{1}{2} \left(\frac{d(d+1)}{\operatorname{vol}_{d-1}(B_2^{d-1})} \right)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x).$$

Thus

$$\operatorname{vol}_d(K^t) - \operatorname{vol}_d(K) \sim t^{\frac{2}{d+1}} d \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x).$$

And by the theorem we have

$$n \sim \frac{1}{dt} \operatorname{vol}_d(K^t \setminus K).$$

Thus

$$\operatorname{vol}_d(K^t) - \operatorname{vol}_d(K) \sim d \Big(\frac{1}{dn} \operatorname{vol}_d(K^t \setminus K) \Big)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x),$$

or

$$\operatorname{vol}_d(K^t \setminus K)^{\frac{d-1}{d+1}} \sim d \Big(\frac{1}{dn} \Big)^{\frac{2}{d+1}} \int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x),$$

$$\operatorname{vol}_d(K^t \setminus K) \sim d \Big(\frac{1}{n} \Big)^{\frac{2}{d-1}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x) \right)^{\frac{d+1}{d-1}}.$$

By Theorem 3.1 we now get

$$\operatorname{vol}_d(P_n \setminus K) \gtrsim \frac{1}{d} \Big(\frac{1}{c_1c_2} \Big)^{1+\frac{d}{d+1}} \Big(\frac{1}{n} \Big)^{\frac{2}{d-1}} \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\, d\mu(x) \right)^{\frac{d+1}{d-1}}.$$

By a theorem of F. John [J] we have $c_1c_2 \leq d$.

The following lemma is due to Bronshtein and Ivanov [BI] and Dudley [D_1, D_2]. It can also be found in [GRS].

LEMMA 3.2. *For all dimensions d, $d \geq 2$, and all natural numbers n, $n \geq 2d$, there is a polytope Q_n that has n vertices and is contained in the Euclidean ball B_2^d such that*

$$d_H(Q_n, B_2^d) \leq \frac{16}{7}\left(\frac{\mathrm{vol}_{d-1}(\partial B_2^d)}{\mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{\frac{2}{d-1}} n^{-\frac{2}{d-1}}.$$

We have

$$\begin{aligned}\mathrm{vol}_{d-1}(\partial B_2^d) &= d\ \mathrm{vol}_d(B_2^d) = d\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\\ &= d\sqrt{\pi}\frac{\Gamma(\frac{d-1}{2}+1)}{\Gamma(\frac{d}{2}+1)}\,\mathrm{vol}_{d-1}(B_2^{d-1}) \leq d\sqrt{\pi}\ \mathrm{vol}_{d-1}(B_2^{d-1}).\end{aligned} \tag{3.1}$$

Since $d^{\frac{2}{d-1}} \leq 4$ and $(1-t)^d \geq 1-dt$, (3.1) yields

$$d_H(B_2^d, Q_n) \leq \frac{16}{7}\left(\frac{d\sqrt{\pi}}{n}\right)^{\frac{2}{d-1}} \leq \frac{64}{7}\pi\, n^{-\frac{2}{d-1}}. \tag{3.2}$$

PROOF OF THEOREM 3.1. We denote the $(d-1)$-dimensional faces of P_n by F_i, for $i = 1, \ldots, n$, and the cones generated by the origin and a face F_i by C_i, for $i = 1, \ldots, n$. Take $x_i \in F_i$ and let ξ_i, with $\|\xi_i\|_2 = 1$, be orthogonal to F_i and pointing to the outside of P_n. Then $H(x_i, \xi_i)$ is the hyperplane containing F_i and $H^+(x_i, \xi_i)$ the halfspace containing P_n. See Figure 4.

We may assume that the hyperplanes $H(x_i, \xi_i)$, $i = 1, \ldots, n$, are supporting hyperplanes of K. Otherwise we can choose a polytope of lesser volume. Let Δ_i be the height of the set

$$K^t \cap H^-(x_i, \xi_i) \cap C_i,$$

that is, the smallest number s such that

$$K^t \cap H^-(x_i, \xi_i) \cap C_i \subset H^+(x_i + s\xi_i, \xi_i).$$

Let z_i be a point in $\partial K^t \cap C_i$ where the height Δ_i is attained. We may assume that $B_2^d \subset K \subset cB_2^d$ where $c = c_1 c_2$. Also we may assume that

$$P_n \subset 2cB_2^d \tag{3.3}$$

if we allow twice as many faces. This follows from (3.2): There is a polytope Q_k such that $\frac{1}{2}B_2^d \subset Q_k \subset B_2^d$ and the number of vertices k is smaller than $(\frac{128}{7}\pi)^{(d-1)/2}$. Thus Q_k^* satisfies $B_2^d \subset Q_k^* \subset 2B_2^d$ and has at most $(\frac{128}{7}\pi)^{(d-1)/2}$ $(d-1)$-dimensional faces. As the new polytope P_n we choose the intersection of cQ_k^* with the original polytope P_n. Since we have by assumption that n is greater than $(\frac{128}{7}\pi)^{(d-1)/2}$ the new polytope has at most

$$\frac{1}{16\ edt}\,\mathrm{vol}_d(K^t \setminus K). \tag{3.4}$$

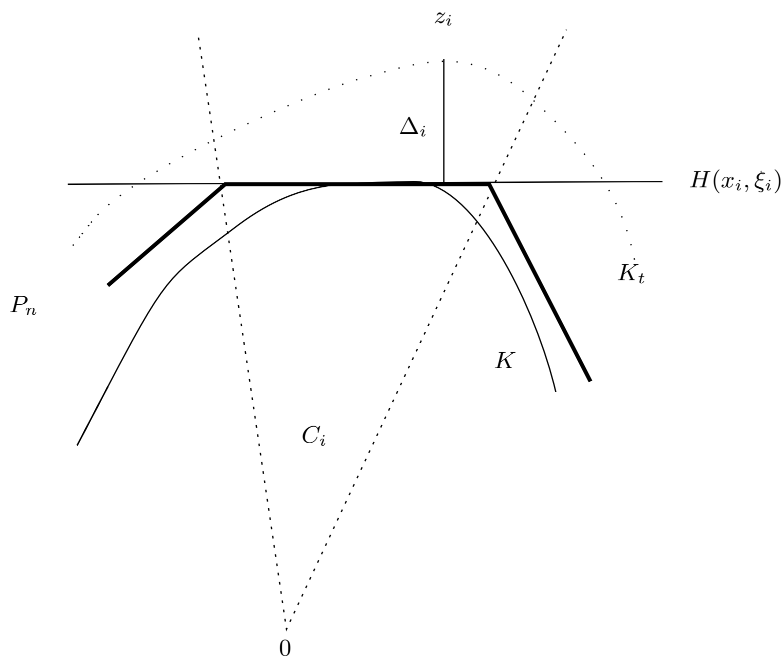

Figure 4.

$(d-1)$-dimensional faces.

We show first that for t with $0 \leq t \leq (5cd)^{-d-1} \operatorname{vol}_d(K)$ and all i, $i=1,\dots,n$ we have

$$\Delta_i \leq \frac{1}{d} \tag{3.5}$$

Assume that there is a face F_i with $\Delta_i > \frac{1}{d}$. Consider the smallest infinite cone D_i having z_i as vertex and containing K. Since $H(x_i,\xi_i)$ is a supporting hyperplane to K and $K \subset c\, B_2^d$ we have

$$K \subset D_i \cap H^+(x_i,\xi_i) \cap H^-(x_i - 2c\xi_i, \xi_i)$$

and

$$D_i \cap H^-(x_i,\xi) = [z_i, K] \cap H^-(x_i,\xi)$$

We have

$$t = \operatorname{vol}_d([z_i,K] \setminus K) \geq \operatorname{vol}_d([z_i,K] \cap H^-(x_i,\xi_i)) = \operatorname{vol}_d(D_i \cap H^-(x_i,\xi_i)) =$$

$$\frac{1}{d}\Delta_i \operatorname{vol}_{d-1}(D_i \cap H(x_i,\xi_i)) \geq \frac{1}{d^2} \operatorname{vol}_{d-1}(D_i \cap H(x_i,\xi_i))$$

Thus

$$\operatorname{vol}_{d-1}(D_i \cap H(x_i,\xi_i)) \leq d^2 t \tag{3.6}$$

Since (3.5) does not hold we have

$$\begin{aligned}\mathrm{vol}_{d-1}(D_i \cap H(x_i - 2c\xi_i, \xi_i)) &= (\frac{2c+\Delta_i}{\Delta_i})^{d-1} \mathrm{vol}_{d-1}(D_i \cap H(x_i, \xi_i)) \\ &\leq (2cd+1)^{d-1} \mathrm{vol}_{d-1}(D_i \cap H(x_i, \xi_i)).\end{aligned}$$

By (3.6) we get

$$\mathrm{vol}_{d-1}(D_i \cap H(x_i - 2c\xi_i, \xi_i)) \leq (2cd+1)^{d-1} d^2 t \leq (3cd)^{d-1} d^2 t.$$

Thus

$$\begin{aligned}\mathrm{vol}_d(K) &\leq \mathrm{vol}_d(D_i \cap H^+(x_i, \xi_i) \cap H^-(x_i - 2c\xi_i, \xi_i)) \\ &\leq 2c(3cd)^{d-1} d^2 t \leq (3cd)^{d+1} t,\end{aligned}$$

and we conclude that

$$t \geq (3cd)^{-d-1} \mathrm{vol}_d(K).$$

This is a contradiction to the assumption on t in the hypothesis of the theorem. Thus we have shown (3.5). We consider now two cases: All those heights Δ_i that are smaller than $2dt/\mathrm{vol}_{d-1}(F_i)$ and those that are greater. We may assume that Δ_i, $i = 1, \dots, k$ are smaller than $2dt/\mathrm{vol}_{d-1}(F_i)$ and Δ_i, $i = k+1, \dots, n$ are strictly greater. We have

$$\mathrm{vol}_d((K^t \setminus P_n) \cap C_i) = \int_0^{\Delta_i} \mathrm{vol}_{d-1}((K^t \setminus P_n) \cap C_i \cap H(x_i + s\xi_i, \xi_i))\, ds.$$

Since $B_2^d \subset K \subset P_n$ we get

$$\mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \leq \int_0^{\Delta_i} \mathrm{vol}_{d-1}(F_i)(1+s)^{d-1}\, ds \leq \Delta_i (1+\Delta_i)^{d-1} \mathrm{vol}_{d-1}(F_i).$$

By (3.5) we get

$$\mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \leq \Delta_i \left(1 + \frac{1}{d}\right)^{d-1} \mathrm{vol}_{d-1}(F_i).$$

For $i = 1, \dots, k$ we get

$$\mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \leq \frac{2dt}{\mathrm{vol}_{d-1}(F_i)} \left(1 + \frac{1}{d}\right)^{d-1} \mathrm{vol}_{d-1}(F_i) \leq 2edt.$$

Thus

$$\mathrm{vol}_d\left((K^t \setminus P_n) \cap \bigcup_{i=1}^k C_i\right) \leq 2kedt \leq 2nedt.$$

By (3.4) we get

$$\mathrm{vol}_d\left((K^t \setminus P_n) \cap \bigcup_{i=1}^k C_i\right) \leq \tfrac{1}{8} \mathrm{vol}_d(K^t \setminus K). \tag{3.7}$$

Now we consider the other faces. For $i = k+1, \dots, n$, we have

$$\Delta_i \geq \frac{2dt}{\mathrm{vol}_{d-1}(F_i)}. \tag{3.8}$$

We show that, for $i = k+1, \dots, n$, we have

$$\Delta_i \le 5c \left(\frac{5c \ \mathrm{vol}_{d-1}(F_i)}{2d \ \mathrm{vol}_d(K)} \right)^{\frac{1}{d-1}} . \tag{3.9}$$

Suppose that there is a face F_i so that (3.9) does not hold. Then

$$t = \mathrm{vol}_d([z_i, K] \setminus K) \ge \mathrm{vol}_d([z_i, K] \cap H^-(x_i, \xi_i)) = \frac{\Delta_i}{d} \mathrm{vol}_{d-1}([z_i, K] \cap H(x_i, \xi_i)).$$

Therefore we get, by (3.8),

$$\mathrm{vol}_{d-1}([z_i, K] \cap H(x_i, \xi_i)) \le \frac{dt}{\Delta_i} \le \frac{1}{2} \mathrm{vol}_{d-1}(F_i). \tag{3.10}$$

Since $K \subseteq B_2^d$ we have

$$K \subset D_i \cap H^+(x_i, \xi_i) \cap H^-(x_i - 2c\xi_i, \xi_i).$$

Thus

$$\mathrm{vol}_d(K) \le \mathrm{vol}_d(D_i \cap H^-(x_i - 2c\xi_i, \xi_i)).$$

The cone $D_i \cap H^-(x_i - 2c\xi_i, \xi_i)$ has a height equal to $2c + \Delta_i$. Therefore

$$\mathrm{vol}_d(K) \le \frac{1}{d}(2c + \Delta_i) \left(\frac{2c + \Delta_i}{\Delta_i} \right)^{d-1} \mathrm{vol}_{d-1}(D_i \cap H(x_i, \xi_i)).$$

By (3.5) we have $\Delta_i \le 1$. Therefore we get

$$\begin{aligned} \mathrm{vol}_d(K) &\le \frac{3c}{d} \left(\frac{3c}{\Delta_i} \right)^{d-1} \mathrm{vol}_{d-1}(D_i \cap H(x_i, \xi_i)) \\ &= \frac{3c}{d} \left(\frac{3c}{\Delta_i} \right)^{d-1} \mathrm{vol}_{d-1}([z_i, K] \cap H(x_i, \xi_i)). \end{aligned}$$

By (3.10) we get

$$\mathrm{vol}_d(K) \le \frac{3c}{2d} \left(\frac{3c}{\Delta_i} \right)^{d-1} \mathrm{vol}_{d-1}(F_i),$$

which implies (3.9).

Let y_i be the unique point

$$y_i = [0, z_i] \cap H(x_i, \xi_i).$$

We want to make sure that $y_i \in F_i \cap [z_i, K]$. This holds since $z_i \in C_i \cap H^-(x_i, \xi_i)$ and $\Delta_i > 0$. Since $y_i \in F_i$ we have

$$\mathrm{vol}_{d-1}(F_i) = \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1} \, d\mu(\eta),$$

where $r_i(\eta)$ is the distance of y_i to the boundary ∂F_i in direction η, $\eta \in \partial B_2^{d-1}$, and, since $y_i \in F_i \cap [z_i, K]$, we have

$$\mathrm{vol}_{d-1}(F_i \cap [z_i, K]) = \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} \rho_i(\eta)^{d-1} \, d\mu(\eta),$$

where $\rho_i(\eta)$ is the distance of y_i to the boundary $\partial(F_i \cap [z_i, K])$. Consider the set

$$A_i = \{\eta \mid (1 - \tfrac{1}{4d}) r_i(\eta) \leq \rho_i(\eta)\,\}.$$

We show that

$$\tfrac{1}{4}\operatorname{vol}_{d-1}(F_i) \leq \frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta) \tag{3.11}$$

We have

$$\begin{aligned}
&\frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta) \\
&\qquad \leq \frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1}(1 - (1 - \tfrac{1}{4d})^{d-1})\, d\mu(\eta) \\
&\qquad \leq \frac{1}{4}\frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1}\, d\mu(\eta) \leq \tfrac{1}{4}\operatorname{vol}_{d-1}(F_i).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta) \\
&\qquad \geq \frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{\partial B_2^{d-1}} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta) \\
&\qquad\quad - \frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta) \\
&\qquad \geq \operatorname{vol}_{d-1}(F_i) - \operatorname{vol}_{d-1}(F_i \cap [z_i, K]) - \tfrac{1}{4}\operatorname{vol}_{d-1}(F_i).
\end{aligned}$$

By (3.10) this is greater than $\frac{1}{4}\operatorname{vol}_{d-1}(F_i)$. This implies

$$\tfrac{1}{4}\operatorname{vol}_{d-1}(F_i) \leq \frac{\operatorname{vol}_{d-1}(B_2^{d-1})}{\operatorname{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}\, d\mu(\eta).$$

Thus we have established (3.11).

We shall show that

$$\operatorname{vol}_d((K^t \setminus P_n) \cap C_i) \leq 10^6\ ed^2 c^{2+\frac{1}{d-1}} \operatorname{vol}_d((P_n \setminus K) \cap C_i). \tag{3.12}$$

We have

$$\operatorname{vol}_d(D_i^c \cap H^+(x_i, \xi_i) \cap C_i) \leq \operatorname{vol}_d((P_n \setminus K) \cap C_i).$$

Compare Figure 5. Therefore, if we want to verify (3.12) it is enough to show that

$$\operatorname{vol}_d((K^t \setminus P_n) \cap C_i) \leq 10^6\ ed^2 c^{2+\frac{1}{d-1}} \operatorname{vol}_d(D_i^c \cap H^+(x_i, \xi_i) \cap C_i).$$

We may assume that y_i and z_i are orthogonal to $H(x_i, \xi_i)$. This is accomplished by a linear, volume preserving map: Any vector orthogonal to ξ_i is mapped onto itself and y_i is mapped to $\langle \xi_i, y_i \rangle \xi_i$. See Figure 6.

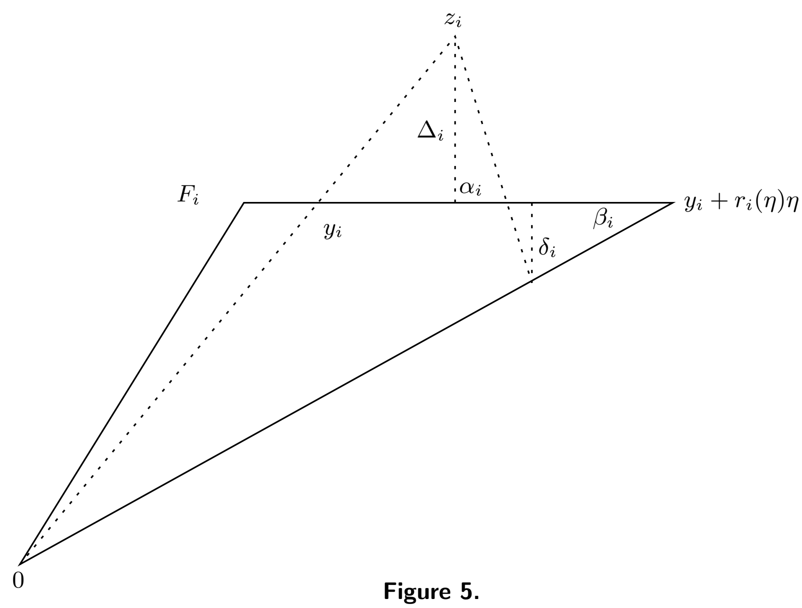

Figure 5.

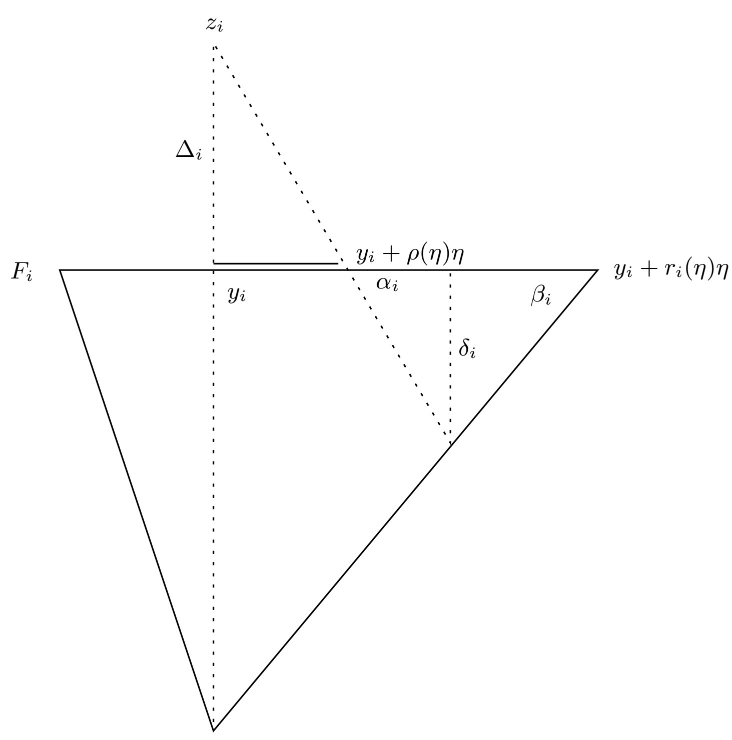

Figure 6.

Let $w_i(\eta) \in D_i^c \cap H^+(x_i,\xi_i) \cap C_i$ such that $w_i(\eta)$ is an element of the 2-dimensional subspace containing 0, y_i, and $y_i+\eta$. Let $\delta_i(\eta)$ be the distance of $w_i(\eta)$ to the plane $H(x_i,\xi_i)$. Then

$$\frac{1}{d}\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})}\int_{A_i^c}(r_i(\eta)^{d-1}-\rho_i(\eta)^{d-1})\delta_i(\eta)\,d\mu(\eta) \leq \mathrm{vol}_d(D_i^c \cap H^+(x_i,\xi_i)\cap C_i).$$

Thus, in order to verify (3.12), it suffices to show that

$$\begin{aligned}&\mathrm{vol}_d((K^t\setminus P_n)\cap C_i)\\ &\quad\leq 10^6\ ed^2c^{2+\frac{1}{d-1}}\frac{1}{d}\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})}\int_{A_i^c}(r_i(\eta)^{d-1}-\rho_i(\eta)^{d-1})\delta_i(\eta)\,d\mu(\eta).\end{aligned}\tag{3.13}$$

In order to do this we shall show that for all $i=k+1,\dots,n$ and all $\eta\in A_i^c$ there is $w_i(\eta)$ such that the distance $\delta_i(\eta)$ of w_i from $H(x_i,\xi_i)$ satisfies

$$\frac{\Delta_i}{\delta_i}\leq\begin{cases}32dc & \text{if } 0\leq\alpha_i\leq\frac{\pi}{4},\\ \frac{160\ dc^2}{r_i}\left(\frac{5c\ \mathrm{vol}_{d-1}(F_i)}{2d\ \mathrm{vol}_d(K)}\right)^{\frac{1}{d-1}} & \text{if } \frac{\pi}{4}\leq\alpha_i\leq\frac{\pi}{2}.\end{cases}\tag{3.14}$$

The angles $\alpha_i(\eta)$ and $\beta_i(\eta)$ are given in Figure 6. We have for all $\eta\in A_i^c$

$$\begin{aligned}\delta_i &=(r_i-\rho_i)\frac{\sin(\alpha_i)\sin(\beta_i)}{\sin(\pi-\alpha_i-\beta_i)},\\ \Delta_i &=\rho_i\tan\alpha_i,\end{aligned}\tag{3.15}$$

with $0\leq\alpha_i,\beta_i\leq\frac{\pi}{2}$. Thus we get

$$\frac{\Delta_i}{\delta_i}\leq\frac{\rho_i}{r_i-\rho_i}\frac{\sin(\pi-\alpha_i-\beta_i)}{\cos(\alpha_i)\sin(\beta_i)}\leq\frac{\rho_i}{(r_i-\rho_i)\cos(\alpha_i)\sin(\beta_i)}.$$

By (3.11) we have $\rho_i\leq(1-\frac{1}{4d})r_i$. Therefore

$$\frac{\Delta_i}{\delta_i}\leq\frac{4d}{\cos(\alpha_i)\sin(\beta_i)}.$$

Since $B_2^d\subset K\subset P_n\subset 2c\ B_2^d$ we have $\tan\beta_i\geq\frac{1}{4c}$: Here we have to take into account that we applied a transform to K mapping y_i to $\langle\xi_i,y_i\rangle\xi_i$. That leaves the distance of F_i to the origin unchanged and $r_i(\eta)$ is less than $4c$. If $\beta_i\geq\frac{\pi}{4}$ we have $\sin\beta_i\geq\frac{1}{\sqrt{2}}$. If $\beta_i\leq\frac{\pi}{4}$ then $\frac{1}{4c}\leq\tan\beta_i=\frac{\sin\beta_i}{\cos\beta_i}\leq\sqrt{2}\sin\beta_i$. Therefore we get

$$\frac{\Delta_i}{\delta_i}\leq\frac{16\sqrt{2}\ dc}{\cos\alpha_i}.$$

Therefore we get, for all $0\leq\alpha_i\leq\frac{\pi}{4}$,

$$\frac{\Delta_i}{\delta_i}\leq 32dc.$$

By (3.9) and (3.15) we get

$$\frac{\Delta_i}{\delta_i} \le \frac{1}{r_i - \rho_i} \frac{\sin(\pi - \alpha_i - \beta_i)}{\sin(\alpha_i)\sin(\beta_i)} 5c \left(\frac{5c \ \mathrm{vol}_{d-1}(F_i)}{2d \ \mathrm{vol}_d(K)} \right)^{\frac{1}{d-1}} .$$

We proceed as in the estimate above and obtain

$$\frac{\Delta_i}{\delta_i} \le \frac{16\sqrt{2}\ dc}{r_i} \frac{5c}{\sin(\alpha_i)} \left(\frac{5c \ \mathrm{vol}_{d-1}(F_i)}{2d \ \mathrm{vol}_d(K)} \right)^{\frac{1}{d-1}} .$$

Thus we get for $\frac{\pi}{4} \le \alpha_i \le \frac{\pi}{2}$

$$\frac{\Delta_i}{\delta_i} \le \frac{32\ dc}{r_i} 5c \left(\frac{5c \ \mathrm{vol}_{d-1}(F_i)}{2d \ \mathrm{vol}_d(K)} \right)^{\frac{1}{d-1}} .$$

We verify now (3.13). By the definition of A_i we get

$$\begin{aligned}\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} &\int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta)\, d\mu(\eta) \\ &\ge (1 - e^{-\frac{1}{8}}) \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} r_i(\eta)^{d-1} \delta_i \, d\mu(\eta).\end{aligned}$$

We get by (3.14) that the last expression is greater than

$$\begin{aligned}\frac{1}{320dc} \Delta_i &\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \\ &\times \left(\int_{\substack{A_i^c \\ \alpha_i \le \frac{\pi}{4}}} r_i^{d-1} \, d\mu + \frac{1}{5c} \left(\frac{2d \ \mathrm{vol}_d(K)}{5c \ \mathrm{vol}_{d-1}(F_i)} \right)^{\frac{1}{d-1}} \int_{\substack{A_i^c \\ \alpha_i > \frac{\pi}{4}}} r_i^d \, d\mu \right).\end{aligned}$$

By (3.11) we get that either

$$\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{\substack{A_i^c \\ \alpha_i \le \frac{\pi}{4}}} r_i^{d-1} \, d\mu \ge \tfrac{1}{8} \ \mathrm{vol}_{d-1}(F_i)$$

or

$$\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{\substack{A_i^c \\ \alpha_i > \frac{\pi}{4}}} r_i^{d-1} \, d\mu \ge \tfrac{1}{8} \ \mathrm{vol}_{d-1}(F_i).$$

In the first case we get for the above estimate

$$\begin{aligned}\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} &\int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1}) \delta_i(\eta)\, d\mu(\eta) \\ &\ge \frac{\Delta_i}{2560dc} \ \mathrm{vol}_{d-1}(F_i) \ge \frac{1}{2560edc} \ \mathrm{vol}_d((K^t \setminus P_n) \cap C_i).\end{aligned}$$

The last inequality is obtained by using (3.5): Since $B_2^d \subset K$ we have, for all hyperplanes H that are parallel to F_i,

$$\mathrm{vol}_{d-1}(K^t \cap H \cap C_i) \le (1 + \Delta_i)^{d-1} \, \mathrm{vol}_{d-1}(F_i).$$

By (3.5) we get $\mathrm{vol}_{d-1}(K^t \cap H \cap C_i) \le e\ \mathrm{vol}_{d-1}(F_i)$. In the second case we have

$$\begin{aligned}
&\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1})\delta_i(\eta)\, d\mu(\eta) \\
&\ge \frac{1}{5c}\left(\frac{2d\ \mathrm{vol}_d(K)}{5c\ \mathrm{vol}_{d-1}(F_i)}\right)^{\frac{1}{d-1}} \frac{1}{320dc}\Delta_i \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{\substack{A_i^c \\ \alpha_i > \frac{\pi}{4}}} r_i^d\, d\mu \\
&\ge \frac{1}{5c}\left(\frac{2d\ \mathrm{vol}_d(K)}{5c\ \mathrm{vol}_{d-1}(F_i)}\right)^{\frac{1}{d-1}} \frac{1}{320dc}\Delta_i \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{(\mathrm{vol}_{d-2}(\partial B_2^{d-1}))^{\frac{d}{d-1}}} \left(\int_{\substack{A_i^c \\ \alpha_i > \frac{\pi}{4}}} r_i^{d-1}\, d\mu\right)^{\frac{d}{d-1}} \\
&\ge \frac{1}{5c}\left(\frac{2d\ \mathrm{vol}_d(K)}{5c\ \mathrm{vol}_{d-1}(F_i)}\right)^{\frac{1}{d-1}} \frac{\Delta_i}{320dc}\ \mathrm{vol}_{d-1}(B_2^{d-1})^{-\frac{1}{d-1}} \left(\tfrac{1}{8}\ \mathrm{vol}_{d-1}(F_i)\right)^{\frac{d}{d-1}} \\
&= \frac{1}{5c}\left(\frac{d\ \mathrm{vol}_d(K)}{20c\ \mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{\frac{1}{d-1}} \frac{\Delta_i}{2560dc}\ \mathrm{vol}_{d-1}(F_i) \\
&\ge \frac{1}{5c}\left(\frac{d\ \mathrm{vol}_d(K)}{20c\ \mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{\frac{1}{d-1}} \frac{1}{2560edc}\ \mathrm{vol}_d((K^t \setminus P_n) \cap C_i).
\end{aligned}$$

Since $B_2^d \subset K$ we get

$$\begin{aligned}
&\frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1})\delta_i(\eta)\, d\mu(\eta) \\
&\qquad \ge \frac{1}{5c}\left(\frac{d\ \mathrm{vol}_d(B_2^d)}{20c\ \mathrm{vol}_{d-1}(B_2^{d-1})}\right)^{\frac{1}{d-1}} \frac{1}{2560edc}\ \mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \\
&\qquad \ge \frac{1}{5c}\left(\frac{1}{20c}\right)^{\frac{1}{d-1}} \frac{1}{2560edc}\ \mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \\
&\qquad \ge (10^6\ edc^{2+\frac{1}{d-1}})^{-1}\ \mathrm{vol}_d((K^t \setminus P_n) \cap C_i).
\end{aligned}$$

The second case gives a weaker estimate. Therefore we get for both cases

$$\begin{aligned}
&\mathrm{vol}_d((K^t \setminus P_n) \cap C_i) \\
&\qquad \le 10^6\ edc^{2+\frac{1}{d-1}} \frac{\mathrm{vol}_{d-1}(B_2^{d-1})}{\mathrm{vol}_{d-2}(\partial B_2^{d-1})} \int_{A_i^c} (r_i(\eta)^{d-1} - \rho_i(\eta)^{d-1})\delta_i\, d\mu(\eta).
\end{aligned}$$

Thus we have verified (3.13) and thereby also (3.12). By (3.12) we get

$$\begin{aligned}
\mathrm{vol}_d\left((K^t \setminus P_n) \cap \bigcup_{i=k+1}^{n} C_i\right) &\le 10^6\ ed^2c^{2+\frac{1}{d-1}}\ \mathrm{vol}_d\left(\left(\bigcup_{i=k+1}^{n} C_i\right) \cap (P_n \setminus K)\right) \\
&\le 10^6\ ed^2c^{2+\frac{1}{d-1}}\ \mathrm{vol}_d((P_n \setminus K)). \qquad (3.16)
\end{aligned}$$

If the assertion of the theorem does not hold we have

$$\mathrm{vol}_d((P_n \setminus K)) \le \frac{\mathrm{vol}_d(K^t \setminus K)}{10^7\ ed^2 c^{2+\frac{1}{d-1}}}\,. \tag{3.17}$$

Thus we get

$$\mathrm{vol}_d\left((K^t \setminus P_n) \cap \bigcup_{i=k+1}^{n} C_i\right) \le \tfrac{1}{10}\,\mathrm{vol}_d(K^t \setminus K).$$

Together with (3.7) we obtain

$$\mathrm{vol}_d(K^t \setminus P_n) \le \tfrac{1}{4}\,\mathrm{vol}_d(K^t \setminus K) \le \tfrac{1}{4}\{\mathrm{vol}_d(K^t \setminus P_n) + \mathrm{vol}_d(P_n \setminus K)\}. \tag{3.18}$$

By (3.17) we have

$$\begin{aligned}\mathrm{vol}_d(P_n \setminus K) &\le \frac{\mathrm{vol}_d(K^t \setminus K)}{10^7\ ed^2 c^{2+\frac{1}{d-1}}}\\ &\le \tfrac{1}{2}\,\mathrm{vol}_d(K^t \setminus K) \le \tfrac{1}{2}\,\mathrm{vol}_d(K^t \setminus P_n) + \tfrac{1}{2}\,\mathrm{vol}_d(P_n \setminus K).\end{aligned}$$

This implies

$$\mathrm{vol}_d(P_n \setminus K) \le \mathrm{vol}_d(K^t \setminus P_n).$$

Together with (3.18) we get now the contradiction

$$\mathrm{vol}_d(K^t \setminus P_n) \le \tfrac{1}{2}\,\mathrm{vol}_d(K^t \setminus P_n). \qquad \square$$

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