Tannakian Fundamental Groups
Associated to Galois Groups

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ABSTRACT. The goal of this paper is to give background and motivation for
several conjectures of Deligne and Goncharov concerning the action of the
absolute Galois group on the fundamental group of the thrice punctured
line, and to sketch solutions, complete and partial, of several of them. A
major ingredient in these is the theory of weighted completion of profinite
groups. An exposition of weighted completion from the point of view of
tannakian categories is included.

1. Introduction

The goal of this paper is to provide background, heuristics and motivation for
several conjectures of Deligne [14, 8.2, p. 163], [14, 8.9.5, p. 168] and [27, p. 300]
and Goncharov [19, Conj. 2.1], presumably along the lines used to arrive at them.
A complete proof of the third of these conjectures, and partial solutions of the
remaining three are given in [23].[1] A second goal of this paper is to show that the
weighted completion of a profinite group, developed in [23], and a key ingredient
in the proofs referred to above, can be defined as the tannakian fundamental
group[2] of certain categories of modules of the group. This should help clarify
the role of weighted completion in [23].

[1] After writing that paper, we learned from Goncharov that proofs of ℓ-adic versions of [14,
8.2, p. 163] and [14, 8.9.5, p. 168] had previously been given in the unpublished manuscript [2]
of Beilinson and Deligne.

[2] A tannakian category $\mathcal{T}$ with fiber functor ω is equivalent to the category of representations
of the automorphism group of ω. We shall refer to this proalgebraic group as the *tannakian fundamental group* of $\mathcal{T}$ with respect to the base point ω. Basic material on tannakian categories,
such as their definition, can be found in [13].

2. Motivic Cohomology

It is believed that there is a universal cohomology theory, called *motivic cohomology*. It should be defined for all schemes X. It is indexed by two integers m and n. The coefficient ring Λ is typically $\mathbb{Z}$, $\mathbb{Z}/N$, $\mathbb{Z}_\ell$, $\mathbb{Q}$ or $\mathbb{Q}_\ell$; the corresponding motivic cohomology group is denoted

$$H^m_{\mathrm{mot}}(X, \Lambda(n)).$$

There should be cup products

$$H^{m_1}_{\mathrm{mot}}(X, \Lambda(n_1)) \otimes H^{m_2}_{\mathrm{mot}}(X, \Lambda(n_2)) \to H^{m_1+m_2}_{\mathrm{mot}}(X, \Lambda(n_1 + n_2)). \qquad (2\text{--}1)$$

Motivic cohomology should have the following universal mapping property: if $H^\bullet_{\mathcal{C}}(\ , \Lambda(\ ))$ is any Bloch-Ogus cohomology theory [9] (such as étale cohomology, Deligne cohomology, Betti (i.e., singular) cohomology, de Rham cohomology, and crystalline cohomology) there should be a unique natural transformation

$$H^m_{\mathrm{mot}}(\ , \Lambda(n)) \to H^m_{\mathcal{C}}(\ , \Lambda(n))$$

compatible with products and Chern classes

$$c_n : K_m(X) \to H^{2n-m}_{\mathcal{C}}(X, \Lambda(n)),$$

where $K_\bullet$ denotes Quillen's algebraic K-group functor, [39].

Beilinson's definition.   Beilinson [1] observed that the motivic cohomology with $\mathbb{Q}$ coefficients of a large class of schemes could be defined in terms of Quillen's algebraic K-theory [39].

Suppose that X is the spectrum of the ring of S-integers in a number field or a smooth scheme over a perfect field. Denote the algebraic K-theory of X by $K_\bullet(X)$. As in the case of topological K-theory, there are Adams operations ([25], [31])

$$\psi^k : K_\bullet(X) \to K_\bullet(X),$$

defined for all $k \in \mathbb{Z}_+$. They can be simultaneously diagonalized:

$$K_m(X) \otimes \mathbb{Q} = \bigoplus_{n \in \mathbb{Z}} K_m(X)^{(n)}$$

where ψ^k acts as k^n on $K_m(X)^{(n)}$.

DEFINITION 2.1 (BEILINSON).   For a ring Λ containing $\mathbb{Q}$, define the motivic cohomology groups of X by

$$H^m_{\mathrm{mot}}(X, \Lambda(n)) = K_{2n-m}(X)^{(n)} \otimes_{\mathbb{Q}} \Lambda.$$

The ring structure of $K_\bullet(X)$ induces a cup product (2–1) as

$$\psi^k(xy) = \psi^k(x)\psi^k(y) \quad x, y \in K_\bullet(X).$$

Motivation for Beilinson's definition comes from topological K-theory and can be found in the introduction of [4].

If X is smooth, then it follows from a result of Grothendieck (see [12]) that

$$H^{2n}_{\mathrm{mot}}(X, \mathbb{Q}(n)) \cong CH^n(X) \otimes \mathbb{Q}.$$

In the next section, we present the well-known computation of the motivic cohomology of the ring of S-integers in a number field.

PROPOSITION 2.2. *There are Chern classes*

$$c_j^{\mathrm{mot}} : K_m(X) \to H^{2j-m}_{\mathrm{mot}}(X, \mathbb{Q}(j))$$

such that for each Bloch-Ogus cohomology theory $H^\bullet_{\mathcal{C}}(\ , \Lambda(\ ))$, where Λ contains $\mathbb{Q}$, there is a natural transformation

$$H^\bullet_{\mathrm{mot}}(\ , \Lambda(\ )) \to H^\bullet_{\mathcal{C}}(\ , \Lambda(\ ))$$

that is compatible with Chern classes.

PROOF. The basic tool needed to construct the natural transformations to other cohomology theories is the theory of Chern classes

$$c_j : K_m(X) \to H^{2j-m}_{\mathcal{C}}(X, \mathbb{Z}(j))$$

constructed by Beilinson [1] and Gillet [17] for a very large set of cohomology theories $H^\bullet_{\mathcal{C}}$ that includes all Bloch-Ogus cohomology theories. These give rise to the Chern character maps

$$ch : K_m(X) \to \prod_{j \geq 0} H^{2j-m}_{\mathcal{C}}(X, \mathbb{Q}(j)).$$

The degree j part of this

$$ch_j : K_m(X) \to H^{2j-m}_{\mathcal{C}}(X, \mathbb{Q}(j))$$

is a homogeneous polynomial of degree j in the Chern classes, just as in the topological case. The key point is the compatibility with the Adams operations which implies that the restriction of ch_k to $K_m(X)^{(j)}$ vanishes unless $k = j$. It follows that ch_j factors through the projection onto $K_m(X)^{(j)}$:

$$
\begin{array}{ccc}
K_m(X) & \xrightarrow{ch_j} & H^{2j-m}_{\mathcal{C}}(X, \mathbb{Q}(j)) \\
{\scriptstyle \mathrm{proj}}\downarrow & \nearrow & \\
K_m(X)^{(j)} & &
\end{array}
$$

Thus the Chern character induces a natural transformation

$$H_{\mathrm{mot}}^m(X, \mathbb{Q}(n)) \to H_{\mathcal{C}}^m(X, \mathbb{Q}(n)).$$

It is a ring homomorphism as the Chern character is. Define

$$ch_j^{\mathrm{mot}} : K_m(X) \to H_{\mathrm{mot}}^{2j-m}(X, \mathbb{Q}(j))$$

to be the projection $K_m(X) \to K_m(X)^{(j)}$. From this, one can inductively construct Chern classes $c_j^{\mathrm{mot}} : K_m(X) \to H_{\mathrm{mot}}^{2j-m}(X, \mathbb{Q}(j))$. Compatibility with the Chern classes $c_j : K_m(X) \to H_{\mathcal{C}}^{2j-m}(X, \Lambda(j))$ is automatic and guarantees the uniqueness of natural transformation $H_{\mathrm{mot}}^\bullet \to H_{\mathcal{C}}^\bullet$. $\qquad\square$

The quest for cochains.   Beilinson's definition raises many questions and problems such as:

(i) How does one define motivic cohomology with *integral* coefficients?
(ii) Find natural cochain complexes (a.k.a., motivic complexes) whose homology groups are motivic cohomology.
(iii) Compute motivic cohomology groups.

Bloch's higher Chow groups [5] provide an integral version of motivic cohomology as well as a chain complex whose homology is motivic cohomology. (See also [6] and [33].) One difficulty with this approach is that, being based on algebraic cycles and rational equivalence, it is difficult to compute with.

More fundamentally, one would also like motivic cohomology groups to be the ext or hyper-ext groups associated to a suitable category of motives. In the ideal case, this category will be tannakian after tensoring all its objects with $\mathbb{Q}$, so that the category of $\mathbb{Q}$-motives will be equivalent to the category of representations of a proalgebraic group defined over $\mathbb{Q}$. These goals have been achieved to some degree. For all fields k, Voevodsky [44] and Levine [34] have each constructed a triangulated tensor category of "mixed motives over k". For each scheme X, smooth and quasi-projective over k, there is an object $M(X)$ in this category such that $\mathrm{Ext}^\bullet(\mathbb{Z}(-n), M(X))$ is isomorphic to the integral motivic cohomology groups of X (i.e., Bloch's higher Chow groups). However, the categories obtained from the categories of Levine and Voevodsky by tensoring their objects with $\mathbb{Q}$ are not tannakian.

One can also propose that there should be a tannakian category of *mixed Tate motives* over a field k. The motivic cohomology of k should be an ext in this category. In the case where k is a number field (or any field satisfying Beilinson-Soulé vanishing), Levine [32] has constructed such a tannakian category of mixed Tate motives. Goncharov [18, p. 611] later proved a result similar to Levine's and proved, in addition, that the bounded derived category of this tannakian category of mixed Tate motives is equivalent to the full subcategory of mixed Tate motives of the category of mixed motives over k.

An older and less fundamental approach to constructing categories of motives, proposed by Deligne [14] and Jannsen [29], is to view them as "compatible systems of realizations". These also form a tannakian category. We shall take this approach in this paper as it is more accessible and is more consistent with our point of view.

3. The Motivic Cohomology of the Spectrum of a Ring of S-integers

Basic results of Quillen [37] and Borel [10] give the computation of the motivic cohomology of the spectra of rings of S-integers in number fields. Suppose that F is a number field with ring of integers $\mathcal{O}_F$ and that S is a finite subset of $\operatorname{Spec}\mathcal{O}_F$. Set $X_{F,S} = \operatorname{Spec}\mathcal{O}_F - S$. Set

$$d_n = \operatorname{ord}_{s=1-n}\zeta_F(s) = \begin{cases} r_1 + r_2 - 1 & \text{when } n = 1, \\ r_1 + r_2 & \text{when } n \text{ is odd and } n > 1, \\ r_2 & \text{when } n \text{ is even,} \end{cases} \qquad (3\text{–}1)$$

where $\zeta_F(s)$ denotes the Dedekind zeta function of F and r_1, and r_2 denote the number of real and complex places of F, respectively.

THEOREM 3.1. *For all n and m, $H^m_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n))$ is a finite dimensional rational vector space whose dimension is given by*

$$\dim H^m_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n)) = \begin{cases} d_1 + \#S & \text{when } m = n = 1, \\ d_n & \text{when } m = 1 \text{ and } n > 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. First suppose that S is empty. Quillen [37] showed that each K-group $K_m(X_{F,S})$ is a finitely generated abelian group. It follows that each of the groups $H^j_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n))$ is finite dimensional. The rank of $K_0(X_{F,S})$ is 1 and the rank of $K_1(X_{F,S})$ is $r_1 + r_2 - 1$ by the Dirichlet Unit Theorem. The ranks of the remaining $K_m(X_{F,S})$ were computed by Borel [10]. It is zero when m is even and > 0, and d_n when $m = 2n - 1 > 1$. It is easy to see that

$$H^0_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(0)) = K_0(X_{F,S}) \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Borel [11] constructed regulator mappings

$$K_{2n-1}(X_{F,S}) \to \mathbb{R}^{d_n}, \quad n > 0,$$

and showed that each is injective mod torsion. Beilinson [1] showed that Borel's regulator is a non-zero rational multiple of the regulator mapping

$$ch_n : K_{2n-1}(X_{F,S}) \to H^1_{\mathcal{D}}(X_{F,S}, \mathbb{R}(n)) \cong \mathbb{R}^{d_n}$$

to Deligne cohomology. The properties of the Chern character and Borel's injectivity together imply that

$$H^1_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n)) = K_{2n-1}(X_{F,S})^{(n)} = K_{2n-1}(X_{F,S}) \otimes \mathbb{Q}$$

and that $H^m_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n))$ vanishes when $m > 1$, and when $m = 0$ and $n \neq 0$. The result when S is non-empty follows from this using the localization sequence [39], and the fact, due to Quillen [38], that the K-groups of finite fields are torsion groups in positive degree. Together these imply that each prime removed adds one to the rank of K_1 and does not change the rank of any other K-group. □

Denote the Galois group of the maximal algebraic extension of F, unramified outside S, by $G_{F,S}$. In this paper, a finite dimensional $G_{F,S}$-module means a finite dimensional $\mathbb{Q}_\ell$-vector space with continuous $G_{F,S}$-action. Denote the category of $\mathbb{Q}$ mixed Hodge structures by $\mathcal{H}$. Denote the ext functor in the category of finite dimensional $G_{F,S}$-modules by $\mathrm{Ext}_{G_{F,S}}$, and the ext functor in $\mathcal{H}$ by $\mathrm{Ext}_{\mathcal{H}}$. The results on regulators of Borel [11] and Soulé [41] can be stated as follows.

THEOREM 3.2. *The natural transformation from motivic to étale cohomology induces isomorphisms*

$$H^1_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}_\ell(n)) \cong H^1_{\text{ét}}(X_{F,S}, \mathbb{Q}_\ell(n)) \cong \mathrm{Ext}^1_{G_{F,S}}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n))$$

for all $n \geq 1$. The natural transformation from motivic to Deligne cohomology induces injections

$$H^1_{\mathrm{mot}}(X_{F,S}, \mathbb{Q}(n)) \hookrightarrow H^1_{\mathcal{D}}(X_{F,S}, \mathbb{Q}(n)) = \left[\bigoplus_{\nu : F \hookrightarrow \mathbb{C}} \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}, \mathbb{Q}(n)) \right]^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}. \quad \square$$

Thus each element x of $K_{2n-1}(X_{F,S})$ determines an extension

$$0 \to \mathbb{Q}_\ell(n) \to E_{\ell,x} \to \mathbb{Q}_\ell(0) \to 0$$

of ℓ-adic local systems over $X_{F,S}$ and a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant extension

$$0 \to \mathbb{Q}(n) \to E_{\mathrm{Hodge},x} \to \mathbb{Q}(0) \to 0$$

of mixed Hodge structures over $X_{F,S} \otimes \mathbb{C}$. One can think of these as the étale and Hodge realizations of $x \in K_{2n-1}(X_{F,S})$.

4. Mixed Tate Motives

As mentioned earlier, one approach to motivic cohomology is to postulate that to each sufficiently nice scheme X (say, smooth and quasi-projective over a field, or regular over a ring $\mathcal{O}_{F,S}$ of S-integers in a number field) one can associate a category $\mathcal{T}(X)$ of *mixed Tate motives over X*. This should satisfy the following conjectural properties.

(i) $\mathcal{T}(X)$ is a (neutral) tannakian category over $\mathbb{Q}$ with a fiber functor $\omega : \mathcal{T}(X) \to \mathrm{Vec}_{\mathbb{Q}}$ to the category of finite dimensional rational vector spaces.

(ii) Each object M of $\mathcal{T}(X)$ has an increasing filtration called the weight filtration

$$\cdots \subseteq W_{m-1}M \subseteq W_m M \subseteq W_{m+1}M \subseteq \cdots,$$

whose intersection is 0 and whose union is M. Morphisms of $\mathcal{T}(X)$ should be strictly compatible with the weight filtration — that is, the functor

$$\mathrm{Gr}^W_\bullet : M \mapsto \bigoplus_m \mathrm{Gr}^W_m M := \bigoplus_m W_m M / W_{m-1} M$$

to graded objects in $\mathcal{T}(X)$ should be an exact tensor functor.

(iii) $\mathcal{T}(X)$ contains "the Tate motive $\mathbb{Q}(1)$" over $\mathrm{Spec}\, R$ where R is the base ring (here either a field or $\mathcal{O}_{F,S}$). This can be considered as the dual of the local system $R^1 f_*(\mathbb{Q})$ over $\mathrm{Spec}\, R$, where f is the structure morphism of the multiplicative group $\mathbb{G}_m$, i.e., $f : \mathbb{G}_m \otimes R \to \mathrm{Spec}\, R$. Put $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ for $n \in \mathbb{Z}$. (Negative tensor powers are defined by duality.)

(iv) There should be *realization functors* to various categories such as ℓ-adic étale local systems over $X[1/\ell] := X \otimes_R R[1/\ell]$ (where R is the base ring and ℓ does not divide the characteristic of R), variations of mixed Hodge structure over X, etc. These functors should be faithful, exact tensor functors. These functors are related by natural comparison transformations. The Betti, de Rham, ℓ-adic and crystalline realizations of $\mathbb{Q}(1)$ should be the Betti, de Rham, ℓ-adic and crystalline versions of $H_1(\mathbb{G}_m)$.

(v) For each object M, $\mathrm{Gr}^W_{2m+1} M$ is trivial and $\mathrm{Gr}^W_{2m} M$ is isomorphic to the direct sum of a finite number of copies of $\mathbb{Q}(-m)$.

The last property characterizes *mixed Tate* motives among mixed motives. The category $\mathcal{T}(X)$, being tannakian, is equivalent to the category of finite dimensional representations of a proalgebraic $\mathbb{Q}$-group $\pi_1(\mathcal{T}(X), \omega)$, which represents the tensor automorphism group of the fiber functor ω. We denote it simply by $\pi_1(\mathcal{T}(X))$, if the selection of ω does not matter.

There are several approaches to constructing the category $\mathcal{T}(X)$, at least when X is the spectrum of a field or $X = X_{F,S}$, such as those of Bloch-Kriz [8], Levine [32], and Goncharov[18].

We follow Deligne [14] and Jannsen [29], who define a motive to be a "compatible set of realizations" of "geometric origin." This is a tannakian category. Deligne does not define what it means to be of geometric origin, but wants it to be broad enough to include those compatible realizations that occur in the unipotent completion of fundamental groups of varieties in addition to subquotients of cohomology groups. We refer the reader to Section 1 of Deligne's paper [14] for the definition of compatible set of realizations. One example is $\mathbb{Q}(1)$, defined as $H_1(\mathbb{G}_{m/\mathbb{Z}})$, another is the extension E_x of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ coming from $x \in K_{2n-1}(X_{F,S})$ described in the previous section.

The hope is that

$$H^m_{\mathrm{mot}}(X, \mathbb{Q}(n)) \cong \mathrm{Ext}^m_{\mathcal{T}(X)}(\mathbb{Q}(0), \mathbb{Q}(n))$$

holds when X is the spectrum of a field or $X_{F,S}$.[3] This covers the cases of interest for us. In general, one expects that motivic cohomology groups of X can be computed as hyper-exts:

$$H^m_{\mathrm{mot}}(X, \mathbb{Q}(n)) \cong \mathbb{H}^m(X, \mathcal{E}xt^\bullet_{\mathcal{T}}(\mathbb{Q}(0), \mathbb{Q}(n))).$$

Deligne's conjecture (Conjecture 5.5) will be a consequence of:

POSTULATE 4.1. If $X = X_{F,S}$, there is a category of mixed Tate motives $\mathcal{T}(X)$ over X with the above mentioned properties. It has the property that there is a natural isomorphism

$$H^m_{\mathrm{mot}}(X, \mathbb{Q}(n)) \cong \mathrm{Ext}^m_{\mathcal{T}(X)}(\mathbb{Q}(0), \mathbb{Q}(n)),$$

which is compatible with Chern maps.

Examples of mixed Tate motives over $\mathrm{Spec}\,\mathbb{Z}$. One of the main points of [14] is to show that the unipotent completion of the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is an example of a mixed Tate motive (actually a pro-mixed Tate motive), smooth over $\mathrm{Spec}\,\mathbb{Z}$.

As base point, take $\overrightarrow{01}$, the tangent vector of $\mathbb{P}^1$ based at 0 that corresponds to $\partial/\partial t$, where t is the natural parameter on $\mathbb{P}^1 - \{0, 1, \infty\}$. Deligne [14] shows that the unipotent completion of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ is a mixed Tate motive over $\mathrm{Spec}\,\mathbb{Z}$ by exhibiting compatible Betti, étale, de Rham and crystalline realizations of it. It is smooth over $\mathrm{Spec}\,\mathbb{Z}$ essentially because the pair $(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ has everywhere good reduction.

There is an interesting relation to classical polylogarithms which was discovered by Deligne (cf. [14], [3], [20]). There is a polylog local system P, which is a motivic local system over $\mathbb{P}^1_{\mathbb{Z}} - \{0, 1, \infty\}$ in the point of view of compatible realizations. Its Hodge-de Rham realization is a variation of mixed Hodge structure over the complex points of $\mathbb{P}^1 - \{0, 1, \infty\}$ whose periods are given by $\log x$ and the classical polylogarithms: $\mathrm{Li}_1(x) = -\log(1 - x)$, $\mathrm{Li}_2(x)$ (Euler's dilogarithm), $\mathrm{Li}_3(x)$, and so on. Here $\mathrm{Li}_n(x)$ is the multivalued holomorphic function on $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ whose principal branch is given by

$$\mathrm{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$$

in the unit disk.[4]

The fiber $P_{\overrightarrow{01}}$ of P over the base point $\overrightarrow{01}$ is a mixed Tate motive over $\mathrm{Spec}\,\mathbb{Z}$ and has periods the values of the Riemann zeta function at integers $n > 1$. In

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[3]If this is true, then $H^m_{\mathrm{mot}}(\mathrm{Spec}\,F, \mathbb{Q}(n))$ will vanish when $n < 0$ and $m = 0$, and when $n \le 0$ and $m > 0$. This vanishing is a conjecture of Beilinson and Soulé. It is known for number fields, for example.

[4]This goes back to various letters of Deligne. Accounts can be found, for example, in [3] and [20].

fact, $P_{\overrightarrow{01}}$ is an extension

$$0 \to \bigoplus_{n \geq 1} \mathbb{Q}(n) \to P_{\overrightarrow{01}} \to \mathbb{Q}(0) \to 0$$

and thus determines a class

$$(e_n)_n \in \bigoplus_{n \geq 1} \operatorname{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), \mathbb{Q}(n)).$$

The class e_n is trivial when $n = 1$ and is the coset of $\zeta(n)$ in

$$\mathbb{C}/(2\pi i)^n \mathbb{Q} \cong \operatorname{Ext}^1_{\mathcal{H}}(\mathbb{Q}(0), \mathbb{Q}(n))$$

when $n > 1$. Since $\zeta(2n)$ is a rational multiple of π^{2n}, each e_{2n} is trivial.

Deligne computes the ℓ-adic realization of $P_{\overrightarrow{01}}$ in [14] and shows that the polylogarithm motive is a canonical quotient of the enveloping algebra of the Lie algebra of the unipotent completion of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$. (See also [3] and [20].)

5. The Motivic Lie Algebra of $X_{F,S}$ and Deligne's Conjectures

Assume that X is as in Section 4, and that there is a category of mixed Tate motives $\mathcal{T}(X)$ with properties (i)–(v) in Section 4. Since $\mathcal{T}(X)$ is tannakian, it is determined by its tannakian fundamental group $\pi_1(\mathcal{T}(X))$, which is an extension of $\mathbb{G}_m$ by a prounipotent $\mathbb{Q}$-group

$$1 \to \mathcal{U}_X \to \pi_1(\mathcal{T}(X)) \to \mathbb{G}_m \to 1 \tag{5–1}$$

as we shall now explain.

The category of pure Tate motives is the tannakian subcategory of $\mathcal{T}(X)$ generated by $\mathbb{Q}(1)$. By the faithfulness of realization functors, it is equivalent to the category of finite dimensional graded $\mathbb{Q}$-vector spaces, and hence to the category of finite dimensional representations of $\mathbb{G}_m$; $\mathbb{Q}(n)$ corresponds to the nth power of the standard representation. This induces a group homomorphism between the tannakian fundamental groups

$$\pi_1(\mathcal{T}(X)) \to \pi_1(\text{pure Tate motives}) \cong \mathbb{G}_m.$$

Since the category of pure Tate motives is a full subcategory and every subobject of a pure Tate motive is pure, this morphism is surjective (cf. [13, Proposition 2.21a]), and the properties of the weight filtration imply the unipotence of its kernel $\mathcal{U}_X$, thus we have (5–1). The Lie algebra $\mathfrak{t}_X$ of $\pi_1(\mathcal{T}(X))$ is an extension

$$0 \to \mathfrak{u}_X \to \mathfrak{t}_X \to \mathbb{Q} \to 0$$

where $\mathfrak{u}_X$ is pronilpotent. This $\mathfrak{t}_X$ is called the motivic Lie algebra of X. We shall see that the knowledge of the cohomologies of $\mathfrak{u}_X$ (as $\mathbb{G}_m$-modules) is equivalent

to the knowledge of the extension groups $\mathrm{Ext}^\bullet_{\mathcal{T}(X)}(\mathbb{Q}(0), \mathbb{Q}(m))$ for all m in the next section.

5.1. Extension groups in a tannakian category. We start with a general setting. Let K be a field of characteristic zero. Let $\mathcal{G}$ be a proalgebraic group (in this paper a proalgebraic group means an affine proalgebraic group) over K, or equivalently, an affine group scheme over K (cf. [13]). A $\mathcal{G}$-module V is a (possibly infinite dimensional) K-vector space with algebraic $\mathcal{G}$-action (cf. [30]). The category of $\mathcal{G}$-modules is abelian with enough injectives, and hence we have the cohomology groups

$$H^m(\mathcal{G}, V) := \mathrm{Ext}^m_{\mathcal{G}}(K, V)$$

defined as the extension groups, where K denotes the trivial representation. The right hand side has an interpretation as Yoneda's extension groups, i.e., as the set of equivalence classes of m-step extensions (see [45]). Since each $\mathcal{G}$-module is locally finite [30, 2.13], every m-step extension representing an element of $\mathrm{Ext}^m_{\mathcal{G}}(K, V)$ can be replaced by an equivalent extension consisting of finite dimensional modules when V is finite dimensional. Thus, the right hand side does not change when the category of $\mathcal{G}$-modules is replaced by the category of finite dimensional $\mathcal{G}$-modules.

Let $\mathcal{T}$ be a neutral tannakian category over K with a fiber functor ω, and let $\mathcal{G}$ be its tannakian fundamental group with base point ω. Since $\mathcal{T}$ is isomorphic to the category of finite dimensional $\mathcal{G}$-modules, we have the following.

LEMMA 5.1. *Let $\mathcal{T}$ be a neutral tannakian category and $\mathcal{G}$ be its tannakian fundamental group. Then, for any object V, we have*

$$\mathrm{Ext}^m_{\mathcal{T}}(K, V) \cong H^m(\mathcal{G}, V). \qquad \square$$

Suppose that $\mathcal{G}$ is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

of proalgebraic groups over K. Then, for any $\mathcal{G}$-module V, we have the Lyndon-Hochschild-Serre spectral sequence (cf. [30, 6.6 Proposition]):

$$E_2^{s,t} = H^s(R, H^t(\mathcal{U}, V)) \Rightarrow H^{s+t}(\mathcal{G}, V).$$

If R is a reductive algebraic group, then every R-module is completely reducible. Consequently, $H^s(R, V)$ vanishes for $s \geq 1$ for all V, and

$$H^m(\mathcal{G}, V) \cong H^0(R, H^m(\mathcal{U}, V)).$$

If, in addition, the action of $\mathcal{G}$ on V factors through R, then one has an R-module isomorphism

$$H^m(\mathcal{U}, V) \cong H^m(\mathcal{U}, K) \otimes V.$$

Moreover, if we assume that $\mathcal{U}$ is prounipotent, then its Lie algebra $\mathfrak{u}$ is a projective limit

$$\mathfrak{u} \cong \varprojlim_{\mathfrak{n}} \mathfrak{u}/\mathfrak{n}$$

of finite dimensional nilpotent Lie algebras. It has a topology as a projective limit, where each $\mathfrak{u}/\mathfrak{n}$ is viewed as a discrete topological space.

Let V be a continuous $\mathfrak{u}$-module over K. The continuous cohomology $H^m_{\mathrm{cts}}(\mathfrak{u}, V)$ is defined as the extension group $\mathrm{Ext}^m(K, V)$ in the category of continuous $\mathfrak{u}$-modules. We denote $H^m_{\mathrm{cts}}(\mathfrak{u}, K)$ by $H^m_{\mathrm{cts}}(\mathfrak{u})$. It is easy to show that

$$H^m_{\mathrm{cts}}(\mathfrak{u}) \cong \varinjlim_{\mathfrak{n}} H^m(\mathfrak{u}/\mathfrak{n}),$$

where $H^m(\mathfrak{u}/\mathfrak{n})$ can be computed as the cohomology of the complex of cochains

$$\mathrm{Hom}(\Lambda^\bullet(\mathfrak{u}/\mathfrak{n}), K).$$

The following is standard.

PROPOSITION 5.2. *Let $\mathfrak{u}$ be a pronilpotent Lie algebra, and let $H_1(\mathfrak{u})$ denote the abelianization of $\mathfrak{u}$. Then*

$$H_1(\mathfrak{u}) \cong \mathrm{Hom}(H^1_{\mathrm{cts}}(\mathfrak{u}), K).$$

If $H^2(\mathfrak{u}) = 0$, then $\mathfrak{u}$ is free.

It is also well known that the category of $\mathcal{U}$-modules is equivalent to the category of continuous $\mathfrak{u}$-modules. Hence we have

$$H^m(\mathcal{U}, K) \cong H^m_{\mathrm{cts}}(\mathfrak{u}).$$

Putting this together, we have the following.

THEOREM 5.3. *Suppose that $1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$ is a short exact sequence of pro-algebraic groups over a field K of characteristic zero. Assume that R is a reductive algebraic group, and that $\mathcal{U}$ is a prounipotent group. Let $\mathfrak{u}$ be the Lie algebra of $\mathcal{U}$. If V is an R-module, considered as a $\mathcal{G}$-module, then*

$$H^m(\mathcal{G}, V) \cong (H^m_{\mathrm{cts}}(\mathfrak{u}) \otimes V)^R.$$

Consequently, we have the R-module isomorphism

$$H^m_{\mathrm{cts}}(\mathfrak{u}) \cong \bigoplus_{\alpha} (H^m(\mathcal{G}, V_\alpha) \otimes V_\alpha^*),$$

where $\{V_\alpha\}$ is a set of representatives of the isomorphism classes of irreducible R-modules, and $(\ )^*$ denotes $\mathrm{Hom}(\ , K)$.

5.2. Deligne's conjecture. By applying Theorem 5.3 to (5–1), we have

$$\mathrm{Ext}^m_{\mathcal{T}(X)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong [H^m_{\mathrm{cts}}(\mathfrak{u}_X) \otimes \mathbb{Q}(n)]^{\mathbb{G}_m}$$

and $\mathbb{G}_m$-module isomorphisms

$$H^m_{\mathrm{cts}}(\mathfrak{u}_X) \cong \bigoplus_{n \in \mathbb{Z}} \mathrm{Ext}^m_{\mathcal{T}(X)}(\mathbb{Q}(0), \mathbb{Q}(n)) \otimes \mathbb{Q}(-n), \qquad (5\text{–}2)$$

where $\mathfrak{u}_X$ is the Lie algebra of $\mathcal{U}_X$, the prounipotent radical of $\pi_1(\mathcal{T}(X), \omega)$. By a weight argument, each extension on the right hand side vanishes if $n \leq m - 1$. Postulate 4.1 says that these Ext groups should be the motivic cohomology groups of X, and Theorem 3.1 says that they should be isomorphic to the Adams eigenspaces of the K-groups of X:

PROPOSITION 5.4. *Assume the existence of a category $\mathcal{T}(X_{F,S})$ of mixed Tate motives over $X_{F,S}$ with properties (i)–(v) as in Section 4. Suppose that Postulate 4.1 holds for all $n \geq 1$. Let $\mathcal{U}_{X_{F,S}}$ be the unipotent radical of $\pi_1(\mathcal{T}(X_{F,S}), \omega)$, and $\mathfrak{u}_{X_{F,S}}$ be its Lie algebra. Then there is a natural $\mathbb{G}_m$-module isomorphism*

$$H^1_{\mathrm{cts}}(\mathfrak{u}_{X_{F,S}}) \cong \bigoplus_{n \geq 1} K_{2n-1}(X_{F,S}) \otimes_{\mathbb{Z}} \mathbb{Q}(-n),$$

and $H^m_{\mathrm{cts}}(\mathfrak{u}_{X_{F,S}}) = 0$ whenever $m \geq 2$. Moreover, the exactness of $\mathrm{Gr}^W_\bullet$ implies that

$$H_1(\mathrm{Gr}^W_\bullet \mathfrak{u}_{X_{F,S}}) = \bigoplus_{n \geq 1} K_{2n-1}(X_{F,S})^* \otimes \mathbb{Q}(n)$$

and that

$$H^m(\mathrm{Gr}^W_\bullet \mathfrak{u}_{X_{F,S}}) = 0 \text{ when } m > 1.$$

It follows from this that $\mathrm{Gr}^W_\bullet \mathfrak{u}_{X_{F,S}}$ is isomorphic to the free Lie algebra generated by $H_1(\mathrm{Gr}^W_\bullet \mathfrak{u}_{X_{F,S}})$.

Let us assume that there is a category $\mathcal{T}(X_{F,S})$ satisfying (i)–(v) in Section 4. Then Postulate 4.1 is equivalent to the following conjecture of Deligne:

CONJECTURE 5.5 (DELIGNE). (i) [14, 8.2.1] For the category $\mathcal{T}(X_{F,S})$ of mixed Tate motives smooth over $X_{F,S}$ one has a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{T}(X_{F,S})}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(X_{F,S}) \otimes \mathbb{Q} \text{ for all } n,$$

which is compatible with the Chern mappings.
(ii) [14, 8.9.5] The group $\pi_1(\mathcal{T}(X_{F,S}))$ is an extension of $\mathbb{G}_m$ by a free prounipotent group.

Note that by Definition 2.1, Theorem 3.1 and the isomorphism (5–2), (i) is equivalent to Postulate 4.1 for $m = 1$, and that (ii) is equivalent to Postulate 4.1 for $m \geq 2$.

Consequences of Deligne's conjecture. Deligne's conjecture suggests restrictions on the action of Galois groups on pro-ℓ completions of fundamental groups of curves. Here is a sketch of how this should work.

As in the beginning of Section 4, there should be a Betti realization functor

$$\mathrm{real}_B : \mathcal{T}(X_{F,S}) \to \{\mathbb{Q}\text{-vector spaces}\}$$

to the category of $\mathbb{Q}$-vector spaces, and an ℓ-adic realization functor

$$\mathrm{real}_\ell : \mathcal{T}(X_{F,S}) \to \{\ell\text{-adic } G_F\text{-modules}\},$$

to the category of the $\mathbb{Q}_\ell$-vector spaces with a continuous G_F-action. The Galois modules should be unramified outside $S \cup [\ell]$, where $[\ell]$ denotes the set of primes of F over ℓ. We choose real_B as our fiber functor ω. Let ω_ℓ denote the functor real_ℓ which forgets the G_F-action. Conjecturally, there is a comparison isomorphism

$$\omega \otimes \mathbb{Q}_\ell \cong \omega_\ell,$$

so we shall identify these two. Define $\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell$ to be the tannakian category whose objects are the same as those of $\mathcal{T}(X_{F,S})$ and whose hom-sets are those of $\mathcal{T}(X_{F,S})$ tensored with $\mathbb{Q}_\ell$. The ℓ-adic realization functor induces a functor

$$\mathrm{real}_\ell : \mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell \to \{\ell\text{-adic } G_F\text{-modules}\} \tag{5–3}$$

(by an abuse of notation we denote it by real_ℓ again), and by forgetting the Galois action a fiber functor $\omega_\ell : \mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell \to \mathrm{Vec}_{\mathbb{Q}_\ell}$ (under a similar abuse of notation). Through the comparison isomorphism, it is easy to show that

$$\pi_1(\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell, \omega_\ell) \cong \pi_1(\mathcal{T}(X_{F,S}), \omega) \otimes \mathbb{Q}_\ell.$$

The following is closely related to the Tate conjecture on Galois modules.

POSTULATE 5.6. The realization functor real_ℓ in (5–3) is fully faithful, and its image is closed under taking subobjects.

The first condition is that every Galois compatible morphism comes from a morphism of motives up to extension of scalars, and the second condition is that every Galois submodule arises as an ℓ-adic realization of a motive. We shall see that this postulate follows from Deligne's Conjecture 5.5 and our Theorem 9.2 (see Corollary 9.4).

Every element of G_F gives an automorphism of the forgetful fiber functor of the category of G_F-modules (i.e. forgetting the Galois action), and hence an automorphism of ω_ℓ. Thus we have a homomorphism

$$G_F \to \pi_1(\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell, \omega_\ell)(\mathbb{Q}_\ell) \cong \pi_1(\mathcal{T}(X_{F,S}), \omega)(\mathbb{Q}_\ell). \tag{5–4}$$

In addition, the G_F-action on the ℓ-adic realization of any (pro)object M of $\mathcal{T}(X_{F,S})$ factors through $\pi_1(\mathcal{T}(X_{F,S})) \otimes \mathbb{Q}_\ell$ via the morphism (5–4).

PROPOSITION 5.7. *Postulate 5.6 is equivalent to the statement that the above morphism (5–4) has Zariski dense image.*

PROOF. Let $\mathcal{G}$ denote the tannakian fundamental group of the category of finite dimensional ℓ-adic G_F-modules. By [13, Prop. 2.21a], the conditions in Postulate 5.6 are equivalent to the surjectivity of $\mathcal{G} \to \pi_1(\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell, \omega_\ell)$. It is a general fact that the image of $G_F \to \mathcal{G}(\mathbb{Q}_\ell)$ is Zariski dense. $\qquad\square$

Assuming Postulate 5.6, the Zariski density of the image of (5–4) implies that for any object M of $\mathcal{T}(X_{F,S})$, the Zariski closure of the image of G_F in $\mathrm{Aut}(M)$ should be a quotient of $\pi_1(\mathcal{T}(X_{F,S})) \otimes \mathbb{Q}_\ell$. We can define a filtration $J_M^\bullet$ on G_F (which depends on M) by

$$J_M^m G_F := \text{ the inverse image of } W_m \mathrm{Aut}\, M.$$

The image of the Galois group $G_{F(\mu_{\ell^\infty})}$ of $F(\mu_{\ell^\infty})$ in $\pi_1(\mathcal{T}(X_{F,S}), \omega_\ell)$ will lie in its prounipotent radical and should be Zariski dense in it. The exactness of $\mathrm{Gr}_\bullet^W$ will then imply that

$$\left(\bigoplus_{m<0} \mathrm{Gr}_{J_M}^m G_F \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

(a Lie algebra) is a quotient of $\mathrm{Gr}_\bullet^W \mathfrak{u}_{X_{F,S}}$, and hence generated by

$$\bigoplus_{m \geq 1} \mathrm{Hom}(K_{2m-1}(X_{F,S}), \mathbb{Q}_\ell(m)).$$

For example, the pronilpotent Lie algebra $\mathfrak{p}$ of the unipotent completion of $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \overrightarrow{01})$ should be a pro-object of $\mathcal{T}(\mathrm{Spec}\,\mathbb{Z})$. One should therefore expect that the graded Lie algebra

$$\left(\bigoplus_{m<0} \mathrm{Gr}_{J_\mathfrak{p}}^m G_\mathbb{Q} \right) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

is generated by elements $z_3, z_5, z_7, \ldots$, where z_m has weight $-2m$.

Following Ihara [26], we define

$$I_\ell^m G_\mathbb{Q} = \ker\{G_\mathbb{Q} \to \mathrm{Out}(\pi_1^{(\ell)}(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, \overrightarrow{01})/L^{m+1})\}$$

where L^m denotes the mth term of the lower central series of the pro-ℓ completion of $\pi_1(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}, \overrightarrow{01})$. This is related to the filtration $J_\mathfrak{p}^\bullet$ by

$$I_\ell^m G_\mathbb{Q} = J_\mathfrak{p}^{-2m} G_\mathbb{Q} = J_\mathfrak{p}^{-2m+1} G_\mathbb{Q}.$$

Making this substitution, we are led to the following conjecture, stated by Ihara in [27, p. 300] and which he attributes to Deligne.

CONJECTURE 5.8 (DELIGNE). The Lie algebra

$$\left[\bigoplus_{m>0} \mathrm{Gr}_{I_\ell}^m G_\mathbb{Q} \right] \otimes \mathbb{Q}_\ell$$

is generated by generators $s_3, s_5, s_7, \ldots$, where $s_m \in \mathrm{Gr}_{I_\ell}^m G_\mathbb{Q}$.

Deligne also asked whether this Lie algebra is free. A related conjecture of Goncharov [19, Conj. 2.1], stated below, and the questions of Drinfeld [15] can be 'derived' from Deligne's Conjecture 5.5 in a similar fashion. The freeness questions are more optimistic and are equivalent to the statement that the representation of the motivic Galois group $\pi_1(\mathcal{T}(\mathrm{Spec}\,\mathbb{Z}))$ in the automorphisms of the ℓ-adic unipotent completion of the fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$ is faithful. The computational results [27], [35] and [43] give support to the belief that this Lie algebra is free. Indeed, these computations show that $\mathrm{Gr}_{I_\ell}^{>0} G_\mathbb{Q}$ is free up to $\mathrm{Gr}_{I_\ell}^{12} G_\mathbb{Q}$.

CONJECTURE 5.9 (GONCHAROV). The Lie algebra of the Zariski closure of the Galois group of $\mathbb{Q}(\mu_{\ell^\infty})$ in the automorphism group of the ℓ-adic unipotent completion of $\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, \overrightarrow{01})$ is a prounipotent Lie algebra freely generated by elements $z_3, z_5, z_7, \ldots$, where z_m has weight $-2m$.

Deligne's Conjecture 5.8 above and the generation part of Goncharov's conjecture are proved in [23]. A brief sketch of their proofs is given in Section 10. Modulo technical details, the main point is the computation of the tannakian fundamental group of the candidate for $\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell$ given in the next section.

Polylogarithms revisited. Assuming the existence of $\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}}$ (i.e. the Lie algebra of $\pi_1(\mathcal{T}(\mathrm{Spec}\,\mathbb{Z}))$), we can give another interpretation of the fiber $P_{\overrightarrow{01}}$ of the polylogarithm local system. Being a motive over $\mathrm{Spec}\,\mathbb{Z}$, it is a $\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}}$-module. Note that since

$$W_{-1}P_{\overrightarrow{01}} = \bigoplus_{n \geq 1} \mathbb{Q}(n),$$

a direct sum of Tate motives (no nontrivial extensions), the restriction of the $\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}}$-action on $W_{-1}P_{\overrightarrow{01}}$ to $\mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}$ is trivial and $[\mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}, \mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}]$ annihilates $P_{\overrightarrow{01}}$. Since $P_{\overrightarrow{01}}$ is an extension of $\mathbb{Q}(0)$ by $W_{-1}P_{\overrightarrow{01}}$, this implies that there is a homomorphism

$$\psi : \left(\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}} / [\mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}, \mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}] \right) \otimes \mathbb{Q}(0) \longrightarrow P_{\overrightarrow{01}}$$

such that the diagram

$$
\begin{array}{ccc}
\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}} \otimes P_{\overrightarrow{01}} & \xrightarrow{\ \mathrm{action}\ } & P_{\overrightarrow{01}} \\
{\scriptstyle \mathrm{quotient}} \downarrow & \nearrow {\scriptstyle \psi} & \\
\mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}} / [\mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}, \mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}] & &
\end{array}
$$

commutes. By comparing graded quotients, it follows that ψ is an isomorphism

$$P_{\overrightarrow{01}} \cong \mathfrak{t}_{\mathrm{Spec}\,\mathbb{Z}} / [\mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}, \mathfrak{u}_{\mathrm{Spec}\,\mathbb{Z}}]$$

of motives over $\mathrm{Spec}\,\mathbb{Z}$.

6. ℓ-adic Mixed Tate Modules over $X_{F,S}$

In this section, we describe a candidate for the category of ℓ-adic realizations of objects and morphisms of $\mathcal{T}(X_{F,S})$. This is essentially the category constructed by Deligne and Beilinson in their unpublished manuscript [2]. It is purely Galois-representation theoretic, and requires no postulates. For technical reasons, we assume that S contains $[\ell]$, the set of primes over ℓ. This condition will be removed in Section 11. By a finite dimensional $G_{F,S}$-module, we shall mean a finite dimensional $\mathbb{Q}_\ell$-vector space on which $G_{F,S}$ acts continuously.

We define the category $\mathcal{T}_\ell(X_{F,S})$ of ℓ-adic mixed Tate modules which are smooth over $X_{F,S}$ to be the category whose objects are finite dimensional $G_{F,S}$-modules M that are equipped with a *weight filtration*

$$\cdots \subseteq W_{m-1}M \subseteq W_m M \subseteq W_{m+1}M \subseteq \cdots$$

of M by $G_{F,S}$-submodules. The weight filtration satisfies:

(i) all odd weight graded quotients of M vanish: $\mathrm{Gr}^W_{2m+1} M = 0$;
(ii) $G_{F,S}$ acts on its $2m$th graded quotient $\mathrm{Gr}^W_{2m} M$ via the $(-m)$th power of the cyclotomic character,
(iii) the intersection of the $W_m M$ is trivial and their union is all of M.

Morphisms are $\mathbb{Q}_\ell$-linear, $G_{F,S}$-equivariant mappings. These will necessarily preserve the weight filtration, so that $\mathcal{T}_\ell(X_{F,S})$ is a full subcategory of the category of $G_{F,S}$-modules.

The category $\mathcal{T}_\ell(X_{F,S})$ is a tannakian category over $\mathbb{Q}_\ell$ with a fiber functor ω' that takes an object to its underlying $\mathbb{Q}_\ell$-vector space. We shall denote the tannakian fundamental group of this category by $\mathcal{A}^\ell_{F,S} := \pi_1(\mathcal{T}_\ell(X_{F,S}), \omega')$. Every element of $G_{F,S}$ acts on ω', which induces a natural, continuous homomorphism

$$\rho : G_{F,S} \to \mathcal{A}^\ell_{F,S}(\mathbb{Q}_\ell).$$

This has Zariski-dense image as $\mathcal{T}_\ell(X_{F,S})$ is a full subcategory of the category of $G_{F,S}$-modules, closed under taking subobjects (cf. [13, Proposition 2.21a]).

Relation to mixed Tate motives over $X_{F,S}$. As explained in Section 4, the existence of a category $\mathcal{T}(X_{F,S})$ of mixed Tate motives over $X_{F,S}$ satisfying (i)–(v) in Section 4 implies the existence of an ℓ-adic realization functor

$$\mathrm{real}_\ell : \mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell \to \mathcal{T}_\ell(X_{F,S}).$$

This will induce a morphism of tannakian fundamental groups

$$\mathcal{A}^\ell_{F,S} = \pi_1(\mathcal{T}_\ell(X_{F,S}), \omega') \to \pi_1(\mathcal{T}(X_{F,S}), \omega) \otimes \mathbb{Q}_\ell.$$

The main result of [23] may be regarded as saying that $\mathcal{A}^\ell_{F,S} = \pi_1(\mathcal{T}_\ell(X_{F,S}), \omega')$ is isomorphic to the conjectured value of the $\mathbb{Q}_\ell$-form $\pi_1(\mathcal{T}(X_{F,S}), \omega) \otimes \mathbb{Q}_\ell$ of the motivic fundamental group of $X_{F,S}$. We shall explain this in Section 9.

It is interesting to note that we have not restricted to $G_{F,S}$-modules of geometric origin as Deligne would like to. So one consequence of our result is that, if Deligne's Conjecture 5.5 is true, then all objects and morphisms of $\mathcal{T}(X_{F,S})$ will be of geometric origin.

7. Weighted Completion of Profinite Groups

In this and the subsequent two sections we will sketch how to compute the tannakian fundamental group $\pi_1(\mathcal{T}_\ell(X_{F,S}), \omega')$ of the category of ℓ-adic mixed Tate modules smooth over $X_{F,S}$, which was defined in Section 6. It is convenient to work in greater generality.[5]

Suppose that R is a reductive algebraic group over $\mathbb{Q}_\ell$ and that $w : \mathbb{G}_m \to R$ is a central cocharacter — that is, its image is contained in the center of R. It is best to imagine that w is nontrivial as the theory of weighted completion is uninteresting if w is trivial.

Suppose that Γ is a profinite group and that a homomorphism $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ has Zariski dense image and is continuous where we view $R(\mathbb{Q}_\ell)$ as an ℓ-adic Lie group.

By a *weighted* Γ-module with respect to ρ and w we shall mean a finite dimensional $\mathbb{Q}_\ell$-vector space with continuous Γ-action together with a weight filtration

$$\cdots \subseteq W_{m-1}M \subseteq W_m M \subseteq W_{m+1}M \subseteq \cdots$$

by Γ-invariant subspaces. These should satisfy:

(i) the intersection of the $W_m M$ is 0 and their union is M,
(ii) for each m, the representation $\Gamma \to \operatorname{Aut} \operatorname{Gr}_m^W M$ should factor through ρ and a homomorphism $\phi_m : R \to \operatorname{Aut} \operatorname{Gr}_m^W M$,
(iii) $\operatorname{Gr}_m^W M$ has weight m when viewed as a $\mathbb{G}_m$-module via

$$\mathbb{G}_m \xrightarrow{\ w\ } R \xrightarrow{\ \phi_m\ } \operatorname{Aut} \operatorname{Gr}_m^W M.$$

That is, $\mathbb{G}_m$ acts on $\operatorname{Gr}_m^W M$ via the mth power of the standard character.

The category of weighted Γ-modules consists of the Γ-equivariant morphisms between weighted Γ-modules. These morphisms automatically preserve the weight filtration and are strict with respect to it; that is, the functor $\operatorname{Gr}_\bullet^W$ is exact.

One can show that the category of weighted Γ-modules is tannakian, with fiber functor ω' given by forgetting the Γ-action.

DEFINITION 7.1. The weighted completion of Γ with respect to $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ and $w : \mathbb{G}_m \to R$ is the tannakian fundamental group of the category of weighted Γ-modules with respect to ρ and w.

---

[5]We may generalize further: weighted completion and its properties in this section are unchanged even if we replace $\mathbb{Q}_\ell$ by an arbitrary topological field of characteristic zero and Γ by an arbitrary topological group.

Denote the weighted completion of Γ with respect to ρ and w by $\mathcal{G}$. There is a natural homomorphism $\Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ which has Zariski dense image as we shall see below.

This definition differs from the one given in [23, Section 5], but is easily seen to be equivalent to it. (See below.) In particular, we can apply it when:

- Γ is $G_{F,S}$,
- R is $\mathbb{G}_m$ and $w : \mathbb{G}_m \to \mathbb{G}_m$ takes x to x^{-2},
- ρ is the composite of the ℓ-adic cyclotomic character $\chi_\ell : G_{F,S} \to \mathbb{Z}_\ell^\times$ with the inclusion $\mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$.

In this case, the category of weighted Γ-modules is nothing but the category of mixed Tate modules $\mathcal{T}_\ell(X_{F,S})$. Recall that we denote the corresponding weighted completion by $\mathcal{A}_{F,S}^\ell := \pi_1(\mathcal{T}_\ell(X_{F,S}), \omega')$.

Equivalence of definitions. Here we show that the definition of weighted completion given in [23] agrees with the one given here.

Suppose that G is a linear algebraic group over $\mathbb{Q}_\ell$ which is an extension

$$1 \to U \to G \to R \to 1$$

of R by a unipotent group U. Note that $H_1(U)$ is naturally an R-module, and therefore a $\mathbb{G}_m$-module via the given central cocharacter $w : \mathbb{G}_m \to R$. We can decompose $H_1(U)$ as a $\mathbb{G}_m$-module:

$$H_1(U) = \bigoplus_{n \in \mathbb{Z}} H_1(U)_n$$

where $\mathbb{G}_m$ acts on $H_1(U)_n$ via the nth power of the standard character. We say that G is a *negatively weighted extension of R* if $H_1(U)_n$ vanishes whenever $n \geq 0$.

Given a continuous homomorphism $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ with Zariski dense image, we can form a category of pairs $(\tilde{\rho}, G)$, where G is a negatively weighted extension of R and $\tilde{\rho} : \Gamma \to G(\mathbb{Q}_\ell)$ is a continuous homomorphism that lifts ρ. Morphisms in this category are given by homomorphisms between the Gs that respect the projection to R and the lifts $\tilde{\rho}$ of ρ. The objects of this category, where $\tilde{\rho}$ is Zariski dense, form an inverse system. Their inverse limit is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

of R by a prounipotent group. There is a natural homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$, which is continuous in a natural sense. It has the following universal mapping property: if $\tilde{\rho} : \Gamma \to G(\mathbb{Q}_\ell)$ is an object of this category, then there is a unique homomorphism $\phi : \mathcal{G} \to G$ that commutes with the projections to R and with the homomorphisms $\tilde{\rho} : \Gamma \to G(\mathbb{Q}_\ell)$ and $\hat{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$. In [23], the weighted completion is defined to be this inverse limit. The equivalence of the two definitions follows from the following result.

PROPOSITION 7.2. *The inverse limit defined above is naturally isomorphic to the weighted completion of* Γ *relative to* ρ *and* w.

PROOF. Denote the inverse limit by $\mathcal{G}$ and by $\mathcal{M} = \mathcal{M}(\rho, w)$ the category of weighted Γ-modules with respect to $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ and w. We will show that $\mathcal{M}$ is the category of finite dimensional $\mathcal{G}$-modules, from which the result follows.

Suppose that M is an object of $\mathcal{M}$. Then the Zariski closure of Γ in $\operatorname{Aut} M$ is an extension

$$1 \to U \to G \to R' \to 1$$

of a quotient of R by a unipotent group. Here R' is the Zariski closure of the image of Γ in $\operatorname{Aut} \operatorname{Gr}^W_\bullet M$. Because the action of Γ on each weight graded quotient factors through ρ, and because $\mathbb{G}_m$ acts on the mth weight graded quotient of M with weight m, it follows that this is a negatively weighted extension of R'. Pulling back this extension along the projection $R \to R'$, we obtain a negatively weighted extension

$$1 \to U \to \widetilde{G} \to R \to 1$$

of R and a continuous homomorphism $\Gamma \to \widetilde{G}(\mathbb{Q}_\ell)$ that lifts both ρ and the homomorphism $\Gamma \to R'(\mathbb{Q}_\ell)$. By the universal mapping property of $\mathcal{G}$, there is a natural homomorphism $\mathcal{G} \to \widetilde{G}$ compatible with the projections to R and the homomorphisms from Γ to $\mathcal{G}(\mathbb{Q}_\ell)$ and $\widetilde{G}(\mathbb{Q}_\ell)$. Thus every object of $\mathcal{M}$ is naturally a $\mathcal{G}$-module. It is also easy to see that every morphism of $\mathcal{M}$ is $\mathcal{G}$-equivariant.

Conversely, suppose that M is a finite dimensional $\mathcal{G}$-module. Composing with the natural homomorphism $\hat{\rho} : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$ gives M the structure of a Γ-module. In [23, Sect. 4], it is proven that every $\mathcal{G}$-module has a natural weight filtration with the property that the action of $\mathcal{G}$ on each weight graded quotient factors through the projection $\mathcal{G} \to R$ and that $\mathbb{G}_m$ acts with weight m on the mth weight graded quotient. It follows that M is naturally an object of $\mathcal{M}$. Since $\mathcal{G}$-equivariant mappings are naturally Γ-equivariant, this proves that $\mathcal{M}$ is naturally the category of finite dimensional $\mathcal{G}$-modules, which completes the proof. $\qquad\square$

8. Computation of Weighted Completions

Suppose that Γ, R, $\rho : \Gamma \to R(\mathbb{Q}_\ell)$ and $w : \mathbb{G}_m \to R$ are as above. The weighted completion $\mathcal{G}$ of Γ is controlled by the low-dimensional cohomology groups $H^\bullet_{\mathrm{cts}}(\Gamma, V)$ of Γ with coefficients in certain irreducible representations V of R. If one knows these cohomology groups, as we do in the case of $G_{F,S}$, one can sometimes determine the structure of the weighted completion. These cohomological results are stated in this section.

The weighted completion of Γ with respect to ρ and w is an extension

$$1 \to \mathcal{U} \to \mathcal{G} \to R \to 1$$

where $\mathcal{U}$ is prounipotent. Now we are in the situation of Theorem 5.3. Denote the Lie algebra of $\mathcal{U}$ by $\mathfrak{u}$. Since $\mathfrak{u}$ is a $\mathcal{G}$-module by the adjoint action, the natural weight filtration on $\mathfrak{u}$ induces one on $H_{\mathrm{cts}}^\bullet(\mathfrak{u})$. By looking at cochains, it is not difficult to see that if $\mathfrak{u} = W_{-N}\mathfrak{u}$ for some $N > 0$, then

$$W_n H_{\mathrm{cts}}^m(\mathfrak{u}) = 0 \text{ if } n < Nm. \tag{8--1}$$

Let V_α be an irreducible R-module. Since w is central in R, the $\mathbb{G}_m$-action commutes with the R-action, so Schur's Lemma implies that there is an integer $n(\alpha)$ such that $\mathbb{G}_m$ acts on V_α via the $n(\alpha)$th power of the standard character. This is the weight of V_α as a $\mathcal{G}$-module. Now (8--1) and Theorem 5.3 imply

$$H^m(\mathcal{G}, V_\alpha) = 0$$

if $n(\alpha) > -Nm$. Note that always $\mathfrak{u} = W_{-1}\mathfrak{u}$.

Suppose that V is an ℓ-adic Γ-module, i.e., a $\mathbb{Q}_\ell$-vector space with continuous Γ-action. We shall need the continuous cohomology $H_{\mathrm{cts}}^\bullet(\Gamma, V)$, which is defined as the cohomology of a suitable complex of continuous cochains as in [42, Sect. 2]. A $\mathcal{G}$-module V can be considered as an ℓ-adic Γ-module through $\hat\rho : \Gamma \to \mathcal{G}(\mathbb{Q}_\ell)$. There is a natural group homomorphism

$$\Phi^m : H^m(\mathcal{G}, V) \to H_{\mathrm{cts}}^m(\Gamma, V)$$

for each $m \geq 0$.

Let $\{V_\alpha\}_\alpha$ be as in Theorem 5.3. These are considered as Γ-modules via ρ. The following theorem is our basic tool for computing $\mathfrak{u}$ when the appropriate continuous cohomology groups $H_{\mathrm{cts}}^i(\Gamma, V_\alpha)$ are known for $i = 1, 2$.

THEOREM 8.1. *For $m = 1, 2$, the mappings Φ^m defined above satisfy*:

(i) $\Phi^1 : H^1(\mathcal{G}, V_\alpha) \to H_{\mathrm{cts}}^1(\Gamma, V_\alpha)$ *is an isomorphism if $n(\alpha) < 0$;*
(ii) $\Phi^2 : H^2(\mathcal{G}, V_\alpha) \to H_{\mathrm{cts}}^2(\Gamma, V_\alpha)$ *is injective.*

This and Theorem 5.3 imply the following, by using (8--1) and the comment following it.

COROLLARY 8.2. (i) *There is a natural R-equivariant isomorphism*

$$H_{\mathrm{cts}}^1(\mathfrak{u}) \cong \bigoplus_{\{\alpha : n(\alpha) \leq -1\}} H_{\mathrm{cts}}^1(\Gamma, V_\alpha) \otimes V_\alpha^*.$$

(ii) *If N is an integer such that $H_{\mathrm{cts}}^1(\Gamma, V_\alpha) = 0$ for $0 > n(\alpha) > -N$, then there is a natural R-equivariant inclusion*

$$\Phi : H_{\mathrm{cts}}^2(\mathfrak{u}) \hookrightarrow \bigoplus_{\{\alpha : n(\alpha) \leq -2N\}} H_{\mathrm{cts}}^2(\Gamma, V_\alpha) \otimes V_\alpha^*.$$

This is proved in [23]. Below we shall give another more categorical proof, similar to that in [2].

COROLLARY 8.3. (i) *If $H_{\mathrm{cts}}^1(\Gamma, V_\alpha) = 0$ whenever $n(\alpha) < 0$, then $\mathfrak{u} = 0$.*

(ii) *Let N be as in Corollary 8.2. If $H^2_{\text{cts}}(\Gamma, V_\alpha) = 0$ whenever $n(\alpha) \leq -2N$, then $\mathfrak{u}$ is free as a pronilpotent Lie algebra.* $\qquad\qquad\qquad\square$

In the proof of Theorem 8.1 we shall use Yoneda extensions. Let V be a finite dimensional ℓ-adic Γ-module. For each $m \geq 0$, define

$$\text{Ext}^m_\Gamma(\mathbb{Q}_\ell, V)$$

to be the m-th Yoneda extension group in the category of finite dimensional ℓ-adic Γ-modules, where $\mathbb{Q}_\ell$ denotes the trivial Γ-module. For each $m \geq 1$, there is a natural homomorphism

$$\text{Ext}^m_\Gamma(\mathbb{Q}_\ell, V) \to H^m_{\text{cts}}(\Gamma, V),$$

which, by Theorem A.6, is an isomorphism when $m = 1$, and injective when $m = 2$.

There is an exact functor from the category of weighted $\mathcal{G}$-modules to the category of ℓ-adic Γ-modules. It induces morphisms between the extension groups, and hence homomorphisms

$$\Psi^m : H^m(\mathcal{G}, V) \to \text{Ext}^m_\Gamma(\mathbb{Q}_\ell, V), \qquad m \geq 0.$$

The homomorphisms Φ^m above factor through these:

$$H^m(\mathcal{G}, V) \xrightarrow[\Psi^m]{} \text{Ext}^m_\Gamma(\mathbb{Q}_\ell, V) \xrightarrow{} H^m_{\text{cts}}(\Gamma, V).$$
$$\overbrace{\phantom{H^m(\mathcal{G}, V) \xrightarrow{} \text{Ext}^m_\Gamma(\mathbb{Q}_\ell, V) \xrightarrow{} H^m_{\text{cts}}(\Gamma, V)}}^{\Phi^m}$$

In fact, this is one of several equivalent ways to define the natural mappings Φ^m.

PROOF OF THEOREM 8.1. In view of Theorem A.6, it suffices to prove that Ψ^1 is an isomorphism, and that Ψ^2 is injective.

Since the functor from the category of weighted Γ-modules to the category of Γ-modules is fully faithful, a 1-step extension of weighted Γ-modules splits if it splits as an extension of Γ-modules. This establishes the injectivity of Ψ^1.

To prove surjectivity of Ψ^1, we define a natural weight filtration on each Γ-module extension E of $\mathbb{Q}_\ell$ by V_α. Simply set $W_0 E = E$ and $W_{n(\alpha)} E = V_\alpha$. Since $n(\alpha) < 0$, this makes E a weighted Γ-module.

To prove that Ψ^2 is injective, we need to show that if a 2-step extension

$$1 \to V_\alpha \to E_2 \to E_1 \to \mathbb{Q}_\ell \to 1 \tag{8–2}$$

lies in the trivial class of extensions of Γ-modules, then it also lies in the trivial class of extensions of *weighted* Γ-modules.

If $n(\alpha) \geq -1$, then $H^2(\mathcal{G}, V_\alpha) = 0$, and there is nothing to prove. Thus we may assume $n(\alpha) \leq -2$. Since W_m is an exact functor, we may apply W_0 to (8–2) to obtain another 2-step extension, without changing the extension class. Then, taking Gr^W_0, we have a short exact sequence

$$0 \to \text{Gr}^W_0 E_2 \to \text{Gr}^W_0 E_1 \to \mathbb{Q}_\ell \to 0$$

of R-modules. Since R is reductive, this has a splitting $\mathbb{Q}_\ell \hookrightarrow \mathrm{Gr}_0^W E_1$. Taking the inverse images of this copy of $\mathbb{Q}_\ell$ along $E_2 \to E_1 \to \mathrm{Gr}_0^W E_1$ in E_2 and in E_1, we obtain a 2-step extension

$$0 \to V_\alpha \to E_2' \to E_1' \to \mathbb{Q}_\ell \to 0$$

equivialent to (8–2) satisfying $W_{-1}E_2' = E_2'$ and $W_0 E_1' = E_1'$. Using the dual argument, we may assume that (8–2) satisfies $W_0 E_1 = E_1$, $W_{-1}E_2 = E_2$, $W_{n(\alpha)-1}E_2 = 0$, and $W_{n(\alpha)}E_1 = 0$.

By Yoneda's characterization [45, p. 575] of trivial m-step extensions, the extension (8–2) represents the trivial 2-step extension class as Γ-modules if and only if there is a Γ-module E and exact sequences

$$0 \to E_2 \to E \to \mathbb{Q}_\ell \to 0 \text{ and } 0 \to V_\alpha \to E \to E_1 \to 0$$

which are compatible with the existing mappings $V_\alpha \hookrightarrow E_2$ and $E_1 \twoheadrightarrow \mathbb{Q}_\ell$. To establish the injectivity of Ψ^2, it suffices to prove that E is a weighted Γ-module. But E has the weight structure $W_0 E = E$ and $W_{-1}E = E_2$. This completes the proof of Theorem 8.1. $\qquad\qquad\square$

EXAMPLE 8.4. Suppose that $\Gamma = \mathbb{Z}_\ell^\times$, that $R = \mathbb{G}_{m/\mathbb{Q}_\ell}$ and that $\rho : \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{G}_m(\mathbb{Q}_\ell) = \mathbb{Q}_\ell^\times$ is the natural inclusion. Take w to be the inverse of the square of the standard character. (With this choice, representation theoretic weights coincide with the weights from Hodge and Galois theory.) In this example we compute the weighted completion of $\mathbb{Z}_\ell^\times$ with respect to ρ and w. Note that

$$H^1_{\mathrm{cts}}(\mathbb{Z}_\ell^\times, \mathbb{Q}_\ell(n)) = 0,$$

for all non-zero $n \in \mathbb{Z}$, where $\mathbb{Q}_\ell(n)$ denotes the nth power of the standard representation of $\mathbb{G}_m$. It has weight $-2n$ under the central cocharacter.

Corollary 8.3 tells us that the unipotent radical $\mathcal{U}$ of the weighted completion of $\mathbb{Z}_\ell^\times$ is trivial, so that the weighted completion of $\mathbb{Z}_\ell^\times$ with respect to ρ is just $\rho : \mathbb{Z}_\ell^\times \to \mathbb{G}_m(\mathbb{Q}_\ell)$. More generally, if Γ is an open subgroup of $\mathbb{Z}_\ell^\times$, then the weighted completion of Γ, relative to the restriction $\Gamma \to \mathbb{Q}_\ell^\times$ of the homomorphism ρ above and the same w, is simply $\mathbb{G}_m$.

EXAMPLE 8.5. Let $\mathcal{M}_g$ be the moduli stack of genus g curves over $\mathrm{Spec}\,\mathbb{Z}$. Suppose that there is a $\mathbb{Z}[1/\ell]$-section $x : \mathrm{Spec}\,\mathbb{Z}[1/\ell] \to \mathcal{M}_g$. We allow tangential sections, and then such x exist for all g.

Let $\bar{x} : \mathrm{Spec}\,\overline{\mathbb{Q}} \to \mathrm{Spec}\,\mathbb{Z}[1/\ell] \to \mathcal{M}_g$ be a geometric point on the generic point of x. Let $C_{\bar{x}}$ be the curve corresponding to $\bar{x}$.

There is a short exact sequence of algebraic fundamental groups

$$1 \to \pi_1(\mathcal{M}_g \otimes \overline{\mathbb{Q}}, \bar{x}) \to \pi_1(\mathcal{M}_g \otimes \mathbb{Q}, \bar{x}) \to G_\mathbb{Q} \to 1, \qquad (8\text{–}3)$$

where the left group is isomorphic to the profinite completion $\hat{\Gamma}_g$ of the mapping class group Γ_g of a genus g surface. We fix such an isomorphism. We have the

natural representation

$$\pi_1(\mathcal{M}_g \otimes \mathbb{Q}, \bar{x}) \to \operatorname{Aut} H^1_{\text{ét}}(C_{\bar{x}}, \mathbb{Q}_\ell). \tag{8–4}$$

It is known that the image of (8–4) is isomorphic to $\operatorname{GSp}_g(\mathbb{Z}_\ell)$, where GSp_g denotes the group of symplectic similitudes of a symplectic module of rank $2g$.

By considering the action of the mapping class group on the $\mathbb{Z}/\ell\mathbb{Z}$ homology of the surface, we obtain a natural representation $\hat{\Gamma}_g \to \operatorname{Sp}_g(\mathbb{Z}/\ell)$. Let Γ_g^ℓ be the largest quotient of $\hat{\Gamma}_g$ that also maps to $\operatorname{Sp}_g(\mathbb{Z}/\ell)$ and such that the kernel of the induced mapping $\Gamma_g^\ell \to \operatorname{Sp}_g(\mathbb{Z}/\ell)$ is a pro-ℓ group.

One can construct a quotient

$$1 \to \Gamma_g^\ell \to \Gamma_g^{\text{arith},\ell} \to G_{\mathbb{Q},\{\ell\}} \to 1$$

of the short exact sequence (8–3) such that the homomorphism (8–4) induces a homomorphism

$$\rho : \Gamma_g^{\text{arith},\ell} \to \operatorname{GSp}_g(\mathbb{Q}_\ell)$$

from (8–4).

Define the central cocharacter $\omega : \mathbb{G}_m \to \operatorname{GSp}_g$ by $x \to x^{-1}I_{2g}$. In [24] we show that the weighted completion $\mathcal{G}_g^{\text{arith},\ell}$ of $\Gamma_g^{\text{arith},\ell}$ is an extension

$$\mathcal{G}_g \otimes \mathbb{Q}_\ell \to \mathcal{G}_g^{\text{arith},\ell} \to \mathcal{A}_{\mathbb{Q},\{\ell\}}^\ell \to 1,$$

where $\mathcal{G}_g$ is the completion of Γ_g relative to the standard homomorphism $\rho : \Gamma_g \to \operatorname{Sp}_g(\mathbb{Q})$, which is studied in [21] and for which a presentation is given in [22]. We expect that the left homomorphism is injective.

9. Computation of $\mathcal{A}_{F,S}^\ell$

In this section, we compute $\mathcal{A}_{F,S}^\ell$, the tannakian fundamental group of the category of ℓ-adic mixed Tate modules over $X_{F,S}$. An equivalent computation was done by Beilinson and Deligne in [2]. We shall need the following result of Soulé [41] when ℓ is odd. The case $\ell = 2$ follows from [40]. Recall that d_n is defined in (3–1).

THEOREM 9.1 (SOULÉ [41]). *With notation as above,*

$$K_{2n-1}(X_{F,S}) \otimes \mathbb{Q}_\ell \cong H^1_{\text{cts}}(G_{F,S}, \mathbb{Q}_\ell(n))$$

and hence

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{cts}}(G_{F,S}, \mathbb{Q}_\ell(n)) = \begin{cases} d_1 + \#S & \text{when } n = 1, \\ d_n & \text{when } n > 1. \end{cases}$$

In addition, $H^2_{\text{cts}}(G_{F,S}, \mathbb{Q}_\ell(n))$ vanishes for all $n \geq 2$.

Denote the unipotent radical of $\mathcal{A}^\ell_{F,S}$ by $\mathcal{K}^\ell_{F,S}$. We have the exact sequence

$$1 \to \mathcal{K}^\ell_{F,S} \to \mathcal{A}^\ell_{F,S} \to \mathbb{G}_m \to 1,$$

and the corresponding exact sequence of Lie algebras

$$0 \to \mathfrak{k}^\ell_{F,S} \to \mathfrak{a}^\ell_{F,S} \to \mathbb{Q}_\ell \to 0.$$

The Lie algebra $\mathfrak{a}^\ell_{F,S}$, being the Lie algebra of a weighted completion, has a natural weight filtration. Note that since w is the inverse of the square of the standard character, all weights are even. Thus the weight filtration of $\mathfrak{a}^\ell_{F,S}$ satisfies

$$\mathfrak{a}^\ell_{F,S} = W_0 \mathfrak{a}^\ell_{F,S}, \quad \mathfrak{k}^\ell_{F,S} = W_{-2} \mathfrak{a}^\ell_{F,S} \text{ and } \operatorname{Gr}^W_{2n+1} \mathfrak{a}^\ell_{F,S} = 0 \text{ for all } n \in \mathbb{Z},$$

and we may take $N = 2$ in Corollary 8.2. The basic structure of $\mathcal{A}^\ell_{F,S}$ now follows from Corollary 8.2, Corollary 8.3 and Soulé's computation above.

THEOREM 9.2 (HAIN-MATSUMOTO [23]). *The Lie algebra* $\operatorname{Gr}^W_\bullet \mathfrak{k}^\ell_{F,S}$ *is a free Lie algebra and there is a natural* $\mathbb{G}_m$-*equivariant isomorphism*

$$H^1_{\mathrm{cts}}(\mathfrak{k}^\ell_{F,S}) \cong \bigoplus_{n=1}^\infty H^1_{\mathrm{cts}}(G_{F,S}, \mathbb{Q}_\ell(n)) \otimes \mathbb{Q}_\ell(-n) \cong \mathbb{Q}_\ell(-1)^{d_1+\#S} \oplus \bigoplus_{n>1} \mathbb{Q}_\ell(-n)^{d_n},$$

where d_n *is defined in* (3–1). *Any lift of a graded basis of* $H_1(\operatorname{Gr}^W_\bullet \mathfrak{k}^\ell_{F,S})$ *to a graded set of elements of* $\operatorname{Gr}^W_\bullet \mathfrak{k}^\ell_{F,S}$ *freely generates* $\operatorname{Gr}^W_\bullet \mathfrak{k}^\ell_{F,S}$.

As a corollary of the proof, we have:

COROLLARY 9.3. *There are natural isomorphisms*

$$\operatorname{Ext}^m_{\mathcal{T}_\ell(X_{F,S})}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n)) \cong \begin{cases} \mathbb{Q}_\ell & \text{when } m = n = 0, \\ H^1_{\mathrm{cts}}(G_{F,S}, \mathbb{Q}_\ell(n)) & \text{when } m = 1 \text{ and } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, for all $n \in \mathbb{Z}$, *there are natural isomorphisms*

$$\operatorname{Ext}^1_{\mathcal{T}_\ell(X_{F,S})}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n)) \cong K_{2n-1}(\operatorname{Spec} \mathcal{O}_{F,S}) \otimes \mathbb{Q}_\ell.$$

COROLLARY 9.4. *Suppose that there is a category of mixed Tate motives* $\mathcal{T}(X_{F,S})$ *with properties* (i)–(v) *as in Section 4. If Deligne's conjecture 5.5 is true, then, the image of the* ℓ-*adic realization functor* real_ℓ *in* (5–3) *is equivalent to the category of weighted* $G_{F,S}$-*modules. In particular, Postulate 5.6 follows.*

PROOF. Deligne's conjecture 5.5 implies that $\pi_1(\mathcal{T}(X_{F,S}))$, the tannakian fundamental group, is an extension

$$1 \to \mathcal{U}_{X_{F,S}} \to \pi_1(\mathcal{T}(X_{F,S})) \to \mathbb{G}_m \to 1,$$

where $\mathcal{U}_{X_{F,S}}$ is a free prounipotent group generated by $K_{2n-1}(X_{F,S})^*$. This and Theorem 9.2 show that the natural map $\pi_1(\mathcal{T}_\ell(X_{F,S})) \to \pi_1(\mathcal{T}(X_{F,S})) \otimes \mathbb{Q}_\ell$ is an isomorphism, and it follows that

$$\mathrm{real}_\ell : \mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell \to \ell\text{-adic } G_F\text{-modules}$$

is fully faithful and its image is equivalent to the category of weighted $G_{F,S}$-modules. $\qquad\square$

Note that these theorems can be generalized to the case where S may not contain all the primes above ℓ, see Section 11.

Another example. Suppose that S is a finite set of rational primes containing ℓ. Suppose that F is a finite Galois extension of $\mathbb{Q}$ with Galois group G, which is unramified outside S. Define

$$\rho : G_{\mathbb{Q},S} \to \mathbb{G}_m(\mathbb{Q}_\ell) \times G$$

by

$$\rho(\sigma) = (\chi_\ell(\sigma), f(\sigma))$$

where $f : G_{\mathbb{Q},S} \to G$ is the quotient homomorphism and χ_ℓ is the ℓ-adic cyclotomic character. Define

$$w : \mathbb{G}_m \to \mathbb{G}_m \times G$$

by $w : x \mapsto (x^{-2}, 1)$. It is a central cocharacter. Denote the weighted completion of $G_{\mathbb{Q},S}$ with respect to ρ and w by $\mathcal{G}_{\mathbb{Q},S}$.

Denote the set of primes in $\mathcal{O}_F$ that lie over $S \subset \mathrm{Spec}\,\mathbb{Z}$ by T.

PROPOSITION 9.5. *There is a natural inclusion $\iota : \mathcal{A}_{F,T}^\ell \to \mathcal{G}_{\mathbb{Q},S}$ and an exact sequence*

$$1 \longrightarrow \mathcal{A}_{F,T}^\ell \overset{\iota}{\longrightarrow} \mathcal{G}_{\mathbb{Q},S} \longrightarrow G \longrightarrow 1.$$

PROOF. If $\{V_\alpha\}$ is a set of representatives of the isomorphism classes of irreducible representations of G, then $\{\mathbb{Q}_\ell(m) \boxtimes V_\alpha\}$ is a set of representatives of the isomorphism classes of irreducible representations of $\mathbb{G}_m \times G$, where $W \boxtimes V$ denotes the exterior tensor product of a representation W of $\mathbb{G}_m$ and V of G. Consider the restriction mapping

$$\phi : H_{\mathrm{cts}}^i(G_{\mathbb{Q},S}, \mathbb{Q}_\ell(m) \boxtimes V_\alpha) \to H_{\mathrm{cts}}^i(G_{F,T}, \mathbb{Q}_\ell(m) \boxtimes V_\alpha)^G$$

and the transfer mapping [42]

$$\psi : H_{\mathrm{cts}}^i(G_{F,T}, \mathbb{Q}_\ell(m) \boxtimes V_\alpha)^G \to H_{\mathrm{cts}}^i(G_{\mathbb{Q},S}, \mathbb{Q}_\ell(m) \boxtimes V_\alpha).$$

A direct computation on cocycles shows that $\phi \circ \psi$ and $\psi \circ \phi$ are both multiplication by the order of G, and are thus isomorphisms.

Therefore $H_{\mathrm{cts}}^i(G_{\mathbb{Q},S}, \mathbb{Q}_\ell(m) \boxtimes V_\alpha)$ vanishes if V_α is nontrivial, and it equals $H_{\mathrm{cts}}^i(G_{F,T}, \mathbb{Q}_\ell(m))$ if V_α is trivial. This shows that the unipotent radical of the completion $\mathcal{G}_{\mathbb{Q},S}$ is isomorphic to that of $\mathcal{A}_{F,T}^\ell$.

By functoriality of weighted completion, we have a homomorphism $\mathcal{A}^\ell_{F,T} \to \mathcal{G}_{\mathbb{Q},S}$ which induces the isomorphism on the unipotent radical. The statement follows. $\qquad\qquad\square$

10. Applications to Galois Actions on Fundamental Groups

Let G_ℓ denote $G_{\mathbb{Q},\{\ell\}}$. In this section, we sketch how our computation of the weighted completion of G_ℓ can be used to prove Deligne's Conjecture 5.8 about the action of the absolute Galois group $G_\mathbb{Q}$ on the pro-ℓ completion of the fundamental group of $\mathbb{P}^1(\mathbb{C}) - \{0,1,\infty\}$. Modulo a few technical details, which are addressed in [23], the proof proceeds along the expected lines suggested in Section 5.2 given the computation of $\mathcal{A}^\ell_{F,S}$.

We begin in a more general setting. Suppose that F is a number field and that X is a variety over F. Set $\overline{X} = X \otimes \overline{\mathbb{Q}}$ and denote the absolute Galois group of F by G_F. Suppose that the étale cohomology group $H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$ is a trivial G_F-module. Let S be a set of finite primes of F, containing those above ℓ. Suppose that X has a model $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_{F,S}$ which has a base point section $x : \operatorname{Spec}\mathcal{O}_{F,S} \to \mathcal{X}$ (possibly tangential) such that $(\mathcal{X},x)$ has good reduction outside S.[6] Then the G_F-action on the pro-ℓ fundamental group $\pi_1^\ell(\overline{X},x)$ factors through $G_{F,S}$.

Denote the ℓ-adic unipotent completion of $\pi_1^\ell(\overline{X},x)$ by $\mathcal{P}$ (see [23, Appendix A]) and its Lie algebra by $\mathfrak{p}$. The lower central series filtration of $\mathfrak{p}$ gives it the structure of a pro-object of the category $\mathcal{T}_\ell(X_{F,S})$ of ℓ-adic mixed Tate modules over $X_{F,S}$. It follows that the G_F-action on $\mathcal{P}$ induces a homomorphism

$$\mathcal{A}^\ell_{F,S} \to \operatorname{Aut}\mathcal{P} \cong \operatorname{Aut}\mathfrak{p}$$

and that the action of G_F on $\mathcal{P}$ factors through the composition of this with the natural homomorphism $G_F \to G_{F,S} \to \mathcal{A}^\ell_{F,S}(\mathbb{Q}_\ell)$. One can show (see [23, Sect. 8]) that the image of $G_{F(\mu_{\ell^\infty})}$ in $\mathcal{A}^\ell_{F,S}$ lies in and is Zariski dense in $\mathcal{K}^\ell_{F,S}$.

For the rest of this section, we consider the case where $X = \mathbb{P}^1 - \{0,1,\infty\}$, $F = \mathbb{Q}$, $S = \{\ell\}$ and x is the tangential base point $\overrightarrow{01}$. Goncharov's conjecture [19, Conj. 2.1] (cf. the generation part of Conjecture 5.9) follows immediately, since $\mathfrak{k}_{\mathbb{Q},\{\ell\}}$ is generated by $z_1, z_3, z_5, \ldots$, where z_j has weight $-2j$. The image of z_1 can be shown to be trivial.

We are now ready to give a brief sketch of the proof of Conjecture 5.8. One can define a filtration $\mathcal{I}_\ell^\bullet$ on $G_\mathbb{Q}$ similar to $I_\ell^\bullet$ using the lower central series $L^\bullet\mathcal{P}$ of $\mathcal{P}$ instead:

$$\mathcal{I}_\ell^m G_\mathbb{Q} = \ker\{G_\mathbb{Q} \to \operatorname{Out}\mathcal{P}/L^{m+1}\mathcal{P}\}$$

---

[6]What we mean here is that there is a scheme $\widetilde{\mathcal{X}}$, proper over $\operatorname{Spec}\mathcal{O}_{F,S}$, and a divisor D in $\widetilde{\mathcal{X}}$ which is relatively normal crossing over $\operatorname{Spec}\mathcal{O}_{F,S}$ such that $\mathcal{X} = \widetilde{\mathcal{X}} - D$, and D does not intersect with x. In the tangential case, the tangent vector should be non-zero over each point of $\operatorname{Spec}\mathcal{O}_{F,S}$.

where $L^m\mathcal{P}$ is the mth term of its lower central series. The lower central series of $\mathcal{P}$ is related to its weight filtration by

$$W_{-2m}\mathcal{P} = L^m\mathcal{P}, \quad \mathrm{Gr}^W_{2m+1}\mathcal{P} = 0.$$

There is a natural isomorphism (see [23, Sect. 10])

$$[\mathrm{Gr}^m_{I_\ell} G_\mathbb{Q}] \otimes \mathbb{Q}_\ell \cong [\mathrm{Gr}^m_{\mathcal{I}_\ell} G_\mathbb{Q}] \otimes \mathbb{Q}_\ell.$$

Thus it suffices to prove that $[\mathrm{Gr}^m_{\mathcal{I}_\ell} G_\mathbb{Q}] \otimes \mathbb{Q}_\ell$ is generated by elements $s_3, s_5, s_7, \ldots$, where s_j has weight $-2j$.

As above, the homomorphism $G_\mathbb{Q} \to \mathrm{Out}\,\mathcal{P}$ factors through the sequence

$$G_\mathbb{Q} \to G_{\mathbb{Q},\{\ell\}} \to \mathcal{A}_{\mathbb{Q},\{\ell\}} \to \mathrm{Out}\,\mathcal{P}$$

of natural homomorphisms. A key point ([23, Sect. 8]) is that the image of $I_\ell^1 G_\mathbb{Q}$ in $\mathcal{K}^\ell_{\mathbb{Q},\ell}$ is Zariski dense. This and the strictness can be used to establish isomorphisms

$$Gr^m_{I_\ell} G_\mathbb{Q} \otimes \mathbb{Q}_\ell \cong \mathrm{Gr}^W_{-2m}(\mathrm{im}\{\mathfrak{k}^\ell_{\mathbb{Q},\ell} \to \mathrm{OutDer}\,\mathfrak{p}\})$$
$$\cong \mathrm{im}\{\mathrm{Gr}^W_{-2m} \mathfrak{k}^\ell_{\mathbb{Q},\ell} \to \mathrm{Gr}^W_{-2m} \mathrm{OutDer}\,\mathfrak{p}\}$$

for each $m > 0$.

Theorem 9.2 implies that $\mathrm{Gr}^W_\bullet \mathfrak{k}^\ell_{\mathbb{Q},\ell}$ is freely generated by $\sigma_1, \sigma_3, \sigma_5, \ldots$ where $\sigma_{2i+1} \in \mathrm{Gr}^W_{-2(2i+1)} \mathfrak{k}^\ell_{\mathbb{Q},\ell}$. It is easy to show that the image of σ_1 vanishes in $\mathrm{Gr}^W_\bullet \mathrm{OutDer}\,\mathfrak{p}$. It follows that the image of $\mathrm{Gr}^W_\bullet \mathfrak{k}^\ell_{\mathbb{Q},\ell}$ is generated by the images of $\sigma_3, \sigma_5, \sigma_7, \ldots$, which completes the proof.

REMARK 10.1. Ihara proves the openness of the group generated by σ_{2i+1} in a suitable Galois group, see [28]. He also establishes the non-vanishing of the images of the σ_{2i+1} and some of their brackets in [27].

11. When ℓ is not contained in S

Let $[\ell]$ denote the set of all primes above ℓ in $\mathcal{O}_F$. In this section, we generalize the definition of the category $\mathcal{T}_\ell(X_{F,S})$ of ℓ-adic mixed Tate modules smooth over $X_{F,S} = \mathrm{Spec}\,\mathcal{O}_F - S$ (see Section 6) to the case where S does not necessarily contain $[\ell]$.

For this, we define the category $\mathcal{T}_\ell(X_{F,S})$ of ℓ-adic mixed Tate modules over $X_{F,S}$ to be the full subcategory of $\mathcal{T}_\ell(X_{F,S \cup [\ell]})$ (defined in Section 6) consisting of the Galois modules which are crystalline at every prime $\mathfrak{p} \in [\ell] - S$. (Recall that an ℓ-adic G_F-module M is crystalline at a prime $\mathfrak{p}$ of F if it is crystalline as $G_{F_\mathfrak{p}}$-module, where $F_\mathfrak{p}$ is the completion of F at $\mathfrak{p}$ and $G_{F_\mathfrak{p}}$ is identified with the decomposition group of G_F at $\mathfrak{p}$, see [16; 7] for crystalline representations.)

It is known that the crystalline property is closed under tensor products, direct sums, duals, and subquotients [16], so that $\mathcal{T}_\ell(X_{F,S})$ is a tannakian category. Denote its tannakian fundamental group by $\mathcal{A}^\ell_{F,S}$. We have a short exact

sequence

$$1 \to \mathcal{K}_{F,S}^\ell \to \mathcal{A}_{F,S}^\ell \to \mathbb{G}_m \to 1,$$

and the corresponding exact sequence of Lie algebras

$$0 \to \mathfrak{k}_{F,S}^\ell \to \mathfrak{a}_{F,S}^\ell \to \mathbb{Q}_\ell \to 0.$$

Let V be a $G_{F,S}$-module. The *finite part* of the first degree Galois cohomology $H_{\mathrm{cts}f}^1(G_{F,S}, V) \subset H_{\mathrm{cts}}^1(G_{F,S}, V)$ is defined in [7, (3.7.2)]. This corresponds to those extensions of $\mathbb{Q}_\ell$ by V as $G_{F,S}$-modules, which are crystalline at every prime in $[\ell]$ outside S. By a remark on p. 354 in [7], $H_{\mathrm{cts}f}^1(G_{F,S}, \mathbb{Q}_\ell) = (\mathcal{O}_{F,S}^\times) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, so its dimension is $d_1 + \#S = r_1 + r_2 + \#S - 1$. Theorem 9.2 is generalized as follows, by replacing H_{cts}^1 with $H_{\mathrm{cts}f}^1$ [23]. We shall give a categorical proof below.

THEOREM 11.1. *The Lie algebra* $\mathrm{Gr}_\bullet^W \mathfrak{k}_{F,S}^\ell$ *is a free Lie algebra and there is a natural* $\mathbb{G}_m$*-equivariant isomorphism*

$$H_{\mathrm{cts}}^1(\mathfrak{k}_{F,S}^\ell) \cong \bigoplus_{n=1}^\infty H_{\mathrm{cts}f}^1(G_{F,S}, \mathbb{Q}_\ell(n)) \otimes \mathbb{Q}_\ell(-n) \cong \mathbb{Q}_\ell(-1)^{d_1 + \#S} \oplus \bigoplus_{n>1} \mathbb{Q}_\ell(-n)^{d_n},$$

where d_n *is defined in* (3–1). *Any lift of a graded basis of* $H_1(\mathrm{Gr}_\bullet^W \mathfrak{k}_{F,S}^\ell)$ *to a graded set of elements of* $\mathrm{Gr}_\bullet^W \mathfrak{k}_{F,S}^\ell$ *freely generates* $\mathrm{Gr}_\bullet^W \mathfrak{k}_{F,S}^\ell$.

COROLLARY 11.2. *There are natural isomorphisms*

$$\mathrm{Ext}_{\mathcal{T}_\ell(X_{F,S})}^m(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n)) \cong \begin{cases} \mathbb{Q}_\ell & \text{when } m = n = 0, \\ H_{\mathrm{cts}f}^1(G_{F,S}, \mathbb{Q}_\ell(n)) & \text{when } m = 1 \text{ and } n > 0, \\ 0 & \text{otherwise}. \end{cases}$$

Consequently, for all $n \in \mathbb{Z}$, *there are natural isomorphisms*

$$\mathrm{Ext}_{\mathcal{T}_\ell(X_{F,S})}^1(\mathbb{Q}_\ell, \mathbb{Q}_\ell(n)) \cong K_{2n-1}(\mathrm{Spec}\, \mathcal{O}_{F,S}) \otimes \mathbb{Q}_\ell. \qquad \square$$

This shows that $\mathcal{T}_\ell(X_{F,S})$ has all the properties of the category $\mathcal{T}(X_{F,S}) \otimes \mathbb{Q}_\ell$, where $\mathcal{T}(X_{F,S})$ is the category whose existence is conjectured by Deligne. In particular, $\mathrm{Gr}_\bullet^W \mathfrak{k}_{\mathbb{Q},\varnothing}^\ell$ is free with generators $\sigma_3, \sigma_5, \ldots$.

PROOF OF THEOREM 11.1. It suffices to show that the natural mapping

$$\Phi^1 : H^1(\mathcal{A}_{F,S}^\ell, \mathbb{Q}_\ell(n)) \to H_{\mathrm{cts}f}^1(G_{F,S}, \mathbb{Q}_\ell(n))$$

is an isomorphism when $n \geq 1$ and that the natural mapping

$$\Phi^2 : H^2(\mathcal{A}_{F,S}^\ell, \mathbb{Q}_\ell(n)) \to H_{\mathrm{cts}}^2(G_{F,S}, \mathbb{Q}_\ell(n))$$

is injective when $n \geq 2$. The proof is similar to that of Theorem 8.1. To show that Φ^1 is an isomorphism, it suffices to show that an extension E of $\mathbb{Q}_\ell$ by $\mathbb{Q}_\ell(n)$ corresponding to an element of $H_{\mathrm{cts}f}^1(G_{F,S}, \mathbb{Q}_\ell(n))$ is crystalline, which is well-known. So the first assertion follows.

We now consider the case of Φ^2. Set $V_\alpha = \mathbb{Q}_\ell(n)$. We may assume $n \geq 2$. It suffices to show that E in the proof of Theorem 8.1 is crystalline provided

E_1 and E_2 are crystalline. But this follows from the next result, which will be proved below.

PROPOSITION 11.3. *Let*

$$0 \longrightarrow V \longrightarrow U \longrightarrow \mathbb{Q}_\ell(1)^n \longrightarrow 0$$

be a short exact sequence of crystalline ℓ-adic representations of $G_{F_\mathfrak{p}}$. Assume that V is a successive extension of direct sums of a finite number of copies of $\mathbb{Q}_\ell(r)$ with $r \geq 2$. Then, for any extension

$$0 \longrightarrow U \longrightarrow E \longrightarrow \mathbb{Q}_\ell \longrightarrow 0$$

of ℓ-adic representations of $G_{F_\mathfrak{p}}$, E is crystalline if and only if its pushout by the surjection $U \to \mathbb{Q}_\ell(1)^n$ is crystalline.

Let U be E_2 as in the proof of Theorem 5.3. Since $W_{-2}E_2 = E_2$, U is an extension of $\mathbb{Q}_\ell(1)^n$ for some n. Since $m \geq 2$, the pushout of E along $U \to \mathbb{Q}_\ell(1)^n$ is a quotient of E_1, and hence is crystalline. Thus the proposition says that E is crystalline, which completes the proof of Theorem 11.1. $\qquad\square$

Proposition 11.3 follows from the following two lemmas.

LEMMA 11.4. *Let*

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

be a short exact sequence of crystalline ℓ-adic representations of $G_{F_\mathfrak{p}}$. Then we have a long exact sequence

$$0 \to H^0(G_{F_\mathfrak{p}}, V_1) \to H^0(G_{F_\mathfrak{p}}, V_2) \to H^0(G_{F_\mathfrak{p}}, V_3)$$
$$\to H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V_1) \to H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V_2) \to H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V_3) \to 0. \quad \square$$

This follows from [7, Cor. 3.8.4].

LEMMA 11.5. *Let V be a crystalline ℓ-adic representation of $G_{F_\mathfrak{p}}$. If V is a successive extension of $\mathbb{Q}_\ell(r)$ ($r \geq 2$), then $H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V) = H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, V)$.*

PROOF. The proof is by induction on the dimension of V. In the case $\dim(V) = 1$, this is well-known (loc. cit. Example 3.9). Assume $\dim V = n \geq 2$ and the claim is true for $n - 1$. By assumption, there exists an exact sequence of ℓ-adic representations of $G_{F_\mathfrak{p}}$:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow \mathbb{Q}_\ell(r) \longrightarrow 0$$

for some integer $r \geq 2$ such that V' satisfies the assumption of the lemma. By Lemma 11.4, we have the following commutative diagram whose two rows are

exact:

$$0 \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V') \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V) \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, \mathbb{Q}_\ell(r)) \longrightarrow 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, V') \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, V) \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, \mathbb{Q}_\ell(r))$$

The right vertical arrow is an isomorphism and the left one is also an isomorphism by the induction hypothesis. Hence the middle one is also an isomorphism. $\square$

PROOF OF PROPOSITION 11.3. By Lemma 11.4, we have the following commutative diagram whose two rows are exact:

$$0 \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, V) \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, U) \longrightarrow H^1_{\mathrm{cts}f}(G_{F_\mathfrak{p}}, \mathbb{Q}_\ell(1)^n) \longrightarrow 0$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0 \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, V) \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, U) \longrightarrow H^1_{\mathrm{cts}}(G_{F_\mathfrak{p}}, \mathbb{Q}_\ell(1)^n)$$

and the left vertical arrow is an isomorphism by Lemma 11.5. Hence the right square is cartesian. $\square$

Appendix: Continuous Cohomology and Yoneda Extensions

In this appendix we prove a result about the relation between continuous cohomology and Yoneda extension groups in low degrees. It is surely well known, but we know of no reference.

Suppose that K is a topological field, and Γ a topological group. A *continuous Γ-module* is a Γ-module V, where V is a finite dimensional K-vector space. The action $\Gamma \to \mathrm{GL}(V)$ is required to be continuous, where $\mathrm{GL}(V)$ is given the topology induced from that of K.

Denote by $\mathcal{C}(\Gamma, K)$ the category of finite dimensional continuous Γ-modules. Since any K-linear morphism between finite dimensional vector spaces is continuous, this is an abelian category. For continuous Γ-modules A and B, define $\mathrm{Ext}^\bullet_\Gamma(A, B)$ to be the graded group of Yoneda extensions of B by A in the category $\mathcal{C}(\Gamma, K)$.

For a continuous Γ-module A, one also has the continuous cohomology groups $H^\bullet_{\mathrm{cts}}(\Gamma, A)$ defined by Tate [42], which are defined using the complex of continuous cochains.

THEOREM A.6. *If A is a continuous Γ-module, there is a natural isomorphism $\mathrm{Ext}^1_\Gamma(K, A) \cong H^1_{\mathrm{cts}}(\Gamma, A)$ and a natural injection $\mathrm{Ext}^2_\Gamma(K, A) \hookrightarrow H^2_{\mathrm{cts}}(\Gamma, A)$.*

PROOF. It is well known that an extension $0 \to A \to E \to K \to 0$ in $\mathcal{C}(\Gamma, K)$ gives a continuous cocycle $f : \Gamma \to A$ by choosing a lift $e \in E$ of $1 \in K$ and

defining $f(\sigma) = \sigma(e) - e$. Conversely, for a given continuous cocycle f, we may define continuous Γ-action on $A \oplus K$ by $\sigma : (a, k) \mapsto (\sigma(a) + kf(\sigma), k)$. These are mutually inverse, which establishes the first claim.

To prove the second claim, we first define a K-linear mapping

$$\varphi : \operatorname{Ext}_\Gamma^2(K, A) \to H_{\mathrm{cts}}^2(\Gamma, A)$$

as follows. For $c \in \operatorname{Ext}_\Gamma^2(K, A)$, choose a 2-fold extension $0 \to A \to E_2 \to E_1 \to K \to 0$ that represents it. By [45], c is the image under the connecting homomorphism

$$\delta : \operatorname{Ext}_\Gamma^1(K, E_2/A) \to \operatorname{Ext}_\Gamma^2(K, A)$$

of the class $\tilde{c}$ of the extension $0 \to E_2/A \to E_1 \to K \to 0$.

We shall construct φ so that the diagram

$$
\begin{array}{ccccc}
\operatorname{Ext}_\Gamma^1(K, E_2) & \longrightarrow & \operatorname{Ext}_\Gamma^1(K, E_2/A) & \xrightarrow{\ \delta\ } & \operatorname{Ext}_\Gamma^2(K, A) \\
\Big\downarrow{\simeq} & & {\psi}\Big\downarrow{\simeq} & & \Big\downarrow{\varphi} \\
H_{\mathrm{cts}}^1(\Gamma, E_2) & \longrightarrow & H_{\mathrm{cts}}^1(\Gamma, E_2/A) & \xrightarrow{\ \delta_{\mathrm{cts}}\ } & H_{\mathrm{cts}}^2(\Gamma, A)
\end{array}
$$

commutes, where the rows are parts of the standard long exact sequences constructed in [45] and [42, Sect. 2]. Define $\varphi(c)$ to be $\delta_{\mathrm{cts}}(\psi(\tilde{c}))$.

To prove $\varphi(c)$ is well-defined, it suffices to show that two 2-fold extensions that fit into a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E_2' & \longrightarrow & E_1' & \longrightarrow & K & \longrightarrow & 0 \\
& & \Big\| & & \Big\downarrow & & \Big\downarrow & & \Big\| \\
0 & \longrightarrow & A & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & K & \longrightarrow & 0
\end{array}
$$

give a same element of $H_{\mathrm{cts}}^2(\Gamma, A)$. But this follows from the functoriality of the connecting homomorphism for $H_{\mathrm{cts}}^\bullet$, i.e., the commutativity of

$$
\begin{array}{ccc}
H_{\mathrm{cts}}^1(\Gamma, E_2'/A) & \longrightarrow & H_{\mathrm{cts}}^2(\Gamma, A) \\
\Big\downarrow & & \Big\| \\
H_{\mathrm{cts}}^1(\Gamma, E_2/A) & \longrightarrow & H_{\mathrm{cts}}^2(\Gamma, A).
\end{array}
$$

The K-linearity of φ is easily checked. Finally, the injectivity of φ follows from the fact that for each extension as above, φ is injective on the image of the connecting homomorphism $\delta : \operatorname{Ext}_\Gamma^1(K, E_2/A) \to \operatorname{Ext}_\Gamma^2(K, A)$. $\qquad\square$

Note that one may define

$$\operatorname{Ext}_\Gamma^m(K, A) \to H_{\mathrm{cts}}^m(G, A)$$

by induction on m in the same way.

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