Several Complex Variables
MSRI Publications
Volume **37**, 1999

Subelliptic Estimates and Finite Type

JOHN P. D'ANGELO AND JOSEPH J. KOHN

Abstract. This paper surveys work in partial differential equations and several complex variables that revolves around subelliptic estimates in the $\bar{\partial}$-Neumann problem. The paper begins with a discussion of the question of *local regularity*; one is given a bounded pseudoconvex domain with smooth boundary, and hopes to solve the inhomogeneous system of Cauchy–Riemann equation $\bar{\partial}u = \alpha$, where α is a differential form with square integrable coefficients and satisfying necessary compatibility conditions. Can one find a solution u that is smooth wherever α is smooth? According to a fundamental result of Kohn and Nirenberg, the answer is yes when there is a *subelliptic estimate*. The paper sketches the proof of this result, and goes on to discuss the history of various finite-type conditions on the boundary and their relationships to subelliptic estimates. This includes finite-type conditions involving iterated commutators of vector fields, subelliptic multipliers, finite type conditions measuring the order of contact of complex analytic varieties with the boundary, and Catlin's multitype.

The paper also discusses additional topics such as nonpseudoconvex domains, Holder and L^p estimates for $\bar{\partial}$, and finite-type conditions that arise when studying holomorphic extension, convexity, and the Bergman kernel function. The paper contains a few new examples and some new calculations on CR manifolds. The paper ends with a list of nine open problems.

Contents

1. Introduction

The solution of the Levi problem during the 1950's established the fundamental result in function theory characterizing domains of holomorphy. Suppose that Ω is a domain in complex Euclidean space $\mathbb{C}^n$. The solution establishes that three conditions on Ω are identical: Ω is a domain of holomorphy, Ω is pseudoconvex, and the sheaf cohomology groups $H^q(\Omega, O)$ are trivial for each $q \geq 1$. The first property is a global function-theoretic property, the second is a local property of the boundary, and the third tells us that certain overdetermined systems of linear partial differential equations (the inhomogeneous Cauchy–Riemann equations) always have smooth solutions.

After the solution of the Levi problem, research focused upon domains with smooth boundaries and mathematicians hoped to establish deeper connections between partial differential equations and complex analysis. This led to the study of the Cauchy–Riemann equations on the closed domain and to many questions relating the boundary behavior of the Cauchy–Riemann operator $\bar{\partial}$ to the function theory on Ω. We continue the introduction by describing the question of local regularity for $\bar{\partial}$, and how its study motivated various geometric notions of "finite type".

Suppose that Ω is a bounded domain and that its boundary $b\Omega$ is a smooth manifold. We define $\bar{\partial}$ in the sense of distributions. Let α be a differential $(0, q)$ form with square-integrable coefficients and satisfying the compatibility condition $\bar{\partial}\alpha = 0$. What geometric conditions on $b\Omega$ guarantee that we can solve the Cauchy–Riemann equation $\bar{\partial}u = \alpha$ so that the $(0, q-1)$ form u must be smooth wherever α is? Here smoothness up to the boundary is the issue.

One approach to regularity results is the $\bar{\partial}$-Neumann problem. See [Folland and Kohn 1972; Kohn 1977; 1984] for extensive discussion. Let $L^2_{(0,q)}(\Omega)$ denote the space of $(0, q)$ forms with square-integrable coefficients. The $\bar{\partial}$-Neumann problem generalizes Hodge theory; careful attention to boundary conditions is now required. Under certain geometric conditions on $b\Omega$, Kohn constructed an operator N on $L^2_{(0,q)}(\Omega)$ such that $u = \bar{\partial}^* N\alpha$ gives the unique solution to $\bar{\partial}u = \alpha$ that is orthogonal to the null space of $\bar{\partial}$ on Ω. This is called the *canonical solution* or the $\bar{\partial}$-Neumann solution. In particular the Neumann operator N exists on bounded pseudoconvex domains. What additional geometric conditions on $b\Omega$ guarantee that N is a pseudo-local operator, and hence yield *local regularity* for the canonical solution u? By local regularity we mean that u is smooth wherever α is smooth. We shall see that pseudolocality for N follows from subelliptic estimates.

Kohn [1963; 1964] solved the $\bar{\partial}$-Neumann problem on strongly pseudoconvex domains in 1962. Subsequent work by Kohn and Nirenberg [1965] exposed clearly the subelliptic nature of the problem. Local regularity holds on strongly pseudoconvex domains because there is a subelliptic estimate; in this case one can take ε equal to $\frac{1}{2}$ in Definition 3.4 of this paper. Local regularity follows from a subelliptic estimate for any positive ε (see Theorem 3.5); this led Kohn to seek geometric conditions for subelliptic estimates. For domains in two dimensions he introduced in [1972] a finite-type condition (called "finite commutator-type" in this paper) enabling him to prove a subelliptic estimate. Greiner [1974] established the necessity of finite commutator-type in two dimensions. These theorems generated much work concerned with intermediate conditions between pseudoconvexity and strong pseudoconvexity. Different analytic problems lead to different intermediate, or finite-type, conditions on $\mathrm{b}\Omega$. After contributions by many authors, Catlin [1983; 1984; 1987] completely solved one major problem of this kind. He proved that a certain finite-type condition is both necessary and sufficient for subelliptic estimates on $(0, q)$ forms for $q \geq 1$ for the $\bar{\partial}$-Neumann problem on smoothly bounded pseudoconvex domains. The finite-type condition is that the maximum order of contact of q-dimensional complex-analytic varieties with the boundary be finite at each point.

In this paper we survey those finite-type conditions arising from subelliptic estimates for the $\bar{\partial}$-Neumann problem and we indicate their relationship to function theory, geometry, and partial differential equations. We provide greater detail when we discuss subelliptic multipliers; we consider their use both on domains that are not pseudoconvex and on domains in CR manifolds. We indicate directions for further research and end the paper with a list of open problems.

2. The Levi Form

We begin by considering the geometry of the boundary of a domain in complex Euclidean space and its relationship to the function theory on the domain, using especially the Cauchy–Riemann operator $\bar{\partial}$ and the $\bar{\partial}$-Neumann problem. Let Ω denote a domain in $\mathbb{C}^n$ whose boundary is a smooth manifold denoted by $\mathrm{b}\Omega$ or by M. Pseudoconvexity is a geometric property of $\mathrm{b}\Omega$ that is necessary and sufficient for Ω to be a domain of holomorphy; for domains with smooth boundaries, pseudoconvexity is determined by the Levi form.

We recall an invariant definition of the Levi form that makes sense also for CR manifolds of hypersurface type. Thus we suppose that M is a smooth real manifold of dimension $2n-1$ and that $\mathbb{C}TM$ denotes its complexified tangent bundle.

We say that M is a CR manifold of hypersurface type if there is a subbundle $T^{1,0}M \subset \mathbb{C}TM$ such that the following conditions hold:

1. $T^{1,0}M$ is integrable (closed under the Lie bracket operation).
2. $T^{1,0}M \cap \overline{T^{1,0}M} = \{0\}$.
3. The bundle $T^{1,0}M \oplus \overline{T^{1,0}M}$ has codimension one in $\mathbb{C}TM$.

For real submanifolds in $\mathbb{C}^n$ the bundle $T^{1,0}M$ is defined by $\mathbb{C}TM \cap T^{1,0}\mathbb{C}^n$, and thus local sections of $T^{1,0}M$ are complex $(1,0)$ vector fields tangent to M.

The bundle $T^{1,0}M$ is closed under the Lie bracket, or commutator, [,]. For CR manifolds this integrability condition is part of the definition.

On the other hand, the bundle $T^{1,0}M \oplus \overline{T^{1,0}M}$ is generally not integrable. The Levi form measures the failure of integrability. To define it, we denote by η a purely imaginary non-vanishing 1-form that annihilates $T^{1,0}M \oplus \overline{T^{1,0}M}$. When M is a hypersurface, and r is a local defining function, we may put $\eta = \frac{1}{2}(\partial - \overline{\partial})r$. We write $\langle\ ,\ \rangle$ for the contraction of a one-form and a vector field.

Definition 2.1. The *Levi form* λ is the Hermitian form on $T^{1,0}M$ defined (up to a multiple) by

$$\lambda(L, \overline{K}) = \langle \eta, [L, \overline{K}] \rangle. \tag{1}$$

The CR manifold M is called strongly pseudoconvex when λ is definite, and is called weakly pseudoconvex when λ is semi-definite but not definite. We say that the domain lying on one side of a real hypersurface is pseudoconvex when λ is positive semi-definite on the hypersurface.

We can also interpret the Levi form as the restriction of the complex Hessian of a defining function to the space $T^{1,0}M$. To see this we use the Cartan formula for the exterior derivative of η. Because $L, \overline{K}$ are annihilated by η and are tangent to M, we can write

$$\langle \partial\overline{\partial} r, L \wedge \overline{K} \rangle = \langle -d\eta, L \wedge \overline{K} \rangle = -L\langle \eta, \overline{K} \rangle + \overline{K}\langle \eta, L \rangle + \langle \eta, [L, \overline{K}] \rangle = \lambda(L, \overline{K}).$$

It is also useful to express the entries of the matrix λ with respect to a special local basis of the $(1, 0)$ vector fields. Suppose that r is a defining function, and that we are in a neighborhood where $r_{z_n} \neq 0$. We put

$$T = \frac{1}{r_{z_n}} \frac{\partial}{\partial z_n} - \frac{1}{r_{\bar{z}_n}} \frac{\partial}{\partial \bar{z}_n}.$$

Then $\langle \eta, T \rangle = 1$. For $i = 1, 2, \ldots, n-1$ we define L_i by

$$L_i = \frac{\partial}{\partial z_i} - \frac{r_{z_i}}{r_{z_n}} \frac{\partial}{\partial z_n}.$$

Then the L_i, for $i = 1, 2, \ldots n-1$, form a commuting local basis for sections of $T^{1,0}M$. Furthermore $[L_i, \overline{L}_j] = \lambda_{ij} T$. Using subscripts for partial derivatives we have

$$\lambda_{ij} = \frac{r_{i\bar{j}}|r_n|^2 - r_{i\bar{n}} r_n r_{\bar{j}} - r_{n\bar{j}} r_i r_{\bar{n}} + r_{n\bar{n}} r_i r_{\bar{j}}}{|r_n|^2}.$$

Strong pseudoconvexity is a non-degeneracy condition: if λ is positive-definite at a point $p \in M$, then it is positive-definite in a neighborhood. Furthermore strong pseudoconvexity is "finitely determined": if M' is another hypersurface containing p and osculating M to second order there, then M' is also strongly pseudoconvex at p. In seeking generalizations of strong pseudoconvexity that have applications in analytic problems we expect that generalizations will be both open and finitely determined conditions.

As a simple example we compute the Levi form for domains in $\mathbb{C}^n$ defined locally by the equation

$$r(z, \bar{z}) = 2\,\mathrm{Re}(z_n) + \sum_{k=1}^{N} |f^k(z)|^2 < 0. \tag{2}$$

Here the functions f^k are holomorphic near the origin, vanish there, and depend only on the variables $z_1, z_2, \ldots, z_{n-1}$. The domain defined by (2) is pseudoconvex. Its Levi form near the origin has the nice expression

$$(\lambda_{ij}) = \left(\sum_{k=1}^{N} f^k_{z_i} \overline{f^k_{z_j}} \right) = (\partial f)^*(\partial f). \tag{3}$$

It follows immediately from (3) that the origin will be a weakly pseudoconvex point if and only if the rank of ∂f (as a mapping on $\mathbb{C}^{n-1}$) is less than full there. It is instructive to consider finite-type notions in this case and compare them with standard notions of singularities from algebraic and analytic geometry. For example, we will see that the origin is a point of finite D_1-type if and only if the germs of the functions f^k define a trivial variety, and more generally a point of finite D_q-type if and only if the functions define a variety of dimension less than q. The origin is a point of finite commutator-type if and only if some f^k is not identically zero; we see that this is the same as being finite D_{n-1}-type. This simple example allows us to glimpse the role of commutative algebra in later discussions, and it illustrates why different finiteness conditions arise.

It will be important to understand the *determinant* of the Levi form. To do so we make some remarks about restricting a linear map to a subspace. Suppose that $A : \mathbb{C}^n \to \mathbb{C}^n$ is a self-adjoint linear map, and that $\zeta \in \mathbb{C}^n$ is a unit vector. We form two new linear transformations using this information.

First we extend A to a map $(E_\zeta A) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n \times \mathbb{C}$ given by

$$(E_\zeta A)(z, t) = (Az + t\zeta, \langle z, \zeta \rangle). \tag{4}$$

Second we restrict A to a map on the orthogonal complement of the span of ζ, and identify this with a map $R_\zeta A : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$, by composing with an isometry in the range. Then, assuming $n \geq 2$, we have

$$\det(R_\zeta A) = \det(E_\zeta A).$$

One way to see this is to choose coordinates so that $\zeta = (0, 0, \ldots, 0, 1)$ and the matrix of the map $E_\zeta A$ has lots of zeroes. Expanding by cofactors (twice) shows that the determinant equals the determinant of the $n-1$ by $n-1$ principal minor of A, which equals the determinant of R_ζ by the same computation that one does to write the Levi form as an $n-1$ by $n-1$ matrix.

According to this result we may express the determinant of the Levi form as the determinant of the $n+1$ by $n+1$ bordered Hessian matrix

$$E = \begin{pmatrix} r_{z_1\bar{z}_1} & r_{z_2\bar{z}_1} & \dots & r_{z_n\bar{z}_1} & r_{\bar{z}_1} \\ r_{z_1\bar{z}_2} & r_{z_2\bar{z}_2} & \dots & r_{z_n\bar{z}_2} & r_{\bar{z}_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{z_1\bar{z}_n} & r_{z_2\bar{z}_n} & \dots & r_{z_n\bar{z}_n} & r_{\bar{z}_n} \\ r_{z_1} & r_{z_2} & \dots & r_{z_n} & 0 \end{pmatrix}. \tag{5}$$

It will be convenient later to write (5) in simpler notation. To do so, we imagine ∂r as a row, and $\bar{\partial} r$ as a column. We get

$$E = \begin{pmatrix} \partial\bar{\partial} r & \bar{\partial} r \\ \partial r & 0 \end{pmatrix}. \tag{6}$$

Finally we remark that when the defining equation is given by (3), there is a simple formula for the determinant of the Levi form. We have

$$\det(\lambda) = \sum |J(f_{i_1}, \dots, f_{i_{n-1}})|^2,$$

where the sum is taken over all choices of $n-1$ of the functions f_k, and J denotes the Jacobian determinant in $n-1$ dimensions. Thus the determinant of the Levi form is the squared norm of a holomorphic mapping in this case.

3. Subelliptic Estimates for the $\bar{\partial}$-Neumann Problem

From the introduction we have seen that the $\bar{\partial}$-Neumann problem constructs a particular solution to the inhomogeneous Cauchy–Riemann equations. The $\bar{\partial}$-Neumann problem is a boundary value problem; the equation is elliptic, but the boundary conditions are not elliptic. One of the most important results, due to Kohn and Nirenberg, states that local regularity for the canonical solution to the inhomogeneous Cauchy–Riemann equations follows from a subelliptic estimate. In this section we define subelliptic estimates, and sketch a proof of the Kohn–Nirenberg result.

We begin by recalling the definition of the tangential Sobolev norms. We write $\mathbb{R}^m_-$ for the subset of $\mathbb{R}^m$ whose last coordinate is negative. For convenience we denote the first $m-1$ components by t and the last component by r.

Definition 3.1 (Partial Fourier Transform). Suppose that $u \in C_0^\infty(\mathbb{R}^m_-)$. The partial Fourier transform of u is given by

$$\tilde{u}(\xi, r) = \int_{\mathbb{R}^{m-1}} e^{-it\cdot\xi} u(t, r)\, dt.$$

Definition 3.2. Suppose that $u \in C_0^\infty(\mathbb{R}^m_-)$. We define the tangential pseudodifferential operator Λ^s and the tangential Sobolev norm $|||u|||_s$ by

$$(\widetilde{\Lambda^s u})(\xi, r) = (1 + |\xi|^2)^{s/2} \tilde{u}(\xi, r), \qquad |||u|||_s = \|\Lambda^s u\|.$$

Note that the L^2 norm is computed over $\mathbb{R}^m_-$. Suppose that Ω is a smoothly bounded domain in $\mathbb{C}^n$, and $p \in \mathrm{b}\Omega$. On a sufficiently small neighborhood U of p we introduce coordinates $(t_1, \dots, t_{2n-1}, r)$ where r is a defining function for Ω. We may also assume that $\omega_1, \dots \omega_n$ form an orthonormal basis for the $(1,0)$ forms on U and that $\omega_n = (\partial r)/|\partial r|$.

Thus a $(0,1)$ form ϕ defined on U may be written

$$\phi = \sum_1^n \phi_j \overline{\omega}_j$$

We write

$$|||\phi|||_s^2 = \sum_1^n |||\phi_j|||_s^2.$$

We denote by $\bar{\partial}^*$ the L^2-adjoint of (the maximal extension) of $\bar{\partial}$ and let $\mathcal{D}(\bar{\partial}^*)$ denote its domain. In terms of the ω_j there is a simple expression for the boundary condition required for a form to be in $\mathcal{D}(\bar{\partial}^*)$. If $\phi \in C^\infty(U \cap \overline{\Omega})$, then ϕ is in $\mathcal{D}(\bar{\partial}^*)$ if and only if $\phi_n = 0$ on $U \cap \mathrm{b}\Omega$.

We define (in terms of the $L^2(\Omega)$ inner product) the quadratic form Q by

$$Q(\phi, \psi) = (\bar{\partial}\phi, \bar{\partial}\psi) + (\bar{\partial}^*\phi, \bar{\partial}^*\psi).$$

Integration by parts yields the following formula for $Q(\phi, \phi)$ on (0,1)-forms, where r is a local defining function for $\mathrm{b}\Omega$.

Lemma 3.3. *The quadratic form Q satisfies*

$$Q(\phi, \phi) = \sum_{i,j=1}^n \int_\Omega |(\phi_i)_{\bar{z}_j}|^2 dV + \sum_{i,j=1}^n \int_{\mathrm{b}\Omega} r_{z_i \bar{z}_j} \phi_i \bar{\phi}_j dS = \|\phi\|_{\bar{z}}^2 + \int_{\mathrm{b}\Omega} \lambda(\phi, \phi)\, dS. \quad (7)$$

This formula reveals an asymmetry between the barred and unbarred derivatives; this is a consequence of the boundary conditions. Observe also that the integral of the Levi form appears. This term is non-negative when Ω is pseudoconvex. The *basic estimate* asserts that the terms on the right of (7) are dominated by a constant times $Q(\phi, \phi)$. For pseudoconvex domains in $\mathbb{C}^n$ we also have the estimate

$$\|\phi\|^2 \leq CQ(\phi, \phi). \quad (8)$$

This estimate does not hold generally for domains in manifolds, unless the manifold admits a strongly plurisubharmonic exhaustion function.

In order to prove local regularity for the $\bar{\partial}$-Neumann solution to the inhomogeneous Cauchy–Riemann equations, we use a stronger estimate, called a subelliptic estimate.

Definition 3.4. Suppose that $\Omega \Subset \mathbb{C}^n$ is smoothly bounded and pseudoconvex. Let $p \in \overline{\Omega}$ be any point in the closure of the domain. The $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate at p on (0,1) forms if there exist positive constants C, ε and a neighborhood $U \ni p$ such that

$$|||\phi|||_{\varepsilon}^{2} \leq C(\|\bar{\partial}\phi\|^{2} + \|\bar{\partial}^{*}\phi\|^{2}) \tag{9}$$

for every (0,1)-form ϕ that is smooth, compactly supported in U, and in $\mathcal{D}(\bar{\partial}^{*})$.

We usually say simply *a subelliptic estimate holds* when the definition applies. Although the definition of $\bar{\partial}^{*}$ (and hence that of Q) depends on the Hermitian metric used, whether a subelliptic estimate holds is independent of the metric [Sweeney 1972].

We begin with the connection to local regularity. Suppose that α is a $(0,1)$ form in $L^2(\Omega)$ and that $\alpha|_{U\cap\overline{\Omega}}$ is smooth. Let ϕ in $\mathcal{D}(\bar{\partial}^{*})$ be the unique form that satisfies

$$Q(\phi,\psi) = (\alpha,\psi)$$

for all ψ in $\mathcal{D}(\bar{\partial}) \cap \mathcal{D}(\bar{\partial}^{*})$. Then $\phi = N\alpha$ and we have $\bar{\partial}(\bar{\partial}^{*}\phi) = \alpha$. A subelliptic estimate implies that $\phi|_{U\cap\overline{\Omega}} \in C^{\infty}(U \cap \overline{\Omega})$. The basic theorem of Kohn and Nirenberg [1965] gives this and additional consequences of a subelliptic estimate.

THEOREM 3.5 (KOHN AND NIRENBERG). *Suppose that a subelliptic estimate holds. Then ϕ restricted to $U \cap \overline{\Omega}$ is smooth. More generally the Neumann operator N is pseudolocal. We also have, in terms of local Sobolev norms H_s,*

$$\begin{aligned} \alpha \in H_s \ &\Rightarrow\ N\alpha \in H_{s+2\varepsilon}, \\ \alpha \in H_s \ &\Rightarrow\ \bar{\partial}^{*}N\alpha \in H_{s+\varepsilon}. \end{aligned} \tag{10}$$

SKETCH OF PROOF. Suppose that a subelliptic estimate holds, and that D is an arbitrary first order partial differential operator. The first step is to prove the estimate

$$|||D\phi|||_{\varepsilon-1}^{2} \leq Q(\phi,\phi) \tag{11}$$

for all $\phi \in C_0^{\infty}(U\cap\overline{\Omega}) \cap \mathcal{D}(\bar{\partial}^{*})$. This is clear when D is tangential, so it suffices to consider $D = \frac{\partial}{\partial r}$. Observe that $b\Omega$ is non-characteristic for the quadratic form Q (in fact Q is elliptic, although the boundary conditions are not). Therefore we have an estimate

$$\left\|\frac{\partial\phi}{\partial r}\right\|^{2} \leq C\,(Q(\phi,\phi) + |||\phi|||_{1}^{2}). \tag{12}$$

After using cut-off functions to give a meaning to $Q(\Lambda^{\varepsilon-1}\phi, \Lambda^{\varepsilon-1}\phi)$, we replace ϕ by $\Lambda^{\varepsilon-1}\phi$ in (12). This yields

$$|||\frac{\partial\phi}{\partial r}|||_{\varepsilon-1}^{2} \leq C\,(Q(\Lambda^{\varepsilon-1}\phi, \Lambda^{\varepsilon-1}\phi) + |||\phi|||_{\varepsilon}^{2}). \tag{13}$$

We next require some calculations involving the commutators $[\bar{\partial}, \Lambda^{\varepsilon-1}]$ and $[\bar{\partial}^{*}, \Lambda^{\varepsilon-1}]$. We omit the proofs, but both $\|[\bar{\partial}, \Lambda^{\varepsilon-1}]\phi\|$ and $\|[\bar{\partial}^{*}, \Lambda^{\varepsilon-1}]\phi\|$ can be estimated in terms of a constant times $|||\phi|||_{\varepsilon-1}$. Given this we can estimate

$$Q(\Lambda^{\varepsilon-1}\phi, \Lambda^{\varepsilon-1}\phi) \leq cQ(\phi,\phi). \tag{14}$$

Combining (13) and (14) with the subelliptic estimate proves (11) when $D = \partial/\partial r$.

Assume that ϕ is smooth. Let ζ and ζ' be cutoff functions with $\operatorname{supp}(\zeta) \Subset \operatorname{supp}(\zeta')$ and suppose that $\zeta' = 1$ on a neighborhood of the support of ζ. We need an estimate involving higher derivatives:

$$\sum_{|\gamma| \le m+2} |||D^\gamma \zeta\phi|||_{(k+2)\varepsilon - |\gamma|} \le C_{mk}\Big(\sum_{|\gamma|\le m} |||D^\gamma \zeta'\alpha|||_{m\varepsilon-|\gamma|} + \|\phi\|\Big).$$

The proof of this is complicated, and we omit it.

The next step is to introduce elliptic regularization. For $\delta > 0$ we consider the quadratic form Q_δ defined by

$$Q_\delta(\phi,\psi) = Q(\phi,\psi) + \delta \sum_{|\gamma|\le 1} (D^\gamma\phi, D^\gamma\psi).$$

The form Q_δ is elliptic. We can solve

$$Q_\delta(\phi_\delta,\psi) = (\alpha,\psi)$$

so that ϕ_δ is smooth wherever α is smooth. From estimate (8) we obtain $\|\phi_\delta\| \le C\|\alpha\|$ where C is independent of δ. One then proves that a subsequence of the ϕ_δ converges in the C^∞ topology to a solution ϕ of the original problem. □

We close the section by making a few remarks about the definition of a subelliptic estimate. Observe that the set of points for which a subelliptic estimate holds must be an open subset of the closed domain. For interior points, the estimate (9) is elliptic, and holds with $\varepsilon = 1$. At strongly pseudoconvex boundary points, the estimate holds for $\varepsilon = \frac{1}{2}$. Catlin has found necessary and sufficient conditions for a subelliptic estimate for some $\varepsilon > 0$ to hold. See Theorem 7.1. In the weakly pseudoconvex case there is no general result giving the largest possible value of the parameter ε in terms of the geometry of $\mathrm{b}\Omega$ at the boundary point p.

4. Ideals of Subelliptic Multipliers

We assume that Ω is a smoothly bounded pseudoconvex domain. The estimate (9) holds at interior points; we next let x be a boundary point of Ω. For a neighborhood U containing x, consider the set of all functions $f \in C_0^\infty(U \cap \Omega$ such that there are $C, \varepsilon > 0$ for which

$$|||f\phi|||_\varepsilon^2 \le C(\|\bar\partial\phi\|^2 + \|\bar\partial^*\phi\|^2) \tag{15}$$

for all $\phi \in C_0^\infty(U\cap\Omega) \cap \mathcal{D}(\bar\partial^*)$. Here both constants may depend on f. Let $\mathcal{I}_x$ denote the collection of all germs of such functions at x; its elements are called subelliptic multipliers. We see immediately that a subelliptic estimate holds precisely when the constant function 1 is a subelliptic multiplier.

Lemma 4.1. *Suppose that λ_{ij} are the components of the Levi matrix with respect to the local basis $\{L_1, \dots, L_{n-1}\}$ of $T^{10}(\mathrm{b}\Omega)$. Then there is a constant C so that*

$$\sum_{i,j=1}^{n-1} (\lambda_{ij}\Lambda^{1/2}\phi_i, \Lambda^{1/2}\phi_j) \leq CQ(\phi,\phi). \tag{16}$$

We omit the proof, which uses the expression

$$\det\begin{pmatrix} \partial\bar\partial r & \bar\partial r \\ \partial r & 0 \end{pmatrix}$$

for the determinant of the Levi form (see the discussion between (4) *and* (6)) *and also requires properties of commutators of tangential pseudodifferential operators.*

Proposition 4.2. *Suppose that Ω is pseudoconvex. The defining function r is a subelliptic multiplier, with $\varepsilon = 1$. The determinant of the Levi form is a subelliptic multiplier, with $\varepsilon = \frac{1}{2}$.*

Proof. To show that r is a subelliptic multiplier with $\varepsilon = 1$ is easy. It follows from integration by parts that $\|(r\phi_k)_{z_i}\|^2 = \|(r\phi_k)_{\bar z_i}\|^2$. Therefore it suffices to estimate the first order barred derivatives. To do so we replace ϕ by $r\phi$ in Lemma 3.3 and observe that $Q(r\phi, r\phi) \leq CQ(\phi,\phi)$.

That $\det(\lambda_{ij})$ is a subelliptic multiplier with $\varepsilon = \frac{1}{2}$ follows from Lemma 4.1. □

Starting with Proposition 4.2, Kohn [1979] developed an algorithmic procedure for constructing new multipliers, for which the corresponding value of epsilon is typically smaller. We now discuss a slight reformulation of this procedure.

Proposition 4.3. *Let x be a boundary point of the pseudoconvex domain Ω. Then the collection of subelliptic multipliers $\mathfrak{J}_x$ on* (0,1) *forms is a radical ideal. In particular,*

$$f \in \mathfrak{J}_x, |g|^N \leq f \quad \Rightarrow \quad g \in \mathfrak{J}_x. \tag{17}$$

When $m\varepsilon \leq 1$, we also have the estimate

$$|||g\phi|||_\varepsilon^2 \leq c|||g^m\phi|||_{m\varepsilon}^2 + c\|\phi\|^2 \tag{18}$$

Proposition 4.4. *Suppose that f is a subelliptic multiplier, and that*

$$|||f\phi|||_\varepsilon^2 \leq cQ(\phi,\phi) \tag{19}$$

for all appropriate ϕ and for $0 < \varepsilon \leq 1$. Then there is a constant $c > 0$ so that

$$|||\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\phi_j|||_{\varepsilon/2}^2 \leq cQ(\phi,\phi) \tag{20}$$

We will use Proposition 4.4 by augmenting the Levi matrix by adding the rows ∂f and the column $\bar\partial f$ in the same way we did this for ∂r and $\bar\partial r$. More precisely, suppose that $f_1, \dots, f_N$ are subelliptic multipliers. We define the $n+1+N$ by $n+1+N$ matrix $A(f)$ by

$$A(f) = \begin{pmatrix} \partial\bar\partial r & \bar\partial r & \bar\partial f_1 & \dots & \bar\partial f_N \\ \partial r & 0 & 0 & \dots & 0 \\ \partial f_1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \partial f_N & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{21}$$

PROPOSITION 4.5. *Suppose that f_i are subelliptic multipliers. Then the determinant of $A(f)$ is a subelliptic multiplier.*

Define I_0 to be the real radical of the ideal generated by r and the determinant of the Levi form $\det(\lambda)$. For $k \geq 1$, define I_k to be the real radical of the ideal generated by I_{k-1} and all determinants $\det(A(f))$ for $f_j \in I_{k-1}$.

By Proposition 4.2 we know that r and $\det(\lambda)$ are subelliptic multipliers. By Proposition 4.3 all the elements in I_0 are subelliptic multipliers. By Propositions 4.4 and 4.5, and induction, for each k all the elements of I_k are subelliptic multipliers. Thus a subelliptic estimate holds whenever 1 lies in some I_k.

DEFINITION 4.6. The point p in a pseudoconvex real hypersurface M is of *finite ideal-type* if there is an integer k such that $1 \in I_k$. (Equivalently I_k is the ring of germs of smooth functions at p.)

As for the subelliptic estimate, whether p is of finite ideal-type is independent of the Hermitian metric used. Next we prove directly that the existence of a complex analytic variety V in $\mathrm{b}\Omega$ prevents points on V from being of finite ideal-type. This theorem motivates Section 6.

THEOREM 4.7. *Suppose that Ω is pseudoconvex and that there is a complex analytic variety V lying in* $\mathrm{b}\Omega$. *Then points of V cannot be of finite ideal-type.*

PROOF. The condition of finite ideal-type is an open condition, so we may assume that p is a smooth point of V. We may find a non-zero vector field L that is tangent to V and is a holomorphic combination of the usual L_i. Then L is in the kernel of the Levi form along V, so $\det(\lambda)$ vanishes along V. Therefore all elements of I_0 vanish on V. We proceed by induction. Suppose that all elements of I_{k-1} vanish along V. Choosing $f_j \in I_{k-1}$ we have $L(f_j) = 0$ because L is tangent. Therefore the matrix whose entries are $L_i(f_j)$ must have a non-trivial kernel, and hence $\det(A(f))$ must vanish on V, and thus all elements of I_k vanish on V also. □

For real-analytic pseudoconvex domains, the sequence of ideals stabilizes after finitely many steps [Kohn 1979]. Either $1 \in I_k$ for some k, or the process uncovers a real-analytic real subvariety in the boundary of "positive holomorphic dimension". A CR submanifold of M has positive holomorphic dimension when it has a non-zero tangent vector field annihilated by the Levi form of M. Diederich and Fornaess [1978] then proved (assuming M is pseudoconvex and real-analytic) that a variety with positive holomorphic dimension can lie in M and pass through p only when there are complex-analytic varieties in the boundary passing through points arbitrarily close by. This is equivalent to the statement that there are no complex-analytic varieties in the boundary passing through p. See [D'Angelo 1993; 1991] for a proof of this last equivalence that applies without the hypothesis of pseudoconvexity. Conversely by Theorem 4.7 the estimate cannot hold when there is a complex-analytic variety passing through p and lying in the boundary.

This gives the result in the pseudoconvex real-analytic case.

THEOREM 4.8. *Let $\Omega \Subset \mathbb{C}^n$ be pseudoconvex, and suppose that its boundary is real-analytic near p. Then there is a subelliptic estimate at p on $(0,q)$ forms if and only if there is no germ of a complex-analytic variety of dimension q lying in $\mathrm{b}\Omega$ and passing through p.* (*and thus*, *in the language of Section* 6, *if and only if $\Delta_q(M,p)$ is finite*).

5. Finite Commutator-Type

The definition of finite commutator-type for a point p on a CR manifold involves only the CR structure. For imbedded hypersurfaces finite commutator-type is equivalent to regular $(n-1)$-type, namely, the order of tangency of every complex hypersurface with M at p is finite. See Section 6. For domains in $\mathbb{C}^2$, finite commutator-type, finite ideal-type, and finite D_1-type are equivalent conditions.

Suppose that $p \in M$, and that L is a local section of $T^{1,0}M$. We define the type of L at p by

$$t(L,p) = \min\{k : \text{there is a commutator } X = [\dots[L_1, L_2], \dots L_k] \text{ such that} \langle X, \eta\rangle(p) \neq 0\}.$$

In this definition each L_i equals either L or $\overline{L}$. Thus the type of a vector field at p equals two precisely when the Levi form $\lambda(L,\overline{L})(p)$ is non-zero. Taking higher commutators is closely related to but not precisely the same as taking higher derivatives of $\lambda(L,\overline{L})$ in the directions of L and $\overline{L}$. Because of the distinction it is worth introducing a related number. We define

$$c(L,p) = \min\{k : Y\langle[L,\overline{L}],\eta\rangle(p) \neq 0\},$$

where Y is a monomial differential operator $Y = \prod_{j=1}^{k-2} L_j$ and again each L_j equals either L or $\overline{L}$. Thus $c(L,p) = 2$ precisely when the Levi form $\lambda(L,\overline{L})(p)$ is non-zero. By computing higher commutators, we observe that some but not all of the terms arising are those in the definition of $c(L,p)$. For points in a CR manifold where the Levi form has eigenvalues of opposite sign, there are vector fields for which these numbers are different. It is believed to be true, but not proved in the literature, that these two numbers are the same for all vector fields in the pseudoconvex case. See [Bloom 1981; D'Angelo 1993] for what is known.

Next we define the commutator-type of a point on a CR manifold of hypersurface type.

Definition 5.1. The point p on a CR manifold M of hypersurface type is a point of finite commutator-type if $t(L,p)$ is finite for some local section L of $T^{1,0}M$. The commutator-type of p is the minimum of the types of all such $(1,0)$ vector fields L.

We next discuss some geometric aspects of this notion. For a 3-dimensional CR manifold such as a hypersurface in $\mathbb{C}^2$, the space $T_p^{1,0}(M)$ is 1-dimensional, so the types of all vector fields non-zero at p are the same. In this case we also have $t(L,p) = c(L,p)$ for all L and p. When the Levi form has $n-2>0$ positive eigenvalues and one vanishing eigenvalue on a pseudoconvex CR manifold of dimension $2n-1$, the minimum value of $t(L,p)$ is two, but furthermore there can be only one possible value for $t(L,p)$ other than 2 and again $t(L,p) = c(L,p)$ for all L and p. For real hypersurfaces, the commutator-type equals the maximum order of tangency of a complex hypersurface. The geometry becomes more complicated when the Levi form has several vanishing eigenvalues, and the types of vector fields give incomplete information. In particular the condition that all $(1,0)$ vector fields L satisfy $t(L,p) < \infty$ does not prevent complex-analytic varieties from lying in a hypersurface.

Remark. We discuss the geometric interpretation of the type of a single vector field. Suppose that M is a real hypersurface in $\mathbb{C}^n$ and that V is a complex manifold osculating M to order N at p. Then there is a $(1,0)$ vector field L with $L(p) \neq 0$ and $t(L,p) \geq N$. We may take L to be tangent to V. The converse is not generally true, but the first author believes that it may be true in the pseudoconvex case. We give an example due to Bloom [1981].

Example 5.2. Put $r(z,\bar{z}) = 2\operatorname{Re}(z_3) + (z_2 + \bar{z}_2 + |z_1|^2)^2$, and let M denote the zero set of r. Let p be the origin. Put $L_j = \partial/\partial z_j - r_{z_j}\partial/\partial z_3$ for $j=1,2$. In this case L_1 and L_2 form a global basis for sections of $T^{1,0}M$. We put $L = L_1 - \bar{z}_1 L_2$. Then $\lambda(L,\bar{L})(0) = 0$, and the iterated bracket $[[L,\bar{L}],L]$ vanishes identically. Consequently $t(L,p) = \infty$. On the other hand, it is easy to check that the maximum order of contact of a complex-analytic curve (whether singular or not) with M at p is 4; in the notation of the next section, $\Delta_1^{\mathrm{Reg}}(M,p) = \Delta_1(M,p) = 4$.

Singularities create a new difficulty. Suppose that $t(L,p)$ is finite for every local vector field that is non-zero at p. There may nevertheless be a complex *variety* lying in M and passing through p. Thus the notion of type of a vector field does not detect singularities.

Example 5.3. Put $r(z,\bar{z}) = 2\operatorname{Re}(z_3) + |f(z_1,z_2)|^2$ and let M denote its zero set. Here f is a holomorphic polynomial with $f(0,0)=0$. The complex subvariety of $\mathbb{C}^3$ defined by the vanishing of z_3 and f lies in M and passes through the origin. Depending on f we can exhibit several phenomena. Rather than giving a complete discussion, we choose several different f to illustrate the possibilities:

Consider the real hypersurfaces in $\mathbb{C}^3$ defined by $r(z,\bar{z}) = 2\,\mathrm{Re}(z_3) + |f(z_1,z_2)|^2$ when f is as follows:

1. $f(z_1,z_2) = z_1^m$
2. $f(z_1,z_2) = z_1^2 - z_2^3$
3. $f(z_1,z_2) = z_1 z_2$

The first hypersurface contains the complex manifold defined by $z_1 = z_3 = 0$. We detect it by commutators because the type of $L = \partial/\partial z_2$ is infinity. The second hypersurface contains an irreducible complex variety V that has a singularity at the origin. (The variety is not normal). All non-zero vector fields have type either 4 or 6 there. Consider the $(1,0)$ vector field defined by

$$L = 3z_1 L_1 + 2z_2 L_2 = 3z_1 \frac{\partial}{\partial z_1} + 2z_2 \frac{\partial}{\partial z_2} - 6|f(z_1,z_2)|^2 \frac{\partial}{\partial z_3}.$$

A simple calculation shows that L is tangent to V, has infinite type along V except at 0, but vanishes at 0. The third hypersurface contains a reducible complex variety W. Commutators detect this, because each irreducible branch is a complex manifold. These examples motivated the first author to express notions of finite-type directly in terms of orders of contact and the resulting commutative algebra.

6. Orders of Contact and Finite D_q-type

D'Angelo defined several numerical functions measuring the order of contact of possibly singular complex varieties of dimension q with a real hypersurface M. For each q with $1 \le q \le n-1$, we have the functions $\Delta_q(M,p)$ and $\Delta_q^{\mathrm{Reg}}(M,p)$. The first measures the maximum order of contact of all q-dimensional complex-analytic varieties, and the second measures the maximum order of contact of all q-dimensional complex manifolds. Catlin's necessary and sufficient condition for subellipticity for $(0,q)$ forms on a pseudoconvex domain is equivalent to $\Delta_q(M,p)$ being finite. Understanding these functions defining orders of contact requires some elementary commutative algebra. The idea is first to consider Taylor polynomials of the defining function to reduce to the algebraic case. The methods of [D'Angelo 1993; 1982] show how to express everything using numerical invariants of families of ideals of holomorphic polynomials. In this section we give the definition of these functions and state some of the geometric results known.

Suppose first that J is an ideal in the ring of germs of smooth functions at $p \in \mathbb{C}^n$. We wish to assign a numerical invariant called the order of contact to J that mixes the real and complex categories. Often J will be $I(M,p)$, the germs of smooth functions vanishing on a hypersurface M near p. A local defining function r for M at p then generates the principal ideal $I(M,p)$.

It is convenient for the definition to write $(\mathbb{C}^k, x)$ for the germ of $\mathbb{C}^k$ at the point x, and to write $z : (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$ when z is the germ of a holomorphic mapping with $z(0) = p$. To define the order of contact of J with such a z, we pull back the ideal J to one dimension. We write $\nu(z) = \nu_p(z)$ for the multiplicity of z; this is the minimum of the orders of vanishing of the mappings $t \to z_j(t) - p_j$. We write $\nu(z^*r)$ for the order of vanishing of the function $t \to r(z(t))$ at the origin. The ratio $\Delta(J, z) = \inf_{r \in J} \nu(z^*r)/\nu_p(z)$ is called the *order of contact* of J with the holomorphic curve z. Note that the germ of a curve z is non-singular if $\nu(z) = 1$. The crucial point is that we allow the curves to be singular. For a hypersurface we have the following definition.

Definition 6.1. The order of contact of (the germ at 0 of) a holomorphic curve z with the real hypersurface M at p is the number

$$\Delta(M, p, z) = \inf_{r \in I(M,p)} \frac{\nu(z^*r)}{\nu_p(z)}.$$

We can compute $\Delta(M, p, z)$ by letting r in the definition be a defining function; this gives the infimum.

There are several ways to generalize to singular complex varieties of higher dimension. Below we do this by pulling back to holomorphic curves after we have restricted to subspaces of the appropriate dimension. Thus we let $\phi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n$ be a linear embedding, and we consider the subset $\phi^*M \subset \mathbb{C}^{n-q+1}$. For generic choices of ϕ this will be a hypersurface; when it is not we work with ideals. We are now prepared to define the numbers $\Delta_q(M, p)$ and $\Delta_q^{\mathrm{Reg}}(M, p)$.

Definition 6.2. Let M be a smooth real hypersurface in $\mathbb{C}^n$. For each integer q with $1 \le q \le n$ we define $\Delta_q(M, p)$ and $\Delta_q^{\mathrm{Reg}}(M, p)$ as follows:

$$\Delta_1(M, p) = \sup_z \Delta(M, p, z),$$

where the supremum is taken over non-constant germs of holomorphic curves;

$$\Delta_q(M, p) = \inf_\phi \Delta_1(\phi^*M, p),$$

where the infimum is taken over linear imbeddings $\phi : \mathbb{C}^{n-q+1} \to \mathbb{C}^n$; and

$$\Delta_1^{\mathrm{Reg}}(M, p) = \sup_{z : \nu(z) = 1} \Delta(M, p, z),$$

where and the supremum is taken over the non-singular germs of holomorphic curves. The last expression is called the *regular order of contact*.

For $q = 1, \ldots, n-1$ we take the supremum over all germs $z : (\mathbb{C}^q, 0) \to (\mathbb{C}^n, p)$ for which $dz(0)$ is injective:

$$\Delta_q^{\mathrm{Reg}}(M, p) = \sup_z \inf_{r \in I(M,p)} \nu(z^*r)$$

We also put $\Delta_n(M,p) = \Delta_n^{\mathrm{Reg}}(M,p) = 1$ for convenience.

EXAMPLE 6.3. Put $r(z,\bar{z}) = \mathrm{Re}(z_4) + |z_1 z_2 - z_3^5|^2$, and let p be the origin. Note that the image of the map $(s,t) \to (s^5, t^5, st, 0)$ lies in M, but that its derivative is not injective at 0. This shows that $\Delta_2(M,p) = \infty$. On the other hand $\Delta_2^{\mathrm{Reg}}(M,0) = 10$; the map $(s,t) \to (s,0,t,0)$ for example gives the supremum. We have

$$(\Delta_4(M,0), \Delta_3(M,0), \Delta_2(M,0), \Delta_1(M,0)) = (1, 4, \infty, \infty),$$
$$(\Delta_4^{\mathrm{Reg}}(M,0), \Delta_3^{\mathrm{Reg}}(M,0), \Delta_2^{\mathrm{Reg}}(M,0), \Delta_1^{\mathrm{Reg}}(M,0)) = (1, 4, 10, \infty).$$

DEFINITION 6.4. Let M be a smooth real hypersurface in $\mathbb{C}^n$. The point $p \in M$ is of *finite D_q- type* if $\Delta_q(M,p)$ is finite. It is of *finite regular D_q-type* is $\Delta_q^{\mathrm{Reg}}(M,p)$ is finite.

One of the main geometric results is local boundedness for the function $p \to \Delta_q(M,p)$. This shows that finite D_q-type is an open non-degeneracy condition. The condition is also finitely determined. See [D'Angelo 1993] for a complete discussion of these functions.

THEOREM 6.5. *Let M be a smooth real hypersurface in $\mathbb{C}^n$. The function $p \to \Delta_q(M,p)$ is locally bounded; if p is near p_o then*

$$\Delta_q(M,p) \leq 2(\Delta_q(M,p_o))^{n-q}$$

Suppose additionally that M is pseudoconvex. For each q with $1 \leq q \leq n-1$ the function $p \to \Delta_q(M,p)$ satisfies the following sharp bounds: if p is near p_o then

$$\Delta_q(M,p) \leq \Delta_q(M,p_o)^{n-q}/2^{n-1-q}.$$

COROLLARY. *For each $q \geq 1$, the set of points of finite D_q-type is an open subset of M.*

The set of points of finite *regular* D_q-type is not generally open when $q < n-1$. See Example 5.3.2.

We remark also on additional information available in the real-analytic case [D'Angelo 1993; 1991; Diederich and Fornaess 1978] and sharper information in the algebraic case (when there is a polynomial defining equation) [D'Angelo 1983].

THEOREM 6.6. *Let M be a real-analytic real hypersurface in $\mathbb{C}^n$. Then either $\Delta_1(M,p)$ is finite or there is a 1-dimensional complex-analytic variety contained in M and passing through p. If M is compact, then the first alternative must hold.*

When the defining equation is a polynomial there is quantitative information depending only on the dimension and the degree of the polynomial [D'Angelo 1983].

Theorem 6.7. *Let M be a real hypersurface in $\mathbb{C}^n$ defined by a polynomial equation of degree d. Then either $\Delta_1(M,p) \le 2d(d-1)^{n-1}$ or there is a complex-analytic* 1*-dimensional variety contained in M and passing through p. Furthermore, there is an explicit way to find the defining equations of the complex variety directly from the defining equation for M.*

Theorems 6.5 and 6.7 rely upon writing real-valued polynomials as differences of squared norms of holomorphic mappings; it is easy to decide when the zero sets of such expressions contain complex analytic varieties. The method enables one to work in the category of holomorphic polynomials and to use elementary commutative algebra.

We mention briefly what this entails. We consider the ring of germs of holomorphic functions at a point and its maximal ideal $\mathcal{M}$. Saying that a proper ideal I of germs of holomorphic functions is primary to the maximal ideal $\mathcal{M}$ is equivalent to saying that its elements vanish simultaneously only at the origin (Nullstellensatz). It is then possible to assign numerical invariants that measure the singularity defined by the primary ideal, such as the order of contact, the smallest power of $\mathcal{M}$ contained in I, the codimension of I, etc. Inequalities among these invariants are crucial to the proofs of Theorems 6.5 and 6.7. Consider again the domains defined by (2); the origin is of finite D_1-type if and only if the ideal $(f_1, \dots f_N, z_n)$ is primary to $\mathcal{M}$. One sees that the passage from strongly pseudoconvex points to points of finite D_1-type precisely parallels the passage from the maximal ideal $\mathcal{M}$ to ideals primary to it.

7. Catlin's Multitype and Sufficient Conditions for Subelliptic Estimates

Catlin generalized Theorem 4.8 to the smooth case. In [Catlin 1983; 1984; 1987] he established that finite type is a necessary and sufficient condition for subellipticity on pseudoconvex domains. In most of this section we consider the results for $(0,1)$ forms.

Theorem 7.1. *Let $\Omega \Subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary. Then there is a subelliptic estimate at p if and only if $\Delta^1(\mathrm{b}\Omega, p) < \infty$. The parameter epsilon from Definition* 3.4 *must satisfy $\varepsilon \le \frac{1}{\Delta^1(\mathrm{b}\Omega,p)}$.*

We start by discussing the proof that finite type implies that subelliptic estimates hold. Catlin applies the method of weight functions used earlier by Hörmander [1966]. Rather than working with respect to Lebesgue measure dV, consider the measure $e^{-\Phi}\, dV$ where Φ will be chosen according to the needs of the problem. After this choice is properly made, one employs, as a substitute for Lemma 3.3, the inequality

$$\int_\Omega \sum_{i,j=1}^{n} \Phi_{z_i \bar{z}_j} a_i \bar{a}_j \, dV + \sum_{j,k=1}^{n} \|\bar{L}_j a_k\|^2 \le CQ(a,a), \tag{22}$$

where $|\Phi| \leq 1$. Here $\overline{L}_j$ are $(0,1)$ vector fields on $\mathbb{C}^n$. There could be also a term on the left side involving the boundary integral of the Levi form, but such a term does not need to be used in this approach to the estimates. Instead, one needs to choose Φ with a large Hessian. One step in Catlin's proof is the following reduction:

Theorem 7.2. *Suppose that $\Omega \Subset \mathbb{C}^n$ is a pseudoconvex domain defined by $\Omega = \{r < 0\}$, and that $p \in \mathrm{b}\Omega$. Let U be a neighborhood of p. Suppose that for all $\delta > 0$ there is a smooth real-valued function Φ_δ satisfying the properties*:

$$\begin{aligned} |\Phi_\delta| &\leq 1 && \text{on } U, \\ (\Phi_\delta)_{z_i \bar{z}_j} &\geq 0 && \text{on } U, \\ \sum_{i,j=1}^{n} (\Phi_\delta)_{z_i \bar{z}_j} a_i \bar{a}_j &\geq c\frac{|a|^2}{\delta^{2\varepsilon}} && \text{on } U \cap \{-\delta < r \leq 0\}. \end{aligned} \tag{23}$$

Then there is a subelliptic estimate of order ε at p.

Theorem 7.2 reduces the problem to constructing such bounded smooth plurisubharmonic functions whose Hessians are at least as large as $\delta^{-2\varepsilon}$. One of the crucial ingredients is the use of an n-tuple of rational numbers ($+\infty$ is also allowed) called the multitype. This n-tuple differs from both the n-tuples of orders of contact or of regular orders of contact. There are inequalities in one direction; in simple geometric situations there may be equality. Later we mention the work of Yu in this direction.

We give some motivation for the use of the multitype. Suppose that $W \subset M$ is a manifold of holomorphic dimension zero. Recall that W is a CR submanifold of M, and that the Levi form for M does not annihilate any $(1,0)$ vector fields tangent to W. It follows from the discussion in Section 3 that the distance d_W is a subelliptic multiplier. Suppose that we have a subelliptic estimate away from W. We then obtain a subelliptic estimate (with a smaller epsilon) on W as well, because d_W is a subelliptic multiplier. Hence manifolds of holomorphic dimension zero are small sets as far as the estimates are concerned. This suggests a stratification of M.

Suppose now that p is a point of finite D_q-type, and that U is a neighborhood of p in $\mathrm{b}\Omega$ where $\Delta_q(M,p) \leq 2(\Delta_q(M,p_o))^{n-q}$. Catlin defines the multitype as an n-tuple of rational numbers, and shows that it assumes only finitely many values in U. The stratification is then given by the level sets of the multitype function. Catlin proves that each such level set is locally contained in a manifold of holomorphic dimension at most $q-1$. Establishing the properties of the multitype is difficult, and involves showing that the multitype equals another n-tuple called the commutator type. The commutator type generalizes the notions of Section 3. See [Catlin 1984] for this material.

We next define the multitype. Let $\mu = (\mu_1, \ldots, \mu_n)$ be an n-tuple of numbers (or plus infinity) with $1 \leq \mu_j \leq \infty$ and such that $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$. We demand that, whenever μ_k is finite, we can find integers n_j so that

$$\sum_{j=1}^{k} \frac{n_j}{\mu_j} = 1.$$

We call such n-tuples weights, and order them lexicographically. Thus, for example, $(1, 2, \infty)$ is considered smaller than $(1, 4, 6)$. A weight is called distinguished if we can find local coordinates so that p is the origin and such that

$$\sum_{j=1}^{n} \frac{a_j + b_j}{\mu_j} < 1 \;\Rightarrow\; D^a \overline{D}^b r(0) = 0, \tag{24}$$

where $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are multi-indices. The multitype $m(p)$ is the smallest weight that dominates (in the lexicographical ordering) every distinguished weight μ. In some sense we are assigning weights m_j to the coordinate direction z_j and measuring orders of vanishing. The following statements are automatic from the definition. If the Levi form has rank $q-1$ at p, then $m_j(p) = 2$ for $2 \le j \le q$. In general $m_1(p) = 1$, and $m_2(p) = \Delta_{n-1}(M, p) = \Delta_{n-1}^{\mathrm{Reg}}(M, p)$.

Example 7.3. Let M be the hypersurface in $\mathbb{C}^3$ defined by

$$r(z, \bar{z}) = 2\,\mathrm{Re}(z_3) + |z_1^2 - z_2^3|^2.$$

The multitype at the origin is $(1, 4, 6)$ and $(\Delta_3(M, 0), \Delta_2(M, 0), \Delta_1(M, 0)) = (1, 4, \infty)$. Thus a finite multitype at p does not guarantee that p is of finite D_1-type. At points of the form $(t^3, t^2, 0)$ for a non-zero complex number t, the multitype will be $(1, 2, \infty)$. This illustrates the upper semicontinuity of the multitype in the lexicographical sense, because $(1, 2, \infty)$ is smaller than $(1, 4, 6)$.

Catlin proved that the multitype on a pseudoconvex hypersurface is upper semicontinuous in this lexicographical sense. He also proved the collection of inequalities given, for $1 \le q \le n$, by

$$m_{n+1-q}(p) \le \Delta_q(M, p). \tag{25}$$

Yu [1994] defined a point to be *h-extendible* if equality in (25) holds for each q. This class of boundary points exhibits simpler geometry than the general case. Yu proved that convex domains of finite D_1-type are h-extendible, after McNeal [1992] had proved for convex domains with boundary M that $\Delta_1(M, p) = \Delta_1^{\mathrm{Reg}}(M, p)$. Yu then gave a nice application, that h-extendible boundary points must be peak points for the algebra of functions holomorphic on the domain and continuous up to the boundary. McNeal applied his result to the boundary behavior of the Bergman kernel function on convex domains.

Sibony has studied the existence of strongly plurisubharmonic functions with large Hessians as in Theorem 7.2. He introduced the notion that a compact subset $X \subset \mathbb{C}^n$ be *B-regular*. The intuitive idea is that such a subset is B-regular when it contains no analytic structure in a certain strong sense. Sibony has given several equivalent formulations of this notion; one is that the algebra of continuous functions on X is the same as the closure of the algebra of continuous plurisubharmonic functions defined near the set. Another equivalence is the existence, given a real number s, of a plurisubharmonic function, defined near X and bounded by unity, whose Hessian has minimum eigenvalue at least s everywhere on X. Catlin proved for example that a submanifold of holomorphic dimension 0 in a pseudoconvex hypersurface is necessarily B-regular. See [Sibony 1991] for considerable discussion of B-regularity and additional applications.

8. Necessary Conditions and Sharp Subelliptic Estimates

We next discuss necessity results for subelliptic estimates. Greiner [1974] proved that finite commutator-type is necessary for subelliptic estimates in two dimensions. Rothschild and Stein [1976] proved in two dimensions that the largest possible value for ε is the reciprocal of $t(L,p)$, where L is any $(1,0)$ vector field that doesn't vanish at p. In higher dimensions finite commutator-type does not guarantee a subelliptic estimate on $(0,1)$ forms. Furthermore, Example 5.3.2 shows that $t(L,p)$ can be finite for every $(1,0)$ vector field L while subelliptic estimates fail.

Although finite D_1-type is necessary and sufficient, an example of D'Angelo shows that one cannot in general choose epsilon as large as the reciprocal of the order of contact [D'Angelo 1982; 1980]. The result is very simple. The function $p \to \Delta^1(\mathrm{b}\Omega, p)$ is not in general upper semicontinuous, so its reciprocal is not lower semicontinuous. Definition 3.4 reveals that, if there is a subelliptic estimate of order epsilon at one point, then there also is one at nearby points. Catlin has shown that the parameter value cannot be determined by information based at one point alone [Catlin 1983]. Nevertheless Theorem 6.5 shows that the condition of finite type does propagate to nearby points. This suggests that one can always choose epsilon as large as $\varepsilon = 2^{n-2}/(\Delta^1(\mathrm{b}\Omega, p))^{n-1}$. A more precise conjecture is that we may always choose epsilon as large as $\varepsilon = 1/B(\mathrm{b}\Omega, p)$. The denominator is the "multiplicity" of the point, defined in [D'Angelo 1993], where it is proved that the function $p \to B(M,p)$ is upper semicontinuous.

Determining the precise largest value for ε seems to be difficult. See Example 8.1 and Proposition 8.3 below. An example from [D'Angelo 1995] considers domains of the form (2), where for $j = 1, \dots, n-1$ the functions f_j are arbitrary Weierstrass polynomials of degree m_j in z_j that depend only on $(z_1, \dots, z_j)$. The multiplicity in this case is $B(\mathrm{b}\Omega, p) = 2\prod m_j$. It is possible, using the method of subelliptic multipliers, to obtain a value of epsilon that works uniformly over all such choices of Weierstrass polynomials and depends only upon these exponents. The result is much smaller than the reciprocal of the multiplicity.

We next illustrate the difficulty in obtaining sharp subelliptic estimates. The existence of the estimate at a point can be decided by examining a finite Taylor polynomial of the defining function there, because finite D_1-type is a finitely determined condition. This Taylor polynomial does not determine the sharp value of epsilon. Suppose that l, m are integers with $m \geq l \geq 2$.

Example 8.1. Consider the pseudoconvex domain, defined near the origin, by the function r, where

$$r(z, \bar{z}) = 2\,\mathrm{Re}(z_3) + |z_1^2 - z_2 z_3^l|^2 + |z_2|^4 + |z_1 z_3^m|^2.$$

We have $\Delta_1(\mathrm{b}\Omega, 0) = 4$ and $B(\mathrm{b}\Omega, 0) = 8$. Catlin [1983] proved that the largest ε for which there is a subelliptic estimate in a neighborhood of the origin equals $\frac{m+2l}{4(2m+l)}$. This number takes on values between $\frac{1}{4}$ (when $m = l$) and $\frac{1}{8}$. This information supports the conjecture that the value of the largest ε satisfies

$$\frac{1}{B(\mathrm{b}\Omega, 0)} \leq \varepsilon \leq \frac{1}{\Delta_1(\mathrm{b}\Omega, 0)}.$$

In order to avoid singularities and obtain precise results, Catlin [1983] considers families of complex manifolds. Suppose that T is a collection of positive numbers whose limit is 0. For each $t \in T$ we consider a biholomorphic image $M_t = g_t(B_t)$ of the ball of radius t about 0 in $\mathbb{C}^q$. We suppose that the derivatives dg_t satisfy appropriate uniformity conditions. In particular we need certain q by q minor determinants of dg_t to be uniformly bounded away from 0. We may then pull back r to g_t and define the phrase "The order of contact of the family M_t with $\mathrm{b}\Omega$ at the origin is at least η" by decreeing that $\sup |g_t^* r| \leq C t^\eta$.

Catlin proved the following precise necessity result.

Theorem 8.2. *Suppose that* $\mathrm{b}\Omega$ *is smooth and pseudoconvex and that there is a family* M_t *of* q*-dimensional complex manifolds whose order of contact with* $\mathrm{b}\Omega$ *at a boundary point* p *is at least* η. *If there is a subelliptic estimate at* p *on* $(0, q)$ *forms for some* ε, *then* $\varepsilon \leq \frac{1}{\eta}$.

Catlin has also proved the following unpublished result.

Proposition 8.3. *Suppose that* ε *is a real number with* $0 < \varepsilon \leq \frac{1}{4}$. *Then there is a smooth pseudoconvex domain in* $\mathbb{C}^3$ *such that a subelliptic estimate holds with parameter* ε *but for no larger number. If* ε *is a rational number in this interval, then there is a pseudoconvex domain in* $\mathbb{C}^3$ *with a polynomial defining equation such that a subelliptic estimate holds with parameter* ε *but for no larger number.*

Sketch of proof. Consider the pseudoconvex domain Ω defined by the following generalization of Example 8.1. We suppose that f, g are holomorphic functions, vanishing at the origin, and satisfying $|g(z)| \leq |f(z)|$. We put

$$r(z, \bar{z}) = 2\,\mathrm{Re}(z_3) + |z_1^m - z_2 f(z_3)|^2 + |z_2|^{2m} + |g(z_3) z_2|^2.$$

It is easy to see that $\Delta_1(\mathrm{b}\Omega, p) = 2m$ and that $B(\mathrm{b}\Omega, p) = 2m^2$ no matter what the choices of f, g are. It is possible to explicitly compute the largest possible value of the parameter ε in many cases. By putting $f(z_3) = z_3^p$ and $g(z_3) = z_3^q$ one can show that $\varepsilon = (q + p(m-1))/(2m^2 q)$, exhibiting the entire range of rational numbers between the reciprocals of the the D_1-type $2m$ and the multiplicity $2m^2$. To see that ε is at most this number one considers the family of complex one-dimensional manifolds M_t defined by the parametric curves $\zeta \to (\zeta,\, \zeta^m/(it)^p,\, it)$ for $\zeta \in \mathbb{C}$ satisfying $|\zeta| \leq |t|^\alpha$ for some exponent α. By choosing α appropriately one can compute the contact of this family of complex manifolds, and obtain $\varepsilon \leq (q + p(m-1))/(2m^2 q)$. To show that equality holds one must construct explicitly the functions needed in Theorem 7.2. More complicated choices of f, g enable us to obtain any real number in this range. □

Remark. Catlin has made other choices of f, g in the examples from Proposition 8.3 to draw a remarkable conclusion. For any η with $0 < \eta \leq \frac{1}{4}$, there is a smoothly bounded pseudoconvex domain in $\mathbb{C}^3$ such that a subelliptic estimate holds for all ε less than η, but not for η.

9. Domains in Manifolds

Suppose now that X is a complex manifold with Hermitian metric g_{ij}. Let Ω be a pseudoconvex domain in X with compact closure and smooth boundary. We still have the notions of defining function, vector fields and forms of type (1,0) as before. We have $|\partial r|^2 = \sum g_{ij} r_{z_i} r_{\bar{z}_j}$ in a local coordinate system. In a small neighborhood U of a point $p \in \mathrm{b}\Omega$ we suppose that $\omega_1, \ldots \omega_n$ form an orthonormal basis for the (1,0) forms. We may suppose that $\omega_n = (\partial r)/|\partial r|$. Let $\{L_i\}$ be a basis of (1,0) vector fields dual to $\{\omega_j\}$ in U. We can write a (0,1)-form ϕ as $\phi = \sum \phi_i \bar{\omega}_i$. When u is a function we have $\bar{\partial} u = \sum \bar{L}_i(u) \overline{w}_i$. Applying $\bar{\partial}$ to ϕ we have

$$\bar{\partial}\phi = \sum_{i<j} \bar{L}_i(\phi_j)\overline{w}_i \wedge \overline{w}_i + \sum \phi_i \bar{\partial}\bar{\omega}_i.$$

Suppose now that ϕ is supported in U, and that $\phi \in \mathcal{D}(\bar{\partial}^*)$. We can write

$$(\bar{\partial}^* \phi, u) = \sum \bar{L}_i{}^* \phi_i, u + \sum \int_{\mathrm{b}\Omega} L_i(r)\phi_i \bar{u}\, dS. \tag{26}$$

We have $L_i(r) = 0$ unless $i = n$. From (26) we see that the boundary condition is given by $\phi_n = 0$, and that

$$\bar{\partial}^* \phi = -\sum L_i \phi_i + \sum a_i \phi_i$$

for smooth functions a_i.

Following the proof of Lemma 3.3, and absorbing terms appropriately we obtain the *basic estimate*. Note that we require $\|\phi\|^2$ on the right hand side:

$$\sum_{i,j} \|\bar{L}_i \phi_j\|^2 + \sum_{i,j=1}^{n} \int_{b\Omega} \lambda_{ij} \phi_i \bar{\phi}_j dS \leq C(Q(\phi, \phi) + \|\phi\|^2)$$

Recall our earlier remark that, when $X = \mathbb{C}^n$, we can estimate $\|\phi\|^2 \leq CQ(\phi, \phi)$. This implies that the space of harmonic (0,1) forms $\mathcal{H}^{0,1}$ consists of 0 alone. For a general Hermitian manifold X, this will not be true.

The definition of the tangential Sobolev norms in the manifold setting uses partitions of unity. Assuming this definition, suppose that Ω is a domain in a Hermitian complex manifold X. We say that the $\bar{\partial}$-Neumann problem satisfies a subelliptic estimate at $p \in b\Omega$ if, for a sufficiently small neighborhood U of p, there are positive constants C, ε so that

$$|||\phi|||_{\varepsilon}^2 \leq C(\|\bar{\partial}\phi\|^2 + \|\bar{\partial}^*\phi\|^2) \tag{27}$$

for every (0,1)-form ϕ that is smooth, compactly supported in U, and in $\mathcal{D}(\bar{\partial})$. Note that (8) holds for ϕ supported in sufficiently small neighborhoods, so we do not require putting $\|\phi\|^2$ on the right side of (27).

Suppose that ϕ^ν is a bounded sequence in the $|||\phi|||_\varepsilon$ norm. Then there is a convergent subsequence in L^2. In other words, the inclusion mapping is a compact operator. Hence the harmonic space $\mathcal{H}^{0,1}$ is finite-dimensional. Furthermore harmonic forms are smooth on $\bar{\Omega}$. Finally we have the usual Hodge decomposition. See [Kohn and Nirenberg 1965] for the details.

10. Domains That Are Not Pseudoconvex

Suppose now that Ω is a domain in $\mathbb{C}^n$ with smooth but not pseudoconvex boundary. Let λ denote the Levi form, considered at each boundary point p as a linear transformation from $T_p^{10}(M)$ to itself. We write $\mathrm{Tr}(\lambda)$ for the trace of this linear mapping. (Since the Levi form is defined up to a multiple, the trace is also defined up to a multiple.) We write Id for the identity operator on $T_p^{10}(M)$.

For a point $p \in b\Omega$, we consider two possible positivity conditions.

Condition 1 (Pseudoconvexity). There is a neighborhood of p on which $\lambda \geq 0$.
Condition 2. There is a neighborhood of p on which $\lambda \geq \mathrm{Tr}(\lambda)\,\mathrm{Id}$.

In case 1 holds we have already defined finite ideal-type and seen that finite ideal-type implies that a subelliptic estimate holds. We now define ideals of subelliptic multipliers in case condition 2 holds. (See [Kohn 1985]).

We let J_0 be the real radical of the ideal generated by a defining function r and by the determinant of the mapping $\lambda - \mathrm{Tr}(\lambda)\,\mathrm{Id}$. Given a collection of functions $f = f_1, \ldots, f_N$ we define a linear transformation $B(f)$ on $T^{10}(M)$ and corresponding Hermitian form by

$$\langle B(f)\zeta,\zeta\rangle = \langle(\lambda - \mathrm{Tr}(\lambda)\,\mathrm{Id})\zeta,\zeta\rangle + \sum_{j=1}^{N}(|\bar\partial_b f_j\otimes\zeta|^2 - |\langle\bar\partial_b f_j,\bar\zeta\rangle|^2).$$

In coordinates we have

$$\sum_{m,l} B_{ml}(f)\zeta_m\bar\zeta_l = \sum_{i,j}\lambda_{ij}\zeta_i\bar\zeta_j - \mathrm{Tr}(\lambda)\sum_i|\zeta_i|^2 + \sum_{i,j,k}|\overline{L}_j(f_k)\zeta_i - \overline{L}_i(f_k)\zeta_j|^2.$$

When condition 2 holds we define the ideals J_k inductively. We let J_k be the real radical of the ideal generated by J_{k-1} and the determinants of all matrices $B(f)$ for $f_j\in J_{k-1}$. When condition 2 holds we say that p is of *finite ideal-type* if there is an integer k for which $1\in J_k$, that is, the ideal J_k is the full ring of germs of smooth functions at p.

Proposition 10.1. *Suppose that condition* 2 *holds*, *and that* p *is of finite ideal-type. Then there is a subelliptic estimate in the* $\bar\partial$*-Neumann problem on* $(0,1)$ *forms*.

Proof. We begin with some lemmas.

Lemma 10.2. *Suppose that* i,j *are less than* n. *Then*

$$\|\overline{L}_i(\phi_j)\|^2 = \|L_i(\phi_j)\|^2 - \int_{b\Omega}\lambda_{ii}|\phi_j|^2 dS + 0(\|\phi_j\|\sum_{k<n}\|L_k\phi_j\| + \|\overline{L}_n\phi_j\|^2) + \|\phi_j\|^2) \tag{28}$$

Sketch of proof. Begin with $\|\overline{L}_i(\phi_j)\|^2 = (\overline{L}_i(\phi_j),\overline{L}_i(\phi_j))$ and integrate by parts twice using Stokes's theorem. At one point write $\overline{L}_iL_i = L_i\overline{L}_i - [L_i,\overline{L}_i]$. Then note that the T component of $[L_i,\overline{L}_i]$ equals λ_{ii}, and integrate the term containing this by parts again to get a boundary integral of $\lambda_{ii}|\phi_j|^2$. The other terms get estimated by the Schwartz inequality. □

Lemma 10.3. *There is a positive constant* C *such that*, *for smooth* $\phi\in\mathcal{D}(\bar\partial^*)$,

$$\sum_{i,j<n}\|L_i(\phi_j)\|^2 + \sum_j\|\overline{L}_n\phi_j\|^2 + \sum_{i,j=1}^{n}\int_{b\Omega}\lambda_{ij}\phi_i\bar\phi_j\,dS - \int_{b\Omega}\mathrm{Tr}(\lambda)|\phi|^2\,dS \le C(Q(\phi,\phi)+\|\phi\|^2).$$

Proof. Take the sum over i,j in (28) and substitute in the basic estimate. Estimate the other terms by the small constant large constant trick. □

To finish the proof of Proposition 10.1, we suppose that f is a subelliptic multiplier, so that $|||f\phi|||_\varepsilon^2 \le CQ(\phi,\phi)$. Next we verify that

$$\sum_{i,j<n}|||\overline{L}_i(f)\phi_j - \overline{L}_j(f)\phi_i|||^2_{\frac{\varepsilon}{2}} \le CQ(\phi,\phi).$$

This inequality is *dual* to the estimate of Proposition 4.4. Suppose that f is a subelliptic multiplier. Given any Hermitian form $W(\phi,\phi)$ whose determinant is a subelliptic multiplier, we form a new form W' defined by

$$W'(\phi,\phi) = W(\phi,\phi) + \sum_{i,j<n} \|\overline{L}_i(f)\phi_j - \overline{L}_j(f)\phi_i\|^2.$$

As before we see that the determinant of the coefficient matrix of W' is also a subelliptic multiplier.

Proposition 10.1 follows by iterating this operation. □

Ho [1991] has proved sharp subelliptic estimates on $(0,n-1)$ forms at p for domains that are not pseudoconvex, under the following assumption: there is a $(1,0)$ vector field L for which $t(L,p)$ is finite and for which $\lambda(L,\overline{L}) \geq 0$ near p.

11. A Result for CR Manifolds

We next study subellipticity on a pseudoconvex CR manifold M of hypersurface type and of dimension $2n-1$. (See [Kohn 1985].) We replace $\bar\partial$ by the tangential Cauchy–Riemann operator $\bar\partial_b$ and the quadratic form Q by Q_b defined by

$$Q_b(\phi,\phi) = (\bar\partial_b\phi, \bar\partial_b\phi) + (\bar\partial_b^*\phi, \bar\partial_b^*\phi).$$

We say that Q_b is subelliptic at p if there is a neighborhood U of p and positive constants C,ε such that

$$\|\phi\|_\varepsilon^2 \leq CQ_b(\phi,\phi)$$

for all smooth forms supported in U.

As before we suppose that the L_i, for $i=1,\dots,n-1$, form a local basis for $T^{1,0}M$ and that $L_1,\dots,L_{n-1},\overline{L}_1,\dots,\overline{L}_{n-1},T$ form a local basis for $\mathbb{C}TM$. We also assume that $\overline{T}=-T$. Definition 2.1 of the Levi form shows that its components λ_{ij} with respect to this local basis are equal to the T coefficient of $[L_i,\overline{L}_j]$. Since M is assumed to be pseudoconvex we may choose the signs so that λ is positive semi-definite. We also define the matrix $\beta=(\beta_{ij})$ by $\beta = \operatorname{Tr}(\lambda)\operatorname{Id} - \lambda$. Recall that condition 2 from Section 10 is that β is negative semi-definite. Both λ and β are size $n-1$ by $n-1$. A simple inequality holds when $n>2$.

LEMMA 11.1. *Suppose that* $n>2$. *Then* $\det(\beta) \geq \det(\lambda)$.

PROOF. For completeness we first observe that $\beta=0$ when $n=2$, and the inequality fails. When $n=3$ the two determinants are equal. Otherwise we suppose that we are working at one point, and that λ is diagonal. We may suppose that the eigenvalues of λ satisfy $0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$. We then have

$$\det(\lambda) = \prod \lambda_j \leq \lambda_{n-2}(\lambda_{n-1})^{n-2}.$$

We have $\beta_j = \mathrm{Tr}(\lambda) - \lambda_j = \sum_{k \neq j} \lambda_k$. Since all the λ_j are non-negative, we can drop terms and easily obtain

$$\det(\beta) = \prod \beta_j \geq \lambda_{n-2}(\lambda_{n-1})^{n-2},$$

and the result follows. □

Proceeding as in Section 10 we obtain the two basic estimates

$$\sum \|\bar{L}_i \phi_j\|^2 + \sum (\lambda_{ij} T\phi_i, \phi_j) \leq Q_b(\phi, \phi) + 0(\|\phi\|^2 + \|\phi\| \sum \|\bar{L}_k \phi_j\|),$$
$$\sum \|L_i \phi_j\|^2 - \sum (\beta_{ij} T\phi_i, \phi_j) \leq Q_b(\phi, \phi) + 0(\|\phi\|^2 + \|\phi\| \sum \|L_k \phi_j\|).$$

Proposition 11.2. *Suppose that $n > 2$. Let $\lambda = (\lambda_{ij})$ be the Levi matrix with respect to the local basis $\{L_1, \dots, L_{n-1}\}$ of $T^{1,0}(M)$. Then there is a constant C so that, for all smooth ϕ supported in U,*

$$(\det(\lambda)\Lambda^{1/2}\phi, \Lambda^{1/2}\phi) \leq CQ_b(\phi, \phi).$$

Proof. We need to microlocalize the two basic estimates. We suppose that we are working in a coordinate neighborhood U of a point p, where our coordinates are denoted by $x_1, \dots, x_{2n-2}, t$. We may assume that these coordinates have been chosen so that, at p, we have $T = (1/\sqrt{-1})(\partial/\partial t)$ and $L_j = \frac{1}{2}(\partial/\partial x_{2j-1} - \sqrt{-1}\,\partial/\partial x_{2j})$.

Let $\xi_1, \dots, \xi_{2n-2}, \tau$ denote the dual coordinates in the Fourier transform space. We may also assume that $T = (1/\sqrt{-1})\partial/\partial t$ in the full neighborhood.

Suppose that u is smooth and supported in U. Write

$$u = u^+ + u^- + u^0,$$

where $\hat{u}^+$ is supported in a conical neighborhood of 0 with $\tau > 0$, $\hat{u}^-$ is supported in a conical neighborhood of 0 with $\tau < 0$, and u^0 is supported outside of such neighborhoods.

Since Q_b is elliptic on the support of $\hat{u}^0$, we have the estimate

$$\|\det(\lambda)\phi^0\|_{\frac{1}{2}}^2 \leq C\|\phi^0\|_1^2 \leq CQ_b(\phi, \phi)$$

By Gårding's inequality we have the estimates

$$(\det(\lambda)T\phi^+, \phi^+) \geq -c\|\phi\|^2, \qquad (\det(\beta)T\phi^-, \phi^-) \geq -c\|\phi\|^2.$$

We also have

$$(\det(\lambda)T\phi^+, \phi^+) = (\det(\lambda)\Lambda^{1/2}\phi^+, \Lambda^{1/2}\phi^+) + \cdots,$$
$$(\det(\beta)T\phi^-, \phi^-) = (\det(\beta)\Lambda^{1/2}\phi^-, \Lambda^{1/2}\phi^-) + \cdots,$$

Here the dots denote error terms. Using the basic estimates we obtain

$$\sum_{i,j=1}^{n-1} (\lambda_{ij}\Lambda^{1/2}\phi_i^+, \Lambda^{1/2}\phi_j^+) \leq CQ_b(\phi, \phi)$$

and

$$\sum_{i,j=1}^{n-1}(b_{ij}\Lambda^{1/2}\phi_i^-, \Lambda^{1/2}\phi_j^-) \le CQ_b(\phi,\phi).$$

Combining the separate estimates for ϕ^0, ϕ^+, and ϕ^- and adding gives Proposition 11.2. □

As before we augment the Levi form. Suppose that $f_1,\dots,f_N$ are subelliptic multipliers. We form the matrix

$$A(f) = \begin{pmatrix} \lambda & \bar\partial_b f_1 & \dots & \bar\partial_b f_N \\ \partial_b f_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_b f_N & 0 & \dots & 0 \end{pmatrix}.$$

Similarly we form matrices $B(f)$ as in Section 10. This gives us sequences of ideals I_k and J_k of germs of smooth functions. Note that the inequality from Lemma 11.1 gives $I_0 \subset J_0$. A simple induction then shows that

$$\det(A(f_1,\dots,f_n)) \le \det(B(f_1,\dots,f_n))$$

Therefore if $n > 2$ and if $1 \in I_k$ for some k, we also have $1 \in J_k$. We obtain the following result.

THEOREM 11.3. *Suppose that M is a pseudoconvex CR manifold of dimension $2n-1$ and of hypersurface type. If $n > 2$, and $1 \in I_k$ for some k, a subelliptic estimate holds.*

REMARK. We mentioned earlier the asymmetry between the L's and the $\bar L$'s in Lemma 3.3. For a CR manifold we eliminate this asymmetry by obtaining two basic estimates, one for the $\bar L$'s using λ and one for the L's using β.

12. Hölder and L^p estimates for $\bar\partial$

Optimal Hölder estimates for $\bar\partial$ and estimates for the Bergman projection and kernel function are known only in two dimensions and in some special cases. See for example [Christ 1988; Chang et al. 1992; McNeal 1989; Nagel et al. 1989; McNeal and Stein 1994]. In the elliptic case Hölder estimates are equivalent to elliptic estimates, but Hölder estimates do not necessarily hold for subelliptic operators. See [Guan 1990] for examples of second order subelliptic operators for which Hölder regularity fails completely.

As before we wish to solve the equation $\bar\partial u = \alpha$, where u and α are in $L^2(\Omega)$. We set Lip(0) to be the set of bounded functions, and, for $0 < s < 1$ we let Lip(s) denote the space of functions u satisfying a Hölder estimate $|u(x) - u(y)| \le C|x-y|^s$. We extend the definition to all real s inductively by applying the definition to first derivatives.

Let p be a boundary point of commutator-type m. Suppose that $\zeta\alpha \in \mathrm{Lip}(s)$ for all smooth cut-off functions ζ for some s. Let u denote the $\bar{\partial}$-Neumann solution to $\bar{\partial}u = \alpha$, so u is orthogonal to the holomorphic functions. Fefferman and Kohn showed that $\zeta u \in \mathrm{Lip}(s + \frac{1}{m})$ for all smooth cut-off functions ζ. This result requires that $s + \frac{1}{m}$ not be an integer, although they obtain the corresponding result in that case as well, by giving a different definition of the Lipschitz spaces for integer values of s. Write $\mathrm{LIP}(s)$ for this class of spaces; for s not an integer $\mathrm{Lip}(s) = \mathrm{LIP}(s)$. They proved also that both the Bergman and Szegő projection preserve Lipschitz spaces.

For bounded pseudoconvex domains in $\mathbb{C}^n$ the range of $\bar{\partial}_b$ in L^2 is closed; see [Kohn 1986; Shaw 1985]. For general CR manifolds (even in the strongly pseudoconvex 3-dimensional case) it is required to assume this. Given the assumption of closed range, all these results follow from the analysis of a second-order pseudodifferential operator A on $\mathbb{R}^3$. Fefferman and Kohn [1988] prove that solutions u They prove that solutions u to the equation $Au = f$ lie in $\mathrm{LIP}(s + \frac{2}{m})$ when $f \in \mathrm{LIP}(s)$ near a point of commutator-type m. See [Fefferman 1995] for a discussion of the operator A and the techniques of microlocal analysis needed. The techniques also work [Fefferman et al. 1990] in the restricted case in higher dimensions where the Levi form is smoothly diagonalizable.

There are many special cases where estimates for the Bergman and Szegő projections and L^p estimates for $\bar{\partial}$ have been proved by other methods. Fornaess and Sibony [1991] construct a smoothly bounded pseudoconvex domain such that, for certain $\alpha \in L^p$ with $p > 2$, the equation $\bar{\partial}u = \alpha$ has no solution in $L^{p'}$ for all p' in a certain range of values $\leq p$. They also prove a positive result for Runge domains. Chang, Nagel and Stein [Chang et al. 1992] give precise estimates in various function spaces for solutions of $\bar{\partial}u = \alpha$ on domains of finite commutator-type in $\mathbb{C}^2$. See [McNeal and Stein 1994] for estimates (Sobolev, Lipschitz, anisotropic Lipschitz) on convex domains of finite type in arbitrary dimensions. We do not discuss these results here.

13. Brief Discussion of Related Topics

Many different finite-type conditions arise in complex analysis. Here we briefly describe some situations where precise theorems are known in various finite-type settings. The reader should consult the bibliographies in the papers we mention for a complete overview.

Hans Lewy [1956] first studied the extension of CR functions from a strongly pseudoconvex real hypersurface. After work by many authors, usually involving commutator finite type, Trepreau [1986] established that every germ of a CR function at a point p extends to one side of the hypersurface M if and only if there is no germ of a complex hypersurface passing through p and lying in M. Tumanov [1988] introduced the concept of minimality that gives necessary and sufficient conditions for the holomorphic extendability to wedges of CR functions defined on generic CR manifolds of higher codimension.

Baouendi, Treves and Jacobowitz [Baouendi et al. 1985] introduced the notion of *essentially finite* for a point on a real-analytic hypersurface. It is a sufficient condition in order that the germ of a CR diffeomorphism between real analytic real hypersurfaces must be a real-analytic mapping. Using elementary commutative algebra, one can extend the definition of essential finiteness to points on smooth hypersurfaces [D'Angelo 1987] and show that the set of such points is an open set. Furthermore, if p is of finite D_1-type, then p is essentially finite. The converse does not hold. It is possible to measure the extent of essential finiteness by computing the multiplicity (codimension) of an ideal of formal power series. This number is called the *essential type*. Baouendi and Rothschild developed the notion of essential type and used it to prove some beautiful results about extension of mappings between real analytic hypersurfaces; see the bibliography in [Baouendi and Rothschild 1991]. We mention one of these results. Let M, M' be real analytic hypersurfaces in $\mathbb{C}^n$ containing 0. Suppose that $f : M \to M'$ is smooth with $f(0) = 0$, and that f extends to be holomorphic on the intersection of a neighborhood of 0 with one side of M. If M' is essentially finite at 0, and f is of *finite multiplicity*, then f extends to be holomorphic on a full neighborhood of 0. In this case the essential type of the point in the domain equals the multiplicity of the mapping times the essential type of the point in the target. The notion of finite multiplicity also comes from commutative algebra; again an appropriate ideal must be of finite codimension.

The essential type also arises when considering infinitesimal CR automorphisms of real analytic hypersurfaces. Stanton [1996] introduced the notion of holomorphic nondegeneracy for a real hypersurface at a point p. A real hypersurface is called *holomorphically nondegenerate* at p if there is no nontrivial ambient holomorphic vector field (a vector field of type $(1, 0)$ on $\mathbb{C}^n$ with holomorphic coefficients) tangent to M near p. If this condition holds at one point on a real analytic hypersurface, it holds at all points. Stanton proves that M is holomorphically nondegenerate if and only if the set of points of finite essential-type is both open and dense. The condition at one point is different from any of the finiteness notions we have discussed so far. Holomorphic nondegeneracy is important because it provides a necessary and sufficient condition for the finite dimensionality of the distinguished subspace of the infinitesimal CR automorphisms consisting of real parts of holomorphic vector fields. We mention this here to emphasize again that different finite-type notions arise in different problems.

A smoothly bounded domain Ω is strongly pseudoconvex if and only if it is locally biholomorphically equivalent to a strictly (linearly) convex domain. We say that the boundary is locally convexifiable. A necessary condition for local convexifiability at a boundary point p is that there is a local holomorphic support function at p. An example of Kohn and Nirenberg [1973] gives a weakly pseudoconvex domain with polynomial boundary for which there is no local holomorphic support function. This means that there is no holomorphic function f on Ω that vanishes at p, but is non-zero for all nearby points of the domain Ω. The existence of a (strict) holomorphic support function at p implies that there is a holomorphic function peaking at p. We have mentioned the result of Yu about peak points in certain finite-type cases. Earlier Bedford and Fornaess [1978] proved that every boundary point of finite type in a pseudoconvex domain in $\mathbb{C}^2$ is a peak point.

One fascinating question we do not consider in this paper is the behavior of the Bergman and Szegő kernels near points of finite type. Only in a few cases are exact formulas for these kernels known, and estimates from above and below are not known in general.

14. Open Problems

1. *Finite ideal-type.* Finite D_1-type is equivalent to subelliptic estimates on $(0,1)$ forms (Theorem 7.1). Finite ideal-type implies subelliptic estimates (Section 4), and the existence of a complex variety in the boundary prevents points along it from being of finite ideal-type. The circle is not complete; does finite D_1-type imply finite ideal-type? This would give a simpler proof of the sufficiency in Theorem 7.1.

2. *Global regularity.* Global regularity for the $\bar\partial$-Neumann problem means that the $\bar\partial$-Neumann solution u to $\bar\partial u = \alpha$ is smooth on the closed domain when α is. Global regularity follows of course from subelliptic estimates, but global regularity holds in some cases when subellipticity does not. Also global regularity fails in general smoothly bounded pseudoconvex domains. (See Christ's article in these proceedings). The necessary and sufficient condition is unknown.

3. *Peak points.* Let Ω be pseudoconvex. Is every point of finite D_1-type a peak point for the algebra of functions holomorphic on Ω and continuous on the closure?

4. *Type conditions for vector fields.* Does $c(L,p)$ equal $t(L,p)$ for each vector field on a pseudoconvex CR manifold of hypersurface type?

5. *Contact of complex manifolds.* Suppose that M is a pseudoconvex real hypersurface, and that $t(L,p) = N$ for some local $(1,0)$ vector field L. Must there be a complex-analytic 1-dimensional manifold tangent to M at p of order m, that is, is it necessarily true that $\Delta_1^{\mathrm{Reg}}(M,p) \geq N$?

6. *Sharp subelliptic estimates.* Suppose that a subelliptic estimate holds at p. Can one express the largest possible ε in terms of the geometry? If this isn't possible, can we always choose ε to be the reciprocal of the multiplicity, as defined in [D'Angelo 1993]?

7. *Hölder estimates.* Extend the results of [Fefferman et al. 1990] to domains of finite type.

8. *Bergman kernel.* Describe precisely the boundary behavior of the Bergman kernel function at a point of finite type.

9. *Hölder continuous CR structures.* Suppose that M is a smooth manifold with a Hölder continuous pseudoconvex CR structure. Discuss the Hölder regularity of solutions to the equation $\bar{\partial}_b u = \alpha$. (Results here would help understand non-linear problems involving $\bar{\partial}$ and $\bar{\partial}_b$.)

References

[Baouendi and Rothschild 1991] M. S. Baouendi and L. P. Rothschild, "Holomorphic mappings of real analytic hypersurfaces", pp. 15–25 in *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), vol. 3, edited by E. Bedford et al., Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI, 1991.

[Baouendi et al. 1985] M. S. Baouendi, H. Jacobowitz, and F. Treves, "On the analyticity of CR mappings", *Ann. of Math.* (2) **122**:2 (1985), 365–400.

[Bedford and Fornaess 1978] E. Bedford and J. E. Fornaess, "A construction of peak functions on weakly pseudoconvex domains", *Ann. of Math.* (2) **107**:3 (1978), 555–568.

[Bloom 1981] T. Bloom, "On the contact between complex manifolds and real hypersurfaces in $\mathbb{C}^3$", *Trans. Amer. Math. Soc.* **263**:2 (1981), 515–529.

[Catlin 1983] D. Catlin, "Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem", *Ann. of Math.* (2) **117**:1 (1983), 147–171.

[Catlin 1984] D. Catlin, "Boundary invariants of pseudoconvex domains", *Ann. of Math.* (2) **120**:3 (1984), 529–586.

[Catlin 1987] D. Catlin, "Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains", *Ann. of Math.* (2) **126**:1 (1987), 131–191.

[Chang et al. 1992] D.-C. Chang, A. Nagel, and E. M. Stein, "Estimates for the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in $\mathbb{C}^2$", *Acta Math.* **169**:3-4 (1992), 153–228.

[Christ 1988] M. Christ, "Regularity properties of the $\bar{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3", *J. Amer. Math. Soc.* **1**:3 (1988), 587–646.

[D'Angelo 1980] J. P. D'Angelo, "Subelliptic estimates and failure of semicontinuity for orders of contact", *Duke Math. J.* **47**:4 (1980), 955–957.

[D'Angelo 1982] J. P. D'Angelo, "Real hypersurfaces, orders of contact, and applications", *Ann. of Math.* (2) **115**:3 (1982), 615–637.

[D'Angelo 1983] J. P. D'Angelo, "A Bézout type theorem for points of finite type on real hypersurfaces", *Duke Math. J.* **50**:1 (1983), 197–201.

[D'Angelo 1987] J. P. D'Angelo, "The notion of formal essential finiteness for smooth real hypersurfaces", *Indiana Univ. Math. J.* **36**:4 (1987), 897–903.

[D'Angelo 1991] J. P. D'Angelo, "Finite type and the intersection of real and complex subvarieties", pp. 103–117 in *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), vol. 3, edited by E. Bedford et al., Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI, 1991.

[D'Angelo 1993] J. P. D'Angelo, *Several complex variables and the geometry of real hypersurfaces*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1993.

[D'Angelo 1995] J. P. D'Angelo, "Finite type conditions and subelliptic estimates", pp. 63–78 in *Modern methods in complex analysis* (Princeton, 1992), edited by T. Bloom et al., Annals of Mathematics Studies **137**, Princeton Univ. Press, Princeton, NJ, 1995.

[Diederich and Fornaess 1978] K. Diederich and J. E. Fornaess, "Pseudoconvex domains with real-analytic boundary", *Ann. of Math.* (2) **107**:2 (1978), 371–384.

[Fefferman 1995] C. Fefferman, "On Kohn's microlocalization of $\bar\partial$ problems", pp. 119–133 in *Modern methods in complex analysis* (Princeton, 1992), edited by T. Bloom et al., Annals of Mathematics Studies **137**, Princeton Univ. Press, Princeton, NJ, 1995.

[Fefferman and Kohn 1988] C. L. Fefferman and J. J. Kohn, "Hölder estimates on domains of complex dimension two and on three-dimensional CR manifolds", *Adv. in Math.* **69**:2 (1988), 223–303.

[Fefferman et al. 1990] C. L. Fefferman, J. J. Kohn, and M. Machedon, "Hölder estimates on CR manifolds with a diagonalizable Levi form", *Adv. in Math.* **84**:1 (1990), 1–90.

[Folland and Kohn 1972] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Annals of Mathematics Studies **75**, Princeton Univ. Press, Princeton, NJ, 1972.

[Fornæss and Sibony 1991] J. E. Fornæss and N. Sibony, "On L^p Estimates for $\bar\partial$", pp. 129–163 in *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), vol. 3, edited by E. Bedford et al., Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI, 1991.

[Greiner 1974] P. Greiner, "Subelliptic estimates for the $\bar\delta$-Neumann problem in $\mathbb{C}^2$", *J. Differential Geometry* **9** (1974), 239–250.

[Guan 1990] P. Guan, "Hölder regularity of subelliptic pseudodifferential operators", *Duke Math. J.* **60**:3 (1990), 563–598.

[Ho 1991] L.-H. Ho, "Subelliptic estimates for the $\bar\partial$-Neumann problem for $n-1$ forms", *Trans. Amer. Math. Soc.* **325**:1 (1991), 171–185.

[Hörmander 1966] L. Hörmander, *An introduction to complex analysis in several variables*, Van Nostrand, Princeton, NJ, 1966.

[Kohn 1963] J. J. Kohn, "Harmonic integrals on strongly pseudo-convex manifolds, I", *Ann. of Math.* (2) **78** (1963), 112–148.

[Kohn 1964] J. J. Kohn, "Harmonic integrals on strongly pseudo-convex manifolds, II", *Ann. of Math.* (2) **79** (1964), 450–472.

[Kohn 1972] J. J. Kohn, "Boundary behavior of δ on weakly pseudo-convex manifolds of dimension two", *J. Differential Geometry* **6** (1972), 523–542.

[Kohn 1977] J. J. Kohn, "Methods of partial differential equations in complex analysis", pp. 215–237 in *Several complex variables* (Williamstown, MA, 1975), vol. 1, edited by R. O. Wells, Jr., Proc. Sympos. Pure Math. **30**, Amer. Math. Soc., 1977.

[Kohn 1979] J. J. Kohn, "Subellipticity of the $\bar\partial$-Neumann problem on pseudo-convex domains: sufficient conditions", *Acta Math.* **142**:1-2 (1979), 79–122.

[Kohn 1984] J. J. Kohn, “A survey of the $\bar\partial$-Neumann problem”, pp. 137–145 in *Complex analysis of several variables* (Madison, WI, 1982), edited by Y.-T. Siu, Proc. Symp. Pure Math. **41**, Amer. Math. Soc., Providence, RI, 1984.

[Kohn 1985] J. J. Kohn, “Estimates for $\bar\partial_b$ on pseudoconvex CR manifolds”, pp. 207–217 in *Pseudodifferential operators and applications* (Notre Dame, IN, 1984), edited by F. Treves, Proc. Symp. Pure Math. **43**, Amer. Math. Soc., Providence, 1985.

[Kohn 1986] J. J. Kohn, “The range of the tangential Cauchy–Riemann operator”, *Duke Math. J.* **53**:2 (1986), 525–545.

[Kohn and Nirenberg 1965] J. J. Kohn and L. Nirenberg, “Non-coercive boundary value problems”, *Comm. Pure Appl. Math.* **18** (1965), 443–492.

[Kohn and Nirenberg 1973] J. J. Kohn and L. Nirenberg, “A pseudo-convex domain not admitting a holomorphic support function”, *Math. Ann.* **201** (1973), 265–268.

[Lewy 1956] H. Lewy, “On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables”, *Ann. of Math.* (2) **64** (1956), 514–522.

[McNeal 1989] J. D. McNeal, “Boundary behavior of the Bergman kernel function in C^2”, *Duke Math. J.* **58**:2 (1989), 499–512.

[McNeal 1992] J. D. McNeal, “Convex domains of finite type”, *J. Funct. Anal.* **108**:2 (1992), 361–373.

[McNeal and Stein 1994] J. D. McNeal and E. M. Stein, “Mapping properties of the Bergman projection on convex domains of finite type”, *Duke Math. J.* **73**:1 (1994), 177–199.

[Nagel et al. 1989] A. Nagel, J.-P. Rosay, E. M. Stein, and S. Wainger, “Estimates for the Bergman and Szegő kernels in $\mathbb{C}^2$”, *Ann. of Math.* (2) **129**:1 (1989), 113–149.

[Rothschild and Stein 1976] L. P. Rothschild and E. M. Stein, “Hypoelliptic differential operators and nilpotent groups”, *Acta Math.* **137**:3-4 (1976), 247–320.

[Shaw 1985] M.-C. Shaw, “L^2-estimates and existence theorems for the tangential Cauchy–Riemann complex”, *Invent. Math.* **82**:1 (1985), 133–150.

[Sibony 1991] N. Sibony, “Some aspects of weakly pseudoconvex domains”, pp. 199–231 in *Several complex variables and complex geometry* (Santa Cruz, CA, 1989), vol. 1, edited by E. Bedford et al., Proc. Sympos. Pure Math. **52**, Amer. Math. Soc., Providence, RI, 1991.

[Stanton 1996] N. K. Stanton, “Infinitesimal CR automorphisms of real hypersurfaces”, *Amer. J. Math.* **118**:1 (1996), 209–233.

[Sweeney 1972] W. J. Sweeney, “Coerciveness in the Neumann problem”, *J. Differential Geometry* **6** (1972), 375–393.

[Trépreau 1986] J.-M. Trépreau, “Sur le prolongement holomorphe des fonctions C–R défines sur une hypersurface réelle de classe C^2 dans $\mathbb{C}^n$”, *Invent. Math.* **83**:3 (1986), 583–592.

[Tumanov 1988] A. E. Tumanov, “Extending CR functions on manifolds of finite type to a wedge”, *Mat. Sb.* (*N.S.*) **136(178)**:1 (1988), 128–139. In Russian; translated in *Math. USSR Sb.* **64**:1 (1989), 129–140.

[Yu 1994] J. Y. Yu, “Peak functions on weakly pseudoconvex domains”, *Indiana Univ. Math. J.* **43**:4 (1994), 1271–1295.

John P. D'Angelo
Deptartment of Mathematics
University of Illinois
Urbana, IL 61801
United States
jpda@math.uiuc.edu

Joseph J. Kohn
Deptartment of Mathematics
Princeton University
Princeton, NJ 08540
United States
kohn@math.princeton.edu