The New Classics

More Games of No Chance
MSRI Publications
Volume **42**, 2002

The 4G4G4G4G4 Problems and Solutions

ELWYN BERLEKAMP

Abstract. This paper discusses a chess problem, a checkers problem, a Go problem, a Domineering problem, and the sum of all four of these problems. These challenging problems were originally entitled *Four Games for Gardner* and presented at *Gathering for Gardner, IV*. The solutions of these problems illustrate the power of extended thermography and the notion of rich environments, the relevance and utility of a broad theory of games which may include kos and other loopy positions, and the robustness of this theory to a variety of interpretations of the rules. It also demonstrates the relevance of this branch of mathematics to the classical board games.

Introduction

An enthusiastic group of puzzlers, magicians, and mathematical game buffs held weekend gatherings in Atlanta in January or February of 1993, 1996, 1998, and 2000. These meetings, which honor Martin Gardner, the well-known author and former Scientific American games columnist, are now called "Gatherings for Gardner". A collection of papers presented at the earlier gatherings was published by [Berlekamp and Rodgers 1999]. The problems shown in Figure 1 were presented at the fourth such gathering in February 2000. The problem statement appears in [Rodgers and Wolfe 2001]. The present paper presents solutions to the problems that appeared in "Four Games for Gardner" for Gathering for Gardner IV. The solutions require some background in combinatorial game theory and thermography, topics with which the readers of this volume are assumed to be familiar.

Superficially, in Figure 1 there appears to be one problem in each of four different well-known games: Go, Domineering, checkers and chess. In each of the four games, the reader is to play white (horizontal in domineering). But lurking below the surface is a more interesting and much deeper problem which occurs if we play the sum of all four games added together.

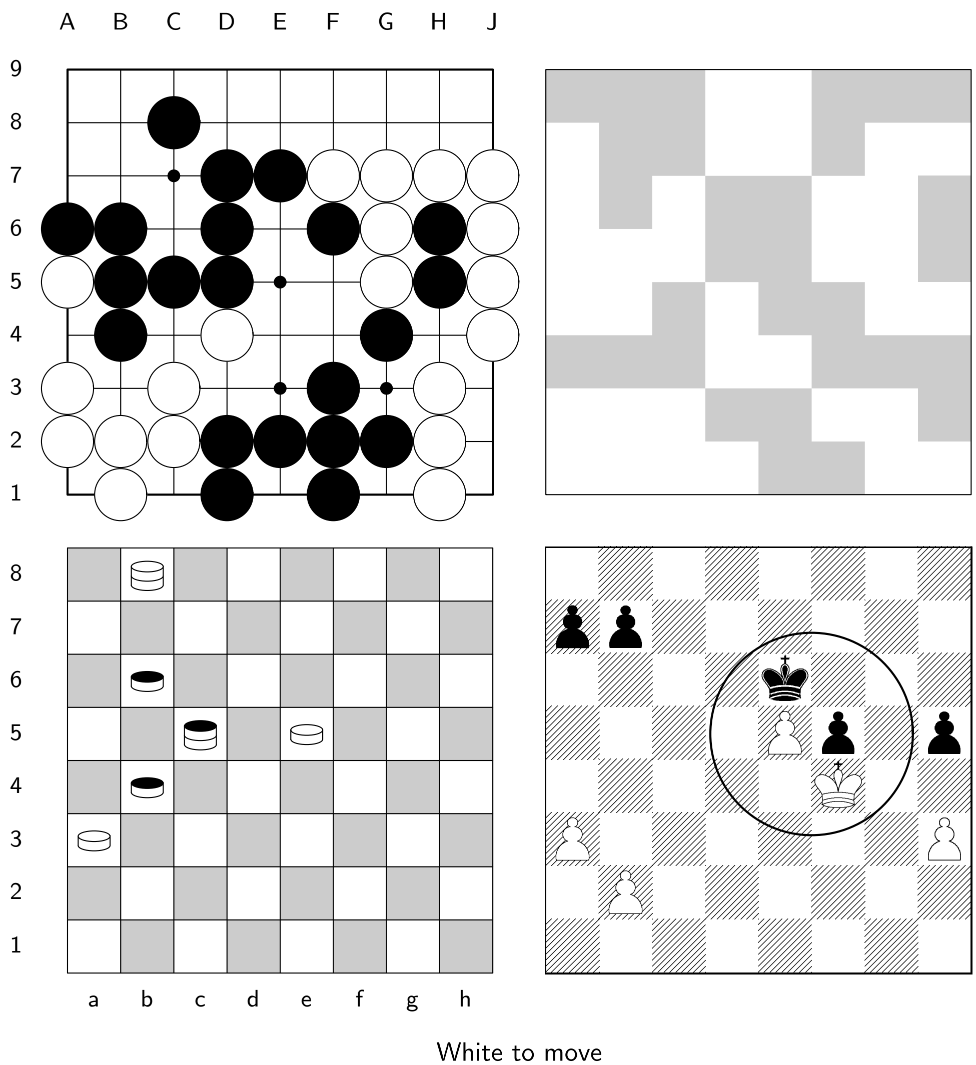


Figure 1. Initial position (in the Domineering board, the shaded areas are out of play).

There may be disputes about how to interpret the rules. Do they matter? In Go, purists might quibble over whether to use the North American, Chinese, or Japanese version of the superko rule. (As in nearly all Go positions, it makes no difference here.) In chess, one might elect either the conventional rules or the simpler rules of "mortal chess", in which the circled kings and pawns are removed from the board and (as in checkers) the game ends when one player is unable to move, at which point his opponent is declared to be the winner. Again, in Figure 1 it makes no difference whether one uses the conventional or the oversimplified rule to determine who wins at chess.

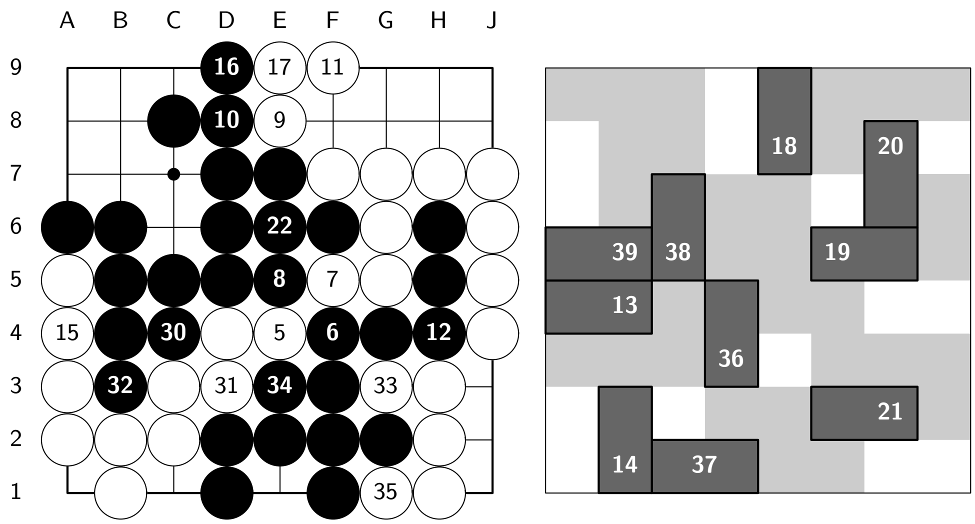
The sum of all four games might also require clarification as to whether certain game-specific rules are to be interpreted locally or globally. In particular:

- The ko rule in Go forbids the recapture of a ko until after another move has been made elsewhere. Must this "elsewhere" be "elsewhere on the Go board", or can one play a chess move as a kothreat?
- The "compulsory capture" rule in checkers forbids noncapturing moves when a capture is available. Does it forbid only noncapturing checkers moves, or moves on all four boards?

Yet another question relates to the overall objective. Do we still seek to checkmate the opposing chess king, or to get the last move as in checkers and Domineering, or to surround the more territory on the Go board? Although Go players are accustomed to passing and then counting score, the American Go Association rules require that a player must pay a one-point fee for each pass. This fee is conventionally paid by returning one of your opponents' stones that you have captured to the pot. After both players have made such passes, they might very well elect to count score by alternately filling in their own territories to see who runs out of moves first. Thus, playing Go with the last-move-wins rule often has no effect on the players' strategies.

In Figure 1, it turns out that *none* of these legalistic questions really matters, because there is a uniform solution. White has a strategy to win the sum no matter which interpretation of the rules one's opponent might select. To simplify the discussion, we assume that the overall goal is to get the last legal move. However, this winning strategy is rather difficult to find without using a substantial body of knowledge about combinatorial game theory.

The notion of temperature turns out to be the key concept underlying all known analyses of problems of this type. Heuristically, the temperature of a move is a measure of the average amount one would gain by playing that move in an asymptotically large game. Rudimentary notions of temperature can be detected in [Milnor 1953; Hanner 1959]. The modern version for loopfree games first appeared in [Conway 1976] and *Winning Ways* [Berlekamp et al. 1982]. Extensions needed to handle kos appeared in [Berlekamp 1996]. Refinements and reformulations continue.



	CHECKERS				CHESS						
White	1:e5 d6		3:a3 a7		23:a4		25:a5		27:a6		29:b3
Black		2:c5 e7		4:e7 d6		24:h4		26:b5		28:b4	

Board	Chk	Go	Go	Go	Do	Do	Go	Do	Do	Do	Go	Chs	Go	Do	Do	Do
Moves	1–4	5–8	9–11	12	13	14	15–17	18	19	20–21	22	23–29	30–35	36–37	38	39
t	∞	>19	3	2	$\frac{9}{8}$	1	1	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{2}$

Figure 2. One orthodox line of play.

The present problem is intended to encourage comparisons of temperatures among different games.

As expounded in [Berlekamp 1996], temperatures provide the basis of a strategy called *sentestrat*, which is optimum when the number of regions is asymptotically large. Any line of play consistent with sentestrat is called *orthodox*. *Orthodox accounting* provides a method for analyzing other lines of play. It turns out that perfect play differs from orthodoxy only very rarely, and then usually only by making very small "sacrifices" that turn out to be investments that yield quick returns.

One orthodox line of play to the problem of Figure 1 is shown in Figure 2.

Although the line of play shown in Figure 2 is not perfect, it provides a baseline from which we are able to analyze the combined 4-board problem. But, let us first consider the four isolated warmup problems individually. We claim that in isolation, White should draw Checkers, and win at each of Go, Chess, and Domineering. When all four boards are treated as a single game played together, we claim that White can win, no matter how one chooses to interpret the rules.

Isolated Checkers

When the checkers game is played alone, White's best first move is from e5 to f6. This leads to a draw. If instead he plays as indicated in the caption below Figure 2, he loses the checkers game in only four moves. However, this position has value 0, which is unusual in checkers. If either player moves again, his opponent will play to a position in which he has an unlimited number of extra moves. In the spirit of Chapter 11 of *Winning Ways*, such a position is called on or off, depending on whether Black or White enjoys the unlimited advantage. (Actually, the on and off used here are bifurcations of the **on** and **off** in *Winning Ways*. Our on allows Left to move back and forth between two states, while Right has no moves. This models a lone checkers king who chooses to remain in a double corner. It also evades any conflict with the ko-ban rule.)

If White opens by playing the checker at e5 to f6 instead of d6, the position eventually becomes a Deathless Universal Draw (dud). The generic dud is the checkers position in which each side has a king located in a double corner. Since the dud always provides each an opportunity to move, neither player can win a game that contains a dud, no matter how strong his position might be elsewhere.

So, if White thinks he can win the sum of the Go, Domineering, and chess positions, then he should play the checkers position to 0 via the first four moves of the line of play depicted in Figure 2. If instead he thinks himself unable to win the sum of those three game positions, then he should open by playing the checkers position to a dud. This decision is more important than any finite number of extra moves that might be acquired by any imaginable capture on the Go board. The temperature of the checkers position is truly infinite.

Isolated Go

The Go moves shown in Figure 2, numbered 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 17, and 22, happen to alternate between White moves and Black moves. This also happens to be the correct sequence for the optimum line of play when the Go board is played in isolation.

Before move 5, the temperature is about 20 — to be precise, $19\frac{5}{12}$. Except for the Black stone at B4, none of the other stones located in rows 1–4 and columns A–G is yet safely connected to life. If either player can make a connection from his lower group through the central region around EF4–5 to his upper group, then he will kill his opponent's lower group and save his own. For example, if Black could play F5 in Figure 1, he would achieve this objective immediately, and all of White stones in the southwestern portion of the board would die.

This big issue is resolved in moves 5–8. Neither side is able to connect, but each side does succeed in preventing his opponent from connecting. The two opposing one-eyed groups are now destined to live, in seki. After move 8, the temperature drops dramatically, from about 20 to exactly 3.

(Alternative attempts to resolve this big issue prove less desirable. Thus, White's 5:F5?? is followed by 6:E4, 7:E5, 8:D3, 9:C4, 10:F4, so Black lives and kills White. If White's 5:E4 is met with Black's 6:F5?, we have 7:F4, 8:G3, 9:H4 ($t > 19$ again), 10:A4 (now $t = 3$), eventually leading to White at E5 and Black at E6; thus Black's small mistake at F5 has cost him $\frac{2}{3}$ point.)

Moves 9–11, at temperature 3, outline the boundary that divides the Black area in the northwest from the White area in the northeast. Move 12, at temperature 2, determines the fate of the two stones at H5–6. After move 12, the temperature of the Go board drops to 1, and after some plays elsewhere (moves 13–14), Go moves 15–17 are played at temperature 1. The node at E6 then has temperature $\frac{1}{3}$. After some more plays elsewhere reduce the ambient temperature to this level, Black 22:E6 reduces the temperature of the Go board to 0. Only dame remain. These are filled at moves 30–35. In this case, each of the Black plays at moves 30, 32, and 34 poses a grave threat that White wisely decides to answer. For example, if White elected to play either 33 or 35 elsewhere, then Black could continue with 36:C1, and then capture eleven Black stones with 38:A1. But this is prevented by White's 33 and 35, after which Black 36:C1? would be followed by White E1, capturing fourteen Black stones.

If this game were actually played in isolation, then after Black plays the stone at 22, White would begin playing at 31, 33, 35 while Black would play 30, 32, 34. Although the order in which these *dame* points are filled might differ, the result would be the same as shown in Figure 2.

After move 35 as shown in Figure 2, the empty points at ACE1 are *seki:* Neither player can play there without being subjected to a large loss. The other empty squares are all territory. Black has 9 points at ABC9, AB8, ABC7, and C6, but White has 10 points at GHJ9, FGHJ8, and J321. So White wins by one point. When the Go board is played in isolation, Black cannot do anything to prevent White from winning in this way.

We would obtain the same result if we insisted on continuing to play until the loser is unable to move. After each player has placed 7 stones in appropriate places in his own territory, only the seki and a few isolated empty nodes, called eyes, will remain. Playing into the opponent's territory only prolongs the game, because the opponent will eventually capture such stones. Following AGA rules, he will subsequently use up one turn returning each of them to the pot. We will eventually reach a position in which White has three eyes, and Black two. White then fills one of his eyes, and then Black's only remaining legal moves are disastrous. If he still refuses to resign, he must either fill in one of his own last two eyes, or play in the seki at nodes C1 or E1. No matter what he does, White will capture many Black stones on the next move. When play continues, White will fill in his new territory except for a few more eyes. Eventually the board will contain nothing but White stones and White eyes, and then Black will finally have no legal moves at all, because playing a suicidal move into an opponent's small eye is illegal in all dialects of Go rules.

Isolated Chess

All of the chess moves that occur in the line of play depicted in Figure 2 occur at an ambient temperature of 0. This means that the temperature of the entire four-board position is zero when the first chess move is made, and it is again zero when the last chess move is completed. The chess position (including the inactive kings and pawns shown in the circle in Figure 1) is among those analyzed mathematically by [Elkies 1996]. The line of play shown is canonical. Black cannot do anything to prevent White from winning in this way.

Isolated Domineering

The Domineering position shown in Figure 1 is the sum of several disjoint pieces, having canonical values $3 \mid \frac{3}{4}$, $1{*} \mid -1{*}$, $1 \mid -1$, $0 \parallel -1 \mid -2$, $0 \mid -1$, and $*$, with respective temperatures $\frac{9}{8}$, 1, 1, $\frac{3}{4}$, $\frac{1}{2}$, and 0. Black (Left) is vertical; White (Right) is horizontal. An orthodox strategy begins playing the regions in order of decreasing temperature, creating smaller regions with values $\frac{3}{4}$, $1{*}$, -1, 0, -1, and 0, respectively. Play will then continue. At $t = 0$, Right will play $1{*}$ to 1, and at $t = -\frac{1}{4}$, Left will play $\frac{3}{4}$ to $\frac{1}{2}$. At $t = -\frac{1}{2}$, Right will play $\frac{1}{2}$ to 1. All regions will then have values that are integers, and the sum of these values is 0. However, it is Left's turn to move, so Right wins.

To prove that Left cannot stop Right's win, we first observe that the values of the initial regions are all switches except for $1{*} \mid -1{*}$ (which is equal to the sum of two switches), and $0 \parallel -1 \mid -2$, whose incentives are confused only with the incentives of $1 \mid -1$ and $0 \mid -1$. So the only region which a canonical Left might possibly want to play out of its temperature-defined order is $0 \parallel -1 \mid -2$, and a simple calculation shows that this does her no good.

The Global Problem

We first notice that the line of play shown in Figure 2 led to a win for White. After move 38, only integers remain. White is one point ahead on the Go board $(9 - 10 = -1)$, but one point behind on the Domineering board $(3 - 2 = +1)$. However, Black's turn is next, so she will run out of moves just before White does.

However, Black made a fatal mistake at move 22. Although that play has temperature $\frac{1}{3}$, which is indeed hotter than any other move then available, Black should not fill this ko as long as she has enough kothreats remaining to ensure that she can win it. Black has several kothreats on the Go board. The sequence of moves later played at 30–35 might serve as Black kothreats, and after that Black has another threat at J3. Black also has two threats at the top of the board: F8 and then G9. And we shall find that Black also has one kothreat on the chessboard! So Black can act as komonster, attempting to leave the ko unfilled until all moves at $t \geq 0$ have been played. In this kofight, Black's goal should be to force White to play a move with negative temperature (such as at 38 in Figure 2), before Black fills the ko.

White, on the other hand, has no need to win this kofight. His goal is merely to force Black to fill the ko before the ambient temperature of the game goes negative. For this purpose, *any* White move that has nonnegative temperature can serve as a kothreat. That includes almost all White moves on the chessboard, two moves in Domineering (at plays numbered 36–37 in Figure 2), and all of the dame on the Go board at which White played during the sequence of moves numbered 30–35 in Figure 2.

Black achieves her goal relatively easily if the ko-ban rule is interpreted locally, because the Chess and Domineering boards contain many more kothreats for White than for Black. But a global interpretation of the ko-ban rule makes Black's problem much more challenging.

In general, some localities provide kothreats for one player or the other, and some offer "two-sided" kothreats that might be played by either player. Good Go strategy requires one to play out all two-sided kothreats *before* starting the ko, in order to prevent one's opponent from using them as kothreats. In the present problem, instead of filling the ko at move 22, Black could play the forcing sequence from 30–35. Detailed infinitesimal values of the various positions aren't very important now, because the bigger consideration is simply the number of moves during which the ambient temperature of the game remains at 0. White would like to drag this out. Black hopes to get this stage of the play (including all moves on chess board) over with while she still has one or more kothreats available on the Go board.

In the corrected line of play, moves 22–29 are skipped, and we will then continue the old numbering from the continuation at 30–35. Then, a good opening chess move for Black is 36:a5! The queenside of the chessboard is a two-sided kothreat. This Black move to a5 heats it up to temperature 1, a value of $2 \mid 0$ to which White must respond. His best response is 37:b3, and Black can then continue with 38:b5. The combinatorial game theorist with no Go experience will surely be surprised at this line of queenside play. From a canonical perspective, White 37:b3 is dominated by White 37:a4, yielding an immediate value of $1 - 1 = 0$. But, because of the kofight, suspense and remoteness are now more important than canonical values, although temperatures remain very important. After Black 38:b5, the canonical value of the queenside is 0, but, until the kofight is concluded, it instead behaves like $*$. *Either* player can play here, and his opponent will then find it desirable to exchange pawn captures rather than to push.

White is then at last free to start the kofight at E6, and the game continues on the chess and Go, and Do (Domineering) boards like this:

Move	39	40	41[1]	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56
White	E6		J2		h4		a3b4		ko		G8		Do		ko		H9	
Black		J3		ko		b4		a5b4		F8		ko		Do		G9		ko

The two zero-temperature moves on the Domineering board were taken at moves 51 and 52, and so at move 57, the hottest move available for White is the Domineering move at temperature $-\frac{1}{4}$. White plays there at move 57, and Black then plays 58:E6, filling the ko and leaving only moves of temperature -1. The score is tied, but White must now play next and lose.

So the line of play shown in Figure 2 is incorrect; Black could have won by adopting a different strategy at move 22.

However, any gurus reading this paper will notice that several other errors were committed before move 22. Moves 16 and 17 were both fatal mistakes, which, against a perfect opponent would have made the difference between winning and losing. Because of the kothreat potential, both players should refrain from playing in the vicinity of DE9 until after all other moves at temperature 1 have been taken, including the domino number 18. But the first fatal mistake was committed even earlier when White played the hanging connection at 11:F9. If he instead plays 11:F8, he denies Black the two kothreats at F8 and G9, and thereby wins the game.

The diligent reader is invited to confirm this assertion, which is not as obvious as it might seem to experienced Go players, most of whom intuitively prefer the solid connection to the hanging connection precisely because the ensuing position gives the opponent fewer kothreats. But other factors can assume more relevance. In particular, the hanging connection at F9 enables White to get the last move at temperature 1; the solid connection allows Black to get the last such move. As explained in detail by [Berlekamp and Wolfe 1994], this can be the decisive consideration, especially when there are no other moves of temperatures between 1 and 0. For example, in the isolated Go problem of Figure 1, the hanging connection wins and the solid connection loses, even though the ko configuration at $t = \frac{1}{3}$ is still present. In this case, the ko turns out to be less important than getting the last move at $t = 1$. However, when the same Go position of Figure 1 is played in the richer environment containing the Domineering positions, which have temperatures $\frac{3}{4}$, $\frac{1}{2}$, 0, and $-\frac{1}{4}$, then getting the last move at temperature 1 provides no advantage. What matters in this case is whether or not the master of the ko at $t = \frac{1}{3}$ can prolong the kofight until the ambient temperature becomes subzero.

[1] The novice Go player who ignores Black 40:J3 falls victim to 41:E6??, 42:J2, 43:J1, 44:J3, 45:J2, and Black 46:J3 captures 7 stones!

The Isolated Go problem and the combined problem both have more than one solution. After Black 10, White has three plausible choices. 11:F8 forms a "solid connection"; 11:F9 forms a "hanging connection", or 11:E9 "descends" to the edge of the board. Any of these three moves would conclude play at $t = 3$. After the descent 11:E9, Black is able to resume playing in this region at a higher temperature than Black, via the following sequence:

Black	F8		D9	
White		G8		F9

Thus, White's three choices at move 11 yield positions with the following properties at temperature 1:

Choice	chilled value	number of Black kothreats at $t < 1$
Solid	$*$	0
Descent	**miny**	1
Hanging	**miny** $\mid 0$	2 (or 0)

So the merits of the descent are seen to lie somewhere between those of the solid connection and the hanging connection. The solid connection wins in the isolated Go problem, but not in the combined problem. The hanging connection wins in the combined problem, but not in the isolated Go problem. Surprisingly enough, the descent happens to win in both problems.

Conclusion

The 4G4G problems illustrate several general points:

(i) When the Go position is played in isolation, the hanging connection dominates because, when chilled, it has the best atomic weight. Playing there ensures getting the last move at temperature 1, which is decisive.

(ii) In the sum of all four games, the environment is sufficiently rich that getting the last move at temperature 1 does not matter. Getting more kothreats is now more important.

(iii) As Elkies [1996] has shown, well-known infinitesimals like $\uparrow$ and $*$ can occur in chess.

(iv) **On**, **off**, and **dud** can occur in some very simple checkers positions.

(v) Temperature-based orthodox analysis is a very powerful tool for analyzing many kinds of combinatorial games.

(vi) Several different analytic methods can all be usefully applied to the same problem. A unified theory encompassing many of these methods appears in [Berlekamp 2002].

Acknowledgements

I am indebted to Bill Spight and Gin Hor Chan for help in composing and debugging these problems, and to Silvio Levy for assistance in the preparation of this paper.

References

[Berlekamp 1996] E. Berlekamp, "The economist's view of combinatorial games", pp. 365–405 in *Games of no chance* (Berkeley, 1994), edited by R. Nowakowski, Math. Sci. Res. Publ. **29**, Cambridge Univ. Press, Cambridge, 1996.

[Berlekamp 2002] E. Berlekamp, "Idempotents among partisan games", in *More games of no chance*, edited by R. Nowakowski, Math. Sci. Res. Publ. **42**, Cambridge Univ. Press, Cambridge, 2002.

[Berlekamp and Rodgers 1999] E. Berlekamp and T. Rodgers (editors), *The mathemagician and pied puzzler: A collection in tribute to Martin Gardner*, edited by E. Berlekamp and T. Rodgers, A K Peters Ltd., Natick (MA), 1999.

[Berlekamp and Wolfe 1994] E. Berlekamp and D. Wolfe, *Mathematical Go: Chilling gets the last point*, A K Peters, Wellesley (MA), 1994.

[Berlekamp et al. 1982] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning ways for your mathematical plays*, Academic Press, London, 1982. Second edition, A K Peters, Natick (MA), 2001.

[Conway 1976] J. H. Conway, *On numbers and games*, London Math. Soc. Monographs **6**, Academic Press, London, 1976. Reprinted by A K Peters, Natick (MA), 2001.

[Elkies 1996] N. Elkies, "On numbers and endgames: combinatorial game theory in chess endgames", pp. 135–150 in *Games of no chance* (Berkeley, 1994), edited by R. Nowakowski, Math. Sci. Res. Publ. **29**, Cambridge Univ. Press, Cambridge, 1996.

[Hanner 1959] O. Hanner, "Mean play of sums of positional games", *Pacific J. Math* **9** (1959), 81–99.

[Milnor 1953] J. Milnor, "Sums of positional games", pp. 291–301 in *Contributions to the theory of games*, edited by H. W. Kuhn and A. W. Tucker, Ann. Math. Studies **28**, Princeton Univ. Press, 1953.

[Rodgers and Wolfe 2001] T. Rodgers and D. S. Wolfe (editors), *A puzzler's tribute*, edited by T. Rodgers and D. S. Wolfe, A K Peters Ltd., Natick (MA), 2001.

Elwyn Berlekamp
Mathematics Department
University of California
Berkeley, CA 94720-3840
berlek@math.berkeley.edu