Galois Groups and Fundamental Groups
MSRI Publications
Volume 41, 2003

Cellular Decomposition of Compactified Hurwitz Spaces

MICHEL IMBERT

Abstract. We describe a cellular decomposition of compactified Hurwitz spaces, generalizing the cellular decomposition of moduli spaces of punctured Riemann surfaces $\mathcal{M}_{g,n}$.

The main motivation for this work is the integration of cohomology classes on compactified Hurwitz spaces, and is provided by Witten's conjecture on moduli spaces of Riemann surfaces with spin, and by the fact (proved in this paper) that these spaces are closely related to Hurwitz spaces of Galois cyclic coverings. This article also aims to give all details of the Harer–Kontsevich theorem.

1. Introduction

In the last ten years, the geometry of moduli spaces of punctured Riemann surfaces have seen an increasing interest and known some striking progress [20] in connection with physic's theories [26]. This also concerns some generalizations of moduli spaces of punctured surfaces, like moduli spaces of stable maps, or moduli spaces of Riemann surfaces with spin [27] [17]. In this paper, we firstly show that the topological framework which have allowed M. Kontsevich to compute in a combinatorial way some Chern classes on $\mathcal{M}_{g,n}$ (a key point of his proof of Witten's conjecture [26]) extends to the setting of Hurwitz spaces, and secondly we sketch an analogy between Hurwitz spaces of cyclic coverings and moduli spaces of Riemann surfaces with spin. In a forthcoming work, our results will be used to study the cohomology of Hurwitz spaces for cyclic coverings.

Hurwitz spaces basically consist in equivalence classes of ramified coverings between two compact Riemann surfaces, where $(p_1 : S_1 \to T_1)$ is equivalent to $(p_2 : S_2 \to T_2)$ if there exists two biholomorphisms $f : S_1 \to S_2$ and $h : T_1 \to T_2$ such that $p_2 \circ f = h \circ p_1$. We often denote by g the genus of the total space, and by g' the genus of the base. They are related by the Riemann–Hurwitz formula: $2g - 2 = d(2g' - 2) + B$, where d is the degree of the covering, and B, the total

Mathematics Subject Classification: 05C10, 14H30, 14E22, 30F30, 30C62, 32G15.

branching number. We only consider cases $d > 1$ and $B > 0$. Let us denote by $\mathcal{H}(g, g', d)$ the set of such equivalence classes with fixed g, g' and d, and call it a Hurwitz space. There are two ways to study it.

The first one is to work with this equivalence relation. This is the approach of W. J. Harvey [11] [15], and A. Kuribayashi [22] in the setting of Teichmüller theory.

The second one is configurative (only branch points are allowed to move on the base). This is the approach of W. Fulton [9] and M. Fried [8].

In both cases, it is necessary to fix more accurate invariants in equivalence classes of coverings, such as number of branch points, multiplicities, monodromy group. It is in fact the notion of ramification data (a way to encode what actually happens in a neighborhood of a ramification point) which allows us a combinatorial study of Hurwitz spaces, using coverings of fat-graphs.

We show that the graphical description of punctured Riemann surfaces with fat-graphs and lengths of edges, extend to holomorphic maps between compact Riemann surfaces. Punctures come as ramification points and branch points. Coverings of fat-graphs are étale on the underlying graphs, the ramification can be read on faces. We give details on the combinatorial model of Kontsevich's compactification of moduli spaces, and compactify Hurwitz spaces in a similar way.

Let $\mathcal{H}(g, g', G, K, R)$ be the subset of $\mathcal{H}(g, g', d)$ where we have fixed a monodromy of type (G, K, R), for a fixed finite group G, some ramification data R of degree b, and a subgroup K of index d such that $\bigcap_{t \in G} t^{-1} K t = \{e_G\}$.

Then, our main result is a generalization of the Harer–Kontsevich theorem (theorem in appendix B of [20]):

THEOREM 1.1. *There exist a combinatorial Hurwitz space $\overline{\mathcal{H}}^{\mathrm{comb}}_{g,g'}(G, K, R)$, a suitable compactification $\overline{\mathcal{H}}'(g, g', G, K, R)$ of $\mathcal{H}(g, g', K, G, R)$ and an homeomorphism*

$$\overline{\mathcal{H}}^{\mathrm{comb}}_{g,g'}(G, K, R) \to \overline{\mathcal{H}}'(g, g', G, K, R) \times \mathbb{P}(\mathbb{R}^b_{>0})$$

leading to a cellular decomposition of $\overline{\mathcal{H}}'(g, g', G, K, R) \times \mathbb{P}(\mathbb{R}^b_{>0})$, compatible with its orbifold structure.

Spaces $\mathcal{H}(g, g', G, K, R)$ arise as finite quotients of Hurwitz spaces $\mathcal{H}_h(G, R)$, made of genus h compact Riemann surfaces with an holomorphic action of G of ramification data R, defined up to G-equivariant biholomorphism. A space $\mathcal{H}_h(G, R)$ generally splits up into many connected components, each one being the quotient of a Teichmüller space.

All these Hurwitz spaces are closely related to moduli spaces. Firstly, forgetting group actions, we have a map from $\mathcal{H}_h(G, R)$ into $\mathcal{M}_h$. See [11] [15] for this viewpoint. Secondly, $\mathcal{H}_h(G, R)$ projects onto moduli spaces of punctured Riemann surfaces (map a Riemann surface with group action onto the quotient Riemann surface with its set of branch points as punctures).

Since Harer–Kontsevich's theorem is useful in explicit calculations of cohomological invariants of moduli spaces [3] [20], we hope that Theorem 1.1 can be useful in this direction, for example integrating cohomology classes on Hurwitz spaces. Actually, this is already the case for pullback of classes defined on the moduli space of punctured Riemann surfaces.

The first motivation is in fact a conjecture of E. Witten [27] which deals with intersection numbers defined on moduli spaces of punctured Riemann surfaces with spin, closely related to Hurwitz spaces of Galois coverings with cyclic groups.

Section 2 is devoted to Hurwitz spaces. It contains a complete treatment of Hurwitz spaces for non Galois coverings. In Section 3, we show how to adapt Strebel's theorem to coverings and to stable Riemann surfaces. Section 4 is first a review on fat-graphs; then we give the graphical construction of holomorphic maps between compact Riemann surfaces to obtain the main theorem in the non-compactified setting (Theorem 4.8). In Section 5, we prove the continuity and describe the cellular decomposition of decorated Hurwitz spaces. The cellular decomposition is compatible with the orbifold structure. The continuity involves a non trivial construction of quasiconformal homeomorphisms. As consequence of the cellular decomposition, we obtain a combinatorial characterization of their connected components. In Section 6 we show how to extend the setting to suitable compactifications, and finish the proof of Theorem 1.1. We conclude in the last section by the analogy between Hurwitz spaces of Galois cyclic coverings and moduli spaces of spin curves.

Acknowledgments. The author expresses his gratitude to Professor J. Bertin.

2. Hurwitz Spaces

In this section, we recall how to define Hurwitz spaces and their orbifold structure, coming from Teichmüller theory. For the sake of clearness, we first consider Hurwitz spaces of Galois coverings. The main and most natural discrete invariant is the ramification data associated to a group and to any holomorphic action of it on a compact Riemann surface. Used by A. Kuribayashi [22], W. J. Harvey [15] in the context of Teichmüller spaces, the ramification data is also useful in the context of arithmetic geometry after M. Fried [8], see also [5]. We conclude by a study of Hurwitz spaces for non Galois coverings, they arise as finite quotients of Galois Hurwitz spaces.

DEFINITION 2.1. Let G be a finite group. Then an abstract *ramification data* of G is a formal sum $R = \sum_{i=1}^{t} r_i C_i$ where $r_i \in \mathbb{N}^* \; \forall i \in \{1, \ldots, t\}$, and $(C_i)_{i \in \{1,\ldots,t\}}$ is a set of non-trivial and pairwise distinct conjugacy classes of G. The degree of R is $\sum_{i=1}^{t} r_i$.

Let C be a compact Riemann surface, and $\varphi : G \hookrightarrow \mathrm{Aut}(C)$ an effective biholomorphic action of a finite group (finiteness is automatic if $g(C) \geq 2$ by the Hurwitz theorem). We only consider actions with some fixed points.

Denote by p the branched cover $C \to C/\varphi(G)$. Let x be a fixed point of some element of G, and $y = p(x)$. There exist local coordinates (U, z) and (U', z') such that $z(x) = 0$, $z'(y) = 0$, and $z' \circ p = z^{e_x}$. The integer e_x is called the multiplicity of x. The stabilizer G_x of x is a cyclic group of order e_x. We consider the privileged generator τ_x of G_x defined by $\tau_x(z) = \exp\left(\frac{2i\pi}{e_x}\right) z$. This generator is independent of z since it is the unique element of G_x which acts on the holomorphic tangent space of C at x in multiplying by $\exp\left(\frac{2i\pi}{e_x}\right)$.

If $y = g \cdot x$, then $\tau_y = g\tau_x g^{-1}$. We say that the conjugacy class $C(\tau_x)$ of τ_x is the *color* of x (and of its orbit). Two distinct orbits can possess the same color. Let $\{O_i, b_i\}_{i\in\{1,\dots,t\}}$ be the colored orbits of fixed points with multiplicities, and $C(\tau_i)$ be the colors.

Then to the action φ is associated the ramification data:

$$R(\varphi) = \sum_{i=1}^{t} b_i C(\tau_i).$$

Note that a G-equivariant biholomorphism $h : (C, \varphi) \to (C', \varphi')$ preserves the colors, but can permute two orbits of the same color.

Definition 2.2. Let G be a fixed finite group with a fixed ramification data R. Then a $(G, R)-$Hurwitz space $\mathcal{H}_g(G, R)$ consists in equivalence classes of couples (C, φ) made of a compact Riemann surface C of genus g together with an action $\varphi : G \hookrightarrow \mathrm{Aut}(C)$ such that $R(\varphi) = R$. Two couples are equivalent if they differ by a G-equivariant biholomorphism.

Looking at the action of a stabilizer G_x on the complexified real tangent space $T_x \otimes \mathbb{C}$ shows that orientation preserving equivariant diffeomorphisms preserve the ramification data.

An important tool for studying Hurwitz spaces is Riemann's existence theorem [8] [9] [10], which furnishes a condition of existence for coverings. We state here a Galois and modular version of this theorem. In the following, it is not necessary to specify the base point for fundamental groups.

Theorem 2.3. *Fix $R = b_1C_1 + \cdots + b_tC_t$ a degree b ramification data. There is a bijective correspondence between $\mathcal{H}_g(G, R)$ and the set of classes $[S - Y, \psi]$, where S is a compact Riemann surface of genus g', Y a finite subset of S shared into t colors, $\psi : \pi_1(S - Y) \to G$ an epimorphism with $\psi(\gamma) \in C_i$ if γ is a canonical loop around some point of color i; $[S - Y, \psi] = [S' - Y', \psi']$ if and only if there exist a biholomorphism $h : S \to S'$ with $h(Y) = Y'$ (preserving the set of colors) and $\theta \in \mathrm{Int}(G)$ such that $\theta \circ \psi = \psi' \circ h_*$.*

For convenience, we recall the correspondence:

The map from $\mathcal{H}_g(G, R)$ associates to $[C, \varphi]$ the quotient surface $S = C/\varphi(G)$, and Y is the set of branch points. The genus $g(S) = g'$ is given by Riemann–Hurwitz. The epimorphism ψ is built as follows: given $\gamma \in \pi_1(S - Y, Q)$, we

consider its lift α on C from a given point of the fiber over Q. Then we associate to γ the unique element of G which sends $\alpha(0)$ on $\alpha(1)$.

Conversely, given $[S-Y,\psi]$, we build C as the universal covering $\tilde{S}$ of S quotiented by $\ker(\psi)$. Then $\theta \in G$ acts on $[x] \in C$ by $\theta \cdot [x] = [\psi^{-1}(\theta) \cdot x]$.

If $[S,Y,\psi]$ corresponds to $[C,\varphi]$, then the centralizer of $\varphi(G)$ in $\mathrm{Aut}(C)$ quotiented by the center $Z(G)$ is isomorphic to the subgroup of $\mathrm{Aut}(S-Y)$ made of the elements h satisfying $\theta \circ \psi = \psi \circ h_*$ for $\theta \in \mathrm{Int}(G)$.

EXAMPLE 1. We consider the case of cyclic coverings of compact Riemann surfaces of genus g'. Put $G = \mathbb{Z}/n$; we fix the ramification data $R = \sum_{i=1}^{t} b_i[m_i]$ of degree b, where $[m_i] \in \mathbb{Z}/n\backslash\{[0]\}$, and $[m_i] \neq [m_j]$ if $i \neq j$. If e_i is the order of m_i, then the Riemann–Hurwitz formula determines the genus g of such coverings: $2g-2 = n(2g'-2) + \sum_{i=1}^{t} nb_i(1-1/e_i)$.

If $[C,\varphi] \in \mathcal{H}_g(\mathbb{Z}/n,R)$, and $\sigma = \varphi([1])$, then we set $S = C/\langle\sigma\rangle$. Let $Q = (Q_1,\ldots,Q_b)$ be the set of branch points of $\pi : C \to S$, $\{P_{i,j}\}_j$ the preimages of Q_i, and $d_i = \#\{P_{i,j}\}$, such that $e_i = n/d_i$. The stabilizers $G_{i,j}$ of all preimage points of Q_i are equal to the unique subgroup of order e_i generated by σ^{d_i}.

If $\sigma^{d_i k_i}$ (with $(e_i,k_i)=1$) is the generator of $G_{i,j}$ which acts on the tangent space $T_{P_{i,j}}(S)$ by $\exp(2i\pi/e_i)$, then we have $R(\varphi) = \sum_i b'_i[d_i k_i]$, and $R(\varphi) = R$ means $m_i = d_i k_i$ and $b_i = b'_i$.

Riemann's existence theorem implies that $\sum_{i=1}^{t} b_i[d_i k_i] \equiv 0(n)$ (in particular $b \geq 2$). Since $[C,\varphi]$ is given by an epimorphism $\pi_1(S\backslash Q) \to \mathbb{Z}/n$, the condition is implied by the surface relation satisfied by $\pi_1(S\backslash Q)$: the image of the product of commutators made of homological loops is trivial.

Using the Theorem 2.3, it is easy to see that the natural projection $\mathcal{H}_g(G,R) \to \mathcal{M}_{g',(b_1,\ldots,b_t)}$ which sends $[C,\varphi]$ to $[C/\varphi(G),Y]$, where Y is the branch locus, is in general a ramified covering. The fiber over $[S,Y]$ is in 1-1 correspondence with the set of epimorphisms $\pi_1(S-Y) \to G$ with ramification data R, up to conjugacy, and up to automorphisms of $[S,Y]$ which do not act trivially on the fundamental group. Generically, there is no such automorphisms, and exceptional Riemann surfaces with such automorphisms make the locus of branching. It may happen that every $[S,Y]$ possess such automorphisms (think to the hyperelliptic involution), then the branch locus is made of exceptional Riemann surfaces with more automorphisms.

In the previous example, if the surjectivity is ensured by the ramification data, then images of homological loops are free, and the degree of $\mathcal{H}_g(\mathbb{Z}/n,R) \to \mathcal{M}_{g',(b_1,\ldots,b_t)}$ is $n^{2g'}$.

We recall how to define the topology of $\mathcal{H}_g(G,R)$ coming from the Teichmüller theory. It turns out that these spaces possess in general many connected components, each one being a quotient of a Teichmüller space.

REMARK. There is a nice description of this by W. J. Harvey [15], in the context of Fuchsian groups. Instead of Teichmüller spaces of Fuchsian groups, we use

Teichmüller spaces of marked Riemann surfaces. This is because, in our context, Riemann surfaces arise as explicit complex structures on smooth surfaces.

DEFINITION 2.4. Let (C, φ) be a compact Riemann surface with a biholomorphic action of G. Then the Hurwitz space $\mathcal{H}_G(C, \varphi)$ of topological type (C, φ) is the subset of $\mathcal{H}_{g(C)}(G, R(\varphi))$ made of couples (T, ρ) such that there is a G-equivariant diffeomorphism between T and C.

We recall that in each Hurwitz space $\mathcal{H}_g(G, R)$, the number $h_g(G, R)$ of distinct topological types (sometimes called the Hurwitz number) is finite. Take $[p_i : C_i \to (S, Y)]_{i=\{1,2\}}$ two elements of $\mathcal{H}_g(G, R)$, and let $\psi_i : \pi_1(S - Y) \to G$ the corresponding epimorphisms (Theorem 2.3). Then the existence of a G-equivariant diffeomorphism $f : C_1 \to C_2$ such that $p_2 \circ f = p_1$ amounts to the existence of an element k in $\mathrm{Diff}^+(S, Y)$ (the subgroup of $\mathrm{Diff}^+(S)$ which preserves Y and its partition in colors), which satisfies $\psi_2 \circ k_* = \psi_1$, up to conjugation. This relies on the theorem on lift of mappings.

Using now Nielsen's theorem (Theorem 1 of [16]), every element of $\mathrm{Diff}^+(S, Y)$ comes from an automorphism of the fundamental group of $S - Y$ which preserves the conjugacy class of loops around points of the same color.

If we denote the quotient of this subgroup by the inner automorphisms by $\mathrm{Out}(\pi_1(S - Y), Y)$, and by $\mathrm{Epi}_R(\pi_1(S - Y), G)$ the set of epimorphims from $\pi_1(S-Y)$ onto G with image of loops around points of Y fixed by the ramification data R, then

$$h_g(G, R) = \#\ \mathrm{Out}(\pi_1(S - Y), Y) \backslash \mathrm{Epi}_R(\pi_1(S - Y), G)/G.$$

To see an element of $\mathrm{Epi}_R(\pi_1(S-Y), G)$ as a collection of elements of G satisfying certain relations (see [8]) allows the computation of $h_g(G, R)$ in some particular case. As an example, the reader can verify that it is one for G abelian and genus zero or one for quotient surfaces. See [6] for examples with two topological types.

Now, Hurwitz spaces with fixed topological type arise as quotients of some Teichmüller spaces.

DEFINITION 2.5. Let G be a fixed finite group, and (C, φ) a reference couple made of a compact Riemann surface C and of a biholomorphic action φ of G on C. The Teichmüller space $\mathcal{T}_G(C, \varphi)$ of C relative to the action φ, consists in classes of 3-tuple (T, ρ, f), where T is a compact Riemann surface, $\rho : G \hookrightarrow \mathrm{Aut}(T)$ is a monomorphism, and $f : C \to T$ is a G-equivariant quasiconformal homeomorphism. Then $(T_i, \rho_i, f_i)_{i=1,2}$ are equivalent if and only if $f_2 \circ f_1^{-1}$ is homotopic to a G-equivariant biholomorphism $h : T_1 \to T_2$.

To illustrate this definition, note that $[T, \rho, f] = [T, \rho, f \circ \varphi(g^{-1})]$ if $g \in Z(G)$ (the center of G). The following lemma justifies the use of quasiconformal homeomorphisms in the previous definition and gives a metric on $\mathcal{T}_G(C, \varphi)$. See [22], Proposition 3.12, for a proof. The author limits itself to cyclic groups, but the proof remains valid for an arbitrary group.

LEMMA 2.6. *Let $[T_i, \rho_i, f_i]_{i=1,2}$ be two points of $\mathcal{T}_G(C, \varphi)$. Then there exist G-equivariant quasiconformal homeomorphisms homotopic to $f_2 \circ f_1^{-1}$. Among them, there is a single one f_0, for which the infimum of the maximal dilatations is reached, almost everywhere on the surface. Moreover its dilatation K_{f_0} is a constant.*

The distance is then defined by $d([T_1, \rho_1, f_1], [T_2, \rho_2, f_2]) = \ln(K_{f_0})$.

The *relative modular group* of a couple (C, φ), denoted by $\mathcal{M}od_G(C, \varphi)$, is the quotient

$$\frac{Z_G(C, \varphi)}{Z_G(C, \varphi) \cap \mathrm{Diff}^0_+(\mathrm{C})},$$

where $Z_G(C, \varphi)$ is itself the quotient of the centralizer of $\varphi(G)$ in $\mathrm{Diff}_+(\mathrm{C})$ by $\varphi(Z(G))$. An element f of $\mathcal{M}od_G(C, \varphi)$ acts on $\mathcal{T}_G(C, \varphi)$ by $f \cdot [T, \rho, h] = [T, \rho, h \circ f^{-1}]$. As said above:

LEMMA 2.7. *The quotient of $\mathcal{T}_G(C, \varphi)$ by the action of $\mathcal{M}od_G(C, \varphi)$ is in bijection with the Hurwitz space $\mathcal{H}_G(C, \varphi)$. The stabilizer of some fixed point $[T, \rho, h]$ under the action of the relative mapping class group is in bijection with the commutant of $\rho(G)$ in $\mathrm{Aut}(T)$ quotiented by $Z(G)$.*

PROOF. Let $[C_i, \varphi_i, f_i]$ map to $[C_i, \varphi_i]$ for $i = 1, 2$ and assume that $f : C_1 \to C_2$ is a G-equivariant biholomorphism. Then $m = f_1^{-1} \circ f^{-1} \circ f_2$ is an element of the relative modular group $\mathcal{M}od_G(C, \varphi)$, and $[C_1, \varphi_1, f_1] = [C_2, \varphi_2, f_2 \circ m^{-1}]$ in $\mathcal{T}_G(C, \varphi)$. □

The Hurwitz space $\mathcal{H}_G(C, \varphi)$ is endowed with the quotient topology. Furthermore, the action of $\mathcal{M}od_G(C, \varphi)$ on Teichmüller space is a discontinuous one (see [16]); hence $\mathcal{H}_G(C, \varphi)$ acquires the structure of an orbifold. On $\mathcal{H}_g(G, R)$ we put the natural topology so that its subsets $\mathcal{H}_G(C, \varphi)$ become its connected components.

THEOREM 2.8. *Let C be a compact Riemann surface equipped with a biholomorphic action $\varphi : G \hookrightarrow \mathrm{Aut}(C)$, and P be the set of branch points of the quotient surface. Then the Teichmüller spaces $\mathcal{T}_G(C, \varphi)$ and $\mathcal{T}(C/\varphi(G), P)$ are analytically equivalent.*

It is a classical theorem, see Corollary 3 of [15], Proposition 8 of [16].

Note however that Teichmüller spaces considered here are slightly different from the ones in [15] and [16], since we consider group actions up equivariance instead of global conjugacy. However, using the lemma in § 2.3 of [7], it is easy to adapt their proof. For, this lemma of C. J. Earle ensures that an orientation preserving diffeomorphism homotopic to the identity map which normalizes the action of G, commutes with the action of G.

In [11], the authors describe the normalization of the locus in the moduli space $\mathcal{M}_h$ of points with automorphism group G. They also work with fixed topological type. This normalization is the quotient of a Teichmüller space by

a relative modular group. Since their relative modular group is a normalizer rather than a centralizer, the Hurwitz spaces $\mathcal{H}_G(C,\varphi)$ defined here are finite Galois coverings of these normalizations (with Galois group $\mathrm{Out}(G)$).

In this work, we restrict ourself to the topological setting on Teichmüller spaces. However, it will be worthwhile to study whether the combinatorial study of Teichmüller spaces (see § 5) extends to the analytic setting.

We now turn to the general case, and define a Hurwitz space for non Galois coverings with fixed monodromy action. Again, we need a version of Riemann's existence theorem [8] [9] [10].

THEOREM 2.9. *There is a 1-1 correspondence between $\mathcal{H}(g,g',d)$ and the space of following couples $[S-Y,\psi]$: S is a compact Riemann surface of genus g', Y is a finite subset of S, $\psi:\pi_1(S-Y,q)\to S_d$ is a group homomorphism with transitive image, defined up to base point change; and $[S'-Y',\psi']=[S-Y,\psi]$ if and only if there exist some biholomorphism $h:S\to S'$ sending Y on Y', and $\theta\in\mathrm{Int}(S_d)$, such that $\theta\circ\psi=\psi'\circ h_*$.*

As usually, the image G of ψ is called the *monodromy group* of the corresponding covering. Let $[p:T\to S]\in\mathcal{H}(g,g',d)$ and $\psi:\pi_1(S-Y,q)\to G\hookrightarrow S_d$ the corresponding homomorphism. Another important data is the conjugacy class of the stabilizer K under the monodromy action of any point in the fiber over the base point q. This is a subgroup of G, of index d, which satisfies: $\bigcap_{t\in G}t^{-1}Kt=\{e_G\}$.

Note that the natural action of G on the coset G/K of right classes (G acts by $(h,Kg)\mapsto Kgh^{-1}$), and the identification between right classes and the set $\{1,\dots,d\}$, furnish the embedding $G\hookrightarrow S_d$: faithfulness of the action is implied by the property of K.

DEFINITION 2.10. We fix G and K a subgroup of index d such that $\bigcap gKg^{-1}=\{e_G\}$. Let $i:G\hookrightarrow S_d$ the embedding given by the action of G on G/K.

- Let $[p:T\to S]$ in $\mathcal{H}(g,g',d)$ with the corresponding $\psi:\pi_1(S-Y,q)\to S_d$. Then the branched covering p is said to have a monodromy of type (G,K) if there exists an isomorphism α between $\mathrm{Im}(\psi)$ and G such that $i\circ\alpha$ is the identity on $\mathrm{Im}(\psi)$ (the corresponding permutation representations are isomorphic).
- $\mathcal{H}(g,g',G,K)$ is the subset of $\mathcal{H}(g,g',d)$ made of elements with monodromy of type (G,K).

For an element of $\mathcal{H}(g,g',G,K)$, the stabilizer K_i of any point x_i in the fiber over the base point q is then conjugated to K via α. We set $\tilde{\psi}=\alpha\circ\psi$.

We cannot define the ramification data in this setting, due to the fact that the equivalence relation involved is the conjugation in the symmetric group, not in G. The solution comes from the operation of Galois closure [10].

Let us consider the embeddings $p_*^i : \pi_1(T - X, x_i) \to \pi_1(S - Y, q)$, and $H_i = p_*^i(\pi_1(T - X, x_i))$. Then these are conjugated subgroups of $\pi_1(S - Y, q)$, $K_i = \psi(H_i)$ and $\ker(\psi) = \cap_i H_i$.

Now $\tilde{\psi} = \alpha \circ \psi$ furnishes, via the theorem 2.3, a Riemann surface C with a well-defined holomorphic action φ of G on C such that $S = C/\varphi(G)$, with a ramification data induced by $\tilde{\psi}$. Clearly, the restriction $\tilde{\psi} : H_i \to K_i$ gives T isomorphic to $C/\varphi(K_i)$.

The Galois covering $C \to S$ is called a Galois closure of $p : T \to S$, this is a minimal Galois covering factorizing by p. We note that if $[\psi_i, \alpha_i]_{i\in\{1,2\}}$represents two elements of $\mathcal{H}(g, g', G, K)$ in the same class, then $\tilde{\psi}_2 = \alpha_2 \circ \theta \circ \alpha_1^{-1} \circ \tilde{\psi}_1$ for θ in $\mathrm{Int}(S_d)$, and $\alpha_2 \circ \theta \circ \alpha_1^{-1}$ is an element of $\mathrm{Aut}(G, K)$, the subgroup of $\mathrm{Aut}(G)$ whose elements preserve the conjugacy class of K.

Thus a Galois closure $[C, \varphi]$ with ramification data R is defined only modulo the action of $\mathrm{Out}(G, K)$ (since $\mathrm{Int}(G)$ acts trivially).

Let $\mathrm{Out}(G, K, R)$ be the subgroup whose elements also preserve the ramification data R.

PROPOSITION 2.11. *Let (G, K) be as in the previous definition, R a ramification data of G, and h an integer. There is a bijection*

$$\mathcal{H}(g, g', G, K) \longleftrightarrow \coprod_{R/\mathrm{Out}(G,K)} \frac{\mathcal{H}_h(G, R)}{\mathrm{Out}(G, K, R)}$$

PROOF. There is a well-defined map from $\mathcal{H}_h(G, R)$ to $\mathcal{H}(g, g', d)$ sending $[C, \varphi]$ to the branched covering $[C/\varphi(K) \to C/\varphi(G)]$.

If $\psi : \pi_1(S - Y, q) \to G$ is the corresponding epimorphism, then using the action of G on G/K, we also have an homomorphism $\check{\psi} : \pi_1(S - Y, q) \to S_d$ with monodromy group G and isotropy group K. Thus we have in fact a map $\mathcal{H}_h(G, R) \to \mathcal{H}(g, g', G, K)$. Note that since the monodromy is determined, we can calculate the genus g, since g' is known from h and R.

This map factorize by the quotient of $\mathcal{H}_h(G, R)$ by $\mathrm{Out}(G, K, R)$. If θ is an element of $\mathrm{Out}(G, K, R)$, then we can extend it to $\bar{\theta} \in \mathrm{Int}(S_d)$. Assuming $\theta(K) = K$, $\bar{\theta}$ is the conjugation by the bijection $Kg \mapsto K\theta(g)$.

We now prove that the map is injective. Assume that $[C_i, \varphi_i] \mapsto [C_i/\varphi_i(K) \to C_i/\varphi_i(G)]$ such that

$$[p_1 : C_1/\varphi_1(K) \to C_1/\varphi_1(G)] = [p_2 : C_2/\varphi_2(K) \to C_2/\varphi_2(G)],$$

i.e., there exist $f : C_1/\varphi_1(K) \to C_2/\varphi_2(K)$ and $h : C_1/\varphi_1(G) \to C_2/\varphi_2(G)$ such that $p_2 \circ f = h \circ p_1$. This means that $\theta \circ \psi_1 = \psi_2 \circ h_*$ for some $\theta \in \mathrm{Int}(S_d)$. Thus we can lift f onto $k : (C_1, \varphi_1) \to (C_2, \varphi_2)$ such that $k \circ (\varphi_1 \circ \theta'(g)) = \varphi_2(g) \circ k$ for $\theta' \in \mathrm{Out}(G, K)$ induced by θ. It remains to show that $\theta'(R) = R$. But k is an equivariant map between $[C_2, \varphi_2]$ and $[C_1, \varphi_1 \circ \theta']$ so that $R(\varphi_2) = R(\varphi_1 \circ \theta')$. Since $R(\varphi_1) = R(\varphi_2)$, we are done.

Finally, the discussion preceding the proposition shows that every element of $\mathcal{H}(g,g',G,K)$ comes exactly from one element of $\mathcal{H}_h(G,R)/\mathrm{Out}(G,K,R)$, provided that R is taken modulo $\mathrm{Out}(G,K)$. □

The image of $\mathcal{H}_h(G,R)/\mathrm{Out}(G,K,R)$ into $\mathcal{H}(g,g',G,K)$ will be denoted by $\mathcal{H}(g,g',G,K,R)$; this is the space involved in our main theorem. We will say that an element of $\mathcal{H}(g,g',G,K,R)$ possess a monodromy of type (G,K,R), but will keep in mind that R is defined modulo $\mathrm{Out}(G,K)$.

Example 2. We look at the case of degree d simple coverings, for which the monodromy around each branch point is given by a single transposition of S_d. Thus the preimage of a branch point consists in $d-2$ points of index one, and one point of index two. The ramification data is $R = b\tau$ where b is the number of branch points and τ is the conjugacy class of transpositions. In this case (see [9] for more details) the monodromy group is S_d, and $K \cong S_{d-1}$, so that $\mathrm{Out}(S_d,K)$ is the trivial group. Also, the spaces $\mathcal{H}_h(S_d,R)$ are connected [9].

3. Strebel Differentials

We recall the main ingredient of cellular decomposition of moduli spaces of pointed Riemann surfaces, i.e., Strebel's theorem [25], and show how to adapt it to the study of Hurwitz spaces.

Let C be a compact Riemann surface of genus g with a set of n punctures $X = \{x_1,\dots,x_n\}$. Let us consider the set $\mathcal{S}t(C,X)$ of holomorphic quadratic differentials q on $C\backslash X$ with the following properties:

(1) Its critical graph covers a set of measure zero (q is a Jenkins–Strebel differential).
(2) Each puncture x_i is a double pole with a real negative coefficient $-p_i^2$.
(3) Its characteristic ring domains are n disks punctured at the x_i, described by the closed trajectories.

The elements of $\mathcal{S}t(C,X)$ are often called *Strebel differentials*.

Recall that the critical graph Γ_q is the union of non closed trajectories, and that the perimeter of a closed trajectory is its length measured with the metric induced by the quadratic differential.

The second condition means that the perimeter of a closed trajectory around x_i is p_i. The third condition says that the critical graph cuts the surface into n punctured disks described by the closed trajectories. In this case, Γ_q consists in segments joining zeroes of the differential. Moreover, all its vertices are at least trivalent. Indeed, vertices are zeroes of q, and this excludes monovalent vertices (corresponding to first order poles) and bivalent vertices (corresponding to some regular points).

Theorem 3.1. [25] *Let C be a compact Riemann surface of genus g with a set $X = (x_1,\dots,x_n)$ of n punctures such that $2g-2+n>0$ and $n>0$. Then*

any n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ of perimeters determines a single element w of $\mathcal{S}t(C, X)$ such that the perimeter around x_i is p_i.

The following proposition shows that Strebel differentials behave well under lifting through ramified coverings.

PROPOSITION 3.2. *Let $[p : C \to S]$ be some class of ramified coverings between two compact Riemann surfaces. Let Y be the set of branch points, b its cardinal, and X be the set of ramification points. Denote by e_x the multiplicity of $x \in X$. Then each b-tuple $(p_1, \dots, p_b)$ of strictly positive real numbers determines a unique couple of Strebel differentials $w_C \in \mathcal{S}t(C, X)$ and $w_S \in \mathcal{S}t(S, Y)$, such that $p^*(w_S) = w_C$, and the perimeter of a closed trajectory of w_C around $x \in X$ which maps onto $y_j \in Y$ is $e_x p_j$.*

PROOF. We first apply Strebel's theorem to the surface S endowed with the b-tuple $(p_j)_{j \in \{1,\dots,b\}}$. We obtain a unique element w_S of $\mathcal{S}t(S, Y)$ such that $w_S = -(p_j)^2 (dz_j/z_j)^2$ around $y_j \in Y$. Then consider the pullback $p^*(w_s) = \Phi(\hat{z})(d\hat{z})^2$. For local coordinates such that $p(\hat{z}) = z$, it is defined by $p^*(w_s) = \Phi(z)(dz)^2$. If precisely $z = \hat{z}^{e_x}$, then $(dz/z)^2 = e_x^2 (d\hat{z}/\hat{z})^2$, so that the only poles of order two of $p^*(w_s)$ are the ramification points. Moreover perimeters of $p^*(w_s)$ are the $e_x p_j$. The critical graph of $p^*(w_S)$ is the pullback of Γ_{w_S}, and its characteristic ring domains are necessarily disks punctured at the ramification points, so that $p^*(w_S) \in \mathcal{S}t(C, X)$. Uniqueness is obtained by a new application of Strebel's theorem. □

If we take $[C, \varphi] \in \mathcal{H}_g(G, R)$, then the same arguments yield a Strebel differential q relative to C and to fixed points, invariant under $\varphi(G)$. Moreover, the action of $\varphi(G)$ on q induce a free isometric action on its critical graph.

We also need a more precise version of Strebel's theorem for the case of Riemann surfaces with nodes. Let T be a compact Riemann surface with $n \geq 1$ punctures $P = (P_1, \dots, P_n)$, and m ties $Q = (Q_1, \dots, Q_m)$ (some other distinguished points, distinct from the punctures). Then we consider the set $\mathcal{S}t'(T, P, Q)$ of quadratic differential, holomorphic on $T \backslash (P \cup Q)$, with the properties 1,2 and 3 stated out above (replace x_i by P_i), but we also demand that ties are vertices or middles of edges of the critical graph. Hence, they are possibly monovalent or bivalent vertices of the critical graph.

THEOREM 3.3. *Let T be a compact Riemann surface of genus g, with a set of $n \geq 1$ punctures $P = (P_1, \dots, P_n)$, and a set of m ties $Q = (Q_1, \dots, Q_m)$, such that $2g - 2 + n + m > 0$ and $n \geq 2$ if $g = 0$. Then each b-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ determines a unique element of $\mathcal{S}t'(T, P, Q)$ with perimeters equal to p_i around P_i.*

PROOF. Assume that $m = 2k$. Using the Riemann's existence theorem (theorem 2.3), we take a double covering $p : R \to T$ ramified only at the ties. The surface R is endowed with the unique complex structure such that the projection

becomes holomorphic. Put $(S_i^1, S_i^2) = p^{-1}(P_i)$ and $T_j = p^{-1}(Q_j)$. We have $g(R) = 2g - 1 + k$ by Riemann–Hurwitz, and the stability condition asserts that $2g(R) - 2 + 2n > 0$. We apply Strebel's theorem to R: there exists a unique element w of $\mathcal{S}t(R, (S_i^1, S_i^2)_i)$ with perimeters equal to $(p_1, p_1, p_2, p_2, \ldots, p_n, p_n)$.

Let α be the order two deck transformation of the covering. Then $\alpha(S_i^1) = S_i^2$, and uniqueness in Strebel's theorem tells us that $\alpha^*(w) = w$. Furthermore, α induces an isometry of the critical graph Γ_w; and since $\alpha(T_j) = T_j$, these points belong to Γ_w. Thus, they are vertices of even valency (possibly bivalent) of Γ_w.

Since $\alpha^*(w) = w$, we get a quadratic differential q on T such that $p^*(q) = w$. The punctures P_i are its poles of order two, the perimeters are given by the p_i, and clearly $q \in \mathcal{S}t'(T, P, Q)$. If the valency of the vertex T_j is $2v$ then the valency of Q_j is v, possibly equal to one. Uniqueness of w gives uniqueness of q.

If $m = 2q+1$, we take a double covering ramified only in $Q_1, \ldots, Q_{2k}$ to bring us back to the previous case. □

4. Graphical Construction of Coverings

DEFINITION 4.1. [18] [3] A *fat-graph* Γ is given by a finite set $A(\Gamma)$ (of oriented edges) and by two permutations σ_0 and σ_1 of this set, where σ_1 is an involution without fixed points.

All fat-graphs will be connected: we assume that the group generated by σ_0 and σ_1 (the so-called cartographic group) acts transitively on the set of oriented edges.

The geometric edges are the orbits of $\sigma_1(\Gamma)$, and the vertices are those of $\sigma_0(\Gamma)$. We denote by $A_g(\Gamma)$ and $V(\Gamma)$ these sets, and by $a(\Gamma)$ and $v(\Gamma)$ their cardinals. Note that Γ is a graph. If $a \in A(\Gamma)$, we denote by $a(0)$ its origin and by $a(1)$ its end. We take the convention that $a(1)$ is the σ_0-orbit of a and $a(0)$ is the σ_0-orbit of $\sigma_1(a)$. For convenience, we often put $\bar{a} = \sigma_1(a)$.

We define $\sigma_2(\Gamma) = \sigma_1(\Gamma)\sigma_0(\Gamma)^{-1}$, such that $\sigma_0\sigma_1\sigma_2 = 1$. The orbits of $\sigma_2(\Gamma)$ are called the *faces* of Γ. We denote the set of faces by $F(\Gamma)$ and its cardinal by $f(\Gamma)$. The length of a cycle of $\sigma_0(\Gamma)$ (resp. $\sigma_2(\Gamma)$) is the valency of the corresponding vertex (resp. face).

Every face is an oriented loop, for, if $b = \sigma_2(a)$ then $b(0) = a(1)$.

As an example, a graph Γ embedded in a compact oriented surface is a topological realization of a fat-graph. The permutation σ_0 of Γ is given by the projection of the neighborhoods of the vertices on the tangent planes at these points.

The genus $g(\Gamma)$ of the fat-graph Γ is then defined by the Euler formula:

$$2 - 2g(\Gamma) = v(\Gamma) - a(\Gamma) + f(\Gamma).$$

A morphism $f : \Gamma \to \Gamma'$ between two fat-graphs is a map $f : A(\Gamma) \to A(\Gamma')$ which satisfies to $f \circ \sigma_i = \sigma_i' \circ f$ for $\in \{0, 1, 2\}$. If f is bijective, then f is an isomorphism. In this definition of an isomorphism, we assume that faces can

be exchanged. If we specify some colors on the set of faces, then we ask that isomorphisms respect these colors. The (full) *automorphism group* $\mathrm{Aut}(\Gamma)$ is the centralizer of σ_1 and σ_2 in the group of all permutations of $A(\Gamma)$.

Figure 1 shows a fat-graph of genus one described by its cartographic group $\sigma_0(\Delta) = (abdc)(\bar{a}\bar{b}\bar{c}\bar{d})$, $\sigma_1(\Delta) = (a\bar{a})(b\bar{b})(c\bar{c})(d\bar{d})$, $\sigma_2(\Delta) = (a\bar{c}b\bar{a}d\bar{b})(c\bar{d})$.

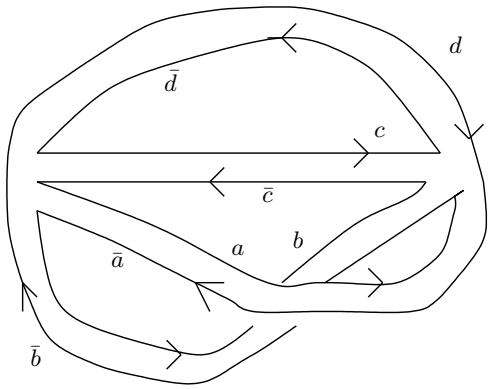

Figure 1. A fat-graph Δ of genus one.

Let Γ be a fat-graph. A metric m on Γ is given by a map $m : A_g(\Gamma) \to \mathbb{P}(\mathbb{R}_{>0}^{a(\Gamma)})$; we say that $m(e)$ is the length of the geometric edge e. We note (Γ, m) a fat-graph endowed with a metric, and we call it a *Riemannian fat-graph*. The perimeter $p(F)$ of a face $F = (a_1, \ldots, a_k)$ of (Γ, m) is defined by $p(F) = \sum_{i=1}^{k} m(\{a_i, \bar{a}_i\})$.

An isomorphism between two Riemannian fat-graphs is called an isometry if it preserves lengths of edges. We denote by $\mathrm{Aut}(\Gamma, m)$ the group of isometric automorphisms of (Γ, m).

Given Γ, we can realize its faces as oriented polygons in the plane, and fill in by punctured disks. Then gluing them with σ_1 give an orientable compact surface $F(\Gamma)$ minus $f(\Gamma)$ points (one for each face), together with an embedding $i : |\Gamma| \hookrightarrow F(\Gamma)$ such that $i(|\Gamma|)$ is a retract by deformation of $F(\Gamma)$. We will see that some metric on Γ determines some complex structure on $F(\Gamma)$.

From the induced isomorphism between first homotopy groups $\pi_1(|\Gamma|)$ and $\pi_1(F(\Gamma))$ we deduce that $g(\Gamma) = g(F(\Gamma))$. For, on one hand, $\pi_1(F(\Gamma))$ is a free group of rank $2g(F(\Gamma))+f(\Gamma)-1$, and on the other hand, $\pi_1(|\Gamma|)$ is isomorphic to the fundamental group of Γ which is free of rank $a(\Gamma)-v(\Gamma)+1 = 2g(\Gamma)+f(\Gamma)-1$.

Faces of a fat-graph Γ provide some particular elements of the fundamental group $\pi_1(\Gamma, p)$ of Γ. Let $(a_1, a_2, \ldots, a_k)$ a face of Γ, with $a_i = \sigma_2^{i-1}(a_1)$. Join $a_1(0)$ to the base vertex p by an oriented path α, and consider the homotopy class of the oriented loop $\gamma = \alpha a_1 \cdots a_k \bar{\alpha}$. Note that another choice of α leads to a conjugate loop. And if the face is given by $(a_i, \ldots, a_{i-1})$ with $i > 1$, we get the ori-

ented loop $\beta a_i \cdots a_{i-1}\bar{\beta}$, homotopic to $(\beta a_i \cdots a_l)(a_1 \cdots a_{i-1}a_i \cdots a_l)(\bar{a}_l \cdots \bar{a}_i\bar{\beta})$, which is a conjugate loop of γ.

Therefore to each face is associated a well defined conjugacy class of elements of $\pi_1(\Gamma, p)$, which will be called a *loop-face*.

Fat-graphs whose vertices are at least trivalent are called *smooth* fat-graphs.

DEFINITION 4.2. A covering of fat-graphs $p : \Gamma \to \Delta$ is a morphism of fat-graphs which satisfies the existence and uniqueness property of lifting oriented paths. The cardinal of vertex's fiber is then constant: this is the degree. A covering of Riemannian fat-graphs is a covering where lengths edges are preserved. Define $\mathcal{H}^{\text{comb}}(g, g', d)$ to be equivalence classes of degree d coverings of smooth Riemannian fat-graphs $[p : \Gamma \to \Delta]$ with $g(\Gamma) = g$ and $g(\Gamma') = g'$.

As in the case of Riemann surfaces (Theorem 2.9) we have:

THEOREM 4.3. *There is a bijection between $\mathcal{H}^{\text{comb}}(g, g', d)$ and the set of classes $[\Delta, l, \psi]$ where (Δ, l) is a genus g' smooth Riemannian fat-graph, and ψ is a group homomorphism $\pi_1(\Delta) \to S_d$ with transitive image; $[\Delta, l, \psi] = [\Delta', l', \psi']$ if there exists an isometry $h : (\Delta, l) \to (\Delta', l')$ and $\theta \in \text{Int}(S_d)$ such that $\theta \circ \psi = \psi' \circ h_*$.*

It will become clear after Theorem 4.8 that both theorems 2.9 and 4.3 are equivalent, using the fact that Riemannian fat-graphs are deformation retract of punctured Riemann surfaces. In fact, it is not difficult to give a direct proof of 4.3. Furthermore, we have the notions of monodromy groups and monodromy actions. Thus we define $\mathcal{H}^{\text{comb}}_{g,g'}(G, K)$ in perfect analogy with $\mathcal{H}(g, g', G, K)$.

We now focus on the case of Galois coverings. Let Γ be a fat-graph, and G be some finite group. We denote by $\varphi : G \hookrightarrow \text{Aut}(\Gamma)$ an action of G on Γ. We call it *quasifree* if $\varphi(G)$ acts freely on the underlying graph (on geometric edges and vertices), but with non trivial isotropy group for each face of Γ (these actions will provide some actions of G on surfaces with fixed points). Then the quotient fat-graph is well defined. In fact, the quotient graph is well defined by freeness of $\varphi(G)$ on edges. And since edges incident on a vertex of the quotient graph are in bijection with edges incident on any vertex of its fiber, we put the induced cyclic ordering. The projection $p : \Gamma \to \Gamma/\varphi(G)$ is a Galois covering of fat-graphs. Two couples (Γ, φ) and (Γ', φ') are equivalent if they differ by a G-equivariant isomorphism.

Inspired by the setting of Riemann surfaces (see the previous section), we look at stabilizers G_F of faces F. Let $\sigma_{2,F}$ be the cycle defining F, of order v_F. Then $g \in G_F$ means that $\varphi(g)\sigma_{2,F}\varphi(g^{-1}) = \sigma_{2,F}$. Hence $\varphi(g) = \sigma_{2,F}^i$ on F, for $i \in \mathbb{Z}/v(F)$, and we have an embedding $\varphi_F : G_F \hookrightarrow \langle \sigma_{2,F} \rangle \cong \mathbb{Z}/v(F)$. Set e_F the order of G_F. Then $\varphi_F^{-1}(\sigma_{2,F}^{v_F/e_F})$ is a privileged generator of G_F, which we denote by τ_F. As before, we say that F and its orbit are colored by $C(\tau_F)$, the conjugacy class of τ_F.

Let $\{O_i, b_i\}_{i \in \{1,\ldots,t\}}$ be the set of colored orbits of faces with multiplicities, and $C(\tau_i)$ be the conjugacy classes of privileged generators.

Then the ramification data $R(\varphi)$ associated to φ is given by

$$R(\varphi) = \sum_{i=1}^{t} b_i C(\tau_i).$$

If $f : (\Gamma, \varphi) \to (\Gamma', \varphi')$ is a G-equivariant isomorphism, then $R(\varphi) = R(\varphi')$. The degree of R is the number of face's orbits, i.e., the number of faces of the quotient fat-graph $\Gamma/\varphi(G)$. Furthermore, note that if $R = \sum_{i=1}^{t} b_i C_i$, and if $e_i = \mathrm{Ord}(C_i)$ is the order of any element in the class C_i, then we simply have $f(\Gamma) = \sum_{i=1}^{t} b_i |G|/e_i$.

Writing the Euler–Poincaré formulas, we recover now the Riemann–Hurwitz formula $2g(\Gamma) - 2 = |G|(2g(\Gamma/\varphi(G)) - 2) + \sum_{i=1}^{b} |G|(1 - 1/e_i)$.

DEFINITION 4.4. Let G be a finite group, and R a ramification data of G. Then $\mathcal{H}_g^{\mathrm{comb}}(G, R)$ is the set of equivalence classes $[\Gamma, l, \varphi]$ where (Γ, l) is a smooth Riemannian fat-graph of genus g, and $\varphi : G \hookrightarrow \mathrm{Aut}(\Gamma, l)$ is a quasifree action of G such that $R(\varphi) = R$. Two elements are equivalent if they differ by a G-equivariant isometry.

We state now a combinatorial version of Riemann's existence theorem for Galois coverings (Theorem 2.3).

THEOREM 4.5. *The space $\mathcal{H}_g^{\mathrm{comb}}(G, R)$ is in 1-1 correspondence with a set made of classes $[\Delta, l, \psi : \pi_1(\Delta) \to G]$, where (Δ, l) is a smooth Riemannian fat-graph of genus g', ψ is an epimorphism, defined up to conjugacy, sending loop-faces on prescribed images (by the ramification data), and such that $[\Delta, l, \psi] = [\Delta', l', \psi']$ if there exists an isometry $h : (\Delta, l) \to (\Delta', l')$ such that $\psi = \psi' \circ h_*$.*

If $\mathrm{Aut}(\Delta, \psi)$ is the subgroup of $\mathrm{Aut}(\Delta)$ made of elements h satisfying $\psi = \psi \circ h_*$, and if $\mathrm{Aut}_G(\Gamma, \varphi)$ is the centralizer of $\varphi(G)$ in $\mathrm{Aut}(\Gamma)$, then we have $\mathrm{Aut}(\Delta, \psi) \cong \mathrm{Aut}_G(\Gamma, \varphi)/Z(G)$.

Again, after Theorem 4.8, it will become clear that Theorems 2.3 and 4.5 are equivalent.

We also have an analog of Proposition 2.11, which allows us to define the set $\mathcal{H}_{g,g'}^{\mathrm{comb}}(G, K, R)$.

EXAMPLE 3. We give a covering of the genus one fat-graph Δ described previously. Given (Δ, ψ) we could build (Γ, φ) such that $\Delta = \Gamma/\varphi(G)$ as the universal covering of Δ quotiented by $\ker(\psi)$. Instead, we use the notion of the Cayley graph, which gives a more tractable method to build (Γ, φ) from (Δ, ψ).

First we choose a presentation of $\pi_1(\Delta, O)$. Take the edge $(b, \bar{b})$ as a maximal tree. Then $\pi_1(\Delta, O) = \langle \delta_a, \delta_c, \delta_d \rangle$ where $\delta_a = a\bar{b}$, $\delta_c = c\bar{b}$, $\delta_d = d\bar{b}$. Take $\gamma_3 = (c\bar{d})$ and $\gamma_4 = (a\bar{c}b\bar{a}d\bar{b})$ for loop-faces. An epimorphism ψ from $\pi_1(\Delta, O)$ to G must satisfy $\psi(\gamma_3) = \psi(\gamma_4)^{-1}$. Choose $\psi(\gamma_3) = [1]$ and $\psi(\gamma_4) = [2]$ as ramification data.

This implies $\psi(\delta_c \delta_d^{-1}) = [1]$ and $\psi(\delta_a \delta_c^{-1} \delta_a^{-1} \delta_d) = [2]$. We choose $\psi(\delta_a) = [1]$, $\psi(\delta_c) = [2]$, $\psi(\delta_d) = [1]$. Note that the corresponding element $\Gamma(\psi)$ of

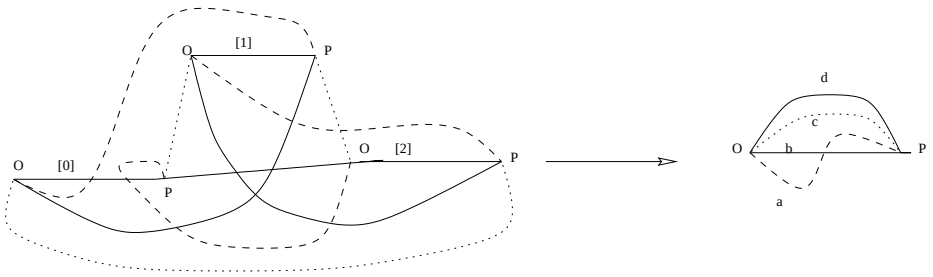

Figure 2. The covering $\Gamma(\Psi) \to \Delta$.

$\mathcal{H}_g^{\mathrm{comb}}(\mathbb{Z}/3, R)$ possess two faces of stabilizers $\mathbb{Z}/3$. Thus $g = 3$ by Riemann–Hurwitz.

To build $\Gamma(\psi)$ (figure 2), we take three copies of the maximal tree, indexed by $[0], [1], [2]$. Join the vertex (O, i) to the vertex (P, j) if $[i] + \psi(\delta_a) = [j]$ to build edges projecting on $(a, \bar{a})$. Idem for c and d. Cyclic orderings around the vertices are giving by these around O and P. The action of G on vertices of $\Gamma(\psi)$ is $g \cdot (O, i) = (O, g \times i)$ (likewise for P).

We describe now the main combinatorial operation on fat-graphs: Whitehead collapses. We generalize them in an obvious way to fat-graphs with a quasifree action of a group G.

Definition 4.6. Let (Γ, φ) a fat-graph with a quasifree action of the (possibly trivial) group G, and $e = (a, \bar{a})$ a geometric edge with $a(0) \neq a(1)$. We denote $O(e)$ the orbit of e under $\varphi(G)$. We assume that $O(e)$ do not contain any loop of Γ. The operation which consists in retracting $O(e)$ and gluing $O(a(0))$ with $O(a(1))$ is called an *equivariant Whitehead collapse* along $O(e)$. We note $W_{O(e)}(\Gamma)$ the new graph acquired in this way.

We group together in a lemma the first properties of this operation.

Lemma 4.7. (i) *The graph $W_{O(e)}(\Gamma)$ is a fat-graph.*
(ii) *The number of faces and the genus are invariant under equivariant Whitehead collapses.*
(iii) *The quasifree action φ of G on Γ restricts to a quasifree action of G on $W_{O(e)}(\Gamma)$. Moreover the ramification data is preserved.*

Proof. For the first point, we describe the cartographic group of $W_{O(e)}(\Gamma)$.

Let $(a_1^j, \ldots, a_k^j)$ and $(b_1^j, \ldots, b_l^j)$ for $j \in \{1, \ldots, |G|\}$ be the G-orbit of $a(0)$ and $a(1)$, such that $\{a_1^j\}_j$ (resp. $\{b_1^j\}_j$) is the orbit of $\bar{a}$ (resp. a). Then retracting the orbit $O(e)$ give the new vertices $s_j = (a_2^j, \ldots, a_k^j, b_2^j, \ldots, b_l^j)$. The graph $W_{O(e)}(\Gamma)$ is the fat-graph with same σ_1 private of $O(e) = \{(a_1^j, b_1^j)\}_j$, and same σ_0 except for the orbits of $a(0)$ and $a(1)$ replaced by the new orbit $\{s_j\}_j$.

As we can retract neither a loop, nor an orbit which contains some faces, we have $f(W_{O(e)}(\Gamma)) = f(\Gamma)$. Since $a(W_{O(e)}(\Gamma)) = a(\Gamma) - |G|$ and $s(W_{O(e)}(\Gamma)) = s(\Gamma) - |G|$, we have $g(W_{O(e)}(\Gamma)) = g(\Gamma)$ by Euler–Poincaré.

The action of G on $W_{O(e)}(\Gamma)$ is defined by restriction of φ on $A(\Gamma)\backslash(O(a)\cup O(\bar{a}))$.

Let $\sigma_{2,k}$ be the cycle of $\sigma_2(\Gamma)$ defining a face F_k, of order v_k. The privileged generator τ_k of the stabilizer is defined such that

$$\sigma_{2,k} = (a_1 a_2 \cdots a_{v_k-1}\tau_k(a_1)\tau_k(a_2)\cdots\tau_k^{(v_k/e_k)-1}(a_1)\cdots\tau_k^{(v_k/e_k)-1}(a_{v_k-1})).$$

Retracting the orbit of a_i do not disturb the action of τ_k on $\sigma_{2,k}$, and thus the ramification data is preserved. □

If $[\Gamma, m, \varphi] \in \mathcal{H}_g^{\mathrm{comb}}(G, R)$, then to each equivariant Whitehead collapses along $O(e)$, we associate $[W_{O(e)}(\Gamma), m, \varphi] \in \mathcal{H}_g^{\mathrm{comb}}(G, R)$, setting $m(\varphi(t)(e)) = 0$ for all $t \in G$.

Recall that $\mathcal{M}^{\mathrm{comb}}_{g',(b_1,\ldots,b_t)}$ consist in isometry classes of smooth Riemannian fat-graphs $[\Gamma, m]$ of genus g', with b faces shared in t colors, where isometries respect these colors. We state our main theorem (without compactifications):

THEOREM 4.8. *Let G be a finite group, $R = b_1C_1 + \cdots + b_tC_t$ a degree b ramification data, and K a subgroup of G such that $\bigcap_{t\in G} tKt^{-1} = \{e_G\}$.*

Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{H}_h^{\mathrm{comb}}(G,R) & \longrightarrow & \mathcal{H}_h(G,R)\times\mathbb{P}(\mathbb{R}^b_{>0}) \\
\downarrow{\scriptstyle q_{\mathrm{comb}}} & & \downarrow{\scriptstyle q} \\
\mathcal{H}^{\mathrm{comb}}_{g,g'}(G,K,R) & \longrightarrow & \mathcal{H}(g,g',G,K,R)\times\mathbb{P}(\mathbb{R}^b_{>0}) \\
\downarrow{\scriptstyle p_{\mathrm{comb}}} & & \downarrow{\scriptstyle p} \\
\mathcal{M}^{\mathrm{comb}}_{g',(b_1,\ldots,b_t)} & \longrightarrow & \mathcal{M}_{g',(b_1,\ldots,b_t)}\times\mathbb{P}(\mathbb{R}^b_{>0})
\end{array}$$

Horizontal maps are homeomorphisms, and vertical ones are ramified coverings.

The vertical maps have been already defined: $q([C,\varphi]) = [C/\varphi(K) \to C/\varphi(G)]$, and $p\circ q([C,\varphi]) = [C/\varphi(G), Y]$. The maps from the right to the left are given by Theorem 3.1 and Proposition 3.2. Next theorem gives the inverse maps. At the level of moduli spaces of punctured Riemann surfaces, the construction is known since [13]. More details are given in [20] and [23]. The continuity is discussed in the next section.

THEOREM 4.9. *Let $[\Gamma, l, \varphi] \in \mathcal{H}_h^{\mathrm{comb}}(G, R)$. There exist a compact Riemann surface $\overline{\|\Gamma\|}$ endowed with $\hat{\varphi} : G \hookrightarrow \mathrm{Aut}(\overline{\|\Gamma\|})$ and $w_\Gamma \in \mathcal{S}t(\overline{\|\Gamma\|}, X)$, where $X = \{x_1, \ldots, x_{f(\Gamma)}\}$ is the set of fixed points of $\hat{\varphi}(G)$, such that:*

(1) *The critical graph of w_Γ is the image of a topological realization of Γ by an embedding $\iota : |\Gamma| \hookrightarrow \|\Gamma\|$ such that $\overline{\|\Gamma\|} - \iota(|\Gamma|)$ is a disjoint union of $f(\Gamma)$ disks, punctured at the x_i. Moreover $l_{w_\Gamma}(\iota(|a|) = l(a)$, where l_{w_Γ} is the metric induced by w_Γ.*

(2) $\varphi(g)^*(w_\Gamma) = w_\Gamma \ \forall g \in G$, *the action of* $\hat{\varphi}(G)$ *on* $\iota(|\Gamma|)$ *is the realization of the action of* $\varphi(G)$ *on* Γ.
(3) *If* $[S, \psi]$ *is another element of* $\mathcal{H}_h(G, R)$ *with preceding properties for a Strebel differential* v, *then there exists a single* G*-equivariant biholomorphism* $h : S \to \overline{\|\Gamma\|}$ *such that* $h^*(w_\Gamma) = v$.

PROOF. We first describe $\overline{\|\Gamma\|}$ in terms of complex local coordinates (see [23] for a discussion along the same lines). If $a \in A(\Gamma)$, we denote by $F(a)$ the unique face in which it lies, and p_a the perimeter of this face. We associate to each $a \in A(\Gamma)$ the following closed strip embedded in the complex plane $\mathbb{C}$: $B_a = [0, p_a] \times [0, \infty[$. Moreover we establish a bijection $|F(a)| \to [0, p_a] \times \{0\}$ such that $|a|$ maps to $[0, l(a)] \times \{0\}$, $|\sigma_2(a)|$ maps to $[l(a), l(a) + l(\sigma_2(a))]$, and so on...

Define $F(\Gamma) = \left(\coprod_{a \in A(\Gamma)} B_a\right) / \sim$, where $\sim$ stands for the following identifications:

(1) $(0, t) \sim (p_a, t) \ \forall t \in [0, \infty[$ (Form half-cylinders).
(2) Let $b = \sigma_2^i(a) \in F(a)$ and $\alpha = \sum_{j=0}^{i-1} l(\sigma_2^j(a))$.
Then $(x, t) \sim (x - \alpha, t) \ \forall x \in [\alpha, p_a]$, and $(x, t) \sim (p_a - \alpha + x, t) \ \forall x \in [0, \alpha]$, $\forall t \in [0, \infty[$.
(3) $\{(x, 0) \in B_a\} \sim \{(l(a) - x, 0) \in B_{\bar{a}}\} \ \forall x \in [0, l(a)]$ (glue the cylinders with σ_1).

So $F(\Gamma)$ is a compact orientable surface with $f(\Gamma)$ punctures: points at infinity of each half-cylinder. It is endowed with $\iota : |\Gamma| \hookrightarrow F(\Gamma)$, a canonical embedding of Γ.

The injection $B_a^0 \hookrightarrow F(\Gamma)$, of image U_a, gives a natural local coordinate $u_a : U_a \to B_a^0$. If $b \in F(a)$, then $U_a \cap U_b$ consists in the disjoint union of two strips, and the transition function is a translation.

Let $V_a \subset F(\Gamma)$ be the image of the infinite strip $]0, l(a)[\times]-\infty, +\infty[$. We define the local coordinate by $v_a = u_a$ on $V_a \cap U_a$, and $v_a = p(F(\bar{a})) - u_{\sigma_2(\bar{a})}$ on $V_a \cap U_{\sigma_2(\bar{a})}$. These coordinates are clearly holomorphically compatible with (U_a, u_a). Moreover, in these coordinates, we have the flat quadratic differential $(du_a)^2 = (dv_a)^2$, denoted by w_Γ.

Let now $s \in V(\Gamma)$ be a k-valent vertex with $k \geq 3$, and $(a_1, \ldots, a_k)$ the oriented edges pointing towards it such that $\sigma_0(a_i) = a_{i+1}$ for $i \in \{1, \ldots, k-1\}$ and $\sigma_0(a_k) = a_1$. Note that $\sigma_2(a_i) = \bar{a}_{i-1}$. Let

$$P_{a_i} =]-l(a_i), 0] \times]-\infty, +\infty[\ \cup \ ]0, l(a_{i-1})[\times]0, +\infty[$$

and let T_{a_i} its image in $F(\Gamma)$. We define $t_{a_i} : T_{a_i} \to P_{a_i}$ by $t_{a_i} = u_{a_i} - l(a_i)$ on the image of $]-l(a_i), l(a_{i-1})[\times]0, +\infty[$ and $t_{a_i} = v_{a_i} - l(a_i)$ on the image of $]-l(a_i), 0[\times]-\infty, +\infty[$.

Let $E_s = \bigcup_{j=1}^k T_{a_j}$, then the coordinate around s is $\xi_s : E_s \to \mathbb{C}$ defined by $\xi_s = \exp(2i\pi j/k) t_{a_j}^{2/k}$ on T_{a_j}.

Next we study the quadratic differential w_Γ. Since $\xi_s = ct^{2/k}$ where c is a constant, we deduce that $(dt)^2 = (k/2c)^2 \xi_s^{k-2} (d\xi_s)^2$, that is to say k-valent vertices are zeroes of order $k-2$ of w_Γ. Furthermore its closed trajectories are images in $\|\Gamma\|$ of the segments $[(0,t);(p_a,t)]$ for $t>0$, and $\iota(|\Gamma|)$ is its critical graph. By construction, lengths of edges measured with the metric induced by w_Γ are these specified by the metric l. Finally, the change of coordinates from $]0,p_a[\times]0,\infty[$ onto a disk centered on the origin: $u_a \mapsto \zeta = \exp(2i\pi u_a/p_a)$ yields the following form of w_Γ near x_a: $(du_a)^2 = -\left(\frac{p_a}{2\pi}\right)^2 (d\zeta/\zeta)^2$. To sum up, we have $w_\Gamma \in \mathcal{S}t(\overline{\|\Gamma\|}, (x_1, \ldots, x_{f(\Gamma)}))$.

If $f : (\Gamma, l) \to (\Gamma_0, l_0)$ is an isometry, and as a consequence, $l_0 = cl$ with $c \in \mathbb{R}_{>0}$, then the homothety $h_c : B_a^0 \to B_{f(a)}^0$ gives a biholomorphism $\hat{f} : \overline{\|\Gamma\|} \to \overline{\|\Gamma_0\|}$. Thus to every isometric automorphism $\varphi(g)$, we associate a biholomorphic automorphism $\hat{\varphi}(g)$. The action on the closed trajectories of level $t>0$ is defined to be the same than the action on the graph. The action on the punctures is specified by that on the corresponding faces. By construction, $\varphi(g)^*(w_\Gamma) = w_\Gamma \ \forall g \in G$.

It remains to prove the uniqueness property. Drop group actions to simplify. Let $k : |\Gamma| \hookrightarrow S$ be the embedding of Γ such that $k(|\Gamma|)$ is the critical graph of v. There is a single isotopy class of homeomorphisms $f : \|\Gamma\| \to S$ satisfying $f \circ \iota = k$ (existence is trivial, and uniqueness comes from the Alexander's lemma, which ensures that two orientation preserving homeomorphisms between topological disks which agree on the boundary of the first one are isotopic). Then we find only one biholomorphism in this isotopy class. Indeed, let y_j be the puncture of S corresponding to some face F_j of Γ. Let $\|F_j\|$ (resp. D_j) be the pointed disk by x_j (resp. y_j). Take $\xi_j : \|F_j\| \to D(0,1)$ a local coordinate as above, i.e., such that $(du_a)^2 = -p_j^2 (d\xi_j/\xi_j)^2$. By hypothesis, there exist a local coordinate $z_j : D_j \to D(0,1)$ such that $v = -p_j^2 (dz_j/z_j)^2$. Then define f restricted to $\|F_j\|$ by $f = z_j^{-1} \circ \xi_j$. This is a biholomorphism. The local coordinates ξ_j and z_j extend to the boundary with $\xi_j \circ \iota(|a|) = z_j \circ k(|a|)$, so that f is globally defined with $f \circ \iota = k$. □

For non galoisian coverings, we have the following version of the theorem: to each $[p : (\Gamma, l) \to (\Delta, m)] \in \mathcal{H}_{g,g'}^{\mathrm{comb}}(G, K, R)$ is associated $[\hat{p} : \overline{\|\Gamma\|} \to \overline{\|\Delta\|}] \in \mathcal{H}_{g,g'}(G, K, R)$ such that $w_\Gamma = \hat{p}^*(w_\Delta)$.

Proof of Theorem 4.8. The geometrical realization map

$$[\Gamma, l, \varphi] \mapsto \{[\overline{\|\Gamma\|}, \hat{\varphi}], (p_1, \ldots, p_b)\}$$

(where p_i is the perimeter in the orbit i) is well-defined, since a G-equivariant isometry yield a G-equivariant biholomorphism. The fact that this is the inverse map is ensured by uniqueness in Strebel theorem, and uniqueness in the geometrical realization (property 3 of the last theorem). Again, uniqueness in

Strebel theorem gives the commutativity of the diagram. It remains to prove the continuity; this is done in Theorem 5.2.

5. Cellular Decomposition

We now describe the cellular decomposition of decorated Hurwitz spaces. This leads to a characterization of its connected components. Also, we show that the computation of their orbifold characteristic relies on the computation of the degree between Hurwitz spaces and moduli spaces.

DEFINITION 5.1. Let $(\Gamma_0, l_0, \varphi_0) \in \mathcal{H}_g^{\mathrm{comb}}(G, R)$ and $(\overline{\|\Gamma_0\|}, \hat{\varphi}_0)$ its geometrical realization. The combinatorial relative Teichmüller space $\mathcal{T}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0)$ is made of quadruplets (Γ, l, φ, f), where $(\Gamma, l, \varphi) \in \mathcal{H}_g^{\mathrm{comb}}(G, R)$, together with a G-equivariant quasiconformal homeomorphism $f : \overline{\|\Gamma_0\|} \to \overline{\|\Gamma\|}$. Two quadruplets $(\Gamma_i, l_i, \varphi_i, f_i)_{i=1,2}$ are equivalent if and only if there exists a G-equivariant isometry $h : (\Gamma_1, l_1, \varphi_1) \to (\Gamma_2, l_2, \varphi_2)$ such that $f_2 \circ f_1^{-1} = \hat{h}$ up to homotopy.

The relative modular group acts obviously on this Teichmüller space. Denote by $\mathcal{H}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0)$ the quotient space.

The bijection $\mathcal{H}_g^{\mathrm{comb}}(G, R) \to \mathcal{H}_g(G, R) \times \mathbb{P}(\mathbb{R}_{>0}^b)$ comes in fact from equivariant bijections $\mathcal{T}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0) \to \mathcal{T}_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \times \mathbb{P}(\mathbb{R}_{>0}^b)$ with respect to the actions of the relative modular group, ((Δ, m, ψ, f) maps to $(\overline{\|\Delta\|}, \hat{\psi}, f)$), and from the induced bijections $\mathcal{H}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0) \to \mathcal{H}_G(\Gamma_0, l_0, \varphi_0) \times \mathbb{P}(\mathbb{R}_{>0}^b)$.

Continuity. The following result gives control on variations of complex structures on $\|\Gamma\|_l$ with the variations of the metric l.

THEOREM 5.2. *If $f : (\Gamma, l, \varphi) \to (\Gamma_0, l_0, \varphi_0)$ is composed of equivariant isomorphisms and equivariant Whitehead collapses, then there exists a G-equivariant K-quasiconformal homeomorphism $\hat{f} : \overline{\|\Gamma\|} \to \overline{\|\Gamma_0\|}$ such that $K(\hat{f}) \to 1$ when $|l - l_0| \to 0$.*

PROOF. We can restrict ourself to the case where either f is an isomorphism which alters only one edge's length, or f is a Whitehead collapse on one edge, keeping constant other lengths. For, in the general case, we decompose f, and use the fact that $g \circ h$ is $K(g)K(h)$-quasiconformal if g (resp. h) is $K(g)$- (resp. $K(h)$-) quasiconformal.

We use complex structures defined in the proof of Theorem 4.9, but with the change of coordinates

$$z_a = \frac{p(F(a))}{2\pi} \exp(2i\pi u_a / p(F(a))).$$

We will always use the power function like this: when the argument is fixed, it consists in multiplying it with the exponent.

First Case: Let $e = (a, \bar{a})$ be the edge such that $l(f(e)) = l(e) + \varepsilon$ and $F(a) = (a_1, a_2, \ldots, a_k = a)$ be the face which contains a. Set $l_k = l(a)$, $p = p(F(a))$, $R = p/(2\pi)$, $R_\varepsilon = (p+\varepsilon)/(2\pi)$. We first assume that $k \geq 2$.

Then we define $\hat{f}_F : D(O, R) \to D(O, R_\varepsilon)$ by

$$\frac{R_\varepsilon}{R} z^{\frac{p}{p+\varepsilon}} |z|^{\frac{\varepsilon}{p+\varepsilon}} \text{ for } 0 \leq \arg(z) \leq 2\pi(p - l_k)/p,$$

$$\frac{R_\varepsilon}{R} \frac{z^{\frac{p}{p+\varepsilon}(1+\varepsilon/l_k)}}{|z|^{\frac{\varepsilon}{p+\varepsilon}(p/l_k - 1)}} \times \exp\left(2i\pi \frac{\varepsilon}{p+\varepsilon}\left(1 - \frac{p}{l_k}\right)\right) \text{ for } 2\pi(p - l_k)/p \leq \arg(z) \leq 2\pi.$$

This is well defined for $\arg(z) = 2\pi$ and $\arg(z) = 2\pi(p - l_k)/p$.

We do the same construction for the face $F(\bar{a})$, and take identity for other faces.

In the case $k = 1$, take $\hat{f}_F(z) = (p+\varepsilon)z/p$, and note that $F(\bar{a})$ can not be monovalent.

We have a similar construction in the case where a and $\bar{a}$ belong to the same face.

We have to check that the $\hat{f}_F$ glue to defined $\hat{f}$ between $\|\Gamma\|$ and $\|\Gamma_0\|$.

Firstly, the homeomorphisms $\hat{f}_F$ extend on boundaries of disks, and vertices maps onto vertices.

Without loss of generality, we consider the case where a and $\bar{a}$ do not belong to the same face. If we set $\theta = \arg(z)$ and $\alpha = \arg(\hat{f}_F(z))$, we have

$$\alpha = \theta \frac{p}{p+\varepsilon} \text{ or } \alpha = \frac{\theta p(1 + \varepsilon/l_k) + c}{p+\varepsilon}.$$

with c some constant. Since $\theta = 2\pi x/p$, and $\alpha = 2\pi y/(p+\varepsilon)$, we deduce that, with canonical coordinates, $\hat{f}_F$ is the identity or an affine map $(x \mapsto y = x(1 + \varepsilon/l_k) + c)$ on the segment $[0, p]$. As values at the vertices are fixed, affine maps induced by $\hat{f}_F$ and $\hat{f}_{F'}$ equal on edges.

We have to calculate the dilatations $K(\hat{f}(z))$. We can restrict the computation to angular sectors where $\hat{f}$ is differentiable, and use following lemma, whose proof is direct.

LEMMA 5.3. *Let T the open set of the complex plane defined by $0 < \alpha < \arg(z) < \beta < 2\pi$ and $0 \leq |z| < 1$. Let $g : T \to T'$ be some diffeomorphism of the form $g(z) = cz^a \bar{z}^b$, where a, b, c are real numbers such that $|a| > |b|$. Then g is K-quasiconformal with $K = (|a|+|b|)/(|a|-|b|)$.*

In the case where a and $\bar{a}$ do not belong to the same face, we obtain for example

$$K(\hat{f}(z)) = \frac{p+\varepsilon}{p} \text{ for } 0 < \arg(z) < 2\pi(p - l_k)/p,$$

$$K(\hat{f}(z)) = \frac{l_k}{l_k + \varepsilon} \frac{p+\varepsilon}{p} \text{ for } 2\pi(p - l_k)/p < \arg(z) < 2\pi.$$

Second case: Let (Γ, l) be a smooth Riemannian fat-graph and $(W_e(\Gamma), l)$ the smooth Riemannian fat-graph obtained by collapsing the edge $e = (a, \bar{a})$. Let

p be the perimeter of the face $F(a) = (a_1, \ldots, a_k = a)$. We set $a'_i = f(a_i)$ for $i \in \{1, \ldots, k-1\}$. We assume $l(a_i) = l(a'_i)$, and denote it by l_i; we also set $l_k = l(a_k)$.

For brevity, we will give details only for $k \geq 3$ and $\bar{a} \notin F(a)$ (the procedure is totally similar for other cases).

Again, we build $\hat{f}$ on each face, but firstly in disks of radii $\eta_F p(F)/2\pi$ with $0 < \eta_F < 1$. Then we should define $\hat{f}$ on the complementary of these disks, a ribbon which contains the geometric realization of the fat-graph.

In the disk corresponding to the face $F(a)$, we define $\hat{f}_{F(a)}$ by

$$\frac{p-l_k}{p}\frac{z^{\frac{p}{p-l_k}}}{|z|^{\frac{l_k}{p-l_k}}} \quad \text{for} \quad 0 \leq \arg(z) \leq 2\pi\frac{p-l_k-l_{k-1}}{p},$$

$$\frac{p-l_k}{p}\frac{z^{\frac{pl_{k-1}}{(p-l_k)(l_{k-1}+l_k)}}}{|z|^{\frac{l_k(l_k+l_{k-1}-p)}{(p-l_k)(l_{k-1}+l_k)}}} \times e^{2i\pi l_k\frac{p-l_k-l_{k-1}}{(p-l_k)(l_{k-1}+l_k)}} \quad \text{for} \quad 2\pi\frac{p-l_k-l_{k-1}}{p} \leq \arg(z) \leq 2\pi.$$

This defines a homeomorphism onto its image, a disk of radius $\eta_{F(a)}(p-l_k)/2\pi$. We make the same construction for $F(\bar{a})$ and take the identity for other faces.

Using Lemma 5.3, the desired properties for $K(\hat{f}(z))$ are immediate.

The transformation along closed trajectories with canonical coordinates is

$$x' = \begin{cases} x & \text{for } 0 \leq x \leq p-l_k-l_{k-1}, \\ \dfrac{xl_{k-1}}{l_{k-1}+l_k} + \dfrac{l_k(p-l_k-l_{k-1})}{l_{k-1}+l_k} & \text{for } p-l_k-l_{k-1} \leq x \leq p. \end{cases} \qquad (*)$$

We define a closed neighborhood R of $(a, \bar{a})$ which contains an open neighborhood of vertices $a(0)$ et $a(1)$, and included in the ribbon defined upon.

Set $b_0 = \sigma_2^{-1}(\bar{a})$ and $b_1 = \sigma_2(\bar{a})$, of lengths m_0 and m_1. Recall that $a_{k-1} = \sigma_2^{-1}(a)$ and $a_1 = \sigma_2(a)$ of lengths l_1 et l_{k-1}. Set $p = p(F(a))$ and $p' = p(F(\bar{a}))$. For shortness we assume that $a(0)$ et $a(1)$ are trivalent. Then $\bar{a}_{k-1} = \sigma_2(\bar{b}_1)$ and $\bar{b}_1$ belong to a common face F_3 of perimeter p_3, and $\bar{a}_1$ and $\bar{b}_0$ belong to a common face F_4 of perimeter p_4. Let $\mu \in]0, \min(m_0, m_1, l_1, l_{k-1})[$ be a real number.

The closed set R is made of four glued quadrangles $(R_i)_{i\in\{1,\ldots,4\}}$ (see Figure 3). The size of R is parametrized by $l_k, \mu, \eta, \eta', \eta_3, \eta_4$.

We send R on the corresponding set of $\|W_e(\Gamma)\|$ in a compatible way with the definition of $\hat{f}$ on the boundary; see formula $(*)$.

We choose as image of the edge e a segment of vertical trajectory, and whose length is $2c(l_k)$, where $c(l_k)$ goes to zero as l_k goes to zero.

The image of the $(R_i)_{i\in\{1,\ldots,4\}}$ are the polygons $(P_i)_{i\in\{1,\ldots,4\}}$ of figure 3.

To estimate the dilatation K_i of $\hat{f}$ on R_i, we can use the notion of modulus of $\Lambda(\Gamma)$ for a family of piecewise smooth paths describing an open disk of the plane, defined by L. V. Ahlfors in chapter I, § D of [2].

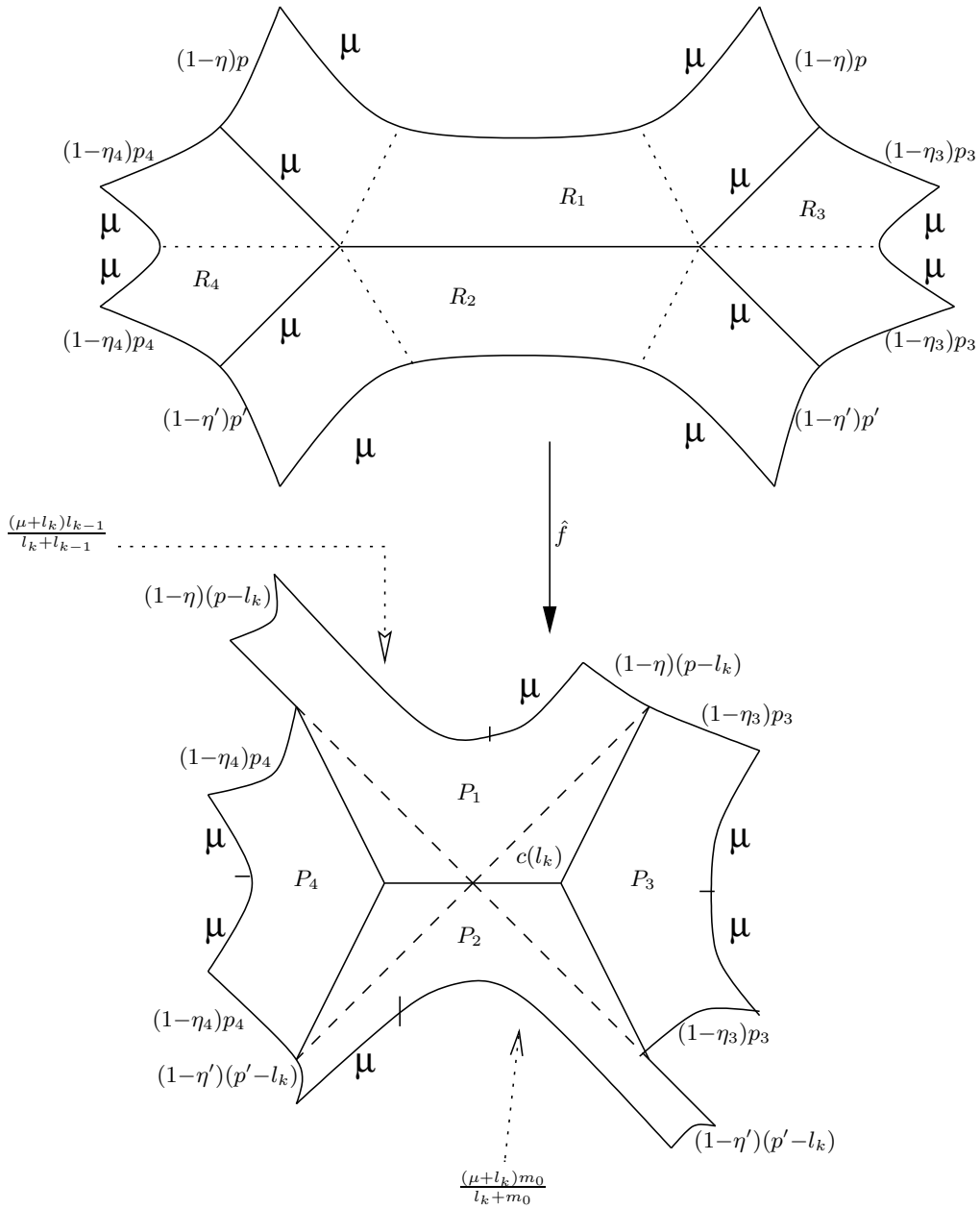

Figure 3. Homeomorphism in a neighborhood of the collapsed edge.

For R_1 and R_3 we choose paths between the sides of lengths $2\mu + l_k$ and 2μ, respectively. We put P_i between rectangles to estimate $\Lambda(P_i)$. We calculate for example

$$\frac{\Lambda(R_3)}{\Lambda(P_3)} \leq K_3 = \frac{(1-\eta_3)p_3}{(1-\eta_3)p_3 - c(l_k)}.$$

If $\Lambda(P_1) \geq \Lambda(R_1)$, then

$$\frac{\Lambda(P_1)}{\Lambda(R_1)} \leq K_1 = \frac{(l_k + l_{k-1})(2\mu + l_k)[(1-\eta)p + c(l_k)]}{[\mu(l_k + 2l_{k-1}) + l_k l_{k-1}](1-\eta)p}. \qquad \square$$

This also concludes the proof of Theorem 4.8. $\square$

Using Harer's techniques [13], we could define a bijection from $\mathcal{T}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0)$ to a subset of a simplicial complex so that the topology on this space is (roughly speaking) the following: two points $(\Gamma_i, l_i, \varphi_i, f_i)_{i=\{1,2\}}$ are very close if and only if there exists a map f between $(\Gamma_1, l_1, \varphi_1)$ and $(\Gamma_2, l_2, \varphi_2)$ composed of equivariant Whitehead collapses and equivariant isomorphisms with small changes of metric. Then we can find an isotopy between $\hat{f}$ (defined in Theorem 5.2) and $f_1^{-1} \circ f_2$, to obtain (using Lemma 2.6) the continuity at level of Teichmüller spaces.

Cellular decomposition

DEFINITION 5.4. Let $J_\varphi : \mathcal{T}_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \times \mathbb{P}(\mathbb{R}_{>0}^b) \to \mathcal{T}_G^{\mathrm{comb}}(\Gamma_0, l_0, \varphi_0)$ the inverse map of the geometrical realization, given by the equivariant Strebel theorem.

Let $(\Gamma, m, \varphi) \in \mathcal{H}_{g(\Gamma_0)}^{\mathrm{comb}}(G, R(\varphi_0))$ and suppose $f : (\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \to (\overline{\|\Gamma\|}_m, \hat{\varphi})$ is a G-equivariant homeomorphism. Then we define $C(\Gamma, f, \varphi)$ as the subset of $\mathcal{T}_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \times \mathbb{P}(\mathbb{R}_{>0}^b)$ made of elements $[R, \psi, g, (p_i)_i]$ satisfying

$$J_\varphi(R, \psi, g, (p_i)_i) = (\Gamma, l, \varphi, \hat{i} \circ f),$$

where l is any metric on Γ such that the action of G is isometric, and $i : (\Gamma, l, \varphi) \to (\Gamma, m, \varphi)$ is the canonical G-equivariant isomorphism.

Note that $C(\Gamma, \varphi, f)$ does not depend on the choice of the metric m, since we can always compose by an equivariant isomorphism. These subsets cut the decorated relative Teichmüller space into (open) cells, homeomorphic to $\mathbb{P}(\mathbb{R}_{>0}^{a(\Gamma)/|G|})$.

THEOREM 5.5. *The stratification of $\mathcal{T}_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \times \mathbb{P}(\mathbb{R}_{>0}^b)$ is a cellular decomposition. The relative modular group $\mathcal{M}od_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0)$ acts in a cellular way. The stabilizer of a cell $C(\Gamma, \varphi, h)$ is isomorphic to the quotient of the automorphism group $\mathrm{Aut}_G(\Gamma, \varphi)$ by $Z(G)$.*

PROOF. Let K_m be the m-skeleton made of cells $C(\Gamma, \varphi, h)$ with $a(\Gamma)/|G| \leq m+1$, and take such a cell with $a(\Gamma)/|G| = m+1$. We build the attachment map of this cell. Let $l = (l_1, \ldots, l_m)$ a metric on Γ, where l_i is the constant length in the orbit of a geometrical edge e_i, and $\sum_i l_i = 1$. We now allow $l_i = 0$, except for orbits those quotient edge is a loop.

If $l_i > 0\ \forall i \in \{1, \dots, m\}$, then we map l to the corresponding element of $C(\Gamma, \varphi, h)$.

If $l_i = 0$ for some i, we map l to the corresponding element of $C(W_{O(e_i)}(\Gamma), \varphi_i, \hat{w} \circ h)$, where φ_i is the induced action of G, and $w : (\Gamma, l, \varphi) \to (W_{O(e_i)}(\Gamma), l, \varphi_i)$ is the equivariant Whitehead collapse.

This defines a continuous map $\prod_{i=1}^{n}]0,1[\times \prod_{i=n+1}^{m} [0,1[\to K_m$, which maps $\prod_{i=1}^{m}]0,1[$ homeomorphically onto $C(\Gamma, \varphi, h)$, and its complementary in K_{m-1}.

The cellular action of the relative modular group is $[\theta, C(\Gamma, \varphi, h)] \mapsto C(\Gamma, \varphi, h \circ \theta^{-1})$.

We consider the center $e(\Gamma, \varphi, h)$ of the cell. This is the point defined by the unitary metric: $l(e) = 1\ \forall e \in A_g(\Gamma)$. For this special metric, all the automorphisms are isometric. The center consists in a compact Riemann surface R marked by $f : \overline{\|\Gamma_0\|} \to R$, equipped with an action of G and a Strebel differential q_R stable under this action. The critical graph of q_R is an embedding of Γ which realizes the unitary metric. Thus we have $\mathrm{Aut}_G(\Gamma, \varphi) \cong \mathrm{Aut}_G(R, q_R) = \mathrm{Aut}(e(\Gamma, \varphi, h))$.

But if $\theta \in \mathcal{M}od_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0)$, then $\theta \cdot e(\Gamma, \varphi, h) = e(\Gamma, \varphi, h \circ \theta^{-1})$, so that $\mathrm{Aut}(e(\Gamma, \varphi, h))/Z(G) \cong \mathrm{Stab}(C(\Gamma, \varphi, h))$, from which we deduce the result. □

On the decorated Hurwitz space $\mathcal{H}_G(\overline{\|\Gamma_0\|}, \hat{\varphi}_0) \times \mathbb{P}(\mathbb{R}^b_{>0})$, we obtain the induced cellular decomposition.

The cellular decomposition is compatible with the orbifold structure in the sense of Thurston: the finite groups associated to each point of the orbifold are constant along each cell. We also emphasize that the notion of cellular decomposition used here coincide with the classical one only for compact spaces.

Cells $C(\Gamma, \varphi)$ are parametrized by metrics on the quotient fat-graph. Thus they are homeomorphic to $C(\Gamma/\varphi(G))$. Moreover, $C(\Gamma', \varphi')$ is in the boundary of $C(\Gamma, \varphi)$ if and only if (Γ', φ') is obtained from (Γ, φ) by equivariant Whitehead collapses. Then $C(\Gamma'/\varphi'(G))$ is in the boundary of $C(\Gamma/\varphi(G))$. Thus, in a way, the projection $\mathcal{H}_g(G, R) \to \mathcal{M}_{g', (b_1, \dots, b_t)}$ is a cellular one; the same fact holds for the natural morphisms between Hurwitz spaces obtained by restriction to a subgroup, or by quotient by a normal subgroup.

Top-dimensional cells of moduli spaces and Hurwitz spaces are yielded by trivalent fat-graphs (maximal number of edges).

Generalizing the notion of flip, we show that equivariant flips characterize connected components of $\mathcal{H}_g(G, R)$, described in § 2. Two top-dimensional cells $C(\Gamma, \varphi)$ and $C(\Gamma', \varphi')$ share a codimension one cell $C(\Gamma_0, \varphi_0)$ if and only if there exists $e \in A_g(\Gamma)$ and $e' \in A_g(\Gamma')$ such that $[W_{O(e)}(\Gamma)] = [\Gamma_0, \varphi_0] = [W_{O(e')}(\Gamma')]$. We say that (Γ, φ) differs from (Γ', φ') by an *equivariant flip*.

The flips are sometimes referred as elementary moves [24], or as Whitehead moves, the result of a Whitehead collapse and a Whitehead inflation.

PROPOSITION 5.6. *Two points of $\mathcal{H}_g(G, R)$ lie in the same connected component if and only if they belong to the adherence of top-dimensional cells indexed by*

trivalent fat-graphs, endowed with a quasifree action of G of ramification data R, which differ one from the other by a finite sequence of equivariant flips.

PROOF. The condition is sufficient since equivariant Whitehead collapses give equivariant homeomorphisms which preserve the topological type.

The condition is necessary: let $C(\Gamma_1, \varphi_1)$ and $C(\Gamma_2, \varphi_2)$ be two top dimensional cells of $\mathcal{H}_G(S_0, \varphi_0)$, a connected component of $\mathcal{H}_g(G, R)$. Let a_i be an adherent point of $C(\Gamma_i, \varphi_i)$. Join a_i to a point b_i of $C(\Gamma_i, \varphi_i)$ if necessary. Then join b_1 to b_2 by a path intersecting only cells of codimension zero and one. Then (Γ_1, φ_1) differs from (Γ_2, φ_2) by a finite sequence of equivariant flips. □

To illustrate the fact that the combinatorial description of Hurwitz spaces encodes their orbifold structure, we link their orbifold Euler characteristic, to those of moduli spaces.

W. Thurston has extended the notion of Euler characteristic to orbifolds $\mathcal{O}$ which possess some cellular decomposition $(C_i)_i$ compatible with its orbifold structure. Then

$$\chi_{\text{orb}}(\mathcal{O}) = \sum_i \frac{(-1)^{\dim(C_i)}}{|G(C_i)|},$$

where $G(C_i)$ is the finite group associated to each cell.

Here, using Theorem 5.5, we have

$$\chi_{\text{orb}}(\mathcal{H}_g(G, R) \times \mathbb{P}(\mathbb{R}^b_{>0})) = \sum_{[\Gamma_j, \varphi_j]} \frac{(-1)^{a(\Delta_j)-1} \times |Z(G)|}{|\text{Aut}_G(\Gamma_j, \varphi_j)|}$$

where $\Delta_j = \Gamma_j/\varphi_j$. Then, since $\chi_{\text{orb}}(\mathbb{P}(\mathbb{R}^b_{>0})) = (-1)^{b-1}$, using the Euler formula, we deduce:

$$\chi_{\text{orb}}(\mathcal{H}_g(G, R)) = \sum_{[\Gamma_j, \varphi_j]} \frac{(-1)^{s(\Delta_j)} \times |Z(G)|}{|\text{Aut}_G(\Gamma_j, \varphi_j)|}.$$

We resum on the isomorphism classes $[\Gamma_j/\varphi_j = \Delta_j]$ indexing the cellular decomposition of $\mathcal{M}_{g',(b_1,\ldots,b_t)}$, and then we use the relation between automorphisms groups (Theorem 4.5) to obtain

$$\chi_{\text{orb}}(\mathcal{H}_g(G, R)) = \sum_{[\Delta]} \frac{(-1)^{s(\Delta)}}{|\text{Aut}(\Delta)|} \sum_{\psi} \frac{|\text{Aut}(\Delta)|}{|\text{Aut}(\Delta, \psi)|}$$

where the ψ describe the set of conjugacy classes of epimorphisms from $\pi_1(\Delta)$ onto G with images of loop-faces fixed by the ramification data. But

$$\sum_{\psi} \frac{|\text{Aut}(\Delta)|}{|\text{Aut}(\Delta, \psi)|}$$

is the class equation for the action of $\text{Aut}(\Delta)$ on this set. Thus this set is independent from Δ. Denote it by $\text{Epi}_{g'}(G, R)$, and let d_{orb} be its cardinal. We have proved:

PROPOSITION 5.7. $\chi_{\rm orb}(\mathcal{H}_g(G,R)) = d_{\rm orb} \times \chi_{\rm orb}(\mathcal{M}_{g',(b_1,\ldots,b_t)})$.

This means that Hurwitz spaces are less singular than moduli spaces. The rational number $\chi_{\rm orb}(\mathcal{M}_{g',(b_1,\ldots,b_t)})$ is calculated in [14] (see also [3] [20]).

The computation of $d_{\rm orb}$ is a non trivial one. The cardinal of $\mathrm{Hom}_{g'}(G,R)$ (replace epimorphisms by homomorphisms) is calculated in [19] (see also [12]), it only depends on the irreducible complex representations of G. For symmetric groups, these cardinals appear as coefficient of a generating series coming from a matrix model [21], closely related to Yang–Mills theory for surfaces [12].

We conclude with the non-galoisian case. We also have a cellular decomposition of $\mathcal{H}(g,g',G,K,R) \times \mathbb{P}(\mathbb{R}^b_{>0})$ into cells $C(p)$, where $[p : \Gamma \to \Delta]$ is an equivalence class of degree d covering with a monodromy of type (G,K,R). These cells are parametrized by edges' lengths of Δ. Performing Whitehead collapses along e on Δ, and thus Whitehead collapses along $p^{-1}(e)$ on Γ, give a cell $C(q)$ incident on $C(p)$, where $q : W_{p^{-1}(e)}(\Gamma) \to W_e(\Delta)$.

Furthermore, we see that $\mathrm{Out}(G,K,R)$ acts cellularly on $\mathcal{H}_h(G,R) \times \mathbb{P}(\mathbb{R}^b_{>0})$ by $\theta \cdot C(\Gamma,\varphi) = C(\Gamma, \varphi \circ \theta^{-1})$. Hence

$$\chi_{\rm orb}(\mathcal{H}(g,g',K,G,R)) = \chi_{\rm orb}(\mathcal{H}_h(G,R))/\mathrm{Out}(G,K,R).$$

REMARK. An important step in Kontsevich results [20] is the explicit computation of cohomology classes on the cellular decomposition of moduli space of punctured Riemann surfaces. The cohomology classes are first Chern classes of line bundles over moduli spaces, whose fibers are the cotangent spaces at the i-th puncture.

It follows from our work, that the pullback of these classes on Hurwitz spaces, can be computed in the same way.

6. Compactification

The convenient tool to describe compactifications of moduli spaces, or of Hurwitz spaces, is again a graphical one: the modular graph (terminology of Y. Manin). To avoid any confusion, we keep the Greek letters for fat-graphs. For a graph E, we denote by $V(E)$ the set of vertices, $v(E)$ the number of vertices, v_s the valency of the vertex s, $a(E)$ the number of edges, and $h^1(E) = a(E) - v(E) + 1$.

Moduli spaces

DEFINITION 6.1. A modular graph E of type (g',b) is a connected graph together with two maps $g : V(E) \to \mathbb{N}$ and $P : V(E) \to \mathcal{P}(\{1,\ldots,b\})$ such that

- $\{1,\ldots,b\} = \coprod_v P(v)$,
- $2g(s) - 2 + v_s + \#P(s) > 0 \ \forall s \in V(E)$,
- $g' = \sum_{s \in V(E)} g(s) + h^1(E)$.

Two such modular graphs (E, g, P) and (H, h, Q) are isomorphic if there exists an isomorphism of graphs $i : E \to H$ with $i \circ g = h \circ i$ and $i \circ P = Q \circ i$.

We have the following well-known description of the Deligne–Mumford–Knudsen compactification of moduli spaces: $\overline{\mathcal{M}}_{g',b} = \coprod_E \mathcal{M}_{g',b}(E)$, where E runs over the modular graphs of type (g', b), and $\mathcal{M}_{g',b}(E)$ consists in all stable Riemann surfaces built as follows:

- Associate to each vertex s a compact Riemann surface R_s of genus $g(s)$, with $\#P(s)$ punctures and v_s ties.
- Associate to each edge an identification of the corresponding ties.

  Such objects are connected compact Riemann surfaces C with punctures and singular points, called nodes, obtained by identification of ties. We denote them by (C, i_C), where i_C is the identification of ties.
- Associate to each isomorphism, an homeomorphism $h : C \to C'$ such that $h \circ i_C = i_{C'} \circ h$, biholomorphic when restricted to each component.

Each element of $\overline{\mathcal{M}}_{g',b}$ comes from pinching some boundary curves of a pants decomposition of a smooth surface [4].

Then the neighborhood of a node looks like $\{(y : U \to D) \times (z : V \to \overline{D})/yz = 0\}$, where U and V are some disks in the surface pointed by the node N, D is the unit disk of the complex plane, $\overline{D}$ is the same disk but with the conjugate complex structure, and $y(N) = z(N) = 0$ [4].

In our context of marked Riemann surfaces, we define the topology of $\overline{\mathcal{M}}_{g',b}$ with quasiconformal deformations.

First we have the following definition of L. Bers [4]. Let S_1 and S_2 be two stable curves of type (g', b), and N_i be the set of nodes of S_i. A *deformation* $f : S_1 \to S_2$ is a surjective map with $f(N_1) \subset N_2$, such that the preimage of a node is either a node or a Jordan curve (non null-homotopic, non homotopic to some puncture), and such that f is an homeomorphism component by component, when restricted to $S_1 - f^{-1}(N_2)$.

Then after the work of W. Abikoff (see § 1.3 and Theorem 1 of [1]), S_1 is close to S_2 if f^{-1} is $(1 + \varepsilon)$-quasiconformal on each component of $S_2 - K$ with K a compact neighborhood of N_2.

Maybe some components of such a stable Riemann surface do not have any puncture. These components cannot be parametrized by means of fat-graphs, and this leads M. Kontsevich [20] to take a quotient of $\overline{\mathcal{M}}_{g',b}$: homeomorphisms $h : (C, i_C) \to (D, i_D)$ restricted to any component without puncture may be just an homeomorphism instead of a biholomorphism. Kontsevich's compactification $\overline{\mathcal{M}}'_{g',b}$ is then the quotient of $\overline{\mathcal{M}}_{g',b}$ by the closure (of the graph) of this new equivalence relation.

At the level of modular graphs, this means that we can retract any edge a joining two vertices s_1 and s_2 with $P(s_i) = \varnothing$ into a new single vertex s, setting $g(s) = g(s_1) + g(s_2)$ if $s_1 \neq s_2$ and $g(s) = g(s_1) + 1$ if a is a loop.

Thus, in $\overline{\mathcal{M}}'_{g',b}$, if a stable Riemann surface possess a component C_s without punctures, we can forget C_s, but keep as data in the modular graph its genus and the number of ties.

If γ is a simple loop which separates a Riemann surface into two stable components C_1 and C_2 such that C_1 is without punctures, then, from a topological viewpoint in $\overline{\mathcal{M}}'_{g',b}$, pinching γ to a point is the same as collapsing C_1 into a point.

We now precise the intricate definition of $\overline{\mathcal{M}}^{\mathrm{comb}}_{g',b}$ given in [20]. The role of ties is played by distinguished vertices.

DEFINITION 6.2. A stable fat-graph of type (g', b) is made of a modular graph (E, g, P) of type (g', b) such that:

- To each vertex s with $P(s) \neq \varnothing$ is associated a Riemannian fat-graph Γ_s, with $\#P(s)$ faces, genus $g(s)$, and v_s distinguished vertices (maybe monovalent or bivalent, the other ones at least trivalent.)
- To each edge joining vertices s_i with $P(s_i) \neq \varnothing$ is associated an identification of the corresponding distinguished vertices.

Two stable fat-graphs $(E, g, P, (\Gamma)_s)$ and $(F, l, Q, (\Delta)_s)$ are equivalent if there exist $h : (E, g, P) \to (F, l, Q)$ or $(F, l, Q) \to (E, g, P)$ composed of isomorphisms and retraction of edges joining vertices s_i with $P(s_i) = \varnothing$, such that Γ_s and $\Delta_{h(s)}$ are isometric if $P(s) \neq \varnothing$.

We now have to precise how smooth fat-graphs of type (g', b) degenerates into stable fat-graphs of same type. We have to distinguish two cases.

Firstly, we can perform Whitehead's collapses on loops which do not bound any face. Let e such a loop incident on a vertex v in a fat-graph Γ. After the retraction, we disconnect the graph on v to produce two distinguished vertices. Precisely, if $e = (a, \bar{a})$, such that $\sigma_2(a) \neq a$ and $\sigma_2(\bar{a}) \neq \bar{a}$, then, as a σ_0-orbit, $v = (a_1, \ldots, a_j, a, b_1, \ldots, b_k, \bar{a})$ with $j, k \geq 1$ (because for example $\sigma_0(a) = \bar{a}$ implies $\sigma_2(\bar{a}) = \bar{a}$, which is forbidden). Then the new distinguished vertices are $(a_1, \ldots, a_j)$ and $(b_1, \ldots, b_k)$. They are possibly monovalent or bivalent.

At the level of modular graphs we create an edge and a new vertex, or a loop, with appropriate data.

Secondly, we can collapse a subgraph Δ of Γ into a vertex which becomes distinguished (provided that the number of faces keeps constant). At the level of modular graphs, we create an edge and a monovalent vertex s with $g(s) = g(\Delta)$ and $P(s) = \varnothing$.

Then, we have Kontsevich's theorem, which furnishes a cellular decomposition of the whole space $\overline{\mathcal{M}}'_{g',b} \times \mathbb{P}(\mathbb{R}^b_{>0})$.

THEOREM 6.3. *There is an homeomorphism* $\overline{\mathcal{M}}^{\mathrm{comb}}_{g',b} \to \overline{\mathcal{M}}'_{g',b} \times \mathbb{P}(\mathbb{R}^b_{>0})$

PROOF. Fix a modular graph (E, g, P). The map from right to left is given by the Theorem 3.3 applied to each component associated to a vertex s with

$P(s) \neq \varnothing$. For the inverse map, we have to say something about monovalent and bivalent vertices.

If s is a bivalent vertex of a Riemannian fat-graph (Γ, l), with edges a and b incident on it, then $\|\Gamma\|$ is biholomorphic to $\|\Gamma'\|$, where Γ' is the Riemannian fat-graph obtained from (Γ, l) in canceling s, edges a and b becoming a new edge of length $l(a) + l(b)$.

Let s be a monovalent vertex of (Γ, l), and a be the unique edge incident on it. If D_s is a small disk centered in s, then $(D_s \cap U_a, u_a^2)$ is a local coordinate at s, holomorphically compatible with the other ones, and s becomes a first order pole of w_Γ.

Continuity can be sketched as follows. We associate to each edge's retraction a deformation whose dilation is controlled in function of edge's length (as in Theorem 5.2). □

Hurwitz spaces. From now on, we fix $R = b_1C_1 + \cdots + b_tC_t$ a degree b ramification data of a finite group G. We first have to extend the ramified covering $\mathcal{H}_h(G, R) \to \mathcal{M}_{g',\{b_1,\ldots,b_t\}}$ to a suitable compactification $\overline{\mathcal{H}}_h(G, R)$.

Since every stable Riemann surface of $\overline{\mathcal{M}}_{g',\{b_1,\ldots,b_t\}}$ is obtained by retraction of some loops on a smooth Riemann surface of $\mathcal{M}_{g',\{b_1,\ldots,b_t\}}$, we build elements of $\overline{\mathcal{H}}_h(G, R)$ from smooth ones in retracting some orbits of loops. The following result is important since it gives the stability condition for actions of G on stable Riemann surfaces.

PROPOSITION 6.4. *Let $[C, \varphi] \in \mathcal{H}_h(G, R)$. Then the stabilizer $G(L)$ of a loop L is either cyclic or dihedral.*

PROOF. Assume that $G(L)$ is non trivial. Then $G(L)$ stabilizes a small part of cylinder whose fundamental group is generated by L. Using uniformization, we can choose C biholomorphic to a regular ring $\{z \in \mathbb{C} : r < |z| < 1\}$ so that L becomes the circle of radius $\sqrt{r}$. Using uniformization again, the finite order automorphisms of this ring stabilizing L are rotations of finite order, and the symmetry $z \mapsto r/z$. □

Note that rotations do not possess fixed points in the ring, and stabilizes each part of this ring. On the contrary, the symmetry exchanges both parts and possess two fixed points at diametrically opposite points of L. In fact, if $G(L) \cong D_n$ the dihedral group of order $2n$, there is $2n$ fixed points on L. We assimilate the case where $G(L)$ is the cyclic group generated by the symmetry described just upon to the case of dihedral stabilizers (thus we exclude this case when we talk of cyclic stabilizer).

Hence, we cannot retract loops with dihedral stabilizers if we want to keep constant the ramification data. The phenomenon of collision of ramification points is also relevant in the context of algebraic geometry [5].

We can say something more in the case of cyclic stabilizers. Let C_1 and C_2 be the components attached by the node N, obtained by retraction of L. Denote

by N_1 (resp. N_2) the corresponding ties, and by ξ_1 (resp. ξ_2) the privileged generators of stabilizer $G(N_1)$ (resp. $G(N_2)$) (here, the privileged generator has the same meaning than in § 2). Then $\xi_1 = \xi_2^{-1}$ in G (just recall the description of the neighborhood of a node).

DEFINITION 6.5. • An admissible symmetric subsurface decomposition D of $[C, \varphi] \in \mathcal{H}_h(G, R)$ is a subsurface decomposition of $C - X$ (where X is the set of fixed points), symmetric with respect to the holomorphic action of G, and such that every curve of the decomposition does not intersect the set X. Such a decomposition is called maximal if the quotient decomposition of $C/\varphi(G)$ is a pants decomposition.

- Let (C, i_C) be a compact Riemann surface with nodes. An action $\varphi : G \hookrightarrow \text{homeo}(C, i_C)$ is said to be stable if $\varphi(G)$ acts holomorphically on each component, if the node stabilizers are all cyclic, with the preceding property on stabilizers of ties, and if $2g_i - 2 + F_i + L_i > 0$ (L_i is the cardinal of nodes on the i-th component, F_i is the cardinal of smooth fixed points).
- $\overline{\mathcal{H}}_h(G, R)$ is the set of equivalence classes of $[(C, i_C), \varphi]$ where (C, i_C) is a genus g compact Riemann surface with nodes, and φ is a stable action with $R(\varphi) = R$. Two elements define the same class if they differ by a G-equivariant homeomorphism which is biholomorphic on each component.

Then every element $[C, \varphi]$ of $\overline{\mathcal{H}}_h(G, R)$ with nodes come from pinching curves orbits of an admissible symmetric and maximal subsurface decomposition of a smooth $[C_t, \varphi_t]$. The condition $2g_i - 2 + F_i + L_i > 0$ correspond to the fact that curves of the decomposition are not homotopic to a smooth point (fixed or not). The cyclicity of node's stabilizers corresponds to the fact that curves of the decomposition do not contain any fixed point, since curves with dihedral stabilizers contain fixed points.

With this definition, $\overline{\mathcal{H}}_h(G, R) \to \overline{\mathcal{M}}_{g', (b_1, \ldots, b_t)}$ is a ramified covering, thus $\overline{\mathcal{H}}_h(G, R)$ is a compact space. The map is well-defined, because $2g_i - 2 + F_i + L_i = |G_i|(2g'_i - 2 + f_i + l_i)$ by Riemann–Hurwitz, where G_i is the stabilizer of C_i, f_i is the number of branch points (resp. ties) on $C_i/\varphi(G_i)$.

Again, elements of $\overline{\mathcal{H}}_h(G, R)$ are well-described in terms of modular graphs (F, g, P), equipped with a G-action. A map P from $V(F)$ orbits to $\mathcal{P}(\{1, \ldots, b\})$ gives the spreading of ramification data. We put (forgetting the multiplicities) $R_{P(s)} = \sum_{i \in P(s)} C_i$.

Hypothesis of stability for the group actions on Riemann surfaces with nodes impose conditions on modular graphs. Denote by G_s and G_a the stabilizer of $s \in V(F)$ and $a \in A(F)$, respectively. The groups G_a are cyclic. We have excluded elements θ such that $\theta(a) = \bar{a}$, so that $G_a \cong G_{\bar{a}}$ and $G_a \hookrightarrow G_{a(0)}$, $G_a \hookrightarrow G_{a(1)}$. Moreover, if we consider ties stabilizers, and a distinguished generator (which acts on the tangent plane in multiplying by a fixed root of unity), then we can assign to each oriented edge e this generator g_e, and $g_{\bar{e}} = g_{e^{-1}}$.

Now $\overline{\mathcal{H}}_h(G,R) = \coprod_{F,g,P,\rho} \mathcal{H}_{h,G,R}(F,g,P,\rho)$ where (F,g,P,ρ) runs over isomorphism classes of decorated modular graphs of genus h with an action $\rho : G \hookrightarrow \mathrm{Aut}(F,g,P)$ with preceding properties, and such that $R = \sum_s R_{P(s)}$. The decoration consists in assigning to each oriented edge e a generator g_e of G_e such that $g_{\bar{e}} = g_{e^{-1}}$. The stratum $\mathcal{H}_{h,G,R}(F,g,P,\rho)$ is made of all stable Riemann surfaces with stable actions whose associated decorated modular graph is (F,g,P,ρ).

DEFINITION 6.6. $\overline{\mathcal{H}}'_h(G,R)$ is the quotient of $\overline{\mathcal{H}}_h(G,R)$ by the closure of the following equivalence relation: two elements are identified if they differ by an equivariant homeomorphism which is biholomorphic only when restricted to components with smooth fixed points.

Again, this means that we can retract edges in the modular graph, in fact, orbits of edges joining vertices s with $P(s) = \varnothing$.

We now define the suitable extension $\overline{\mathcal{H}}_h^{\mathrm{comb}}(G,R)$ of $\mathcal{H}_h^{\mathrm{comb}}(G,R)$.

DEFINITION 6.7. An element of $\overline{\mathcal{H}}_h^{\mathrm{comb}}(G,R)$ of topological type (F,g,P,ρ) is made as follows:

- Associate to each vertex s of F with $P(s) \neq \varnothing$, a Riemannian fat-graph Γ_s of genus $g(s)$, with v_s distinguished vertices, endowed with $\varphi_s : G_s \hookrightarrow \mathrm{Aut}(\Gamma_s)$, quasifree, except for distinguished vertices, and such that $R(\varphi_s) = R_{P(s)}$.
- A distinguished vertex $v \in V(\Gamma_s)$ given by an edge a incident on s has stabilizer G_a. Edges of F yield the identification of distinguished vertices. If θ_a (resp. $\theta_{\bar{a}}$) are the privileged generators of G_a (resp. $G_{\bar{a}}$), then we ask that $\theta_a = \theta_{\bar{a}}^{-1}$.
- If $\rho(\theta)(s) = t$, then $\varphi(\theta) : \Gamma_s \to \Gamma_t$ must be an equivariant isometry if $P(s) \neq \varnothing$.

As in the case of moduli spaces, these stable fat-graphs are all obtained by retractions of edges, in fact retractions of orbits which do not contain any face, or by equivariant retraction of subgraphs.

We obtain the analog of Theorem 6.3 in the case of Galois coverings.

THEOREM 6.8. *There is an homeomorphism* $\overline{\mathcal{H}}_h^{\mathrm{comb}}(G,R) \to \overline{\mathcal{H}}'_h(G,R) \times \mathbb{P}(\mathbb{R}_{>0}^b)$

PROOF. Fix a modular graph (F,g,P,ρ). Then we apply Theorem 4.8 component by component, with the same discussion than for theorem 6.3. □

We define $\overline{\mathcal{H}}(g,g',G,K,R)$ as the coverings of stable Riemann surfaces got from elements of $\mathcal{H}(g,g',G,K,R)$ by retraction of loops on the base (resp. orbits of loops on the total space) which do not contain any branch points (resp. ramification points). We could describe its elements by coverings of modular graphs. Then $\overline{\mathcal{H}}'(g,g',G,K,R)$ is the quotient by the closure of the equivalence relation where coverings need to be holomorphic only on components with ramification points.

Similarly, $\overline{\mathcal{H}}_{g,g'}^{\mathrm{comb}}(G,K,R)$ is defined as coverings of stable fat-graphs obtained by retraction on the base either of loops which do not bound any face or of

subgraphs which do not contain any face, and by suitable retraction of fibers on the total space.

END OF PROOF OF THEOREM 1.1. The action of $\mathrm{Out}(G,K,R)$ on $\mathcal{H}_h(G,R)$ extends on $\overline{\mathcal{H}}'_h(G,R)$. For, a stable Riemann surface (with action of G) of $\overline{\mathcal{H}}'_h(G,R)$ is obtained from a Riemann surface (with action of G) of $\mathcal{H}_h(G,R)$ by pinching some curves orbits of an admissible maximal and symmetric subsurface decomposition (Definition 6.5).

Thus we have an homeomorphism from the quotient $\overline{\mathcal{H}}'_h(G,R)/\mathrm{Out}(G,K,R)$ to $\overline{\mathcal{H}}'(g,g',G,K,R)$. The map is surjective: an element $(p_0 : T_0 \to S_0)$ comes from a smooth $(p_t : T_t \to S_t)$ got from (C_t,φ_t) in $\mathcal{H}_h(G,R)$ (Proposition 2.11). Then $(p_0 : T_0 \to S_0)$ is the image of (C_0,φ_0). Similarly, the injectivity is deduced from injectivity in the smooth case.

Now the same fact holds for combinatorial spaces, so that theorem 6.8 gives Theorem 1.1. □

7. Moduli Spaces of Curves with Cyclic Group Actions

We now emphasize on Hurwitz spaces $\mathcal{H}_g(\mathbb{Z}/n,R)$, a special case of own interest, due to their striking analogy with moduli space of curves with spin ([17] [27]).

Let G be a fixed finite group, together with a fixed ramification data R. Consider the Hurwitz space $\mathcal{H}_g(G,R)$, assumed to be non empty. Let $[C,\varphi] \in \mathcal{H}_g(G,R)$, and $\pi : [C,\varphi] \to S = C/\varphi(G)$, the associated ramified covering.

Denote by $\mathcal{O}_C$ the sheaf of holomorphic functions on C, and let $\pi_*(\mathcal{O}_C)$ the image direct sheaf. This is a locally free sheaf of G-module, with dimension $|G|$. If $\hat{G}$ denotes the set of the irreducible complex representations of G, $\chi : G \to \mathrm{GL}(V_\chi)$, then $\pi_*(\mathcal{O}_C) = \bigoplus_{\chi\in\hat{G}} L_\chi \otimes \mathcal{V}_\chi$, where L_χ is the χ-isotopic subbundle, with rank $\deg(\chi) = \dim(V_\chi)$, and $\mathcal{V}_\chi$ is the constant bundle with fiber V_χ.

Assume now that G is the cyclic group $\mathbb{Z}/n$. We refer to the example 1 in § 2. Put $\xi = \exp(2i\pi/n)$ and $\sigma = \varphi([1])$. Then $\pi_*(\mathcal{O}_C) = \bigoplus_0^{n-1} L_k$, where L_k is the line bundle of holomorphic germs corresponding to the eigenvalue ξ^k. We have $L_0 = \mathcal{O}_S$.

Describing the image of $L_1^{\otimes n} \hookrightarrow L_0$ involves the ramification data $R(\varphi)$. More precisely, we have:

PROPOSITION 7.1. *If $G = \mathbb{Z}/n$, and $R = \sum_1^b [m_i]$ with $[m_i] \neq [m_j]$ for $i \neq j$, then $L_1^{\otimes n} \cong \mathcal{O}_S(-\sum_1^b m_i x_i)$.*

PROOF. We study the line bundle L_j in a neighborhood U of a branch point Q. We refer to the notations of example 1 in § 2. Let us drop the index i. Set $d = (m,n)$, $e = n/d$, and $k = m/d$. By definition, $\pi_*(\mathcal{O}_C)(U) = \mathcal{O}_C(\pi^{-1}(U))$. Since π is a covering, we choose $\pi^{-1}(U) = \coprod_j V_j$. Let $\{P_1,\ldots,P_{d-1}\}$ be the fiber over Q, and z_j a local coordinate at P_j with $Z_j(P_j) = 0$ and such that $(P_j,z_j) = \sigma^j(P_0,z_0)$. Take also a local coordinate u at Q such that $u(Q) = 0$ and $u = z_j^e$.

Furthermore, we have $\sigma^{kd}(z_0) = \xi^d(z_0)$, or $\sigma^d(z_0) = \xi^{\nu d} z_0$ if ν denotes the inverse of k modulo e.

Extend each z_j on $\coprod V_k$ by $z_j = 0$ on V_k if $k \neq j$. We have to decompose in proper subspaces the space of holomorphic germs defined on $\coprod V_k$.

We use the notation $\langle ik \rangle$ for ik modulo e.

Recall that the function $X_i = z_0^{\langle ik \rangle} + \xi^{-i} z_1^{\langle ik \rangle} + \cdots + \xi^{-(d-1)i} z_{d-1}^{\langle ik \rangle}$ is a frame of L_i and $(X_i)_{i \in \{0,\ldots,n-1\}}$ is a frame of $\pi_*(\mathcal{O}_C)$.

Then for $i = 1$, we have $X_1 = z_0^k + \xi^{-1} z_1^k + \cdots + \xi^{-(d-1)} z_{d-1}^k$, and $X_1^j = z_0^{jk} + \xi^{-j} z_1^{jk} + \cdots + \xi^{-(d-1)j} z_{d-1}^{jk}$ (since $z_j = 0$ on V_k for $k \neq j$, products $z_k z_j$ are zero).

Set $jk = [jk/e]e + \langle jk \rangle$, with $[jk/e]$ the integral part of jk modulo e; then we obtain $X_1^j = X_j \times u^{[jk/e]}.*)$, and in particular $X_1^n = u^m$. □

Moduli spaces of genus g and b-pointed curves with data $(n, (m_i)_{i \in \{1,\ldots,b\}})$ are defined to be equivalence classes of couples made of a Riemann surface S and of a line bundle L on S satisfying $L^{\otimes n} \cong K(-\sum m_i x_i)$, where K is the canonical line bundle. As a corollary, if the quotient surfaces are punctured torus, then the Hurwitz space $\mathcal{H}_g(\mathbb{Z}/n, R)$ is isomorphic to a moduli space of genus one curves with spin.

But the main point in the striking analogy in any genus between both spaces, for example by restriction to a subgroup, or considering the behavior of their compactification.

To emphasize, we conclude by a remark (without proof) on the analogy between the compactification of Hurwitz spaces described here, and compactifications of moduli space of curves with spins defined in [27] and [17].

The boundary of the compactification of $\mathcal{H}_g(\mathbb{Z}/n, R)$ divides into two parts, depending on locally freeness of a n-th root L of $\mathcal{O}(-\sum m_i x_i)$ at a node. The case where L is locally free (resp. non locally free) at a node q corresponds to the case where every node in the preimage of q by a stable covering has trivial (resp. non trivial) stabilizer.

These parts should be called Ramond–Ramond and Neveu–Schwarz, as in reference [27].

References

[1] W. Abikoff, "Degenerating families of Riemann surfaces", Ann. of Math. (2) **105** (1977), 29–44.

[2] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Princeton, 1966.

[3] M. Bauer and C. Itzykson, "Triangulations", in *The Grothendieck theory of dessins d'enfants*, edited by L. Schneps, London Math. Soc. Lecture Notes Series **200**, Cambridge Univ. Press, Cambridge, 1995.

[4] L. Bers, "Finite dimensional Teichmüller spaces and generalizations", *Bull. Amer. Math. Soc.* **5** (1981), 131–172.

[5] J. Bertin, "Compactification des schémas de Hurwitz", *C. R. Acad. Sci. Paris* (1) **322** (1996), 1063–1066.

[6] R. Biggers and M. Fried, "Moduli space of covers and the Hurwitz monodromy group", *J. Reine Angew. Math.* **335** (1982), 87–121.

[7] C. J. Earle, "On the moduli of closed Riemann surfaces with symmetries", pp. 119–130 in *Advances in the theory of Riemann surfaces* (Stony Brook, 1969), edited by L. V. Ahlfors et al., Annals of Math. Studies **66**, Princeton Univ. Press, Princeton, 1966.

[8] M. Fried, "Fields of definition of function fields and Hurwitz families. Groups as Galois groups", *Comm. in Alg.* **5**:1 (1977), 17–82.

[9] W. Fulton, "Hurwitz schemes and the irreducibility of moduli of algebraic curves", *Ann. Math.* **90** (1969), 542–575.

[10] W. Fulton, *Algebraic topology: a first course*, Graduate Texts in Mathematics **153**, Springer, New York, 1995.

[11] G. González-Díez and W. J. Harvey, "Moduli of Riemann surfaces with symmetry", in *Discrete groups and geometry*, edited by W. J. Harvey, London Math. Soc. Lecture Notes Series **160**, Cambridge Univ. Press, Cambridge, 1993.

[12] D. Gross and W. Taylor, "Two-dimensional QCD is a string theory", *Nucl. Phys. B* **400** (1993), 181–208.

[13] J. L. Harer, "The cohomology of moduli space of curves", in *Theory of moduli*, edited by E. Sernesi, Lecture Notes in Math. **1337**, Springer, Berlin, 1988.

[14] J. L. Harer and D. Zagier, "The Euler characteristic of the moduli space of curves", *Inv. Math.* **85** (1986), 457–485.

[15] W. J. Harvey, "On branch loci in Teichmüller spaces", *Trans. Amer. Math. Soc.* **153** (1971), 387–399.

[16] W. J. Harvey and C. MacLachlan, "On mapping class groups and Teichmüller spaces", *Proc. London Math. Soc.*, **30** (1975), 496–512.

[17] T. J. Jarvis, T. Kimura, A. Vaintrob, "Moduli spaces of higher spin curves and integrable hierarchies", math-ag/9905034.

[18] G. A. Jones and D. Singerman, "Theory of maps on orientable surfaces", *Proc. London Math. Soc.* **37** (1978), 273–307.

[19] A. Klyachko and E. Kurtaran, "Some identities and asymptotics for characters of the symmetric group", *J. Algebra* **206** (1998), 413–437.

[20] M. Kontsevich, "Intersection theory on the moduli space of curves and the matrix Airy function", *Comm. Math. Phys.* **147** (1992), 1–23.

[21] I. K. Kostov, M. Staudacher, and T. Wynter, "Complex matrix models and statistics of branched coverings of 2D surfaces", *Comm. Math. Phys.* **191** (1998), 283–298.

[22] A. Kuribayashi, "On analytic families of compact Riemann surfaces with non trivial automorphisms", *Nagoya Math J.* **28** (1966), 119–165.

[23] M. Mulase and M. Penkava, "Ribbon graphs, quadratic differentials on Riemann surfaces, and algebraic curves defined over $\overline{\mathbb{Q}}$", *Asian J. Math.* **2**:4 (1998), 875–919.

[24] R. C. Penner, "The decorated Teichmüller space of punctured surfaces", *Comm. Math. Phys.* **113** (1987), 299–339.

[25] K. Strebel, *Quadratic differentials*, Springer, Berlin, 1984.

[26] E. Witten, "Two dimensional gravity and intersection theory on moduli space", *Surveys in Diff. Geom.* **1** (1991), 243–310.

[27] E. Witten, "Algebraic geometry associated with matrix models of two dimensional gravity", pp. 235–269 in *Topological methods in modern mathematics: a symposium in honor of John Milnor's sixtieth birthday* (Stony Brook, 1991), edited by L. Goldberg and A. Phillips, Publish or Perish, Houston (TX), 1993.

Michel Imbert
Michel.Imbert@ujf-grenoble.fr
Institut Fourier
Boîte Postale 74
38402 Saint Martin d'Hères
France