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Phase Transitions and Random Matrices

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ABSTRACT. Phase transitions generically occur in random matrix models as the parameters in the joint probability distribution of the random variables are varied. They affect all main features of the theory and the interpretation of statistical models. In this paper a brief review of phase transitions in invariant ensembles is provided, with some comments to the singular values decomposition in complex non-hermitian ensembles.

1. Phase Transitions in Invariant Hermitian Ensembles

Random matrix ensembles have been extensively studied for several decades, since the early works of E. Wigner and F. Dyson, as effective mathematical reference models for the descriptions of statistical properties of the spectra of complex physical systems. In the past twenty years new applications spurned a large literature both in theoretical physics and among mathematicians. Several monographs review different sides of the physics literature of the past few decades, such as [7; 10; 18; 40; 41; 59; 92; 94]. Their combined bibliography, although very incomplete, exceeds a thousand papers. Sets of lecture notes are [102; 58; 5; 34; 85; 66]. The classic reference is Mehta's book [82].

For a long time studies and applications of random matrix theory in large part were limited to the choice of gaussian random variables for the independent entries of the random matrix. This was due both to the dominant role of the normal distribution in probability theory as well as to the nice analytic results which were obtained. Increasingly, in the past two decades, a wide variety of matrix ensembles were considered, where the joint probability distribution for the random entries depends on a number of parameters. As the latter are allowed to change, the generic occurrence of phase transitions emerged.

In this section, I begin by recalling the problem in the easiest case, the invariant ensemble of hermitian matrices.

Let $H = (H_{ij})_{i,j=1,\dots,N}$ be hermitian random matrix with joint probability density for the independent entries

$$P(H_{11}, H_{12}, \dots, H_{NN})\, dH = e^{-N \operatorname{Tr} V(H)}\, dH \Big/ \int e^{-N \operatorname{Tr} V(H)}\, dH,$$
$$V(H) = \frac{1}{2} a_2 H^2 + \frac{1}{4} a_4 H^4 + \cdots + \frac{1}{2p} a_{2p} H^{2p}, \quad a_{2p} > 0,$$
$$dH = \prod_{i<j} (d \operatorname{Re} H_{ij}\, d \operatorname{Im} H_{ij}) \prod_i dH_{ii}. \tag{1–1}$$

We are interested in the partition function $Z_N(a_2, a_4, \dots, a_{2p})$, the free energy $F_N(a_2, a_4, \dots, a_{2p}) = -\log Z_N(a_2, a_4, \dots, a_{2p})$, the "one point resolvent" $G_N(z)$, the connected correlator $G_N^{(c)}(z_1, z_2)$

$$Z_N(a_2, a_4, \dots, a_{2p}) = \int e^{-N \operatorname{Tr} V(H)}\, dH,$$
$$G_N(z) = \frac{1}{N} \operatorname{Tr}\Big\langle \frac{1}{z-H} \Big\rangle,$$
$$G_N^{(c)}(z_1, z_2) = \Big\langle \operatorname{Tr} \frac{1}{z_1-H} \operatorname{Tr} \frac{1}{z_2-H} \Big\rangle - \Big\langle \operatorname{Tr} \frac{1}{z_1-H} \Big\rangle \Big\langle \operatorname{Tr} \frac{1}{z_2-H} \Big\rangle. \tag{1–2}$$

The name invariant ensemble for the ensemble of these hermitian matrices reminds that since the density $P(H_{11}, H_{12}, \dots, H_{NN})$ is invariant under a similarity transformation $H \to UHU^{-1}$ with arbitrary unitary matrix U, most of the interesting quantities, like those in (1–2), may be evaluated from the joint probability density of the eigenvalues.

Also important are the monic polynomials $P_n(z) = z^n + O(z^{n-1})$, orthogonal on the real line with the weight $e^{-N V(z)}$:

$$\int_{-\infty}^{\infty} P_n(z) P_m(z)\, e^{-N V(z)} dz = h_n\, \delta_{nm},$$
$$z P_n(z) = P_{n+1}(z) + R_n P_{n-1}(z),$$
$$R_n = \frac{h_n}{h_{n-1}} > 0,$$
$$Z_N(a_2, a_4, \dots, a_{2p}) = N!\, (h_0)^N \prod_{n=1}^{N-1} (R_n)^{N-n}.$$

One obtains a non-linear recursion relation for the coefficients R_n. For instance, if

$$V(x) = \frac{a_2}{2} x^2 + \frac{a_4}{4} x^4 + \frac{a_6}{6} x^6,$$

one has the recurrence relation

$$\frac{n}{N} = R_n \Big(a_2 + a_4 (R_{n-1} + R_n + R_{n+1}) + a_6 (R_{n-1} + R_n + R_{n+1})^2 + a_6 (R_{n-2} R_{n-1} - R_{n-1} R_{n+1} + R_{n+1} R_{n+2}) \Big)$$

(see [77; 63; 32]), occasionally called "pre-string equation" or the Freud equation. One also introduces the set of orthonormal functions $\psi_n(z)$ and the two point kernel $K_N(x, y)$, in terms of which all n-point correlation functions are expressible as

$$\psi_n(z) = \frac{1}{\sqrt{h_n}} e^{-NV(z)/2} P_n(z), \quad K_N(x, y) = \sum_{j=0}^{N-1} \psi_j(x) \psi_j(y).$$

If all the coefficients a_{2k} are positive, the statistics of the eigenvalues may be evaluated in the limit $N \to \infty$ (see [14]), $\lim_{N\to\infty} G_N(z) = G(z)$ is holomorphic in the complex z-plane, except for a segment $(-A, A)$ on the real axis. Furthermore

$$G(z) = \int_{-A}^{A} d\mu \, \frac{\rho(\mu)}{z - \mu}, \quad \rho(\mu) = \lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\langle \delta(\mu - H) \rangle = \lim_{N\to\infty} K_N(\mu, \mu).$$

The limiting density of eigenvalues $\rho(\lambda)$, the one point correlation function, is the unique solution of the integral equation

$$V'(\lambda) = 2\mathcal{P} \int_{-A}^{A} d\mu \, \frac{\rho(\mu)}{\lambda - \mu}, \quad \int_{-A}^{A} \rho(\lambda) \, d\lambda = 1. \tag{1–3}$$

We have $\rho(\lambda) > 0$ on its support $(-A, A)$, and it vanishes as a square root at the boundary: $\rho(\lambda) \sim (A - |\lambda|)^{1/2}$. Furthermore the coefficients R_n approach a smooth limit $R(\frac{n}{N}) \sim R(x)$ and

$$\begin{aligned} &F(a_2, a_4, \dots, a_{2p}) \\ &\quad = N^2 \Big(\int_{-A}^{A} d\lambda \, \rho(\lambda) V(\lambda) - \int \int_{-A}^{A} d\lambda \, d\mu \, \rho(\lambda) \, \rho(\mu) \, \log |\lambda - \mu| + O(\frac{1}{N^2}) \Big) \\ &\quad = N^2 \Big(- \int_0^1 dx \, (1 - x) \log R(x) - \frac{1}{N} \log h_0 + O(\frac{1}{N^2}) \Big). \end{aligned}$$

The free energy $F(a_2, a_4, \dots, a_{2p})$ is analytic in the couplings $(a_2, a_4, \dots, a_{2p})$. The saddle point solution is equivalent to the resummation of the planar graphs, it is equivalent to the solution from the recurrence relations for R_n and to other techniques, such as the loop equations. The orthogonal polynomial technique and the loop equations are superior to evaluate in a systematic way the terms in the series in the parameter $(N^2)^{-k}$ corresponding to the resummation of the graphs which are embeddable on orientable surfaces with k handles [101].

It was also proved that the connected two point correlators exhibit two different forms of universality (they are independent of the set of coefficients $\{a_{2k}\}$ but depend only on the endpoints $\pm A$):

Global Universality. The limiting connected density-density correlator, after smoothing over a scale much larger than the level spacing Δ_N, is [4; 20]

$$\rho_c(\lambda,\lambda') = -\frac{1}{2\pi^2(\lambda-\lambda')^2}\frac{A^2-\lambda\,\lambda'}{\sqrt{A^2-\lambda^2}\sqrt{A^2-\lambda'^2}}, \quad \lambda \neq \lambda'.$$

Local Universality. The limiting two point kernel $K(x,y)$ has the sine law [90] for eigenvalues in the bulk of the spectrum, measured in units of Δ_N

$$K_{\text{bulk}}(s,s') = \frac{\sin\big(\pi(s-s')\big)}{\pi(s-s')}, \quad s=\lambda/\Delta_N,\ s'=\lambda'/\Delta_N,$$

or the Airy law, close to the tail of the spectrum (the soft edge) [15; 9]

$$K_{\text{soft}}(s,s') = \frac{\text{Ai}(s)\,\text{Ai}'(s') - \text{Ai}(s')\,\text{Ai}'(s)}{s-s'}, \quad s \sim N^{2/3}\Big(\frac{\lambda}{A}-1\Big).$$

A general derivation of spectral correlators is given in the recent paper by Kanzieper and Freilikher [70]. By following Shohat [97], the recurrence relation for the orthogonal polynomials $P_n(x)$ is turned into an exact second-order differential equation for the orthogonal functions $\psi_n(z)$

$$\frac{d^2\psi_n(\lambda)}{d\lambda^2} - \mathcal{F}_n(\lambda)\frac{d\psi_n(\lambda)}{d\lambda} + \mathcal{G}_n(\lambda)\psi_n(\lambda) = 0$$

Because of the smooth behavior of the coefficients R_n, in the case where the eigenvalues have support on a single segment, the complicated forms $\mathcal{F}_n(\lambda)$, $\mathcal{G}_n(\lambda)$ simplify at large order and the differential equation leads to the global and the local universality results mentioned above. (If a logarithmic singularity is present at the origin, the Bessel law is derived

$$K_{\text{origin}}(s,s') = \frac{\pi}{2}(s\,s')^{1/2}\frac{J_{\alpha+1/2}(\pi s)J_{\alpha-1/2}(\pi s') - J_{\alpha-1/2}(\pi s)J_{\alpha+1/2}(\pi s')}{s-s'},$$

where s and s' are scaled by the level spacing $\Delta_N(0)$ near the spectrum origin, $s=\lambda/\Delta_N(0)$, and $\alpha > -1/2$. For the simpler situations where the orthogonal polynomials are classical the Bessel law had been derived in [22; 87; 53; 88].)

All the quantities above are deeply affected by *phase transitions.*

The limiting eigenvalue density $\rho(\lambda; a_2, a_4, \dots, a_{2p})$ continued to negative values for one or several coefficients a_{2k} (while keeping $a_{2p} > 0$) may not be positive definite. Then the integral equation (1–3) allows new solutions with eigenvalue density positive definite with support on two or more segments of the real axis. The correct solution minimizes the free energy. As one explores the space of the parameters, the free energy is evaluated on different saddle point solutions, which may coincide for certain critical values of the parameters. Then the free energy is usually a continuous but not analytic function of the parameters.

The recursion coefficients R_n no longer have a smooth "continuous" limit, which makes more difficult the orthogonal polynomials solution.

The lack of analyticity of the free energy at critical values of the parameters is analogous to phase transitions related to spontaneous symmetry breaking in classical statistical mechanics. For instance, in the simplest case of potential $V(x) = \frac{a_2}{2}x^2 + \frac{a_4}{4}x^4$, first analysed for negative values of a_2 in papers [96; 25], it is convenient to add a linear term, which explicitly breaks the Z_2 symmetry, $V(x) = a_1 x + \frac{a_2}{2}x^2 + \frac{a_4}{4}x^4$, next perform the "thermodynamic limit" $N \to \infty$, finally remove the symmetry breaking term $a_1 \to 0$, see [62], [26]. This allows the evaluation of the "order parameter" $\langle \mathrm{Tr}\, H \rangle$:

$$\lim_{a_1 \to 0} \lim_{N \to \infty} \langle \mathrm{Tr}\, H \rangle = \mathrm{sign}(a_1)\, \theta(-a_2 - 2\sqrt{a_4})\, f(a_2, a_4)$$

For the simplest case $V(x) = \frac{a_2}{2}x^2 + \frac{a_4}{4}x^4$, it was shown that the correct ansatz for the recursion coefficients R_n, if $a_2 < -2\sqrt{a_4}$ is that the even R_{2n} and odd R_{2n+1} approach two different smooth "continuous" functions [83]. This period-two ansatz requires a little generalization in the case which includes the infinitesimal symmetry breaking term [84; 23] because it leads to recurrence relations

$$z P_n(z) = P_{n+1}(z) + S_n P_n(z) + R_n P_{n-1}(z), \quad R_n = \frac{h_n}{h_{n-1}} > 0,$$

and it explains the origin of the period-two ansatz.

However in the next simplest case, like $V(x) = \frac{1}{2}a_2 x^2 + \frac{1}{4}a_4 x^4 + \frac{1}{6}a_6 x^6$, or higher order polynomials, corresponding to multiple well potentials, which may not be degenerate, the behaviour of the coefficients R_n is erratic and it is difficult to reproduce the results of the saddle point analysis by the orthogonal polynomials [69; 78; 79; 93; 95].

The lines of "phase transition" in the parameter space, related to the continuation to negative values of the coefficient a_{2p} of the monomial of highest order may be found in terms of previously discussed phase transitions by the addition of an infinitesimal monomial εx^{2p+2}; see for instance [27; 68].

Connected correlators, when the support of the eigenvalue density is two segments, were shown [6; 1] to have a different form of global universality, involving elliptic integrals. In the case of multicritical behaviour the local universality form has a modified Bessel law [3].

Important recent works seem to be so powerful and comprehensive to solve the above mentioned ambiguities. The Freud equation is expressed in a matrix Lax representation and the semiclassical asymptotics of the functions $\psi_n(z)$ is obtained in the whole complex z plane by solving a matrix Riemann-Hilbert problem, following earlier works [51; 52]. Very useful is the non-linear steepest descent method devised by Deift and Zhou [35; 36]. The works [12; 37; 38; 39] not only provide rigorous and more general solutions for methods and ansatzes previously used, but it seems to provide answers also for the models previously left unsolved (like the asymptotics of recurrence coefficients R_n in general cases). The recent and difficult developments will require some time to be exploited by physicists.

Finally, while almost all investigations related to the invariant ensemble of random matrices considered a probability distribution of the form of exponential of a polynomial like in (1–1), with possible addition of logarithmic terms, like the Penner model and the Kontsevich model, there exist probability distributions, still invariant under diagonalization by unitary matrices, which lead to different forms for the connected correlators [2]. Then the classification of universality classes perhaps is not yet complete, even in invariant one-matrix hermitian ensembles.

2. Further Matrix Ensembles and Singular Value Decomposition in Complex Non-Hermitian Random Matrices

Random matrix models more general than the hermitian one-matrix invariant ensemble are often more interesting because of the possibility to describe more interesting statistical models. The analytic solution of multi-matrix models both in the "perturbative phase" or in different phases is more complex. While an ensemble of hermitian random matrices describes triangulations of random orientable surfaces, multi-matrix ensembles are suitable to describe models of classical statistical mechanics on a random two-dimensional lattice. After the breakthrough of the Ising model [71; 13], it was possible to study random walks and loops [43], $O(N)$ model [55; 75; 76; 45; 46; 44], the Potts model [72; 33; 105], surfaces with holes [73; 30], a special case of 8-vertex model [74; 106], the chiral random matrix which simulate the spontaneously broken phase transition of QCD [103; 89; 98; 100; 65; 11]. Often it was possible to evaluate the critical exponents at phase transition. Multi-matrix models of hermitian matrices are also a good framework for combinatorial problems like the four-color theorem [28; 24; 47] or the enumeration of meanders [42; 81]. Most influential, for quantum field theorists were the phase transitions in models of random unitary matrices, describing one-plaquette of the lattice formulation of QCD [57; 17; 67], the saddle-point solution of the one-matrix ensemble [16], the Witten conjecture of a master field [104].

In this section I shall recall the singular value decomposition, which plays a role in the analysis of models with rectangular random matrices and square complex non-hermitian matrices.

Let

$$\phi = (\phi_{ij})_{\substack{i=1,\dots,N \\ j=1,\dots,M}}$$

be a rectangular random matrix with entries ϕ_{ij} real or complex numbers, and joint probability distribution $P(\phi)$ invariant under $\phi \to U\phi V$, with U unitary of order N and V unitary of order M:

$$P(\phi)\, d\phi = e^{-N \operatorname{Tr} V(\phi^\dagger \phi)}\, d\phi \Big/ \int e^{-N \operatorname{Tr} V(\phi^\dagger \phi)}\, d\phi,$$

$$V(\phi^\dagger\phi) = \frac{1}{2}a_2(\phi^\dagger\phi) + \frac{1}{4}a_4(\phi^\dagger\phi)^2 + \cdots + \frac{1}{2p}a_{2p}(\phi^\dagger\phi)^p, \quad a_{2p} > 0,$$

$$d\phi = \begin{cases} \displaystyle\prod_{\substack{i=1,\dots,N\\ j=1,\dots,M}} d\phi_{ij} & \text{if } \phi \text{ is real,} \\ \displaystyle\prod_{\substack{i=1,\dots,N\\ j=1,\dots,M}} d\,\mathrm{Re}\,\phi_{ij}\, d\,\mathrm{Im}\,\phi_{ij} & \text{if } \phi \text{ is complex.} \end{cases} \tag{2–1}$$

The hermitian matrices $\phi^\dagger\phi$ and $\phi\phi^\dagger$ are positive semi-definite, have the same non-vanishing eigenvalues $t_i = \sigma_i^2$, where σ_i are the singular values of ϕ, themselves positive definite. It is straightforward to evaluate the statistics of the singular values in the limit $N \to \infty$, $M \to \infty$, while the ratio $N/M = L$ is fixed, by the ordinary saddle point analysis. Let us consider first $L \geq 1$. The probability distribution (2–1) is

$$e^{-N\sum_{i=1,\dots,M} V(\sigma_i^2)} J(\sigma_i) \prod_{i=1,\dots,M} d\sigma_i \bigg/ \int e^{-N\sum_{i=1,\dots,M} V(\sigma_i^2)} J(\sigma_i) \prod_{i=1,\dots,M} d\sigma_i,$$

where the Jacobian $J(\sigma_i)$ may be evaluated with help from [60; 61]:

$$J(\sigma_i) = \begin{cases} \prod_{i=1,\dots,M}(\sigma_i)^{N-M} \prod_{1\leq i<j\leq M} |\sigma_i^2 - \sigma_j^2| & \text{if } \phi \text{ is real,} \\ \prod_{i=1,\dots,M}(\sigma_i)^{2N-2M+1} \prod_{1\leq i<j\leq M} |\sigma_i^2 - \sigma_j^2|^2 & \text{if } \phi \text{ is complex.} \end{cases}$$

For the simple case $V(\phi^\dagger\phi) = \frac{1}{2}a_2(\phi^\dagger\phi) + \frac{1}{4}a_4(\phi^\dagger\phi)^2$ one easily obtains [29] for complex rectangular matrix ϕ

$$G(z) = \lim_{\substack{N\to\infty\\ M\to\infty}} \frac{1}{M}\,\mathrm{Tr}_M\left\langle \frac{1}{z-\phi^\dagger\phi}\right\rangle = \int_A^B dt\,\frac{u(t)}{z-t}, \quad 0 \leq A \leq B$$

$$u(t) = \frac{1}{\pi}\sqrt{(B-t)(t-A)}\left(\frac{a_4}{4} + \frac{a_4(A+B)}{8t} + \frac{a_2}{2t}\right). \tag{2–2}$$

The extrema A, B are given by the usual pair of algebraic equations. Then for $a_4 = 0$ one finds the distribution of singular values for rectangular matrices, which is the generalization of Wigner "semicircle law"

$$u(\sigma) = \frac{a_2}{\pi}\frac{\sqrt{(B-\sigma^2)(\sigma^2-A)}}{\sigma}, \quad \text{for } \sqrt{A} \leq \sigma \leq \sqrt{B},$$

with

$$A = \left(\frac{\sqrt{L}-1}{a_2}\right)^2, \quad B = \left(\frac{\sqrt{L}+1}{a_2}\right)^2.$$

Returning now to equation (2–2) for square complex matrices, $L = 1$, one finds a "perturbative" phase for $a_2 > -2\sqrt{a_4}$ with $A = 0$ where observables correspond to resummation of planar graphs, and a "non-perturbative" phase for $a_2 < -2\sqrt{a_4}$ where $A > 0$, quite similar to the random hermitian case. These results were rediscovered by several authors [8; 86; 48] who introduced the hermitian matrix

$$H = \begin{pmatrix} 0 & \phi \\ \phi^\dagger & 0 \end{pmatrix}$$

and the partition function

$$\mathcal{Z} = \int DH\, e^{-\beta \operatorname{Tr} V(H^\dagger H)}$$
$$\sim \int_{-\infty}^{\infty} \left(\prod_{i=1}^{M} dx_i\, e^{-2\beta V(x_i^2)} \right) \prod_{i=1}^{M} |x_i|^{2N-2M+1} \prod_{1 \le i < j \le M} (x_i^2 - x_j^2)^2.$$

The eigenvalues x_i of the "chiral" matrix H are in two to one correspondence with the singular values σ_i of A: $x_i = \pm\sigma_i$. The technique of using the auxiliary matrix H in the study of square complex non-hermitian matrix ϕ was developed by [49; 50; 21] into a powerful method to obtain the distribution $\rho(x, y)$ of complex eigenvalues λ_i, see also the similar and simultaneous paper [64]. Indeed,

$$\begin{aligned} \rho(x, y) &= \frac{1}{N} \sum_{i=1}^{N} \big\langle \delta(x - \operatorname{Re} \lambda_i) \delta(y - \operatorname{Im} \lambda_i) \big\rangle \\ &= \frac{1}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial z^*} \frac{1}{N} \big\langle \operatorname{Tr}_{(N)} \log(z - \phi)(z^* - \phi^\dagger) \big\rangle \\ &= \frac{1}{\pi} \frac{\partial}{\partial z} \frac{\partial}{\partial z^*} \frac{1}{N} \big(\langle \operatorname{Tr}_{(2N)} \log H \rangle - i\pi N^2 \big), \end{aligned}$$

where now

$$H = \begin{pmatrix} 0 & \phi - z \\ \phi^\dagger - z^* & 0 \end{pmatrix}.$$

Diagrammatic rules may then be used to evaluate the resolvent $\mathcal{G}(\eta; z, z^*)$

$$\mathcal{G}(\eta; z, z^*) = \frac{1}{2N} \left\langle \operatorname{Tr}_{(2N)} \frac{1}{\eta - H} \right\rangle = \frac{\eta}{N} \left\langle \operatorname{Tr}_N \frac{1}{\eta^2 - (z^* - \phi^\dagger)(z - \phi)} \right\rangle$$

Finally in terms of the integrated density of eigenvalues of H, $\Omega(\mu; z, z^*) = (2N)^{-1} \langle \operatorname{Tr}_{(2N)} \theta(\mu - H) \rangle$, Feinberg and Zee [50] obtain

$$\rho(x, y) = -\frac{4}{\pi} \int_0^\infty d\mu\, \frac{\partial}{\partial z} \frac{\partial}{\partial z^*} \frac{\Omega(\mu; z, z^*)}{\mu}$$

As specific examples, Feinberg and Zee consider probability distributions $P(\phi, \phi^\dagger)$ for the complex matrix ϕ of the form (2–1), invariant under $\phi \to e^{i\alpha}\phi$, $\phi^\dagger \to e^{-i\alpha}\phi^\dagger$, where it is natural to expect that the distribution of complex eigenvalues $\rho(x, y)$ has rotational symmetry $\rho(x, y) = \rho(r)/(2\pi)$, as it indeed happens with the Ginibre gaussian ensemble [56]. For the simple model with only two coefficients a_2, a_4 they find that for $a_2 > -2\sqrt{a_4}$ the complex eigenvalues fill (non-uniformly) a disk centered at the origin, while for $a_2 < -2\sqrt{a_4}$ they fill a ring. This is expected from the analogous distribution of the singular values of the matrix ϕ. Next they prove the surprising "single ring theorem" asserting that for a generic polynomial potential of the form (2–1), even in the multiple-well cases where the distribution of the singular values of the matrix ϕ has support on several segments of the positive real axis, the distribution of eigenvalues of the matrix ϕ may only have one ring at most. It seems to me that in such cases the assumption of rotational symmetry should be checked. If it turns out, by using an explicitly symmetry breaking term to be removed after the thermodynamic limit $N \to \infty$, that one obtains different distributions, dependent on the direction of the symmetry breaking probe, it would be a remarkable example of spontaneous symmetry breaking of rotational symmetry.

3. Conclusion

Phase transitions generically occur in the study of ensembles of random matrices, as the parameters in the joint probability distribution of the random variables are varied. They are important for the physics interpretation of statistical models and they affect all the main features of random matrix theory. In invariant one-matrix ensembles recent progress of mathematicians seems to solve long standing problems related to multi-cut solutions with no symmetry. Phase transitions in ensembles of complex non-hermitian matrices were recently explored and it is likely that a richer variety of phase transitions will be discovered. Random matrix ensembles with a preferential basis, like band matrices were studied since the beginning of random matrix theory, see for instance [80; 31; 54; 99]. Even the simplest cases of tridiagonal matrices with random site or random hopping could be analytically solved only for a very limited choice of the probability distribution. Well known discrepancies between the moment method and numerical methods suggest the presence of phase transitions which seem more difficult to analyse than in case of invariant ensembles.

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