Convex Geometric Analysis
MSRI Publications
Volume **34**, 1998

A Note on the M^*-Limiting Convolution Body

ANTONIS TSOLOMITIS

Abstract. We introduce the mixed convolution bodies of two convex symmetric bodies. We prove that if the boundary of a body K is smooth enough then as δ tends to 1 the δ-M^*-convolution body of K with itself tends to a multiple of the Euclidean ball after proper normalization. On the other hand we show that the δ-M^*-convolution body of the n-dimensional cube is homothetic to the unit ball of ℓ_1^n.

1. Introduction

Throughout this note K and L denote convex symmetric bodies in $\mathbb{R}^n$. Our notation will be the standard notation that can be found, for example, in [2] and [4]. For $1 \le m \le n$, $V_m(K)$ denotes the m-th mixed volume of K (i.e., mixing m copies of K with $n-m$ copies of the Euclidean ball $\mathcal{B}_n$ of radius one in $\mathbb{R}^n$). Thus if $m = n$ then $V_n(K) = \mathrm{vol}_n(K)$ and if $m = 1$ then $V_1(K) = w(K)$ the mean width of K.

For $0 < \delta < 1$ we define the m-th mixed δ-convolution body of the convex symmetric bodies K and L in $\mathbb{R}^n$:

Definition. The m-th mixed δ-convolution body of K and L is defined to be the set

$$C_m(\delta; K, L) = \big\{x \in \mathbb{R}^n : V_m\left(K \cap (x+L)\right) \ge \delta V_m(K)\big\}.$$

It is a consequence of the Brunn–Minkowski inequality for mixed volumes that these bodies are convex.

If we write $h(u)$ for the support function of K in the direction $u \in \mathbb{S}^{n-1}$, we have

$$w(K) = 2M_K^* = 2\int_{\mathbb{S}^{n-1}} h(u)\, d\nu(u), \tag{1.1}$$

where ν is the Lebesgue measure of $\mathbb{R}^n$ restricted on $\mathbb{S}^{n-1}$ and normalized so that $\nu(\mathbb{S}^{n-1}) = 1$. In this note we study the limiting behavior of $C_1(\delta; K, K)$

Work partially supported by an NSF grant.

(which we will abbreviate with $C_1(\delta)$) as δ tends to 1 and K has a C^2_+ boundary. For simplicity we will call $C_1(\delta)$ the δ-M^*-convolution body of K.

We are looking for suitable $\alpha \in \mathbb{R}$ so that the limit

$$\lim_{\delta\to 1^-} \frac{C_1(\delta)}{(1-\delta)^\alpha}$$

exists (convergence in the Hausdorff distance). In this case we call the limiting body "the limiting M^*-convolution body of K".

We prove that for a convex symmetric body K in $\mathbb{R}^n$ with C^2_+ boundary the limiting M^*-convolution body of K is homothetic to the Euclidean ball. We also get a sharp estimate (sharp with respect to the dimension n) of the rate of the convergence of the δ-M^*-convolution body of K to its limit. By C^2_+ we mean that the boundary of K is C^2 and that the principal curvatures of $\mathrm{bd}(K)$ at every point are all positive.

We also show that some smoothness condition on the boundary of K is necessary for this result to be true, by proving that the limiting M^*-convolution body of the n-dimensional cube is homothetic to the unit ball of ℓ_1^n.

2. The Case Where the Boundary of K Is a C^2_+ Manifold

THEOREM 2.1. *Let K be a convex symmetric body in $\mathbb{R}^n$ so that $\mathrm{bd}(K)$ is a C^2_+ manifold. Then for all $x \in \mathbb{S}^{n-1}$ we have*

$$\left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M_K^*} \right| \le C \frac{c_n}{M_K^*} \left(M_K^* n(1-\delta)\right)^2, \tag{2.1}$$

where $c_n = \int_{\mathbb{S}^{n-1}} \left|\langle x,u\rangle\right| d\nu(u) \sim 1/\sqrt{n}$ and C is a constant independent of the dimension n. In particular,

$$\lim_{\delta\to 1^-} \frac{C_1(\delta)}{1-\delta} = \frac{M_K^*}{c_n}\mathcal{B}_n.$$

Moreover the estimate (2.1) *is sharp with respect to the dimension n.*

By "sharp" with respect to the dimension n we mean that there are examples (for instance the n-dimensional Euclidean ball) for which the inequality (2.1) holds true if "$\le$" is replaced with "$\ge$" and the constant C changes by a (universal) constant factor.

Before we proceed with the proof we will need to collect some standard notation which can be found in [4]. We write $p : \mathrm{bd}(K) \to \mathbb{S}^{n-1}$ for the Gauss map $p(x) = N(x)$ where $N(x)$ denotes the unit normal vector of $\mathrm{bd}(K)$ at x. W_x denotes the Weingarten map, that is, the differential of p at the point $x \in \mathrm{bd}(K)$. W^{-1} is the reverse Weingarten map and the eigenvalues of W_x and W_u^{-1} are respectively the principal curvatures and principal radii of curvatures of the manifold $\mathrm{bd}(K)$ at $x \in \mathrm{bd}(K)$ and $u \in \mathbb{S}^{n-1}$. We write $\|W\|$ and $\|W^{-1}\|$

for the quantities $\sup_{x\in \mathrm{bd}(K)} \|W_x\|$ and $\sup_{u\in\mathbb{S}^{n-1}} \|W_u^{-1}\|$, respectively. These quantities are finite since the manifold $\mathrm{bd}(K)$ is assumed to be C^2_+.

For $\lambda \in \mathbb{R}$ and $x \in \mathbb{S}^{n-1}$ we write K_λ for the set $K \cap (\lambda x + K)$. $p_\lambda^{-1} : \mathbb{S}^{n-1} \to \mathrm{bd}(K_\lambda)$ is the reverse Gauss map, that is, the affine hyperplane $p_\lambda^{-1}(u) + [u]^\perp$ is tangent to K_λ at $p_\lambda^{-1}(u)$. The normal cone of K_λ at x is denoted by $N(K_\lambda, x)$ and similarly for K. The normal cone is a convex set (see [4]). Finally h_λ will denote the support function of K_λ.

Proof. Without loss of generality we may assume that both the $\mathrm{bd}(K)$ and $\mathbb{S}^{n-1}$ are equipped with an atlas whose charts are functions which are Lipschitz, their inverses are Lipschitz and they all have the same Lipschitz constant $c > 0$.

Let $x \in \mathbb{S}^{n-1}$ and $\lambda = 1/\|x\|_{C_1(\delta)}$; hence $\lambda x \in \mathrm{bd}\left(C_1(\delta)\right)$ and

$$M^*_{K_\lambda} = \delta M^*_K.$$

We estimate now $M^*_{K_\lambda}$. Let $u \in \mathbb{S}^{n-1}$. We need to compare $h_\lambda(u)$ and $h(u)$. Set $Y_\lambda = \mathrm{bd}(K) \cap \mathrm{bd}(\lambda x + K)$.

Case 1. $p_\lambda^{-1}(u) \notin Y_\lambda$.

In this case it is easy to see that

$$h_\lambda(u) = h(u) - |\langle \lambda x, u\rangle|.$$

Case 2. $p_\lambda^{-1}(u) \in Y_\lambda$.

Let $y_\lambda = p_\lambda^{-1}(u)$ and $y'_\lambda = y_\lambda - \lambda x \in \mathrm{bd}(K)$. The set $N(K_\lambda, y_\lambda) \cap \mathbb{S}^{n-1}$ defines a curve γ which we assume to be parametrized on $[0,1]$ with $\gamma(0) = N(K, y_\lambda)$ and $\gamma(1) = N(K, y'_\lambda)$. We use the inverse of the Gauss map p to map the curve γ to a curve $\tilde\gamma$ on $\mathrm{bd}(K)$ by setting $\tilde\gamma = p^{-1}\gamma$. The end points of $\tilde\gamma$ are y_λ (label it with A) and y'_λ (label it with B). Since $u \in \gamma$ we conclude that the point $p^{-1}(u)$ belongs to the curve $\tilde\gamma$ (label this point by Γ). Thus we get

$$0 \le h(u) - h_\lambda(u) = |\langle \vec{A\Gamma}, u\rangle|.$$

It is not difficult to see that the cosine of the angle of the vectors $\vec{A\Gamma}$ and u is less than the largest principal curvature of $\mathrm{bd}(K)$ at Γ times $|\vec{A\Gamma}|$, the length of the vector $\vec{A\Gamma}$. Consequently we can write

$$0 \le h(u) - h_\lambda(u) \le \|W\|\, |\vec{A\Gamma}|^2.$$

In addition we have

$$\begin{aligned} |\vec{A\Gamma}| &\le \mathrm{length}\left(\tilde\gamma|_A^\Gamma\right) \le \mathrm{length}\left(\tilde\gamma|_A^B\right) = \int_0^1 |d_t\tilde\gamma|\, dt = \int_0^1 |d_t p^{-1}\gamma|\, dt \\ &\le \|W^{-1}\| \mathrm{length}(\gamma) \le \frac{2}{\pi}\|W^{-1}\|\, |p(y_\lambda) - p(y'_\lambda)|, \end{aligned}$$

where $|\,.\,|$ is the standard Euclidean norm. Without loss of generality we can assume that the points y_λ and y'_λ belong to the same chart at y_λ. Let φ be the chart mapping $\mathbb{R}^{n-1}$ to a neighborhood of y_λ on $\mathrm{bd}(K)$ and ψ the chart mapping

$\mathbb{R}^{n-1}$ on $\mathbb{S}^{n-1}$. We assume, as we may, that the graph of γ is contained in the range of the chart ψ. It is now clear from the above series of inequalities that

$$|\vec{A\Gamma}| \leq c_0 \|W^{-1}\| \, |\psi^{-1} p\varphi(t) - \psi^{-1} p\varphi(s)|,$$

where t and s are points in $\mathbb{R}^{n-1}$ such that $\varphi(t) = y_\lambda$ and $\varphi(s) = y'_\lambda$ and $c_0 > 0$ is a universal constant. Now the mean value theorem for curves gives

$$|\vec{A\Gamma}| \leq C\|W^{-1}\|\|W\| \, |t-s| \leq C\|W^{-1}\|\|W\| \, |y_\lambda - y'_\lambda| = C\|W^{-1}\|\|W\|\lambda,$$

where C may denote a different constant every time it appears. Thus we have

$$0 \leq h(u) - h_\lambda(u) \leq C\|W\| \left(\|W^{-1}\|\|W\|\right)^2 \lambda^2.$$

Consequently,

$$\int_{\mathbb{S}^{n-1}\setminus p_\lambda(Y_\lambda)} (h(u) - |\langle \lambda x, u\rangle|) \, d\nu(u) + \int_{p_\lambda(Y_\lambda)} \left(h(u) - C\lambda^2\right) \, d\nu(u)$$

$$\leq M^*_{K_\lambda} = \delta M^*_K \leq$$

$$\int_{\mathbb{S}^{n-1}\setminus p_\lambda(Y_\lambda)} (h(u) - |\langle \lambda x, u\rangle|) \, d\nu(u) + \int_{p_\lambda(Y_\lambda)} h(u) \, d\nu(u),$$

where C now depends on $\|W\|$ and $\|W^{-1}\|$.

Rearranging and using c_n for the quantity $\int_{\mathbb{S}^{n-1}} |\langle x, u\rangle| \, d\nu(u)$ and the fact that $\lambda = 1/\|x\|_{C_1(\delta)}$ we get

$$\left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M^*_K} \right| \leq \frac{c_n}{M^*_K} \left(\frac{\int_{p_\lambda(Y_\lambda)} |\langle x, u\rangle| \, d\nu(u)}{c_n} + C\lambda \frac{\mu\left(p_\lambda(Y_\lambda)\right)}{c_n} \right).$$

We observe now that for $u \in p_\lambda(Y_\lambda), |\langle x, u\rangle| \leq \text{length}(\gamma)/2 \leq \|W\|\lambda$. Using this in the last inequality and the fact that $p_\lambda(Y_\lambda)$ is a band around an equator of $\mathbb{S}^{n-1}$ of width at most length$(\gamma)/2$ we get

$$\left| \|x\|_{\frac{C_1(\delta)}{1-\delta}} - \frac{c_n}{M^*_K} \right| \leq \frac{c_n}{M^*_K} Cn\lambda^2 \leq \frac{c_n}{M^*_K} Cn \frac{(1-\delta)^2}{\|x\|^2_{\frac{C_1(\delta)}{1-\delta}}}.$$

Our final task is to get rid of the norm that appears on the right side of the latter inequality. Set

$$T = \frac{\|x\|_{C_1(\delta)/1-\delta}}{c_n/M^*_K}.$$

We have shown that

$$T^2|T-1| \leq C\frac{M^*_K}{c_n} n(1-\delta)^2.$$

If $T \geq 1$ then we can just drop the factor T^2 and we are done. If $T < 1$ we write $T^2\,|T-1|$ as $\left(1-(1-T)\right)^2(1-T)$ and we consider the function

$$f(x) = (1-x)^2 x : (-\infty, \tfrac{1}{3}) \to \mathbb{R}.$$

This function is strictly increasing thus invertible on its range, that is, f^{-1} is well defined and increasing in $(-\infty, \frac{4}{27})$. Consequently, if

$$C\frac{M_K^*}{c_n}n(1-\delta)^2 \leq \frac{4}{27}, \tag{2.2}$$

we conclude that

$$0 \leq 1 - T \leq f^{-1}\left(C\frac{M_K^*}{c_n}n(1-\delta)^2\right) \leq C\frac{M_K^*}{c_n}n(1-\delta)^2.$$

The last inequality is true since the derivative of f^{-1} at zero is 1. Observe also that the convergence is "essentially realized" after (2.2) is satisfied. □

We now proceed to show that some smoothness conditions on the boundary of K are necessary, by proving that the limiting M^*-convolution body of the n-dimensional cube is homothetic to the unit ball of ℓ_1^n. In fact we show that the δ-M^*-convolution body of the cube is already homothetic to the unit ball of ℓ_1^n.

Example 2.2. *Let $P = [-1, 1]^n$. Then for $0 < \delta < 1$ we have*

$$C_1(P) = \frac{C_1(\delta; P, P)}{1-\delta} = n^{3/2}\operatorname{vol}_{n-1}(\mathbb{S}^{n-1})B_{\ell_1^n}.$$

Proof. Let $x = \sum_{j=1}^n x_j e_j$ where $x_j \geq 0$ for all $j = 1, 2, \ldots, n$ and e_j is the standard basis of $\mathbb{R}^n$. Let $\lambda > 0$ be such that $\lambda x \in \operatorname{bd}(C_1(\delta))$. Then

$$P \cap (\lambda x + P) = \left\{y \in \mathbb{R}^n : y = \sum_{j=1}^n y_i e_i, -1 + \lambda x_i \leq y_i \leq 1\right\}.$$

The vertices of $P_\lambda = P \cap (\lambda x + P)$ are the points $\sum_{j=1}^n \alpha_j e_j$ where α_j is either 1 or $-1 + \lambda x$ for all j. Without loss of generality we can assume that $-1 + \lambda x_j < 0$ for all the indices j. Put $\operatorname{sign}\alpha_j = \alpha_j/|\alpha_j|$ when $\alpha_j \neq 0$ and $\operatorname{sign} 0 = 0$. Fix a sequence of α_j's so that the point $v = \sum_{j=1}^n \alpha_j e_j$ is a vertex of P_λ. Clearly,

$$N(P_\lambda, v) = N\left(P, \sum_{j=1}^n (\operatorname{sign}\alpha_j)e_j\right).$$

If $u \in \mathbb{S}^{n-1} \cap N(P_\lambda, v)$ then

$$h_\lambda(u) = h(u) - \left|\left\langle \sum_{j=1}^n (\alpha_j - \operatorname{sign}\alpha_j)e_j,\, u\right\rangle\right|.$$

If $\operatorname{sign}\alpha_j = 1$ then $\alpha_j - \operatorname{sign}\alpha_j = 0$ otherwise $\alpha_j - \operatorname{sign}\alpha_j = \lambda x$.

Let $\mathcal{A} \subseteq \{1, 2, \ldots, n\}$. Consider the "$\mathcal{A}$-orthant"

$$\mathcal{O}_\mathcal{A} = \{y \in \mathbb{R}^n : \langle y, e_j\rangle < 0, \text{ if } j \in \mathcal{A} \text{ and } \langle y, e_j\rangle \geq 0 \text{ if } j \notin \mathcal{A}\}.$$

Then $\mathcal{O}_{\mathcal{A}} = N\left(P, \sum_{j=1}^{n}(\operatorname{sign}\alpha_j)e_j\right)$ if and only if $\operatorname{sign}\alpha_j = 1$ exactly for every $j \notin \mathcal{A}$. Thus we get

$$h_\lambda(u) = h(u) - \left|\left\langle \sum_{j\in\mathcal{A}} \lambda x_j e_j, u\right\rangle\right|,$$

for all $u \in \mathcal{O}_{\mathcal{A}} \cap \mathbb{S}^{n-1}$. Hence using the facts $M^*_{P_\lambda} = \delta M^*_P$ and $\lambda = 1/\|x\|_{C_1(\delta)}$ we get

$$\|x\|_{\frac{C_1(\delta)}{1-\delta}} = -\frac{1}{M^*_P} \sum_{\mathcal{A}\subseteq\{1,2,\ldots,n\}} \sum_{j\in\mathcal{A}} x_j \int_{\mathcal{O}_{\mathcal{A}}\cap\mathbb{S}^{n-1}} \langle e_j, u\rangle \, d\nu(u),$$

which gives the result since

$$\int_{\mathcal{O}_{\mathcal{A}}\cap\mathbb{S}^{n-1}} \langle e_j, u\rangle \, d\nu(u) = \frac{1}{2^{n-1}} \int_{\mathbb{S}^{n-1}} |\langle e_1, u\rangle| \, d\nu(u). \qquad \square$$

Acknowledgement

We want to thank Professor V. D. Milman for his encouragement and his guidance in this research and for suggesting the study of mixed convolution bodies.

References

[1] K. Kiener, “Extremalität von Ellipsoiden und die Faltungsungleichung von Sobolev”, *Arch. Math.* **46** (1986), 162–168.

[2] V. Milman and G. Schechtmann, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Math. **1200**, Springer, 1986.

[3] M. Schmuckenschläger, “The distribution function of the convolution square of a convex symmetric body in $\mathbb{R}^n$”, *Israel Journal of Mathematics* **78** (1992), 309–334.

[4] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications **44**, Cambridge University Press, 1993.

[5] A. Tsolomitis, Convolution bodies and their limiting behavior, *Duke Math. J.* **87**:1 (1997), 181–203.

Antonis Tsolomitis
The Ohio State University
Department of Mathematics
231 W.18th Avenue
Columbus, OH 43210
United States of America
atsol@eexi.gr
Current address: University of Crete, Department of Mathematics, 71409 Heraklion, Crete, Greece