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On the Specialization Homomorphism of Fundamental Groups of Curves in Positive Characteristic

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Introduction

Recall that for proper smooth and connected curves of genus $g \geq 2$ over an algebraically closed field of characteristic 0 the structure of the étale fundamental group π_g is well known and depends only on the genus g. Namely it is the profinite completion of the topological fundamental group of a compact orientable topological surface of genus g. In contrast to this, the structure of the étale fundamental group of proper smooth and connected curves of genus $g \geq 2$ in positive characteristic is unknown, and it depends on the isomorphy type of the curve in discussion. The aim of this paper is to give new evidence for anabelian phenomena for proper curves over algebraically closed fields of characteristic $p > 0$.

Before going into the details of the results we are going to prove, we set some notation and recall well known facts. Let k be an algebraically closed field of characteristic $p > 0$. Let X be a projective smooth and connected curve of genus $g \geq 2$ over k, and let J be the Jacobian of X. We denote by $\pi_1(X)$, $\pi_1^p(X)$, and $\pi_1^{p'}(X)$ the étale fundamental group of X, its pro-p quotient, and its prime to p quotient. Then:

(1) The structure of $\pi_1^p(X)$ is given by Shafarevich's Theorem; see [Sh]. It is isomorphic to the pro-p free group on $r := r_X$ generators, where r_X is the p-rank of J.
(2) The structure of $\pi_1^{p'}(X)$ is well known by Grothendieck's Specialization Theorem; see [SGA-1]. It is the prime to p completion of the topological fundamental group of a compact orientable topological surface of genus g.
(3) In contrast to this, the structure of the whole fundamental group $\pi_1(X)$ is a big mystery! Its structure is not known in any single case. However, by Grothendieck's Specialization Theorem we know that $\pi_1(X)$ is the quotient of

the profinite completion Π_g of the topological fundamental group of a compact orientable topological surface of genus g. In particular $\pi_1(X)$ is topologically finitely generated. Since such groups are completely determined by the set of their finite quotients, another interpretation of (1) is the following:

- If two curves as above have the same p-rank, then there is a bijection between the set of their Galois étale covers with Galois group a p-group.
- In the same way, the interpretation of (2) is that for two curves of the same genus there is a bijection between the set of their Galois étale covers having a Galois group of order prime to p.

In order to approach the complexity of π_1 of proper curves in positive characteristic we introduce the following: Let $M_g \to \operatorname{Spec} \mathbb{F}_p$ be the coarse moduli space of proper and smooth curves of genus g in characteristic p. It is well known that M_g is a quasi-projective and geometrically irreducible variety. Let k be an algebraically closed field of characteristic p; thus $M_g(k)$ is the set of isomorphism classes of curves of genus g over k. For $\bar{x} \in M_g(k)$ let $C_{\bar{x}} \to \operatorname{Spec} k$ be a curve classified by $\bar{x}$, and let $x \in M_g$ such that $\bar{x} : \operatorname{Spec} k \to M_g$ factors through x. We set

$$\pi_1(x) := \pi_1(C_{\bar{x}}), \quad \pi_1^p(x) := \pi_1^p(C_{\bar{x}}), \quad \pi_1^{p'}(x) = \pi_1^{p'}(C_{\bar{x}}).$$

We remark that the structure of $\pi_1(x)$ as a profinite group depends only on x and not on the concrete geometric point $\bar{x} \in M_g(k)$ used to define it. Indeed, let κ be the algebraic closure of the residue field $\kappa(x)$ at x in k. Then, if C_x is the curve classified by $\operatorname{Spec} \kappa \to M_g$, then $C_{\bar{x}}$ is the base change $C_{\bar{x}} \simeq C_x \times_\kappa k$ of C_x to k. Hence $\pi_1(C_{\bar{x}}) \simeq \pi_1(C_x)$ by the geometric invariance of the fundamental group for proper varieties; see [SGA-1]. Second, the isomorphy type of C_x as an $\mathbb{F}_p$-scheme does depend only on x, and not the concrete choice of the algebraic closure κ of $\kappa(x)$.

We further remark that by the comments above, if $J_{\bar{x}}$ is the Jacobian of $C_{\bar{x}}$, then the p-rank of $J_{\bar{x}}$ as well as $J_{\bar{x}}$ being a simple abelian variety depends only on x and not on the geometric point $\bar{x}$. Indeed, in the notations above, if J_x is the Jacobian of C_x, then $J_{\bar{x}} \simeq J_x \times_\kappa k$; and for different choices of the algebraic closure of $\kappa(x)$, the corresponding curves are isomorphic as $\mathbb{F}_p$-schemes. Hence their Jacobians too are isomorphic as $\mathbb{F}_p$-schemes.

Coming back to the fundamental group we thus have maps

$$\pi_1 : M_g \to (\text{Prof.groups}), \quad x \to \pi_1(x),$$

and the induced maps

$$\pi_1^p : M_g \to (\text{Prof.groups}), \quad x \to \pi_1^p(x)$$

and

$$\pi_1^{p'} : M_g \to (\text{Prof.groups}), \quad x \to \pi_1^{p'}(x),$$

where (Prof.groups) are the objects of the category of profinite groups. The last two maps are not very interesting: first, the isomorphy type of the images of π_1^p depends only on the p-rank; and second, the isomorphy type is constant on the image of $\pi_1^{p'}$.

To finish our preparation we remark that for points $x, y \in M_g$ such that x is a specialization of y, by Grothendieck's specialization theorem there exists a surjective continuous homomorphism $\mathrm{Sp} : \pi_1(y) \to \pi_1(x)$. In particular, if η is the generic point of M_g, then C_η is the generic curve of genus g; and every point x of M_g is a specialization of η. Thus, for every $x \in M_g$, there is a surjective homomorphism $\mathrm{Sp}_x : \pi_1(\eta) \to \pi_1(x)$ which is determined up to Galois-conjugacy by the choice of the local ring of x in the algebraic closure of $\kappa(\eta)$. For every $x \in M_g$ we fix such a map once for all; in particular, if x is a specialization of y, there exists a specialization homomorphism $\mathrm{Sp}_{y,x} : \pi_1(y) \to \pi_1(x)$ such that $\mathrm{Sp}_{y,x} \circ \mathrm{Sp}_y = \mathrm{Sp}_x$. (In order to obtain $\mathrm{Sp}_{y,x}$ one has to choose the local ring of x to be contained in the local ring of y.)

Finally, let $S^{\text{a.s.}} \subset M_g$ be the set of *closed points* corresponding to curves C_x having an absolutely simple Jacobian J_x. Further, let $S^{\text{a.s.}}_{\geq g-1} \subset S^{\text{a.s.}}$ be the subset of points $x \in S^{\text{a.s.}}$ such that the p-rank of C_x equals g or $g-1$. Concerning the set $S^{\text{a.s.}}$, Chai and Oort proved the following (see [Se-1] for facts concerning Dirichlet density):

THEOREM ([CH-OO]). *The subset $S^{a.s.}$ is non empty and has a positive Dirichlet density. In particular, $S^{\text{a.s.}}$ is Zariski dense.*

We now come to the main results of the present article. We remark that for genus $g = 2$, even stronger results were proven by Raynaud. This is *Raynaud's theory of the theta divisor* of the sheaf of locally exact differentials for curves in positive characteristic; see [Ra-1] the main tool that we use in our approach.

THEOREM A. *For all points $s \in S^{\text{a.s.}}$, the specialization homomorphism $\mathrm{Sp}_s : \pi_1(\eta) \to \pi_1(s)$ is not an isomorphism.*

More precisely, every cyclic étale cover of X_η of order prime to p is ordinary, whereas there exist such covers of C_s that are not ordinary.

THEOREM B. *If a point $y \in M_g$ specializes to some point $s \in S^{\text{a.s.}}_{\geq g-1}$ with $s \neq y$, then the specialization homomorphism $\mathrm{Sp}_{y,s} : \pi_1(y) \to \pi_1(s)$ is not an isomorphism.*

In particular, for a given point $s \in S^{\text{a.s.}}_{\geq g-1}$ there exist only finitely many points $s' \in S^{\text{a.s.}}_{\geq g-1}$ such that $\pi_1(s') \simeq \pi_1(s)$.

As an application we have the following corollary answering a question raised by David Harbater:

COROLLARY. *There is no nonempty open subset $U \subset M_g$ such that the isomorphy type of the geometric fundamental group $\pi_1(x)$ is constant on U.*

We conclude with a question:

QUESTION. *Is it true that the specialization homomorphism*

$$\mathrm{Sp} : \pi_1(y) \to \pi_1(x)$$

to points $y \neq x$ with x closed is never an isomorphism?

If this is the case the same proof as that of Corollary 4.4 below would imply the following finiteness result: Given a closed point $x \in M_g$ there exists at most finitely many closed points x' in M_g such that $\pi_1(x') \simeq \pi_1(x)$.

One could ask the preceding question more generally, without the condition that the point x be closed. However, the condition that x be closed is essential in the proof of our results.

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1. Preliminaries and Notations

1.1. The sheaf of locally exact differentials in characteristic $\mathbf{p > 0}$ and the associated theta divisor. We recall here the definition of the sheaf of locally exact differentials associated to an algebraic curve in positive characteristic and its associated theta divisor, mainly following Raynaud (see [Ra-1], 4). Let X be a proper smooth and connected algebraic curve of genus $g_X := g \geq 2$, over an algebraically closed field k of characteristic $p > 0$. Consider the Cartesian diagram

$$\begin{array}{ccc} X^1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \xrightarrow{F} & \operatorname{Spec} k \end{array}$$

where F denotes the absolute Frobenius morphism. The projection $X^1 \to X$ is a scheme isomorphism, in particular X^1 is a smooth and proper curve of genus g. The absolute Frobenius morphism $F : X \to X$ induces in a canonical way a morphism $\pi : X \to X^1$ called the *relative Frobenius* which is a radicial morphism of k-curves of degree p. The canonical differential $\pi_* d : \pi_* \mathcal{O}_X \to \pi_* \Omega^1_X$ is a morphism of $\mathcal{O}_{X^1}$-modules. Its image $B_X := B := \mathrm{Im}(\pi_* d)$ is the *sheaf of locally exact differentials*. One has the exact sequence

$$0 \to \mathcal{O}_{X^1} \to \pi_* \mathcal{O}_X \to B \to 0,$$

and B is a vector bundle on X^1 of rank $p-1$. Let $c : \pi_*(\Omega^1_X) \to \Omega^1_{X^1}$ be the *Cartier operator*; this is a morphism of $\mathcal{O}_{X^1}$-modules. The kernel $\ker(c)$ of c is equal to B, and the following sequence of $\mathcal{O}_{X^1}$-modules is exact (see [Se], 10):

$$0 \to B \to \pi_*(\Omega^1_X) \to \Omega^1_{X^1} \to 0$$

Let L be a *universal Poincaré bundle* on $X^1 \times_k J^1$ where $J^1 := \mathrm{Pic}^0(X^1)$ is the Jacobian of X^1. This is a line bundle such that its restriction to $X^1 \times \{a\}$ for any $a \in J^1(k)$ is isomorphic to the invertible sheaf $\mathcal{L}_a$ which is the image of a under the natural isomorphism $J^1(k) \simeq \mathrm{Pic}^0(X^1)$. Let $h : X^1 \times J^1 \to X^1$ and $f : X^1 \times J^1 \to J^1$ be the canonical projections. As $R^i f_*(h^*B \otimes L) = 0$ for $i \geq 2$, the total direct image $Rf_*(h^*B \otimes L)$ of $(h^*B \otimes L)$ by f can be realized by a complex $u : \mathcal{M}^0 \to \mathcal{M}^1$ of length 1, where $\mathcal{M}^0$ and $\mathcal{M}^1$ are vector bundles on J^1, $\ker u = R^0 f_*(h^*B \otimes L)$, and $\operatorname{coker} u = R^1 f_*(h^*B \otimes L)$. Moreover as the Euler-Poincaré characteristic $\chi(h^*B \otimes L) = 0$, the vector bundles $\mathcal{M}^0$ and $\mathcal{M}^1$ have the same rank. In [Ra-1], théorème 4.1.1, it has been proved that the determinant $\det u$ of u is not identically zero on J^1, hence one can consider the divisor $\theta := \theta_X$ on J^1, which is the positive Cartier divisor locally generated by $\det u$, it is the *theta divisor* associated to the vector bundle B (note that the definition of θ_X is independant on the above chosen complex u). By definition a point $a \in J^1(k)$ lies on the support of θ if and only if $H^0(X^1, B \otimes \mathcal{L}_a) \neq 0$.

1.2. p-Rank of cyclic étale covers of degree prime to p. We use the same notations as in 1.1. The *p-rank* r_X of X is the dimension of the maximal subspace of $H^1(X, \mathcal{O}_X)$ on which the absolute Frobenius F acts bijectively. By duality it is also the dimension of the maximal subspace of $H^0(X, \Omega^1_X)$ on which the Cartier operator c is bijective (see [Se-1], 10). The p-rank r_X of X is also the rank of the maximal pro-p-quotient $\pi_1^p(X)$ of the fundamental group $\pi_1(X)$ of X, which is a free pro-p-group (see [Sh]).

The relative Frobenius morphism $\pi : X \to X^1$ induces a "canonical" isomorphism $\pi_1(X) \to \pi_1(X^1)$ between fundamental groups (see [SGA-1]). In particular for any positive integer n which is prime to p one has a one to one correspondence between μ_n-torsors of X^1 and μ_n-torsors of X. More precisely the canonical homomorphism $H^1_{\mathrm{et}}(X^1, \mu_n) \to H^1_{\mathrm{et}}(X, \mu_n)$ induced by π is an isomorphism. Consider a μ_n-torsor $f : Y \to X$ with Y connected. By Kummer theory f is given by an invertible sheaf $\mathcal{L}$ of order n on X, and $Y := \mathrm{Spec}(\oplus_{i=0}^{n-1} \mathcal{L}^{\otimes i})$. Thus there exists an invertible sheaf $\mathcal{L}^1$ on X^1 of order n, such that if $f' : Y^1 \to X^1$ is the associated μ_n-torsor we have a Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle \pi'} \downarrow & & {\scriptstyle \pi} \downarrow \\ Y^1 & \xrightarrow{f'} & X^1 \end{array}$$

Let J_Y (resp. J_X) denote the Jacobian variety of Y (resp. the Jacobian of X). The morphism $f : Y \to X$ induces a homomorphism $f^* : J_X \to J_Y$ between Jacobians. Let $J^{\rm new} := J_{Y/X}$ denote the quotient of J_Y by the image $f^*(J_X)$ of J_X, that is the *new part* of the Jacobian J_Y of Y with respect to the morphism f.

1.3. Definition. The μ_n-torsor $f : Y \to X$ is said to be *new-ordinary* if the new part $J^{\rm new}$ of the Jacobian of Y with respect to the morphism f is an ordinary abelian variety.

Since the dimension of the abelian variety $J^{\rm new}$ is $h = g_Y - g_X$, it follows that $J^{\rm new}$ is ordinary if the étale part of the kernel of the multiplication by p in $J^{\rm new}$ has order p^h. This is also equivalent to the fact that the absolute Frobenius F acts bijectively on $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}})$. One has $H^1(J_Y, \mathcal{O}_{J_Y}) \simeq H^1(Y, \mathcal{O}_Y)$, and $H^1(Y, \mathcal{O}_Y) = H^1(X, f^*\mathcal{O}_Y) = H^1(X, \oplus_{i=0}^{n-1} \mathcal{L}^{\otimes i})$. Moreover $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}}) \simeq H^1(X, \oplus_{i=1}^{n-1} (\mathcal{L})^{\otimes i})$ and these identifications are compatible with the action of Frobenius. Hence the kernel of Frobenius on $H^1(J^{\rm new}, \mathcal{O}_{J^{\rm new}})$ is isomorphic to the kernel of Frobenius acting on $H^1(X, \oplus_{i=1}^{n-1} \mathcal{L}^{\otimes i})$. On the other hand as f' is étale $(f')^*(B_X) = B_Y$, thus also $(f')_*(B_Y) = B_X \otimes (f')_*(\mathcal{O}_{Y^1}) = \oplus_{i=0}^{n-1} (B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Now by duality, the kernel of the Frobenius acting on $H^1(X^1, \oplus_{i=1}^{n-1} {\mathcal{L}^1}^{\otimes i})$ is isomorphic to the kernel of the Cartier operator on $H^0(X^1, \pi_* \Omega^1_X \otimes (\oplus_{i=1}^{n-1} (\mathcal{L}^1)^{\otimes i}))$, which is $\oplus_{i=1}^{n-1} H^0(X^1, B_X \otimes (\mathcal{L}^1)^{\otimes i})$. Thus the above μ_n-torsor $f : Y \to X$ is new-ordinary if and only if $H^0(X^1, B \otimes (\mathcal{L}^1)^{\otimes i}) = 0$ for $i \in \{1, \ldots, n-1\}$, which is also equivalent to the fact that the subgroup $\langle \mathcal{L}^1 \rangle$ generated by $\mathcal{L}^1$ in J^1 intersects the support of the theta divisor θ_X associated to B_X at most at the zero point 0_{J^1} of J^1.

2. $\mu_{\mathbf{n}}$-Torsors of Curves over Finite Fields and Ordinariness

In this section we consider curves over the algebraic closure $\overline{\mathbb{F}}_p$ of the prime field $\mathbb{F}_p$. We establish that after finite étale covers the theta divisor associated to the sheaf B of locally exact differentials contains infinitely many torsion points of order prime to p. This indeed gives information on the fundamental group of these curves.

Proposition 2.1. *Let A be an abelian variety of dimension ≥ 2 over $\bar{\mathbb{F}}_p$, and let Y be a closed sub-variety of A of dimension ≥ 1. Assume either A is a simple abelian variety, or $Y(\bar{\mathbb{F}}_p)$ contains the zero point 0_A of A. Then $Y(\bar{\mathbb{F}}_p)$ contains an infinity of torsion points of pairwise prime order.*

Proof. First note that the abelian group $A(\bar{\mathbb{F}}_p) = A(\bar{\mathbb{F}}_p)^{\rm tor}$ is torsion. We will use the following result from [An-In]:

Proposition. *Let C be a proper smooth and connected curve over $\bar{\mathbb{F}}_p$ of genus $g \geq 1$, and let $J := \operatorname{Pic}^0(C)$ be its Jacobian. Let $\phi : C \to J$ be the embedding of C in J associated to a point $x_0 \in C(\bar{\mathbb{F}}_p)$. For any integer m denote by ${}_mJ(\bar{\mathbb{F}}_p)$ the m-primary part of the torsion group $J(\bar{\mathbb{F}}_p)$ (i.e., ${}_mJ(\bar{\mathbb{F}}_p) := \oplus_l ({}_lJ(\bar{\mathbb{F}}_p))$ the* *sum being taken over all primes ℓ dividing m), and let $\lambda : J(\bar{\mathbb{F}}_p) \to_m J(\bar{\mathbb{F}}_p)$ be the canonical projection. Then the map $\lambda \circ \phi : C(\bar{\mathbb{F}}_p) \to_m J(\bar{\mathbb{F}}_p)$ is surjective.*

This was proved in [An-In] only in the case where $m = l$ is a prime number, but it is easy to check that the proof there works also in the case of any positive integer m. It follows immediately from the above result that $C(\bar{\mathbb{F}}_p)$ contains infinitely many points which have pairwise prime orders, in particular it contains infinitely many points of order prime to p. Indeed, If $\{x_1, \dots, x_n\}$ are finitely many points of $C(\bar{\mathbb{F}}_p)$, r is the least common multiple of the orders of the points $\{x_1, \dots, x_n\}$, and if $s > 1$ is an integer which is relatively prime to r, and $x \neq 0$ is an s-torsion point on J, then by the above result one can find a point on $C(\bar{\mathbb{F}}_p)$ whose s primary part equal x and whose r-primary part equals 0, in particular such a point has an order which is prime to r. □

For the proof of 2.1, let $y \in Y(\bar{\mathbb{F}}_p)$ be a closed point in Y and let C be an irreducible sub-scheme of Y of dimension 1 which contains y. We endow C with its reduced structure. Let $\tilde{C}$ be the normalization of C which is a smooth and connected curve of genus ≥ 1, and let $\tilde{J}$ be its Jacobian. One has a commutative diagram:

$$\begin{array}{ccc} \tilde{J} & \xrightarrow{f} & A \\ {\scriptstyle \tilde{\phi}}\uparrow & & \uparrow{\scriptstyle i} \\ \tilde{C} & \xrightarrow{\tilde{f}} & C \end{array}$$

where $\tilde{f}$ is the normalization morphism, $\tilde{\phi}$ is the embedding of $\tilde{C}$ in its Jacobian associated to a point $\tilde{y}$ above y, and f is the morphism induced by the universal property of $\tilde{J}$, which is a composition of a homomorphism g and a translation τ_y by the point y. If y is a point of order prime to p then the image via f of the points of order prime to p on $\tilde{\phi}(\tilde{C})$ (which exists and are an infinity by the above result) yields infinitely many points in $C(\bar{\mathbb{F}}_p)$ which have pairwise prime orders. Moreover if $0_A \in Y(\bar{\mathbb{F}}_p)$ and one takes $y = 0_A$, then with the same notations as above, the images via f of the points of $\tilde{\phi}(\tilde{C})$ having pairwise prime orders yield infinitely many points on C having pairwise prime orders. Assume now that A is a simple abelian variety. Then the above homomorphism g is necessarily surjective, in particular there exists x in $\tilde{J}$ such that $g(x) = y$, and $C = g(\tau_x(\tilde{C}))$, where τ_x denotes the translation by x inside $\tilde{J}$. On the other hand it is easy to see, using the above result in [An-In] in the same way that was used above, that $\tau_x(\tilde{C})$ also contains infinitely many points which have pairwise prime orders in $\tilde{J}$ hence the result in this case.

PROPOSITION/DEFINITION 2.2. *With the same hypothesis as in Proposition* 2.1 *let Y_i be an irreducible component of Y which has dimension ≥ 1, and denote by $A(\bar{\mathbb{F}}_p)^{(p')}$ the prime to p-part of the torsion group $A(\bar{\mathbb{F}}_p)$. Then*:

(1) *either $Y_i(\bar{\mathbb{F}}_p) \cap A(\bar{\mathbb{F}}_p)^{(p')}$ is Zariski dense in Y_i, in which case we call Y_i an abelian like sub-variety of A, or*

(2) $Y_i(\bar{\mathbb{F}}_p) \cap A(\bar{\mathbb{F}}_p)^{(p')}$ *is empty in which case* Y_i *must be a translate of an abelian like sub-variety of* A *by a point which necessarily has order divisible by* p.

PROOF. After eventually a translation we can assume that Y_i contains the zero point of A and then we can assume by 2.1 that $Y_i(\bar{\mathbb{F}}_p) \cap A(\bar{\mathbb{F}}_p)^{(p')}$ is non empty. Assume that the closure Z_i of $Y_i(\bar{\mathbb{F}}_p) \cap A(\bar{\mathbb{F}}_p)^{(p')}$ is distinct from Y_i. Let x be a point in $Y_i(\bar{\mathbb{F}}_p) \cap A(\bar{\mathbb{F}}_p)^{(p')}$ and $y \in Y_i(\bar{\mathbb{F}}_p)$, but y is not contained in Z_i. Then one can find a curve C which contains both y and x (see [Mu], lemma on p. 56). It follows then from the same argument used in the proof of 2.1 that C contains infinitely many points of order prime to p, hence $Y_i - Z_i$ contains such a point which contradicts the fact that $Z_i \neq Y_i$. □

Here is an immediate consequence of these propositions:

PROPOSITION 2.3. *Let* X *be a proper smooth and connected curve over* $\bar{\mathbb{F}}_p$. *Let* θ_X *be the theta divisor associated to the sheaf* B_X *of locally exact differentials on* X (see Section 1.1). *Assume*: *either the Jacobian* J *of* X *is a simple abelian variety*, *or that the curve* X *is not ordinary which is equivalent to the fact that* $0 \in \theta_X(\bar{\mathbb{F}}_p)$. *Then* $\theta_X(\bar{\mathbb{F}}_p)$ *contains infinitely many torsion points of the Jacobian* J^1 *of* X^1 *having pairwise prime orders. In general*, *if* $\theta_X(\bar{\mathbb{F}}_p)$ *contains a torsion point of order prime to* p, *then* $\theta_X(\bar{\mathbb{F}}_p)$ *contains infinitely many torsion points of order prime to* p. *In both cases* θ_X *has an irreducible component which is an abelian like sub-variety of* J^1.

In the general case where the conditions of 2.3 are not satisfied one has the following.

PROPOSITION 2.4. *Let* X *be a proper smooth and connected curve over* $\bar{\mathbb{F}}_p$. *Then there exists an étale Galois cover* $Y \to X$ *with Galois group* G *of order prime to* p *such that the theta divisor* θ_Y *associated to the sheaf of locally exact differentials on* Y *contains infinitely many* $\bar{\mathbb{F}}_p$*-torsion points of pairwise prime order*.

PROOF. By a result of Raynaud (see [Ra-2]) there exists an étale Galois cover $Y \to X$ with Galois group G of order prime to p such that Y is not ordinary. In particular the theta divisor θ_Y associated to the sheaf of locally exact differentials on Y contains the zero point of J^1_Y. Hence the result follows from 2.3. □

3. On the Theta Divisor θ of Curves with Simple Jacobians

THEOREM 3.1. *Let* A *be a* **simple** *abelian variety of dimension* $g \geq 2$ *over an algebraically closed field* K *of characteristic* $p > 0$. *Assume that* A *is not defined over a finite field*, *and that the* p*-rank of* A *equals* g *or* $g-1$. *Let* D *be a closed sub-variety of codimension* ≥ 1 *of* A. *Then* $D(K)$ *contains at most finitely many torsion points of* $A(K)^{\mathrm{tor}}$ *of order prime to* p.

PROOF. Since A is simple, any K-homomorphism from A to an abelian variety is either trivial or an isogeny. In particular, the $\bar{\mathbb{F}}_p$-trace of A is either trivial or isogenous to A in which case the kernel of such an isogeny is automatically defined over a $\bar{\mathbb{F}}_p$ because of the condition on the p-rank of A (see [Oo], 3.4), hence the $\bar{\mathbb{F}}_p$-trace of A equals 0 necessarily, since A is not defined over a finite field by assumption. Let $D^{(p')}$ be the closure in A of the intersection of D with the prime to p-part of the torsion group $J(K)^{\mathrm{tor}}$. By the results of Hrushovski on the analog of the Mordell-Lang conjecture over function fields in positive characteristic $D^{(p')}$ is a finite union $\cup_i(a_i + A_i)$ of translates of abelian subvarieties A_i of A (see [Hr], Corollary 1.2). As A is simple, $\dim A_i = 0$ and hence $D^{(p')}$ consists of at most finitely many points. □

Corollary 3.2. *Let X be a proper smooth and connected curve of genus $g \geq 2$ over an algebraically closed field K of characteristic $p > 0$. Assume that X is not defined over a finite field. Let θ_X be the theta divisor associated to the sheaf of locally exact differentials B_X on X^1 (see Section 1.1). Assume that the Jacobian J of X is a simple abelian variety and that the p-rank of X equals g or $g-1$. Then $\theta_X(K)$ contains at most finitely many torsion points of order prime to p.*

Proof. Since X is not defined over a finite field this is also the case for its Jacobian J By Torelli's theorem [We]; hence 3.2 follows from 3.1. □

4. Proof of Theorem A, Theorem B, and Corollary

We reformulate the assertions of the theorems as follows:

Let x, y be points of M_g with x a specialization of y. Thus the local ring $\mathcal{O}_{M_g,x}$ of the point x contains a prime ideal $\mathcal{P}_y$ corresponding to y, and $\mathcal{O}_{M_g,y}$ is the localization of $\mathcal{O}_{M_g,x}$ at $\mathcal{P}_y$. Let K be an algebraic closure of $\kappa(y)$. Then there exits a valuation ring R of K dominating the factor ring $\mathcal{O}_{M_g,x}/\mathcal{P}_y$ inside $\kappa(y) \subset K$, such that the residue field of R is an algebraic closure κ of $\kappa(x)$. Thus $\bar{y} = \operatorname{Spec} K$ is the generic point, and $\bar{x} = \operatorname{Spec} \kappa$ is the closed point of $\operatorname{Spec} R$. We choose a smooth projective curve $f : X \to \operatorname{Spec} R$ so that we have a morphism $g : \operatorname{Spec} R \to M_g$ such that the induced morphisms $\bar{y} \to M_g$ and $\bar{x} \to M_g$ define the generic fiber $X_{\bar{y}} \to \operatorname{Spec} K$, respectively the special fiber $X_{\bar{x}} \to \operatorname{Spec} \kappa$ as points in $M_g(K)$, respectively $M_g(\kappa)$. We can identify $\pi_1(X_\kappa)$ with $\pi_1(x)$, and $\pi_1(X_K)$ with $\pi_1(y)$ respectively, in such a way that the Grothendieck's specialization homomorphism $\pi_1(X_K) \to \pi_1(X_\kappa)$ is exactly the specialization homomorphism $\operatorname{Sp} : \pi_1(y) \to \pi_1(x)$.

Now we suppose that the points y and x are of a special nature, as in Theorem A and/or Theorem B. This means in particular, that y might be the generic point η of M_g, and x is a point s in $S^{\text{a.s.}}$ or $S^{\text{a.s.}}_{\geq g-1}$. Assuming that $\operatorname{Sp}_{y,x}$ is an isomorphism, we will get a contradiction by showing that the morphism $g : \operatorname{Spec} R \to M_g$ is constant.

Concerning Theorem A. In the above notations, let $x = s$ and $y = \eta$, thus κ is an algebraic closure of the finite field $\kappa(s)$, and K is the algebraic closure of $\kappa(\eta)$. We denote by $J_s = J_\kappa$ the Jacobian of $X_s := X_\kappa$, respectively by $J_\eta = J_K$ the Jacobian of $X_\eta := X_K$. Further let θ_s, respectively θ_η be the theta divisor in $(J_s)^1$ associated to the sheaf of locally exact differentials on X_s, respectively the theta divisor in $(J_{\bar\eta})^1$ associated to the sheaf of locally exact differentials on X_η. It follows from 2.3 that θ_s contains infinitely many torsion points of order prime to p. Let $\mathcal{L}$ be an invertible sheaf of order n prime to p on $X \to S$. Let $\mathcal{L}_\eta$, respectively $\mathcal{L}_s$ be the restriction of $\mathcal{L}$ to X_η, respectively its restriction to X_s. The assumption that $\mathrm{Sp} : \pi_1(X_\eta) \to \pi_1(X_s)$ is an isomorphism implies in particular: The μ_n-torsor associated to $\mathcal{L}_\eta$ is new-ordinary (in the sense of 1.3) if and only if the μ_n-torsor associated to $\mathcal{L}_s$ is new ordinary. In other words: The subgroup $\langle \mathcal{L}_\eta^1 \rangle$ generated by $\mathcal{L}_\eta^1$ intersects the theta divisor θ_η at a non zero point if and only if the subgroup $\langle \mathcal{L}_s^1 \rangle$ generated by $\mathcal{L}_s^1$ intersects the theta divisor θ_s at a non zero point. Hence we deduce from Proposition 2.3, it follows that θ_η contains infinitely many torsion points of J_η of order prime to p. On the other hand, it is well known that all cyclic étale covers $Y \to X_\eta$ of degree n prime to p (and even without this condition) are new-ordinary (see [Na], for instance). This means that the theta divisor θ_η contains no torsion point of order prime to p. Thus a contradiction in this case.

Concerning Theorem B. One proceeds as above, but without using the assumption that y is the generic point of M_g. In the above notations we then have: Let $J \to \operatorname{Spec} R$ be the Jacobian of the projective smooth curve $X \to \operatorname{Spec} R$. Thus $J \to \operatorname{Spec} R$ is an abelian scheme over $\operatorname{Spec} R$, and $J_s = J \times_R \kappa$ is the special fiber of J, and $J_y = J \times_R K$ is the generic fiber of J. Since J_s is a simple abelian variety (by the hypothesis on s), it follows that its generic fiber J_y is simple too. Since f is non iso-trivial, it follows that $X_y := X_K$ is not defined over a finite field. Hence Corollary 3.2 implies that the theta-divisor θ_y of X_y^1 is such that $\theta_y(\lambda)$ contains at most finitely many torsion points of order prime to p. This is a contradiction, so Sp cannot be an isomorphism in this case.

We next prove the second assertion of Theorem B. Let $x \in S^{\text{a.s.}}_{\geq g-1}$ be a closed point of M_g. By contradiction, suppose that there exists infinitely many points $x' \in S^{\text{a.s.}}_{\geq g-1}$ such that $\pi_1(x) \simeq \pi_1(x')$. Let S_x denote the subset of those points, and $\overline{S}_x$ be the closure of S_x in M_g. Then $\overline{S}_x$ is a closed sub-scheme of M_g of dimension $d \geq 1$. Let z be a point of $\overline{S}_x$ which is not a closed point. By hypothesis there exists a point $x' \in S_x$ such that z specializes in x', and hence there exists a continuous surjective homomorphism $\mathrm{Sp} : \pi_1(z) \to \pi_1(x')$. In particular one has an inclusion of sets $\pi_A(x') \subset \pi_A(z)$. On the other hand it is well known that every finite group $G \in \pi_A(z)$ belongs to π_A in an open neighborhood of z (see [St]), and as each such a neighborhood contains a point of S_x one deduces in fact that one has an equality $\pi_A(x') = \pi_A(z)$, and the above homomorphism $\mathrm{Sp} : \pi_1(z) \to \pi_1(x')$ is an isomorphism (this follows from the

Hopfian property for finitely generated profinite groups; see [Fr-Ja], Prop. 15.4). But this can not be the case by the first half of Theorem B since $x' \in S^{\text{a.s.}}_{\geq g-1}$.

Concerning the Introduction's Corollary. We finally come to the proof of the Corollary. First, the fact that the subset $S^{\text{a.s.}}$ of closed points with absolutely simple Jacobian has positive Dirichlet density implies in particular that $S^{\text{a.s.}} \cap U$ is dense in U for every open (nonempty) subset U of M_g (see [Se-2]). Further, since the Jacobian of the generic curve C_η is ordinary, it follows that every curve C_x with $\pi_1(x) \cong \pi_1(\eta)$ is ordinary too. Thus we have: If π_1 is constant on U, then $S^{\text{a.s.}} \cap U = S^{\text{a.s.}}_{\geq g} \cap U$ is dense in U, in particular infinite. This in turn is a contradiction by the second part of Theorem B.

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