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Some Matrix Integrals Related to Knots and Links

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ABSTRACT. The study of a certain class of matrix integrals can be motivated by their interpretation as counting objects of knot theory such as alternating prime links, tangles or knots. The simplest such model is studied in detail and allows to rederive recent results of Sundberg and Thistlethwaite. The second nontrivial example turns out to be essentially the so-called *ABAB* model, though in this case the analysis has not yet been carried out completely. Further generalizations are discussed. This is a review of work done (in part) in collaboration with J.-B. Zuber.

1. Introduction

Using random matrices to count combinatorial objects is not a new idea. It stems from the pioneering work [Brézin et al. 1978], which showed how the perturbative expansion of a simple nongaussian matrix integral led, using standard Feynman diagram techniques, to the counting of discretized surfaces. It has resulted in many applications: from the physical side, it allowed to define a discretized version of 2D quantum gravity [Di Francesco et al. 1995] and to study various statistical models on random lattices [Kazakov 1986; Kostov 1989; Gaudin and Kostov 1989; Kostov and Staudacher 1992]. From the mathematical side, let us cite the Kontsevitch integral [Kontsevich 1991; Witten 1991; Itzykson and Zuber 1992], and the counting of meanders and foldings [Makeenko 1996; Di Francesco et al. 1997; 1998].

Here we shall try to apply this idea to the field of knot theory. Our basic aim will be to count knots or related objects. The next section defines these objects, and is followed by a brief overview of matrix models and how they can be related to knots. Section 4 explains the counting of alternating links, following [Zinn-Justin and Zuber 1999]; Section 5 presents a generalized model (the *ABAB* model of [Kazakov and Zinn-Justin 1999]), which leads to a digression and consideration of summations over Young diagrams; finally, Section 6 discusses further generalizations.

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2. Knots, Links and Tangles

Let us recall basic definitions of knot theory (from a physicist's point of view; the reader is referred to the literature for more precise definitions). A *knot* is a smooth circle embedded in $\mathbb{R}^3$. A *link* is a collection of intertwined knots. Both kinds of objects are considered up to homeomorphisms of $\mathbb{R}^3$. Roughly speaking, a *tangle* is a knotted structure from which four strings emerge.

In the nineteenth century, Tait introduced the idea to represent such objects by their projection on the plane, with under/over-crossings at each double point and with *minimal number of such crossings* (Figure 1). We shall now consider

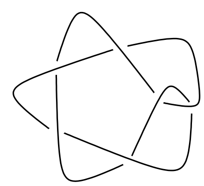

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Figure 1. A reduced diagram with 6 crossings.

such *reduced* diagrams. To a given knot, there corresponds a finite number of (but not necessarily just one) reduced diagrams. We shall come back later to the problem of different reduced diagrams which correspond to the same knot (or link, or tangle).

To avoid redundancies, we can concentrate on *prime* links and tangles, whose diagrams cannot be decomposed as a connected sum of components (Figure 2).

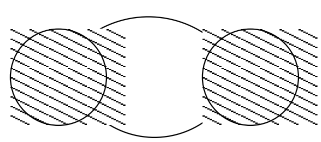


Figure 2. A nonprime diagram.

A diagram is called *alternating* if one meets alternatively under- and over-crossings as one travels along each loop. (Even though it may not seem obvious, there are knots that cannot be drawn in an alternating way — starting with eight crossings.) From now on, we shall concentrate on alternating diagrams only, since they are easier to count. There are two reasons for that.

The first reason is that there is a relatively simple way to characterize whether two reduced alternating diagrams correspond to the same knot or link (simpler than the general Reidemeister theorem [1932] for comparing any two knots). Indeed, a major result conjectured by Tait and proved by Menasco and Thistlethwaite [1991; 1993] is that two alternating reduced knot or link diagrams represent the same object if and only if they are related by a sequence of moves acting on tangles called "flypes" (Figure 3).

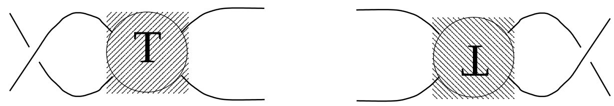


Figure 3. The flype of a tangle.

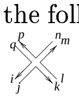
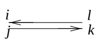
The second reason is that there is a correspondence between alternating diagrams and planar diagrams (see for example [Kauffman 1993]), which will be explained now as we discuss matrix integrals.

3. Matrix Integrals

We now start from a completely different angle and consider the matrix integral

$$Z^{(N)}(g) = \int \mathrm{d}M\, \mathrm{e}^{N\ \mathrm{tr}\left(-\frac{1}{2}M^2 + \frac{g}{4}M^4\right)}, \tag{3–1}$$

where M is a $N \times N$ hermitian matrix and g is a real parameter, which should be chosen negative to make the integral convergent.

As an application of Wick's theorem, the perturbative expansion of $Z^{(N)}$ in powers of g can be made using the following Feynman rules: one should count all diagrams made out of vertices  $= gN\delta_{qi}\delta_{jk}$ and propagators $\langle M_{ij} M^\dagger_{k\ell}\rangle_0 =$  $= \frac{1}{N}\delta_{i\ell}\delta_{jk}$. Due to the double lines, these diagrams form so-called fat graphs which can be identified with triangulated surfaces. Each diagram has a weight

$$(gN)^{\mathrm{V}} N^{-\mathrm{E}} N^{\mathrm{F}} \frac{1}{\text{symmetry factor}},$$

where V, E, F are the number of vertices, edges, faces of the triangulated surface (the factor N^{F} comes from the summation over internal indices). The symmetry factor (the order of the automorphism group of the diagram) is of little importance to us and we shall not discuss it any further. Note that the power of N is simply N^χ where χ is the Euler–Poincaré characteristic of the triangulated surface. If we take the logarithm, which amounts to considering only connected surfaces, we have the genus expansion

$$\log Z^{(N)}(g) = \sum_{h=0}^{\infty} F_h(g) N^{2-2h},$$

where F_h is the sum over surfaces of genus h. In particular, if we consider the large N limit, we see that

$$F(g) = \lim_{N\to\infty} \frac{\log Z^{(N)}(g)}{N^2} = \sum_{\text{planar graphs}} \frac{g^{\mathrm{V}}}{\text{symmetry factor}}$$

is the sum over connected "planar" diagrams (i.e., with spherical topology). $F(g)$ is the quantity we are interested in. The formal power series $F(g) = \sum_p f_p g^p$

turns out to have, as is well-known, a finite radius of convergence (which allows to analytically continue it to positive values of g, as will be explained later). The position and nature of the closest singularity g_c determines the asymptotics of f_p as $p \to \infty$, that is, of the number of planar diagrams with large numbers of vertices.

In order to connect with knot theory, we take any planar diagram and do the following: starting from an arbitrary crossing, we decide it is a crossing of two strings (again there is an arbitrary choice of which is under/over-crossing). Once the first choice is made, we simply follow the strings and form alternating sequences of under- and over-crossings. The remarkable fact is that this can be done consistently (Figure 4). If we identify two alternating diagrams obtained

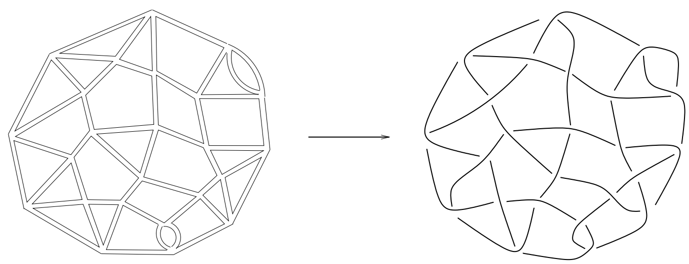


Figure 4. A planar diagram and the corresponding alternating link diagram.

from one another by inverting undercrossings and overcrossings, then there is a one-to-one correspondence between planar diagrams and alternating link diagrams. So the function $F(g)$ also counts alternating link diagrams with a given number of crossings.

A more detailed discussion of the properties of the resulting link diagrams will be made in the next section. For now, we shall address the question of the number of connected components of the link (as a 3-dimension object). Indeed, there is no reason for the diagram to represent a simple knot, and not several intertwined knots. In order to distinguish them, we introduce a more general model, which we shall call the intersecting loops $O(n)$ model. If n is a positive integer, consider the multi-matrix integral

$$Z^{(N)}(n,g) = \int \prod_{a=1}^{n} \mathrm{d}M_a \, \mathrm{e}^{N \operatorname{tr}\left(-\frac{1}{2}\sum_{a=1}^{n} M_a^2 + \frac{g}{4}\sum_{a,b=1}^{n} M_a M_b M_a M_b\right)} \tag{3–2}$$

and the corresponding free energy

$$F(n,g) = \lim_{N\to\infty} \frac{\log Z^{(N)}(n,g)}{N^2}. \tag{3–3}$$

This model has an $O(n)$-invariance where the M_a behave as a vector under $O(n)$. Its Feynman rules are a bit more complicated since we should draw the diagrams with n different colors. The colors "cross" each other at vertices just like strings

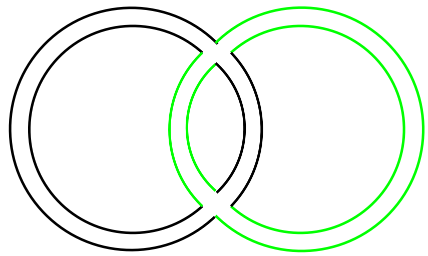


Figure 5. A planar diagram with 2 colors.

in links (Figure 5). So what we have done is allow each loop in the link to have n different colors. This is in itself an interesting generalization of the original counting problem. Indeed, we can write

$$F(n,g) = \sum_{k=1}^{\infty} F^k(g) n^k, \tag{3–4}$$

where $F^k(g)$ is the sum over alternating link diagrams with exactly k intertwined knots. We see that the links are weighted differently according to their number of connected components.

But there is more. The expression (3–4) is an expansion of $F(n,g)$ as a function of n around 0; it provides a definition of $F(n,g)$ for noninteger values of n. In particular, we have the formal expression

$$F^1(g) = \frac{\partial F(n,g)}{\partial n}_{|n=0}$$

for the sum over alternating knots (this is the classical replica trick). Therefore, if one computed $F(n,g)$ for arbitrary (noninteger) values of n, one would have access to the generating function of the number of alternating knots. Of course, it might seem difficult to solve our model for all n; we shall discuss this again in the conclusion.

4. The One Matrix Model and the Counting of Links

Let us now come back to the one-matrix model and show how one can derive explicit formulae for the counting of prime alternating links. We recall the partition function

$$Z^{(N)}(g) = \int \mathrm{d}M \, \mathrm{e}^{N \, \mathrm{tr}\left(-\frac{1}{2}M^2 + \frac{g}{4}M^4\right)} \tag{4–1}$$

and the corresponding free energy

$$F(g) = \lim_{N\to\infty} \frac{1}{N^2} \log Z^{(N)}(g). \tag{4–2}$$

We also define the correlation functions

$$G_{2n}(g) = \lim_{N\to\infty} \left\langle \frac{1}{N} \operatorname{tr} M^{2n} \right\rangle. \tag{4–3}$$

Whereas the perturbative expansion of $F(g)$ generates closed diagrams (and therefore alternating links), the $G_{2n}(g)$ count diagrams with $2n$ external legs. In particular, we shall be interested later in $\Gamma(g) = G_4(g) - 2G_2(g)^2$ which counts connected diagrams with 4 legs, that is, alternating tangles.

There are various methods to compute all these quantities. We shall briefly recall the simplest one: the saddle point method.

4.1. Saddle Point Method for the One-Matrix Model. We start from Equation (4–1) and notice that the action and measure are $U(N)$-invariant; therefore we can go over to the eigenvalues λ_i of M:

$$Z(g) = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta[\lambda_i]^2 \, \mathrm{e}^{N \sum_{i=1}^{N} \left(-\frac{1}{2}\lambda_i^2 + \frac{g}{4}\lambda_i^4\right)}, \tag{4–4}$$

up to an overall constant factor. Here $\Delta[\cdot]$ is the Vandermonde determinant:

$$\Delta[\lambda_i] = \det(\lambda_i^{j-1}) = \prod_{i<j} (\lambda_i - \lambda_j).$$

We are now interested in the large N limit of the N-uple integral (4–4). We would like to justify the fact that it is dominated by a saddle point in this limit. Of course this is not exactly the usual setting for a saddle point method, since not only does the integrand depend on N, but also the number of variables of integrations is equal to N. However, one notes the following: in (4–4), the "action" (that is, the log of the integrated function) is of order N^2 (essentially because the Vandermonde determinant is a product of $\sim N^2$ factors), whereas there are only N variables of integration. Since $N^2 \gg N$, a saddle point analysis does apply. It is easy to see that as $N \to \infty$, the eigenvalues λ_i will condense to form a continuous saddle point density $\rho(\lambda)$ whose support is an interval $[-2a, 2a]$. The density $\rho(\lambda)$ is defined by the property that $N\rho(\lambda)\,\mathrm{d}\lambda$ eigenvalues lie in the interval $[\lambda; \lambda + \mathrm{d}\lambda]$ (note the normalization, which is such that $\int \rho(\lambda)\,\mathrm{d}\lambda = 1$). The saddle point equations are obtained by requiring the derivative with respect to the λ_i of the action of (4–4) to be zero:

$$\frac{2}{N} \sum_{j(\neq i)} \frac{1}{\lambda_i - \lambda_j} - \lambda_i + g\lambda_i^3 = 0 \qquad \text{for all } i. \tag{4–5}$$

As $N \to \infty$, the sum in (4–5) tends to the principal part $\mathcal{P}P \int \frac{\mathrm{d}\lambda'\,\rho(\lambda')}{\lambda_i - \lambda'}$; it is therefore convenient to introduce the resolvent

$$\omega(\lambda) = \lim_{N\to\infty} \left\langle \frac{1}{N} \operatorname{tr} \frac{1}{\lambda - M} \right\rangle = \int_{-2a}^{2a} \mathrm{d}\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} \tag{4–6}$$

for any $\lambda \notin [-2a, 2a]$. Since the λ_i fill the interval $[-2a, 2a]$, the saddle point equations finally read

$$\omega(\lambda + i0) + \omega(\lambda - i0) - \lambda + g\lambda^3 = 0 \qquad \text{for all } \lambda \in [-2a, 2a]. \tag{4–7}$$

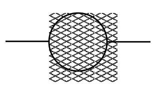
This is a simple (scalar) Riemann–Hilbert problem which can be solved:

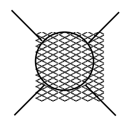
$$\omega(\lambda) = \tfrac{1}{2}\lambda - \tfrac{1}{2}g\lambda^3 - \left(-\tfrac{1}{2}g\lambda^2 + \tfrac{1}{2} - ga^2\right)\sqrt{\lambda^2 - 4a^2}, \tag{4–8}$$

where a is fixed, using the normalization condition $\int \rho(\lambda)\,\mathrm{d}\lambda = 1$, to be $a^2 = (1 - \sqrt{1 - 12g})/(6g)$; equivalently,

$$\rho(\lambda) = \tfrac{1}{2}\pi i(\omega(\lambda - i0) - \omega(\lambda + i0)) = (-\tfrac{1}{2}g\lambda^2 + \tfrac{1}{2} - ga^2)\sqrt{4a^2 - \lambda^2}.$$

What we have found is a generalized semi-circle law (indeed, for $g = 0$ we recover the usual semi-circle law for the Gaussian Unitary Ensemble). From (4–6) it is clear that $\omega(\lambda)$ is a generating function of the G_{2n}, defined by (4–3); so that we can extract

$G_2(g) =$  $= \tfrac{1}{3}a^2(4 - a^2),$

$\Gamma(g) = G_4(g) - 2G_2(g)^2 =$  $= a^4(a^2 - 1)(2a^2 - 5).$

Also, we find

$$F(g) = \tfrac{1}{2}\log a^2 - \tfrac{1}{24}(a^2 - 1)(9 - a^2).$$

All these expressions can now be analytically continued to $g > 0$ all the way to the singularity $g_c = 1/12$. This has a simple interpretation: changing the sign of $g > 0$ corresponds to making the potential in which the eigenvalues lie unstable; however, there is still a local minimum at the origin and since the large N limit is a *classical limit*, the eigenvalues cannot quantum tunnel to the unstable region and therefore remain in the valley (Figure 6). However, as g reaches its critical value g_c, the eigenvalues begin overflowing, which causes the singularity.

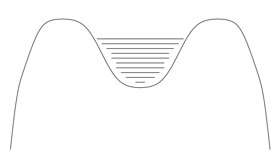


Figure 6. The potential for $g > 0$; analytic continuation is possible as long as the eigenvalues stay trapped inside the central valley.

Of course, $F(g)$ is not yet the counting function of prime alternating links. There are two separate problems to resolve:

1. Are the diagrams *reduced* (do they have minimal crossing number)? Do they correspond to *prime* links?

2. What about the flype equivalence? One should count only once different diagrams which are flype-equivalent.

We shall address them now.

4.2. Primality and Minimality. The diagrams obtained from the matrix model can have "nugatory" crossings or "nonprime" parts (Figure 7). However,

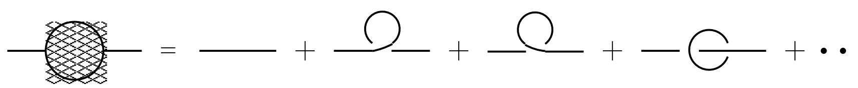


Figure 7. First terms in the perturbative expansion of the 2-point function.

all these unwanted features appear as part of the two-point function. Therefore, in order to remove them, we must simply set the two-point function equal to 1! This is achieved by introducing an additional parameter t in the action:

$$Z^{(N)}(t,g) = \int \mathrm{d}M\, \mathrm{e}^{N\ \mathrm{tr}\left(-\frac{t}{2}M^2 + \frac{g}{4}M^4\right)}. \tag{4–9}$$

Of course, t can be absorbed in a rescaling of M, so the model is essentially unchanged. However we can now ask that t be chosen as a function of g such that

$$G_2(t(g), g) = 1. \tag{4–10}$$

We can solve this equation; the auxiliary function $a(g)$ introduced earlier is now the solution of a third degree equation

$$27g = (a^2 - 1)(4 - a^2)^2, \tag{4–11}$$

equal to 1 when $g = 0$; and $t(g)$ is given by

$$t(g) = \tfrac{1}{3}a^2(g)\big(4 - a^2(g)\big). \tag{4–12}$$

The function $\Gamma(g) := \Gamma(t(g), g)$ is then the counting function for reduced alternating tangle diagrams. Similarly, $F(g)$ defined by $\frac{\mathrm{d}}{\mathrm{d}g}F(g) = \frac{1}{4}G_4(t(g), g)$ counts alternating link diagrams. We find in particular that the singularity of $F(g) = \sum_p f_p g^p$ (given by the equation $g_c/t^2(g_c) = \frac{1}{12}$) has moved to $g_c = \frac{4}{27}$; taking into account the power of the singularity, we find that the rate of growth of the number of alternating diagrams with p crossings is

$$f_p \overset{p\to\infty}{\sim} \text{const}\ 6.75^p\, p^{-7/2}. \tag{4–13}$$

A similar result was found in [Tutte 1963].

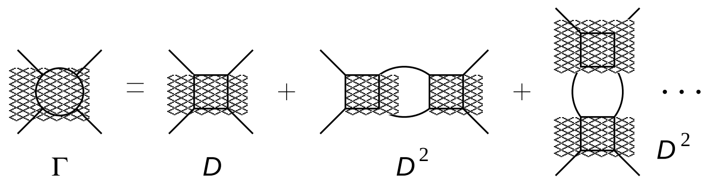


Figure 8. The whole set of diagrams Γ built out of the 2PI diagrams D.

4.3. Flype Equivalence. The more serious problem we have to resolve is that we are not really counting links: we are counting diagrams, and links are flype-equivalence classes of diagrams. Here we shall follow [Sundberg and Thistlethwaite 1998].

Let us take a closer look at the action of a flype (Figure 3). The key remark is that it acts on tangles (four-point functions), but more precisely on *two-particle reducible* (2PR) tangles. This leads naturally to the idea of introducing *skeleton diagrams*: a general connected tangle can be created by putting *two-particle-irreducible* (2PI) diagrams in the "slots" of a *fully two-particle-reducible* skeleton diagram. We then expect that the 2PR skeleton will be modified by the flype-equivalence, whereas the 2PI pieces (or more precisely the corresponding skeletons, see below) will be unaffected. Let $\Gamma(g) = G_4(g) - 2G_2(g)^2$ be the counting function of connected tangles and $D(g)$ of 2PI tangles; then $\Gamma\{D\}$, that is the power series obtained by composing $\Gamma(g)$ and the inverse of $D(g)$, is the counting function of fully 2PR skeleton diagrams (Figure 8). It is easy to see from general combinatorial arguments that $D(g) = \Gamma(g)\big(1-\Gamma(g)\big)/\big(1+\Gamma(g)\big)$, and therefore

$$\Gamma\{D\} = \tfrac{1}{2}\big(1 - D - \sqrt{(1-D)^2 - 4D}\big). \tag{4–14}$$

Inversely if $D(g) = g + \zeta(g)$, then $\zeta[\Gamma]$ is the counting function of *fully 2PI* skeletons diagrams (Figure 9). From the solution of the one-matrix model, one

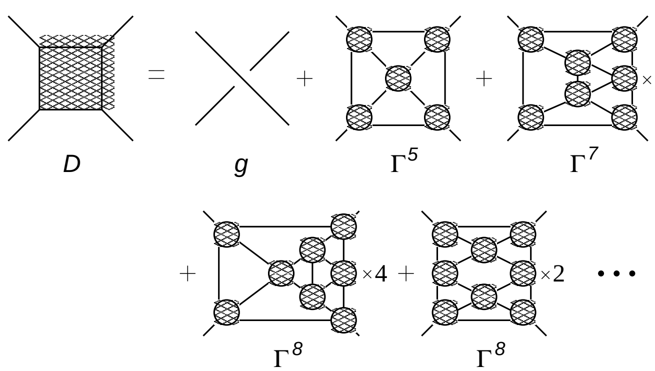


Figure 9. The set of 2PI diagrams built out of general diagrams Γ (plus the single crossing g).

obtains

$$\zeta[\Gamma] = -\frac{2}{1+\Gamma} + 2 - \Gamma - \frac{1}{2}\frac{1}{(\Gamma+2)^3}\big(1 + 10\Gamma - 2\Gamma^2 - (1-4\Gamma)^{3/2}\big). \qquad (4–15)$$

As we mentioned earlier, after taking into account the flyping equivalence, Equation (4–14) will be modified, but not Equation (4–15). To demonstrate how it works, we show how the counting of Figure 8 is redone (Figure 10).

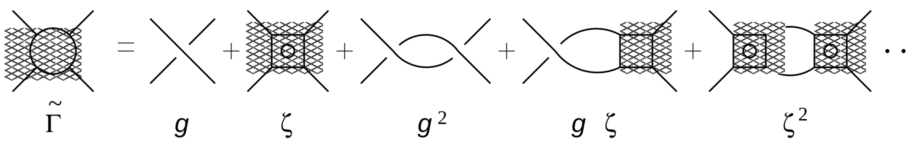


Figure 10. Taking into account the flyping equivalence forces us to distinguish simple crossings from nontrivial 2PI diagrams (marked with a circle). There is only one term $g\zeta$ because the other term is obtained by a flype.

More generally, we can redo the simple combinatorics to find the generating function of the 2PR skeletons, but this time taking into account the flype equivalence. We find

$$\tilde{\Gamma}\{g,\zeta\} = \frac{1}{2}\left((1+g-\zeta) - \sqrt{(1-g+\zeta)^2 - 8\zeta - 8\frac{g^2}{1-g}}\right). \qquad (4–16)$$

This is to be combined with the (unaltered) matrix model data

$$\zeta[\Gamma] = -\frac{2}{1+\Gamma} + 2 - \Gamma - \frac{1}{2}\frac{1}{(\Gamma+2)^3}\big(1 + 10\Gamma - 2\Gamma^2 - (1-4\Gamma)^{3/2}\big). \qquad (4–17)$$

In practice, this means that $\tilde{\Gamma}(g)$ is given by an implicit equation:

$$\tilde{\Gamma}(g) = \tilde{\Gamma}\{g, \zeta[\tilde{\Gamma}(g)]\}, \qquad (4–18)$$

which can be reduced to a fifth degree equation. From the generating function of tangles $\tilde{\Gamma}(g)$ we can go back to the generating function of closed diagrams $\tilde{F}(g)$; we find in particular that the singularity has been displaced again, so that if $\tilde{F}(g) = \sum_{p=0}^{\infty} \tilde{f}_p g^p$, then

$$\tilde{f}_p \overset{p\to\infty}{\sim} \text{const}\left(\frac{101+\sqrt{21001}}{40}\right)^p p^{-7/2}, \qquad (4–19)$$

where the quantity in parentheses equals approximately 6.14793. This result was first obtained in [Sundberg and Thistlethwaite 1998].

5. The $ABAB$ Model and Character Expansion

We shall now inspect the $n = 2$ case of the general $O(n)$ model (3–2). There are various reasons that this model is of particular interest, and we shall discover some of them along the way. Let us rewrite the partition function

$$Z^{(N)}(2,g) = \int \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{N\ \mathrm{tr}\left(-\frac{1}{2}(A^2+B^2)+\frac{g}{4}(A^4+B^4)+\frac{g}{2}(AB)^2\right)}. \quad (5\text{–}1)$$

We see that we could introduce two coupling constants α and β:

$$Z^{(N)}_{ABAB}(\alpha,\beta) = \int \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{N\ \mathrm{tr}\left(-\frac{1}{2}(A^2+B^2)+\frac{\alpha}{4}(A^4+B^4)+\frac{\beta}{2}(AB)^2\right)} \quad (5\text{–}2)$$

(which amounts to introducing "interaction" between the two colors of loops). For $\alpha = \beta = g$ we recover the $O(2)$ model. The more general model with α and β arbitrary is not necessary for the original counting problem, but since it turns out that we can solve it equally easily, we shall keep the two coupling constants. Note that when $\alpha \neq \beta$ the $O(2)$ symmetry of the model is broken. This is even more apparent if we make the change of variables $X = (A+iB)/\sqrt{2}$, $X^\dagger = (A-iB)/\sqrt{2}$:

$$Z^{(N)}_{8v}(b,c,d) = \int \mathrm{d}X\,\mathrm{d}X^\dagger\,\mathrm{e}^{N\ \mathrm{tr}\left(-XX^\dagger+bX^2X^{\dagger 2}+\frac{c}{2}(XX^\dagger)^2+\frac{d}{4}(X^4+X^{\dagger 4})\right)}, \quad (5\text{–}3)$$

with $b = (\alpha+\beta)/2$ and $c = d = (\alpha-\beta)/2$. We recognize in (5–3) the partition function of the *8-vertex model* on random dynamical lattices (more precisely, a two-parameter slice of it, since c and d are not independent). A configuration of the model is defined by a quadr-angulated surface with arrows on the edges of the graph, such that each of the vertices displays one of the eight allowed configurations, which are weighted with the 3 constants b, c, d. For $\alpha = \beta$ the $U(1)$-breaking term $X^4 + X^{\dagger 4}$ vanishes and we recover the *6-vertex model*. (In this case note that the arrows "cross" each other just like strings in links, so that we manifestly recover our link model with the 2 orientations of the loops playing the same role as the 2 colors.)

We shall now show how to solve the model in the planar limit, i.e., compute the large N free energy.

5.1. Character Expansion. All known matrix model solutions are (more or less implicitly) based on the fact that we can reduce the number of degrees of freedom from N^2 to N. Usually the N remaining degrees of freedom are the eigenvalues of the matrices. Unfortunately, from Equation (5–2) one cannot go directly to the eigenvalues of A and B: we do not know how to integrate over the relative angle between A and B. (Only one integral of this type is known exactly, the Harish-Chandra–Itzykson–Zuber integral [Harish-Chandra 1957; Itzykson and Zuber 1980], but it does not apply here.) Therefore, instead of working directly with (5–2), we expand the troublesome part $\exp\left(N\frac{\beta}{2}\,\mathrm{tr}(AB)^2\right)$

in *characters* of $\mathrm{GL}(N)$. Recalling that all class-functions can be expanded on the basis of characters, we write

$$\mathrm{e}^{N\frac{\beta}{2}\operatorname{tr}(AB)^2} = \sum_{\{h\}} c_{\{h\}}\chi_{\{h\}}(AB), \tag{5–4}$$

where $\chi_{\{h\}}(AB)$ is the character taken at AB and $\{h\}$ is the set of shifted highest weights $h_i = m_i + N - i$ (m_i highest weights), $h_1 > h_2 > \cdots > h_n \geq 0$, which parameterize the $\mathrm{GL}(N)$ analytic irreducible representation. The coefficients of the expansion $c_{\{h\}}$ can be determined explicitly:

$$c_{\{h\}} = (N\beta/2)^{\#h/2}\frac{\Delta(h^{\text{even}}/2)\Delta((h^{\text{odd}}-1)/2)}{\prod_i \lfloor h_i/2\rfloor!} \tag{5–5}$$

in terms of the set $\{h^{\text{even}}\}$ and $\{h^{\text{odd}}\}$ of even and odd h_i. The advantage of characters is that they satisfy orthogonality relations, so that we can now integrate over the relative angle between A and B:

$$\int_{U(N)} \mathrm{d}\Omega\, \chi_{\{h\}}(A\Omega B\Omega^\dagger) = \frac{\chi_{\{h\}}(A)\chi_{\{h\}}(B)}{\chi_{\{h\}}(1)} \tag{5–6}$$

where the dimension $\chi_{\{h\}}(1)$ is up to an overall constant the Vandermonde determinant $\Delta[h_i]$.

Once Equations (5–4) and (5–6) are inserted into (5–2), we see that the integrand only depends on the eigenvalues of A and B:

$$Z^{(N)}_{ABAB}(\alpha,\beta) = \sum_{\{h\}} \frac{c_{\{h\}}}{\Delta[h_i]}\left(\int \prod_{i=1}^{N} \mathrm{d}\lambda_i\, \mathrm{e}^{N\sum_{i=1}^{N}\left(-\frac{1}{2}\lambda_i^2+\frac{\alpha}{4}\lambda_i^4\right)}\, \Delta[\lambda_i] \det_{i,j}[\lambda_i^{h_j}]\right)^2 \tag{5–7}$$

The key observation here is that we still have an action of order N^2, but we have N highest weights h_i and N eigenvalues λ_i; therefore a saddle point analysis applies again.

5.2. Saddle Point on Young Diagrams. The notion of a saddle point on Young diagrams first appeared in [Vershik and Kerov 1977] in the context of the asymptotics of the Plancherel measure. It was rediscovered independently in the solution of large N 2D Yang–Mills [Douglas and Kazakov 1993], and was used to deal with character expansions in [Kazakov et al. 1996b; 1996a; 1996c; Kostov et al. 1998]. In the present calculation, the novelty is that we have to deal with a *double* saddle point equation on both eigenvalues and shifted highest weights (that is, the shape of the Young tableau) [Kazakov and Zinn-Justin 1999].

The idea here is to find an appropriate scaling ansatz for the shape of the dominant Young diagram in the large N limit. We find that the highest weights h_i scale as N (the Young diagrams become large both horizontally and vertically), so we can define a continuous density of rescaled h_i/N by

$$\rho(h) = \frac{1}{N}\sum_{i=1}^{N} \delta(h - h_i/N),$$

and the corresponding resolvent

$$H(h) = \int \mathrm{d}h' \frac{\rho(h')}{h-h'}.$$

We also have a density of eigenvalues $\rho(\lambda)$ and the resolvent $\omega(\lambda)$.

If we introduce "slashed" functions by $\not{H}(h) := \frac{1}{2}(H(h+i0)+H(h-i0))$ (and similarly for the other functions), The saddle point equations now read:

$$\begin{cases} -\lambda + \alpha\lambda^3 + \not{\omega}(\lambda) + \not{h}(\lambda)/\lambda = 0, & \lambda \in [-\lambda_0, +\lambda_0], \\ \not{L}(h) - \dfrac{\not{H}(h)}{2} = \dfrac{1}{2}\log(h/\beta), & h \in [h_1, h_2]. \end{cases} \tag{5–8}$$

with $L(h) = \log\lambda^2(h)$. The new unknown functions $h(\lambda)$ and $\lambda(h)$ appear when taking the logarithmic derivative of $\det_{i,j}[\lambda_i^{h_j}]$; this type of functions was analyzed in [Zinn-Justin 1998], where it was shown that $\lambda(h)$ and $h(\lambda)$ are *functional inverses* of each other. Therefore, we have two saddle point equations which are connected by a functional inversion relation. This connection allows to solve them; skipping the details, one can show that one has a well-defined Riemann–Hilbert problem for the auxiliary function $D(h) := 2L(h) - H(h) - 3\log h + \log(h-h_1)$, whose solution can be expressed in terms of Θ functions in an appropriate elliptic parametrization $y(h)$ [Kazakov and Zinn-Justin 1999]:

$$D(h) = \log\frac{h-h_1}{-\alpha h^2} - \frac{\log(\beta/\alpha)}{K} y(h) + 2\log\frac{\Theta_2(x_0 - y(h))}{\Theta_2(x_0 + y(h))}.$$

5.3. Phase Diagram and Discussion. Just as the one-matrix model displayed a singularity at $g_c = \frac{1}{12}$, here the free energy and the various correlation functions have a line of singularities in the (α, β) plane, shown on Figure 11.

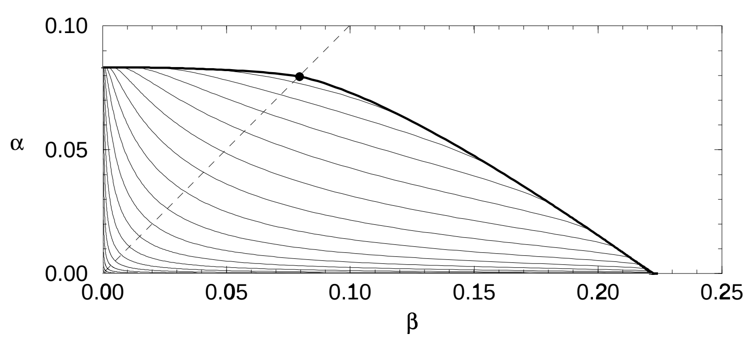


Figure 11. Phase diagram of the $ABAB$ model. The dashed line is the $\alpha = \beta$ line, the curves are equipotentials of the elliptic nome.

We recognize at $\alpha = \frac{1}{12}$, $\beta = 0$ the usual singularity of the one-matrix model. In fact, one can show that everywhere on the critical line except at the critical point $\alpha_c = \beta_c = \frac{1}{4\pi}$ the critical behavior is the same as the one of the one-matrix model ("pure gravity" behavior). This implies the following asymptotics: at fixed slope $s = \beta/\alpha \neq 1$, if the free energy $F(\alpha, \beta = s\alpha) = \sum_p f_p(s)\alpha^p$ then

$$f_p(s) \overset{p\to\infty}{\sim} \text{const } \alpha_c(s)^{-p}\, p^{-7/2}.$$

However, the point $\alpha_c = \beta_c = \frac{1}{4\pi}$, that is, the 6-vertex model point, is very special: it is the point where the elliptic functions degenerate into trigonometric functions, which implies logarithmic corrections:

$$f_p(1) \overset{p\to\infty}{\sim} \text{const } (4\pi)^p\, p^{-3}(\log p)^2.$$

This is characteristic of a $c = 1$ conformal field theory coupled to gravity.

5.4. Application to Reduced Alternating Diagrams. We should remember that the $\alpha = \beta$ line is of special interest to us, since it is the intersecting loops $O(2)$ model (solving a certain counting problem for alternating links) we started from. In order to carry out the program that we have applied to the one-matrix model we should next address the two issues of primality/minimality and of the flype equivalence. We shall only consider the first issue; a discussion of the flype equivalence in the general $O(n)$ case will be made in the next section, and the corresponding calculation for $n = 2$ will appear in a future publication by the author and J.-B. Zuber.

Again, we introduce an additional parameter t in the action:

$$Z^{(N)}(2,t,g) = \int \mathrm{d}A\,\mathrm{d}B\,\mathrm{e}^{N \text{ tr}\left(-\frac{t}{2}(A^2+B^2)+\frac{g}{4}(A^4+B^4)+\frac{g}{2}(AB)^2\right)} \tag{5–9}$$

and we impose the condition that the 2-point function

$$G_2(t,g) = \lim_{N\to\infty}\langle \tfrac{1}{N}\text{ tr } A^2\rangle = \lim_{N\to\infty}\langle \tfrac{1}{N}\text{ tr } B^2\rangle$$

satisfy

$$G_2(t(g),g) = 1. \tag{5–10}$$

Obvious scaling properties imply that $G_2(t,g) = \frac{1}{t}G_2(1, g/t^2)$, and the formula for $G_2(1,g)$ can be found in appendix A of [Kazakov and Zinn-Justin 1999] in terms of complete elliptic integrals. This gives an equation for $t(g)$, which can in principle be solved (at least to an arbitrary order in perturbation theory).

To go further, we notice that at the singularity we must have $g_c/t(g_c)^2 = \frac{1}{4\pi}$; from [Kazakov and Zinn-Justin 1999] we extract $G_2(1, 1/(4\pi)) = \frac{\pi}{2}(4-\pi)$, and therefore using (5–10), $t(g_c) = \frac{\pi}{2}(4-\pi)$, which finally yields

$$g_c = \frac{\pi}{16}(\pi-4)^2. \tag{5–11}$$

We conclude that the number f_p of reduced alternating link diagrams with 2 colors and p crossings has the asymptotics

$$f_p \overset{p\to\infty}{\sim} \text{const}\ \left(\frac{16}{\pi(\pi-4)^2}\right)^p p^{-3}(\log p)^2,$$

where the number $1/g_c = 6.91167\ldots$ is slightly larger than the value 6.75 obtained for only one color.

6. Further Generalizations and Prospects

We have already written a fairly general model, the intersecting loops $O(n)$ model (Equation (3–2)) which should contain in principle all information on the counting of alternating links and knots. We shall now show how this model is in fact not sufficient for our purposes.

Indeed, one should remember that the $O(n)$ model given above counts alternating link diagrams, and not alternating links. For the latter, one should address the problem of the flype equivalence. We have seen in the $n = 1$ case (Section 4) that we needed to do a little surgery on the four-point functions. One can convince oneself that what it amounts to, in more physical terms, is a *finite renormalization* which results in the appearance of quartic counterterms in the action. Generically these counterterms will have the most general form compatible with the symmetry. In our case, we find that there are two independent $O(n)$-symmetric tetravalent vertices, of the form $M_aM_bM_aM_b$ and $M_aM_aM_bM_b$. This results in a generalized $O(n)$ model:

$$Z^{(N)}(n,g,h) = \int \prod_{a=1}^{n} \mathrm{d}M_a\, \mathrm{e}^{N\ \mathrm{tr}\left(-\frac{1}{2}\sum_{a=1}^{n} M_a^2 + \frac{g}{4}\sum_{a,b=1}^{n}(M_aM_b)^2 + \frac{h}{2}\sum_{a,b=1}^{n} M_a^2M_b^2\right)}, \tag{6–1}$$

where h will be given as a function of g by appropriate combinatorial relations of the same form as those of Section 4.

At the moment, the solution of this general model is unknown. We note, however, that for $g = 0$ this model is simply the usual (nonintersecting loops) $O(n)$ model, which has been completely solved [Kostov 1989; Gaudin and Kostov 1989; Kostov and Staudacher 1992]. It is tempting to speculate that there is no phase transition in the (g,h) plane as one moves away from the $g = 0$ line. (This is certainly true for $n = 2$, as shown by the study in [Dalley 1992] and the exact solution in [Zinn-Justin 2000].) Then, one can make predictions on *universal quantities* such as critical exponents of the model. For example, the number $\tilde{f}_p(n)$ of prime alternating links with n colors would have the asymptotics

$$\tilde{f}_p(n) \overset{?}{\sim} \text{const}(n)\ b(n)^p\, p^{-2-1/\nu}, \qquad n = -2\cos(\pi\nu),\ 0 < \nu < 1.$$

In particular the number $\tilde{f}_p$ of prime alternating knots would satisfy

$$\tilde{f}_p \overset{?}{\sim} \text{const}\ b(0)^p\, p^{-4}.$$

One interesting question is to determine the nonuniversal constant $b(0)$. This, of course, requires to really solve the $n = 0$ model (or more precisely to study the $n \to 0$ limit). An alternative option is to note that this model can be recast as a supersymmetric $Osp(2n|2n)$ model, the simplest ($n = 1$) being a supersymmetrized version of the $O(2)$ model considered earlier; using bosonic and fermionic complex matrices X and Ψ, it can be written as

$$Z^{(N)}(g) = \int \mathrm{d}X\, \mathrm{d}X^\dagger\, \mathrm{d}\Psi\, \mathrm{d}\Psi^\dagger \times e^{N\ \mathrm{tr}\left(-XX^\dagger - \Psi\Psi^\dagger + g(XX^\dagger X^\dagger X + \Psi\Psi^\dagger\Psi^\dagger\Psi + \Psi X^\dagger \Psi^\dagger X + X^\dagger \Psi X \Psi^\dagger)\right)}. \tag{6–2}$$

Due to supersymmetry, this partition function is equal to 1, but nonsupersymmetric correlation functions such as $\left\langle \frac{1}{N}\,\mathrm{tr}(XX^\dagger)^n \right\rangle$ are nontrivial and should contain the desired information. Whether this model is solvable or not is an open question.

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References

[Brézin et al. 1978] E. Brézin, C. Itzykson, G. Parisi, and J. B. Zuber, "Planar diagrams", *Comm. Math. Phys.* **59**:1 (1978), 35–51.

[Dalley 1992] S. Dalley, *Mod. Phys. Lett.* **A7** (1992), 1651–.

[Di Francesco et al. 1995] P. Di Francesco, P. Ginsparg, and J. Zinn-Justin, "2D gravity and random matrices", *Phys. Rep.* **254**:1-2 (1995), 133.

[Di Francesco et al. 1997] P. Di Francesco, O. Golinelli, and E. Guitter, "Meanders and the Temperley-Lieb algebra", *Comm. Math. Phys.* **186**:1 (1997), 1–59.

[Di Francesco et al. 1998] P. Di Francesco, B. Eynard, and E. Guitter, "Coloring random triangulations", *Nuclear Phys. B* **516**:3 (1998), 543–587.

[Douglas and Kazakov 1993] M. R. Douglas and V. A. Kazakov, *Phys. Lett.* **B319** (1993), 219–.

[Gaudin and Kostov 1989] M. Gaudin and I. Kostov, "O(n) model on a fluctuating planar lattice. Some exact results", *Phys. Lett. B* **220**:1-2 (1989), 200–206.

[Harish-Chandra 1957] Harish-Chandra, "Differential operators on a semisimple Lie algebra", *Amer. J. Math.* **79** (1957), 87–120.

[Itzykson and Zuber 1980] C. Itzykson and J. B. Zuber, "The planar approximation, II", *J. Math. Phys.* **21**:3 (1980), 411–421.

[Itzykson and Zuber 1992] C. Itzykson and J.-B. Zuber, "Combinatorics of the modular group. II. The Kontsevich integrals", *Internat. J. Modern Phys. A* **7**:23 (1992), 5661–5705.

[Kauffman 1993] L. H. Kauffman, *Knots and physics*, 2nd ed., World Scientific, River Edge, NJ, 1993.

[Kazakov 1986] V. A. Kazakov, "Ising model on a dynamical planar random lattice: exact solution", *Phys. Lett. A* **119**:3 (1986), 140–144.

[Kazakov and Zinn-Justin 1999] V. A. Kazakov and P. Zinn-Justin, "Two-matrix model with $ABAB$ interaction", *Nuclear Phys. B* **546**:3 (1999), 647–668.

[Kazakov et al. 1996a] V. A. Kazakov, M. Staudacher, and T. Wynter, "Almost flat planar diagrams", *Comm. Math. Phys.* **179**:1 (1996), 235–256.

[Kazakov et al. 1996b] V. A. Kazakov, M. Staudacher, and T. Wynter, "Character expansion methods for matrix models of dually weighted graphs", *Comm. Math. Phys.* **177**:2 (1996), 451–468.

[Kazakov et al. 1996c] V. A. Kazakov, M. Staudacher, and T. Wynter, "Exact solution of discrete two-dimensional R^2 gravity", *Nuclear Phys. B* **471**:1-2 (1996), 309–333.

[Kontsevich 1991] M. L. Kontsevich, "Intersection theory on the moduli space of curves", *Funktsional. Anal. i Prilozhen.* **25**:2 (1991), 50–57, 96.

[Kostov 1989] I. K. Kostov, "O(n) vector model on a planar random lattice: spectrum of anomalous dimensions", *Modern Phys. Lett. A* **4**:3 (1989), 217–226.

[Kostov and Staudacher 1992] I. K. Kostov and M. Staudacher, "Multicritical phases of the O(n) model on a random lattice", *Nuclear Phys. B* **384**:3 (1992), 459–483.

[Kostov et al. 1998] I. K. Kostov, M. Staudacher, and T. Wynter, "Complex matrix models and statistics of branched coverings of 2D surfaces", *Comm. Math. Phys.* **191**:2 (1998), 283–298.

[Makeenko 1996] Y. Makeenko, "Strings, matrix models, and meanders", *Nuclear Phys. B Proc. Suppl.* **49** (1996), 226–237. Theory of elementary particles (Buckow, 1995).

[Menasco and Thistlethwaite 1991] W. W. Menasco and M. B. Thistlethwaite, "The Tait flyping conjecture", *Bull. Amer. Math. Soc.* (*N.S.*) **25**:2 (1991), 403–412.

[Menasco and Thistlethwaite 1993] W. Menasco and M. Thistlethwaite, "The classification of alternating links", *Ann. of Math.* (2) **138**:1 (1993), 113–171.

[Reidemeister 1932] K. Reidemeister, *Knotentheorie*, Ergebnisse der Math. **1**, Springer, Berlin, 1932.

[Sundberg and Thistlethwaite 1998] C. Sundberg and M. Thistlethwaite, "The rate of growth of the number of prime alternating links and tangles", *Pacific J. Math.* **182**:2 (1998), 329–358.

[Tutte 1963] W. T. Tutte, "A census of planar maps", *Canad. J. Math.* **15** (1963), 249–271.

[Vershik and Kerov 1977] A. M. Vershik and S. V. Kerov, *Soviet. Math. Dokl.* **18** (1977), 527–.

[Witten 1991] E. Witten, "Two-dimensional gravity and intersection theory on moduli space", pp. 243–310 in *Surveys in differential geometry* (Cambridge, MA, 1990), Lehigh Univ., Bethlehem, PA, 1991.

[Zinn-Justin 1998] P. Zinn-Justin, "Universality of correlation functions of Hermitian random matrices in an external field", *Comm. Math. Phys.* **194**:3 (1998), 631–650. Erratum in **199**:3 (1999), 729.

[Zinn-Justin 2000] P. Zinn-Justin, "The six-vertex model on random lattices", *Europhys. Lett.* **50** (2000), 15–21.

[Zinn-Justin and Zuber 1999] P. Zinn-Justin and J.-B. Zuber, "Matrix integrals and the counting of tangles and links", in *Formal power series and algebraic combinatorics: 11th international conference, FPSAC'99* (Barcelona, 1999), 1999. Available at www.arxiv.org/abs/math-ph/9904019.

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