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Two-player Games on Cellular Automata

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Abstract. Cellular automata games have traditionally been 0-player or solitaire games. We define a two-player cellular automata game played on a finite cyclic digraph $G = (V, E)$. Each vertex assumes a weight $w \in \{0, 1\}$. A move consists of selecting a vertex u with $w(u) = 1$ and *firing* it, i.e., complementing its weight and that of a *selected* neighborhood of u. The player first making all weights 0 wins, and the opponent loses. If there is no last move, the outcome is a draw. The main part of the paper consists of constructing a strategy. The 3-fold motivation for exploring these games stems from complexity considerations in combinatorial game theory, extending the hitherto $\leq$ 1-player cellular automata games to two-player games, and the theory of linear error correcting codes.

1. Introduction

Cellular Automata Games have traditionally been 0-player games such as Conway's *Life*, or solitaire games played on a grid or digraph $G = (V, E)$. (This includes undirected graphs, since every undirected edge $\{u, v\}$ can be interpreted as the pair of directed edges (u, v) and (v, u).) Each cell or vertex of the graph can assume a finite number of possible *states*. The set of all states is the *alphabet*. We restrict attention to the binary alphabet $\{0, 1\}$. A *position* is an assignment of states to all the vertices. There is a local transition rule from one position to another: pick a vertex u and *fire* it, i.e., complement it together with its neighborhood $F(u) = \{v \in V : (u, v) \in E\}$. The aim is to move from a given position (such as all 1s) to a target position (such as all 0s). In many of these games *any* order of the moves produces the same result, so the outcome depends on the set of moves, not on the sequence of moves. Two commercial manifestations are *Lights Out* manufactured by Tiger Electronics, and *Merlin Magic Square* by Parker Brothers (but Arthur–Merlin games are something else again). Quite a bit is known about such solitaire games. Background and theory can be found

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e.g., in [7], [20], [25], [26], [27], [29], [31], [32], [33], [34], [35], [36]. Incidentally, related but different solitaires are *chip firing games*, see e.g., [3], [24], [2].

What seems to be new is to extend such solitaire games to two-player games, where the player first achieving 0s on all the vertices wins and the opponent loses. If there is no last move, the outcome is a draw. In this context it seems best to restrict the players to firing only a vertex in state 1.

Specifically, we play a two-player cellular automata game on a finite cyclic digraph G with an initial distribution of *weights* $w \in \{0,1\}$ on the vertices. For the purposes of the present paper it is convenient to agree that a digraph is cyclic if it *may* contain cycles, but no loops. We put $w(u) = 1$ if u is in state 1, otherwise $w(u) = 0$. The two players alternate in selecting a vertex u with $w(u) = 1$ and *firing* it, i.e., "complementing" it together with a selected neighborhood $N(u)$ of vertices. By complementing we mean that $w(u)$ switches to 0, and $w(v)$ reverses its parity for every vertex $v \in N(u)$. The player making the last move wins (after which all vertices have weight 0), and the opponent loses. If there is no last move, i.e., there is always a vertex with weight 1, the outcome is a draw. A precise definition of the games is given in § 3.

Our aim is to provide a strategy for two-player cellular automata games. The game graph of cellular automata games is exponentially large. For the special case where $N(u)$ is restricted to a single vertex, we can provide a polynomial strategy. For small digraphs, even some cases of large digraphs, as we shall see, the "γ-function" (defined below) can be found by inspection for any fixed size of $N(u)$, leading to an optimal strategy.

2. Preliminaries

For achieving our aim, we need to compute the generalized Sprague–Grundy function γ polynomially on the game graph $\mathbf{G}$ induced by G. The γ function has been defined in [30]. See also [4], Ch. 11. The following simplified definition appears in [18] Definition 1, see also [11] Sect. 3.

Given a cyclic digraph $G = (V, E)$. The set $F(u)$ of *followers* of $u \in V$ is defined by $F(u) = \{v \in V : (u, v) \in E\}$. If $F(u) = ?$, then u is a *leaf*. The *Generalized Sprague–Grundy function* γ is a mapping $\gamma: V \to \mathsf{Z}^0 \cup \{\infty\}$, where the symbol ∞ indicates a value larger than any natural number. If $\gamma(u) = \infty$, we also say that $\gamma(u)$ is infinite. We wish to define γ also on certain subsets of vertices. Specifically: $\gamma\bigl(F(u)\bigr) = \bigl\{\gamma(v) < \infty : v \in F(u)\bigr\}$. If $\gamma(u) = \infty$ and if we denote the set $\gamma\bigl(F(u)\bigr)$ by K for brevity, then we also write $\gamma(u) = \infty(K)$. Next we define equality of $\gamma(u)$ and $\gamma(v)$: if $\gamma(u) = k$ and $\gamma(v) = \ell$ then $\gamma(u) = \gamma(v)$ if one of the following holds: (a) $k = \ell < \infty$; (b) $k = \infty(K)$, $\ell = \infty(L)$ and $K = L$. We also use the notations

$$V_i = \{u \in V : \gamma(u) = i\} \ (i \in \mathsf{Z}^0), \quad V^f = \bigl\{u \in V : \gamma(u) < \infty\bigr\}, \quad V^\infty = V \setminus V^f,$$

$$\gamma'(u) = \operatorname{mex} \gamma\big(F(u)\big) \ ,$$

where for any subset $S \subset \mathsf{Z}_{\geq 0}$, $S \neq \mathsf{Z}_{\geq 0}$, we define $\operatorname{mex} S := \min(\mathsf{Z}_{\geq 0} \setminus S) =$ least nonnegative integer not in S.

We need some device to tell the winner where to move to. This device is a counter function, as used in the following definition. For realizing an optimal strategy, we will normally select a follower of least counter function value with specified γ-value. If only local information is available, or the subgraph is embedded in a larger one, we may not know to which seemingly optimal follower to move. The counter function is the guide in these cases. We remark that it also enables one to prove assertions by induction.

Definition 1. Given a cyclic digraph $G = (V, E)$. A function $\gamma: V \to \mathsf{Z}^0 \cup \{\infty\}$ is a γ-*function* with *counter function* $c: V^f \to J$, where J is any infinite well-ordered set, if the following three conditions hold:

A. If $\gamma(u) < \infty$, then $\gamma(u) = \gamma'(u)$.
B. If there exists $v \in F(u)$ with $\gamma(v) > \gamma(u)$, then there exists $w \in F(v)$ satisfying $\gamma(w) = \gamma(u)$ and $c(w) < c(u)$.
C. If $\gamma(u) = \infty$, then there is $v \in F(u)$ with $\gamma(v) = \infty(K)$ such that $\gamma'(u) \notin K$.

Remarks.

- In **B** we have necessarily $u \in V^f$; and we may have $\gamma(v) = \infty$ as in **C**.
- To make condition **C** more accessible, we state it also in the following equivalent form:

  C'. If for every $v \in F(u)$ with $\gamma(v) = \infty$ there is $w \in F(v)$ with $\gamma(w) = \gamma'(u)$, then $\gamma(u) < \infty$.

- If condition **C'** is satisfied, then $\gamma(u) < \infty$, and so by **A**, $\gamma(w) = \gamma'(u) = \gamma(u)$.
- To keep the notation simple, we write $\infty(0)$, $\infty(1)$, $\infty(0,1)$ etc., for $\infty(\{0\})$, $\infty(\{1\})$, $\infty(\{0,1\})$, etc.
- γ exists uniquely on every finite cyclic digraph.

We next formulate an algorithm for computing γ. Initially a special symbol ν is attached to the label $\ell(u)$ of every vertex u, where $\ell(u) = \nu$ means that u has no label. We also introduce the notation $V_\nu = \{u \in V : \ell(u) = \nu\}$.

Algorithm GSG for computing the Generalized Sprague–Grundy function for a given finite cyclic digraph $G = (V, E)$.

1. (Initialize labels and counter.) Put $i \leftarrow 0$, $m \leftarrow 0$, $\ell(u) \leftarrow \nu$ for all $u \in V$.
2. (Label and counter.) As long as there exists $u \in V_\nu$ such that no follower of u is labeled i and every follower of u which is either unlabeled or labeled ∞ has a follower labeled i, put $\ell(u) \leftarrow i$, $c(u) \leftarrow m$, $m \leftarrow m + 1$.
3. (∞-label.) For every $u \in V_\nu$ which has no follower labeled i, put $\ell(u) \leftarrow \infty$.
4. (Increase label.) If $V_\nu \neq$ **?** , put $i \leftarrow i + 1$ and return to 2; otherwise end.

We then have $\gamma(u) = \ell(u)$. A realization of Algorithm GSG performs a depth-first (endorder) traversal of the digraph for each finite ℓ-value. Letting $\gamma_{\max} = \max\{\gamma(u): u \in V^f\}$, we evidently have $\gamma_{\max} < |V|$. Hence the number of steps of the algorithm is bounded by $O\big((|V|+|E|)|V|\big)$. For a connected digraph the complexity of the entire algorithm is thus $O(|V||E|)$.

Informally, the *sum* of a finite collection of games is a game in which a move consists of selecting one of the component games and making a move in it. Formally, let $\{\Gamma_1, \dots, \Gamma_m\}$ be a finite disjoint collection of games with game-graphs $\big\{G_1 = (V_1, E_1), \dots, G_m = (V_m, E_m)\big\}$, which may have cycles or may be infinite. Then the *sum-game* $\Gamma = \Gamma_1 + \ldots + \Gamma_m$ is the 2-player game in which a position has the form $(u_1, \dots, u_m)$ with $u_i \in V_i$ for all i, and a move consists of selecting some Γ_i and making a legal move $u_i \to v_i$ in it ($(u_i, v_i) \in E_i$).

The *sum-graph* $G = G_1 + \ldots + G_m$ is the digraph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ defined as follows:

$$\mathbf{V} = \big\{(u_1, \dots, u_m):\ u_i \in V_i \quad i \in \{1, \dots, m\}\big\}.$$

If $\mathbf{u} = (u_1, \dots, u_m)$, $\mathbf{v} = (v_1, \dots, v_m) \in \mathbf{V}$, then $(\mathbf{u}, \mathbf{v}) \in \mathbf{E}$ if there is some $j \in \{1, \dots, m\}$ such that $v_j \in F(u_j)$, that is, $(u_j, v_j) \in E_j$, and $u_i = v_i$ for all $i \neq j$.

The *generalized Nim sum* $\oplus$ of a finite number of nonnegative integers is their sum over $\mathbf{GF}(2)$, also called *exclusive or* (XOR). Further, for any nonnegative integer a and subsets $K, L \subseteq \mathsf{Z}^0$ we have $a \oplus \infty(K) = \infty(K) \oplus a = \infty(K \oplus a)$, and $\infty(K) \oplus \infty(L) = \infty(\mathbf{?}\,)$.

Notation. The generalized Nim sum is denoted by $\sum'$. Thus, ${\sum'}_{i=1}^{h} u_i$ is the generalized Nim sum of $u_1, \dots, u_h$.

The important observation is that if $\mathbf{u} = (u_1, \dots, u_m) \in \mathbf{V}$ is any position in a game graph, then $\gamma(\mathbf{u}) = {\sum'}_{i=1}^{m} \gamma(u_i)$ (see [18], Theorem 5). It enables one to compute polynomially the strategy on the exponentially large sum graph for a special case of our games.

Informally, a *P-position* is any position u from which the *P*revious player can force a win, that is, the opponent of the player moving from u. An *N-position* is any position v from which the *N*ext player can force a win, that is, the player who moves from v. The next player can win by moving to a P-position. A *D-position* is any position from which neither player can win, but both have a nonlosing next move, namely, moving to some D-position. Denote the set of all P-positions by P, all N-positions by N and all D-positions by D. The connection between γ on the sum of one or a finite number of disjoint games and P, N, D is given by:

$$\mathsf{P} = \{\mathbf{u} \in \mathbf{V}: \gamma(\mathbf{u}) = 0\}, \qquad \mathsf{D} = \{\mathbf{u} \in \mathbf{V}: \gamma(\mathbf{u}) = \infty(L),\ 0 \notin L\}, \tag{1}$$

$$\mathsf{N} = \{\mathbf{u} \in \mathbf{V}: 0 < \gamma(\mathbf{u}) < \infty\} \cup \{\mathbf{u} \in \mathbf{V}: \gamma(\mathbf{u}) = \infty(L),\ 0 \in L\}, \tag{2}$$

and

for every $\mathbf{u} \in \mathsf{P}$ and every $\mathbf{v} \in F(\mathbf{u})$ there is $\mathbf{w} \in F(\mathbf{v}) \cap \mathsf{P}$ with $c(\mathbf{w}) < c(\mathbf{u})$.

3. Idiosyncrasies of the Exponentially Large Game-Graph

Given a finite digraph $G = (V, E)$, also called *groundgraph*, order V in some way, say

$$V = \{z_0, \ldots, z_{n-1}\}.$$

This ordering, with $|V| = n$, is assumed throughout.

In its general form, the family of two-player cellular automata games played on G depends on integer parameters $(q(0), \ldots, q(n-1))$ such that $1 \le q(k) \le |F(z_k)|$ for $0 \le k \le n-1$. A move from z_k consists of firing some neighborhood of z_k of size $q(k)$. This family will now be modeled by means of a *game graph*.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ denote the following game graph of the two-player cellular automata game played on $G = (V, E)$. The digraph $\mathbf{G}$ is also called the *cellular automata graph* or *game graph* of G. Any vertex in $\mathbf{G}$ can be described in the form $\mathbf{u} = (u^0, \ldots, u^{n-1})$ over the field GF(2), where $u^k = 1$ if $w(z_k) = 1$, $u^k = 0$ if $w(z_k) = 0$. In particular, $\Phi = (0, \ldots, 0)$ is a leaf of $\mathbf{V}$, and $|\mathbf{V}| = 2^n$.

Note that $\mathbf{V}$ is an abelian group under the addition $\oplus$ of GF(2), which is Nim-addition, with identity Φ. Every nonzero element has order 2. Moreover, $\mathbf{V}$ is a vector space over GF(2) satisfying $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{V}$. For $i \in \{0, \ldots, n-1\}$, define unit vectors $\mathbf{z}_i = (z_i^0, \ldots, z_i^{n-1})$ with $z_i^j = 1$ if $i = j$; $z_i^j = 0$ otherwise. They span the vector space. In particular, for any $\mathbf{u} = (u^0, \ldots, u^{n-1}) \in \mathbf{V}$ we can write, $\mathbf{u} = \sum_{i=0}^{n-1} u^i \mathbf{z}_i = \sum_{i=0}^{\prime\, n-1} u^i \mathbf{z}_i$.

For defining $\mathbf{E}$, let $\mathbf{u} \in \mathbf{V}$ and let $0 \le k \le n-1$. For $0 \le q = q(k) \le |F(z_k)|$, let $F^q(z_k) \subseteq F(z_k)$ be any subset of $F(z_k)$ satisfying

$$|F^q(z_k)| = q. \tag{3}$$

Define

$$(\mathbf{u}, \mathbf{v}) \in \mathbf{E} \ \text{ if } u^k = 1,\ q > 0, \text{ and } \mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell, \tag{4}$$

for every $F^q(z_k)$ satisfying (3).

Informally, an edge $(\mathbf{u}, \mathbf{v})$ reflects the firing of u^k in $\mathbf{u}$ (with $u^k = 1$), i.e., the complementing of the weights of z_k and $F^q(z_k)$. Such an edge exists for every $F^q(z_k)$ satisfying (3). Note that if $z_k \in G$ is a leaf, then there is no move from $\mathbf{z}_k$, since then $q = 0$.

If (4) holds, we also write $\mathbf{v} = F_k^q(\mathbf{u})$. The set of all followers of $\mathbf{u}$ is

$$F(\mathbf{u}) = \bigcup_{u^k = 1} \ \bigcup_{F^q(z_k) \subseteq F(z_k)} F_k^q(\mathbf{u}).$$

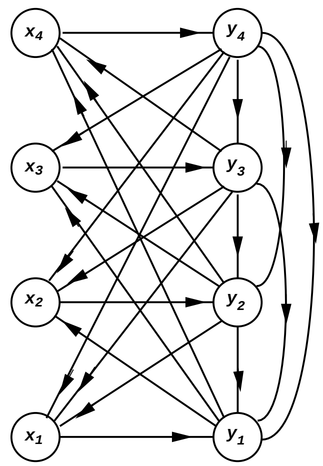

Figure 1. Playing on a parametrized digraph. © 2000

Example 1. Play on the digraph $G(p)$ which depends on a parameter $p \in \mathsf{Z}^+$. It has vertex set $\{x_1, \ldots, x_p, y_1, \ldots, y_p\}$, and edges:

$$F(x_i) = y_i \quad \text{for} \quad i = 1, \ldots, p,$$
$$F(y_k) = \{y_i : 1 \le i < k\} \cup \{x_j : 1 \le j \le p \text{ and } j \ne k\} \text{ for } k = 1, \ldots, p.$$

Figure 1 depicts $G(4)$. Suppose we fire the selected vertex u and complement precisely any *two* of its options. Let's play on $G(7)$ with $w(x_7) = w(y_1) = \ldots = w(y_7) = 1$, where all the other vertices have weight 0. What's the nature of this position?

The reader can verify that though the groundgraph $G(p)$ has no leaf, the game-graph $\mathbf{G}(p)$ has no γ-value ∞. Using step 2 of Algorithm GSG, inspection of $G(p)$ implies that any collection of x_i is in $\mathbf{V}_0$. Moreover, $\gamma(y_i) =$ the ith *odious* number, where the odious numbers are those positive integers whose binary representations have an odd number of 1-bits. Incidentally, odious numbers arise in the analysis of other games, such as Grundy's game, Kayles, Mock Turtles, Turnips. See [1]. They arose earlier in a certain two-way splitting of the nonnegative integers [23] (but without this odious terminology!). Thus $\gamma(x_7 y_1 \ldots y_7) = 0 \oplus 1 \oplus 2 \oplus 4 \oplus 8 \oplus 15 \oplus 16 \oplus 32 = 48$. So either firing y_7 and complementing y_1, y_2, or firing y_6 and complementing y_1, y_3 reduces γ to 0 and so is a winning move.

Definition 2. For $s \in \mathsf{Z}^+$, an *s-game* on a digraph $G = (V, E)$ is a two-player cellular automata game on G satisfying $q \le s$ for all $k \in \{0, \ldots, n-1\}$ (q as in (3)). An *s-regular game* is an s-game such that for all k we have $q = s$ if $|F(z_k)| \ge s$, otherwise $q = |F(z_k)|$. (Thus the game played in Example 1 is a 2-regular game on $G(7)$. Of course if $s \ge \max_{u \in V} |F(u)|$, then firing u in an s-regular game entails complementing all of $F(u)$, for all $u \in V$, in addition to complementing u, unless u is a leaf.)

Remark. The strategy of a cellular automata game on a finite *acyclic* digraph is the same as that of a sum-game on $F^q(z_k)$, i.e., a game without complementation of the 1s of $F^q(z_k)$. This follows from the fact that $a \oplus a = 0$ for any nonnegative integer a. Only the length of play may be longer for the classical case. Thus 1-regular play is equivalent to play without interaction, as far as the strategy is concerned. With increasing s, s-regular games seem to get more difficult. For example, 2-regular play on a Nim-pile gives $g(0) = g(2) = 0$, $g(1) = 1$, $g(i+1) = i$-th odious number ($i \geq 2$).

Our aim is to compute γ on $\mathbf{G}$. This function provides a strategy for sums of games some or all of whose components are two-player cellular automata games.

Lemma 1. *Let* $\mathbf{u}_1, \ldots, \mathbf{u}_h \in \mathbf{V}$, $\mathbf{u} = \sum_{i=1}^{\prime h} \mathbf{u}_i$. *Then,*

(i) $F(\mathbf{u}) \subseteq \bigcup_{j=1}^{h} \big(F(\mathbf{u}_j) \oplus \sum_{i \neq j}' \mathbf{u}_i\big) \subseteq F(\mathbf{u}) \cup F^{-1}(\mathbf{u})$.

(ii) *Let* $\mathbf{v}_j = F_k^q(\mathbf{u}_j)$, $\mathbf{v} = \mathbf{v}_j \oplus \sum_{i \neq j}' \mathbf{u}_i$. *Then* $\mathbf{u} \in F(\mathbf{v})$ *if and only if either*

(a) $u^k = 0$, *or*

(b) *for some* $s \neq k$, $u^s = 0$ *and*

$$\mathbf{z}_k \oplus \sum\nolimits'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell \;=\; \mathbf{z}_s \oplus \sum\nolimits'_{z_t \in F^q(z_s)} \mathbf{z}_t. \tag{5}$$

Before proving the lemma, we single out the special case $h = 2$ of (i).

Corollary 1. *Let* $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{V}$. *Then*

$$\begin{aligned} F(\mathbf{u}_1 \oplus \mathbf{u}_2) &\subseteq \big(\mathbf{u}_1 \oplus F(\mathbf{u}_2)\big) \cup \big(F(\mathbf{u}_1) \oplus \mathbf{u}_2\big) \qquad (6)\\ &\subseteq F(\mathbf{u}_1 \oplus \mathbf{u}_2) \cup F^{-1}(\mathbf{u}_1 \oplus \mathbf{u}_2). \qquad (7) \end{aligned}$$

Notes.

(i) Intuitively, (6) is explained by the similarity between sum-graphs and cellular automata games mentioned in the above remark. The intuition for the appearance of F^{-1} in (7) stems from the observation that $\mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus \sum'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell$ of (4) is consistent with both $\mathbf{v} \in F(\mathbf{u})$ and $\mathbf{u} \in F(\mathbf{v})$. In fact, if $\mathbf{v}$ is in the set on the right hand side of (6), then say, $\mathbf{v} \in F(\mathbf{u}_1) \oplus \mathbf{u}_2$, so for some $k \in \{0, \ldots, n-1\}$ with $u_1^k = 1$ we have,

$$\mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus \sum\nolimits'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell,$$

and there are two cases: (I) $u^k = 1$, then $\mathbf{v} \in F(\mathbf{u})$, or (II) $u^k = 0$. But then $v^k = 1$, so $\mathbf{u} \in F(\mathbf{v})$.

(ii) Equality (5) is consistent with both $z_k \in F^q(z_s)$ and $z_s \in F^q(z_k)$.

Example 2. Consider a 2-play on $G(2)$ (defined in Example 1). Let $\mathbf{u}_1 = x_1 y_2$ (meaning that $w(x_1) = w(y_2) = 1$ and all other weights are 0), $\mathbf{u}_2 = y_1 y_2$. Then $\mathbf{u}_1 \oplus \mathbf{u}_2 = x_1 y_1$, $F(x_1 y_1) = \{\Phi, x_1 x_2\}$, $F^{-1}(x_1 y_1) = \{y_2\}$, $F(\mathbf{u}_1) = \{y_1 y_2, y_1\}$, $F(\mathbf{u}_2) = \{x_2 y_2, x_1\}$, $\mathbf{u}_1 \oplus F(\mathbf{u}_2) = \{x_1 x_2, y_2\}$, $F(\mathbf{u}_1) \oplus \mathbf{u}_2 = \{\Phi, y_2\}$. We see that Corollary 1 is satisfied.

Proof of Lemma 1. Let $\mathbf{v} \in F(\mathbf{u})$. Then $\mathbf{v} = F_k^q(\mathbf{u})$, $u^k = 1$ for some $0 \le k < n$, $F^q(z_k) \subseteq F(z_k)$. Hence $u_j^k = 1$ for some $1 \le j \le h$, and so

$$\begin{aligned}\mathbf{v} &= \mathbf{u} \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell = \mathbf{u}_j \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell \oplus {\sum}'_{i \ne j} \mathbf{u}_i \\ &= \Big(F_k^q(\mathbf{u}_j) \oplus {\sum}'_{i \ne j} \mathbf{u}_i\Big) \in \Big(F(\mathbf{u}_j) \oplus {\sum}'_{i \ne j} \mathbf{u}_i\Big),\end{aligned}$$

proving the left hand side of (i).

Now let $\mathbf{v} \in \bigcup_{j=1}^h \big(F(\mathbf{u}_j) \oplus \sum'_{i \ne j} \mathbf{u}_i\big)$. Then $\mathbf{v} = F_k^q(\mathbf{u}_j) \oplus \sum'_{i \ne j} \mathbf{u}_i$ for some $1 \le j \le h$, $0 \le k < n$, $F^q(z_k) \subseteq F(z_k)$. Substituting $F_k^q(\mathbf{u}_j) = \mathbf{u}_j \oplus \mathbf{z}_k \oplus \sum'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell$, we get,

$$\mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell \qquad (u_j^k = 1). \tag{8}$$

If $u^k = 1$, then $\mathbf{v} \in F(\mathbf{u})$. If $u^k = 0$, then (8) implies $v^k = 1$ and

$$\mathbf{u} = \mathbf{v} \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell, \tag{9}$$

so $\mathbf{u} \in F(\mathbf{v})$, proving the right hand side of (i).

For (ii) we have $\mathbf{v}_j = F_k^q(\mathbf{u}_j)$ for some j, $\mathbf{v} = \mathbf{v}_j \oplus \sum'_{i \ne j} \mathbf{u}_i$. Then both (8) and (9) are valid. Suppose first that (a) holds. Then $v^k = 1$ by (8), and so $\mathbf{u} = F_k^q(\mathbf{v})$ by (9). If (b) holds, then (5) and (9) imply

$$\mathbf{u} = \mathbf{v} \oplus \mathbf{z}_k \oplus {\sum}'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell = \mathbf{v} \oplus \mathbf{z}_s \oplus {\sum}'_{z_t \in F^q(z_s)} \mathbf{z}_t. \tag{10}$$

Therefore $u^s = 0$ implies $v^s = 1$. Hence $\mathbf{u} = F_s^q(\mathbf{v})$.

Conversely, suppose that $\mathbf{u} \in F(\mathbf{v})$, say $\mathbf{u} = F_s^q(\mathbf{v})$. If $s = k$, then $v^k = 1$ so (a) holds by (9). If $s \ne k$, then $\mathbf{u} = F_s^q(\mathbf{v})$ and (9) imply (10), so (5) holds and also $u^s = 0$, i.e., (b) holds.

4. The Additivity of γ

The first key observation for getting a handle at two-player cellular automata games is that γ on $\mathbf{G}$ is, essentially, additive.

Theorem 1. *Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be the cellular automata graph of the finite cyclic digraph $G = (V, E)$. Then $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)$ if $\mathbf{u}_1 \in \mathbf{V}^f$ or $\mathbf{u}_2 \in \mathbf{V}^f$.*

Proof. We use the notation $(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2)$ if either $\mathbf{v}_1 = \mathbf{u}_1$, $\mathbf{v}_2 \in F(\mathbf{u}_2)$, or $\mathbf{v}_1 \in F(\mathbf{u}_1)$, $\mathbf{v}_2 = \mathbf{u}_2$, i.e.,

$$(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2) \ \text{ if } \ (\mathbf{v}_1, \mathbf{v}_2) \in \big(\mathbf{u}_1, F(\mathbf{u}_2)\big) \cup \big(F(\mathbf{u}_1), \mathbf{u}_2\big).$$

Note that F is not a follower in $\mathbf{G}$, but rather in the sum-graph $\mathbf{G} + \mathbf{G}$. It is natural to consider F, because $\sigma = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)$ is the γ-function of the sum $\mathbf{G} + \mathbf{G}$ (see [11], Sect. 3).

(a). We assume $(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}^f \times \mathbf{V}^f$. Let

$$T = \{(\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{V}^f \times \mathbf{V}^f\colon\ \gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) \neq \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)\},$$

$$t = \min_{(\mathbf{u}_1,\mathbf{u}_2)\in T} \big(\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2),\ \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)\big).$$

Further, we define

$$U = \{(\mathbf{u}_1, \mathbf{u}_2) \in T\colon\ \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2) = t\}.$$

We first show that $T \neq$ **?** implies $U \neq$ **?** .

If there is $(\mathbf{u}_1, \mathbf{u}_2) \in T$ such that $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = t$, then $t < \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2) < \infty$. Since σ is a γ-function, **A** of Definition 1, implies that there exists $(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2)$ such that $\gamma(\mathbf{v}_1) \oplus \gamma(\mathbf{v}_2) = t$. Now

$$\mathbf{v}_1 \oplus \mathbf{v}_2 \in \big(\mathbf{u}_1 \oplus F(\mathbf{u}_2)\big) \cup \big(F(\mathbf{u}_1) \oplus \mathbf{u}_2\big) \subseteq F(\mathbf{u}_1 \oplus \mathbf{u}_2) \cup F^{-1}(\mathbf{u}_1 \oplus \mathbf{u}_2)$$

by (7). Since $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = t$, it thus follows that $\gamma(\mathbf{v}_1 \oplus \mathbf{v}_2) > t$, so $(\mathbf{v}_1, \mathbf{v}_2) \in T$.

We have just shown that $T \neq$ **?** implies $U \neq$ **?** . Next we show that $T =$ **?** .

Pick $(\mathbf{u}_1, \mathbf{u}_2) \in U$ with $c(\mathbf{u}_1) + c(\mathbf{u}_2)$ minimum, where c is a monotonic counter function on $\mathbf{V}^f$. Then $c(\mathbf{u}_1) + c(\mathbf{u}_2)$ is a counter function for the γ-function σ on $\mathbf{G} + \mathbf{G}$ (see [18] Theorem 5). Note that

$$\gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2) = t, \qquad \gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) > t. \tag{11}$$

(i) Suppose that there exists $\mathbf{v} \in F(\mathbf{u}_1 \oplus \mathbf{u}_2)$ such that $\gamma(\mathbf{v}) = t$. By (6) there exists $(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2)$ such that $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2$, so $\gamma(\mathbf{v}_1 \oplus \mathbf{v}_2) = t$. Now the definition of F, the minimality of t and the equality of (11) imply $\gamma(\mathbf{v}_1) \oplus \gamma(\mathbf{v}_2) > t$. Since σ is a γ-function, **B** of Definition 1 implies existence of $(\mathbf{w}_1, \mathbf{w}_2) \in \mathsf{F}(\mathbf{v}_1, \mathbf{v}_2)$ such that $\gamma(\mathbf{w}_1) \oplus \gamma(\mathbf{w}_2) = t$, $c(\mathbf{w}_1) + c(\mathbf{w}_2) < c(\mathbf{u}_1) + c(\mathbf{u}_2)$. By (7),

$$\mathbf{w}_1 \oplus \mathbf{w}_2 \in \big(\mathbf{v}_1 \oplus F(\mathbf{v}_2)\big) \cup \big(F(\mathbf{v}_1) \oplus \mathbf{v}_2\big) \subseteq F(\mathbf{v}_1 \oplus \mathbf{v}_2) \cup F^{-1}(\mathbf{v}_1 \oplus \mathbf{v}_2).$$

Since $\gamma(\mathbf{v}_1 \oplus \mathbf{v}_2) = t$, we thus have $\gamma(\mathbf{w}_1 \oplus \mathbf{w}_2) > t$, so $(\mathbf{w}_1, \mathbf{w}_2) \in U$, contradicting the minimality of $c(\mathbf{u}_1) + c(\mathbf{u}_2)$.

(ii) Suppose that $\mathbf{v} \in F(\mathbf{u}_1 \oplus \mathbf{u}_2)$ implies $\gamma(\mathbf{v}) \neq t$. Then **A** of Definition 1 implies $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = \infty$. By **A** applied to σ and the equality in (11), for every $j \in \{0, \dots, t-1\}$ there exists $(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2)$ such that $\gamma(\mathbf{v}_1) \oplus \gamma(\mathbf{v}_2) = j$. By the minimality of t, also $\gamma(\mathbf{v}_1 \oplus \mathbf{v}_2) = j$. As above, by (7), $\mathbf{v}_1 \oplus \mathbf{v}_2 \in F(\mathbf{u}_1 \oplus \mathbf{u}_2) \cup F^{-1}(\mathbf{u}_1 \oplus \mathbf{u}_2)$. If $\mathbf{u}_1 \oplus \mathbf{u}_2 \in F(\mathbf{v}_1 \oplus \mathbf{v}_2)$, then there exists $\mathbf{w} \in F(\mathbf{u}_1 \oplus \mathbf{u}_2)$ with $\gamma(\mathbf{w}) = j$. Hence in any case, $\gamma'(\mathbf{u}_1 \oplus \mathbf{u}_2) = t$. (This holds also if $t = 0$.) By **C** there exists $\mathbf{v} \in F(\mathbf{u}_1 \oplus \mathbf{u}_2)$ such that $\gamma(\mathbf{v}) = \infty(L)$, $t \notin L$. By (6), there exists $(\mathbf{v}_1, \mathbf{v}_2) \in \mathsf{F}(\mathbf{u}_1, \mathbf{u}_2)$ such that $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{v}_2$. Thus,

$$\gamma(\mathbf{v}_1 \oplus \mathbf{v}_2) = \infty(L), \qquad t \notin L, \tag{12}$$

and $\gamma(\mathbf{v}_1) \oplus \gamma(\mathbf{v}_2) > t$ by the equality in (11). As in **(i)** we deduce existence of $(\mathbf{w}_1, \mathbf{w}_2) \in \mathsf{F}(\mathbf{v}_1, \mathbf{v}_2)$ such that $\gamma(\mathbf{w}_1) \oplus \gamma(\mathbf{w}_2) = t$, $c(\mathbf{w}_1) + c(\mathbf{w}_2) < c(\mathbf{u}_1) + c(\mathbf{u}_2)$. By (7) either $\mathbf{w}_1 \oplus \mathbf{w}_2 \in F(\mathbf{v}_1 \oplus \mathbf{v}_2)$ or $\mathbf{v}_1 \oplus \mathbf{v}_2 \in F(\mathbf{w}_1 \oplus \mathbf{w}_2)$. In the former case, $\gamma(\mathbf{w}_1 \oplus \mathbf{w}_2) > t$ by (12). In the latter case, if $\gamma(\mathbf{w}_1 \oplus \mathbf{w}_2) = t$, then **B** implies existence of $\mathbf{y} \in F(\mathbf{v}_1 \oplus \mathbf{v}_2)$ such that $\gamma(\mathbf{y}) = t$, contradicting (12). Thus in either case $(\mathbf{w}_1, \mathbf{w}_2) \in U$, contradicting the minimality of $c(\mathbf{u}_1) + c(\mathbf{u}_2)$. Thus $U = T = \mathbf{?}$.

(b). We now assume, without loss of generality, $\gamma(\mathbf{u}_1) < \infty$, $\gamma(\mathbf{u}_2) = \infty$. If $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) < \infty$, then by **(a)**,

$$\gamma(\mathbf{u}_2) = \gamma\big(\mathbf{u}_1 \oplus (\mathbf{u}_1 \oplus \mathbf{u}_2)\big) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) < \infty,$$

a contradiction. Hence $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = \infty(M)$ for some set M. If the theorem's assertion is false, then there exist $\mathbf{u}_1, \mathbf{u}_2$ with $\gamma(\mathbf{u}_1) < \infty$, $\gamma(\mathbf{u}_2) = \infty(L)$ and $c(\mathbf{u}_1)$ minimum such that $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) \neq \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)$, i.e., $M \neq \gamma(\mathbf{u}_1) \oplus L$.

Let $d \in \gamma(\mathbf{u}_1) \oplus L$. Then $d = \gamma(u_1) \oplus d_1$, where $d_1 \in L$. Let $\mathbf{v}_2 \in F(\mathbf{u}_2)$ satisfy $\gamma(\mathbf{v}_2) = d_1$. Then $\gamma(\mathbf{u}_1 \oplus \mathbf{v}_2) = \gamma(\mathbf{u}_1) \oplus d_1 = d$ by **(a)**. By (7), $\mathbf{u}_1 \oplus \mathbf{v}_2 \in \mathbf{u}_1 \oplus F(u_2) \subseteq F(\mathbf{u}_1 \oplus \mathbf{u}_2) \cup F^{-1}(\mathbf{u}_1 \oplus \mathbf{u}_2)$. Hence **B** implies $d \in M$.

Now let $d \in M$ and $\mathbf{v} \in F(\mathbf{u}_1 \oplus \mathbf{u}_2)$ satisfy $\gamma(\mathbf{v}) = d$. By (6), $\mathbf{v} \in \big(\mathbf{u}_1 \oplus F(\mathbf{u}_2)\big) \cup \big(F(\mathbf{u}_1) \oplus \mathbf{u}_2\big)$. We consider two cases.

(i) $\mathbf{v} = \mathbf{u}_1 \oplus \mathbf{v}_2$ with $\mathbf{v}_2 \in F(\mathbf{u}_2)$. Then by **(a)**, $\gamma(\mathbf{v}_2) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{v}) = \gamma(\mathbf{u}_1) \oplus d \in L$. Hence $d \in \gamma(\mathbf{u}_1) \oplus L$.

(ii) $\mathbf{v} = \mathbf{v}_1 \oplus \mathbf{u}_2$ with $\mathbf{v}_1 \in F(\mathbf{u}_1)$. As at the beginning of **(b)** we conclude $\gamma(\mathbf{v}_1) = \infty$. By **B** there exists $\mathbf{w}_1 \in F(\mathbf{v}_1)$ such that $\gamma(\mathbf{w}_1) = \gamma(\mathbf{u}_1)$, $c(\mathbf{w}_1) < c(\mathbf{u}_1)$. Let $\mathbf{w} = \mathbf{w}_1 \oplus \mathbf{u}_2$. The minimality of $c(\mathbf{u}_1)$ implies

$$\gamma(\mathbf{w}) = \gamma(\mathbf{w}_1 \oplus \mathbf{u}_2) = \gamma(\mathbf{w}_1) \oplus \gamma(\mathbf{u}_2) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2) = \infty\big(\gamma(\mathbf{u}_1) \oplus L\big).$$

By (7), $\mathbf{w} \in F(\mathbf{v}_1) \oplus \mathbf{u}_2 \subseteq F(\mathbf{v}) \cup F^{-1}(\mathbf{v})$. If $\mathbf{v} \in F(\mathbf{w})$, then $\gamma(\mathbf{v}) = d \in \gamma(\mathbf{u}_1) \oplus L$. If $\mathbf{w} \in F(\mathbf{v})$, the same holds by **B**. Thus in all cases $M = \gamma(\mathbf{u}_1) \oplus L$, a contradiction.

5. The Structure of γ

Denote by $\mathrm{GF}(2)^t := (\mathrm{GF}(2))^t$ the vector space of all t-dimensional binary vectors over GF(2) under $\oplus$. It is often convenient to identify $\mathrm{GF}(2)^t$ with the set of integers in the interval $[0, 2^t - 1]$. We now give very precise information about the structure of $\mathbf{V}^f$.

Theorem 2. *Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be the cellular automata graph of the finite cyclic digraph $G = (V, E)$. Then $\mathbf{V}^f$ and $\mathbf{V}_0$ are linear subspaces of $\mathbf{V}$. Moreover, γ is a homomorphism from $\mathbf{V}^f$ onto* $\mathrm{GF}(2)^t$ for some $t \in \mathsf{Z}^0$ with kernel $\mathbf{V}_0$ and quotient space $\mathbf{V}^f/\mathbf{V}_0 = \{\mathbf{V}_i : 0 \le i < 2^t\}$, $\dim(\mathbf{V}^f) = m + t$, where $m = \dim(\mathbf{V}_0)$.

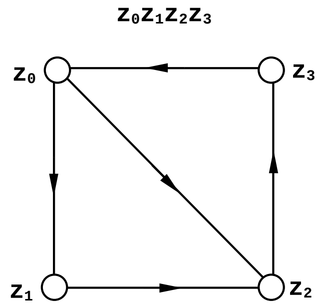

Figure 2. Illustrating Theorem 2. © 2000

Example 3. Consider a 2-regular game played on the digraph depicted in Fig. 2. We adopt a decimal encoding of the vertices of $\mathbf{G}$: $z_i = 2^i$ for all $i \in \mathsf{Z}^0$. Thus 5 means that there are tokens on vertices z_0 and z_2 only. Using inspection and step 2 of Algorithm GSG, we see that $\gamma(1) = 0$ (the position in $\mathbf{G}$ with $w(z_0) = 1$, $w(z_i) = 0$ for $i > 0$). Also $\gamma(4) = \gamma(10) = 0$. Hence by linearity, $\mathbf{V}_0 = \{\Phi, 1, 4, 10, 5, 11, 14, 15\}$. We further note that $\gamma(2) = 1$. Hence we get the coset $\mathbf{V}_1 = 2 \oplus \mathbf{V}_0 = \{2, 3, 6, 7, 8, 9, 12, 13\}$. Also $m = 3$, $t = 1$, and $\mathbf{V}^f$ is spanned by the basis vectors $\beta^q = \{1, 4, 10, 2\}$, $\dim(\mathbf{V}^f) = n = 4$. This illustrates the nice structure of the general case: the number of vertices of $\mathbf{V}$ assuming γ-value i is the same for all i from 0 to some maximum value which is necessarily a power of 2 less 1.

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathbf{V}^f$. Then $\mathbf{u} \oplus \mathbf{v} \in \mathbf{V}^f$ by Theorem 1, and also $\Phi \in \mathbf{V}^f$. Thus $\mathbf{V}^f$ is a subspace of $\mathbf{V}$.

Let t be the smallest nonnegative integer such that $\gamma(\mathbf{u}) \le 2^t - 1$ for all $\mathbf{u} \in \mathbf{V}^f$. Hence, if $t \ge 1$, there is some $\mathbf{v} \in \mathbf{V}^f$ such that $\gamma(\mathbf{v}) \ge 2^{t-1}$. Then the "1's complement" $2^t - 1 - \gamma(\mathbf{v}) < \gamma(\mathbf{v})$. By $\mathbf{A}$ of Definition 1, there exists $\mathbf{w} \in F(\mathbf{v})$ such that $\gamma(\mathbf{w}) = 2^t - 1 - \gamma(\mathbf{v})$. By Theorem 1, $\gamma(\mathbf{v} \oplus \mathbf{w}) = \gamma(\mathbf{v}) \oplus \gamma(\mathbf{w}) = 2^t - 1$. Thus by $\mathbf{A}$, every integer in $[0, 2^t - 1]$ is assumed as a γ-value by some $\mathbf{u} \in \mathbf{V}$. This last property holds trivially also for $t = 0$. Hence γ is onto. It is a homomorphism $\mathbf{V}^f \to \mathrm{GF}(2)^t$ by Theorem 1 and since $\gamma(1\mathbf{u}) = \gamma(\mathbf{u}) = 1\gamma(\mathbf{u})$, $\gamma(0\mathbf{u}) = \gamma(\Phi) = 0 = 0\gamma(\mathbf{u})$.

By linear algebra we have the isomorphism $\mathrm{GF}(2)^t \cong \mathbf{V}^f/\mathbf{V}_0$, where $\mathbf{V}_0$ is the kernel, so $t = \dim(\mathbf{V}^f/\mathbf{V}_0)$. Hence $\mathbf{V}_0$ is a subspace of $\mathbf{V}^f$, and so also of $\mathbf{V}$. Let $m = \dim(\mathbf{V}_0)$. Then $\dim(\mathbf{V}^f) = m + t$. The elements of $\mathbf{V}^f/\mathbf{V}_0$ are the cosets $\mathbf{V}_i = \mathbf{w} \oplus \mathbf{V}_0$ for any $\mathbf{w} \in \mathbf{V}_i$ and every integer $i \in [0, 2^t - 1]$.

Clearly $\mathbf{V}^\infty$ is not a linear subspace: $\mathbf{u} \oplus \mathbf{u} = \Phi$ for every $\mathbf{u} \in \mathbf{V}^\infty$, but $\Phi \in \mathbf{V}^f$. For revealing also the structure of $\mathbf{V}^\infty$ we thus have to embark on a different course. We extend the homomorphism $\gamma\colon \mathbf{V}^f \to \mathrm{GF}(2)^t$ to a homomorphism ρ on the entire space $\mathbf{V}$. Since any $\mathbf{u} \in \mathbf{V}$ can be written as a linear combination of the unit vectors, i.e., $\mathbf{u} = \Sigma'^{n-1}_{i=0} \varepsilon_i \mathbf{z}_i$ ($\varepsilon_i \in \{0, 1\}$, $0 \le i < n$), but some or all of the γ-values of the $\mathbf{z}_i$ may be ∞, the extended homomorphism ρ will then permit to compute $\rho(\mathbf{u}) = \Sigma_{i=0}^{\prime n-1}\varepsilon_i\rho(\mathbf{z}_i)$, such that $\mathbf{u} \in \mathbf{V}^f$ if $\rho(\mathbf{u}) = \gamma(\mathbf{u})$; and we will arrange things so that $\rho(\mathbf{u}) > \gamma(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}^f$ if $\mathbf{u} \in \mathbf{V}^\infty$. Since the homomorphism γ maps an $(m+t)$-dimensional space onto a t-dimensional space and we wish to preserve the kernel $\mathbf{V}_0$ in ρ, we will make ρ map the n-dimensional space $\mathbf{V}$ onto an $(n-m)$-dimensional space $\mathbf{W}$, preserving the reduction by m dimensions. (The extended part will then actually be an isomorphism.)

Lemma 2. *Let $\mathbf{V}$ be the n-dimensional vector space over* GF(2) *of the cellular automata game on $G = (V, E)$ with* $\dim(\mathbf{V}_0) = m$. *There exists a homomorphism ρ mapping $\mathbf{V}$ onto* $\mathrm{GF}(2)^{n-m}$ *with kernel $\mathbf{V}_0$ such that $\mathbf{u} \in \mathbf{V}^f$ if $\rho(\mathbf{u}) = \gamma(\mathbf{u})$, and $\mathbf{u} \in \mathbf{V}^\infty$ if $\rho(\mathbf{u}) > \gamma(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}^f$.*

Proof. From linear algebra, there exists an $(n-m-t)$-dimensional subspace $\mathbf{W}$ of $\mathbf{V}$, such that $\mathbf{V}$ is the direct sum of $\mathbf{V}^f$ and $\mathbf{W}$. Thus, every $\mathbf{u} \in \mathbf{V}$ can be written uniquely in the form

$$\mathbf{u} = \mathbf{w} \oplus \mathbf{v}, \qquad \mathbf{w} \in \mathbf{W}, \qquad \mathbf{v} \in \mathbf{V}^f. \tag{13}$$

Let I: $\mathbf{W} \to \mathrm{GF}(2)^{n-m-t}$ be any isomorphism, and define

$$\rho(\mathbf{u}) = \big(I(\mathbf{w}), \gamma(\mathbf{v})\big), \quad \big(I(\mathbf{w}) \in \mathrm{GF}(2)^{n-m-t}, \quad \gamma(\mathbf{v}) \in \mathrm{GF}(2)^t\big),$$

which is well-defined in view of the uniqueness of the representation (13). Then ρ: $\mathbf{V} \to \mathrm{GF}(2)^{n-m}$ is a homomorphism, since if $\mathbf{u}' = \mathbf{w}' \oplus \mathbf{v}'$, $\mathbf{w}' \in \mathbf{W}$, $\mathbf{v}' \in \mathbf{V}^f$, then

$$\mathbf{u} \oplus \mathbf{u}' = (\mathbf{w} \oplus \mathbf{w}') \oplus (\mathbf{v} \oplus \mathbf{v}'), \qquad \mathbf{w} \oplus \mathbf{w}' \in \mathbf{W}, \qquad \mathbf{v} \oplus \mathbf{v}' \in \mathbf{V}^f,$$

so

$$\begin{aligned}\rho(\mathbf{u} \oplus \mathbf{u}') &= \big(I(\mathbf{w} \oplus \mathbf{w}'), \gamma(\mathbf{v} \oplus \mathbf{v}')\big) = \big(I(\mathbf{w}) \oplus I(\mathbf{w}'), \gamma(\mathbf{v}) \oplus \gamma(\mathbf{v}')\big) \\ &= \big(I(\mathbf{w}), \gamma(\mathbf{v})\big) \oplus \big(I(\mathbf{w}'), \gamma(\mathbf{v}')\big) = \rho(\mathbf{u}) \oplus \rho(\mathbf{u}'),\end{aligned}$$

and

$$\rho(1\mathbf{u}) = \rho(\mathbf{u}) = 1\rho(\mathbf{u}), \qquad \rho(0\mathbf{u}) = \rho(\Phi) = \big(I(\Phi), \gamma(\Phi)\big) = 0 = 0\rho(\mathbf{u}).$$

Finally, $\mathbf{u} \in \mathbf{V}^f$ if $\mathbf{u} = \Phi \oplus \mathbf{u}$ with $\Phi \in \mathbf{W}$, $\mathbf{u} \in \mathbf{V}^f$, and then $\rho(\mathbf{u}) = (\Phi, \gamma(\mathbf{u}))$ with numerical value $\gamma(\mathbf{u})$. For $\mathbf{u} \in \mathbf{V}^\infty$, $I(\mathbf{w}) \neq \Phi$, so the numerical value of the binary vector $(I(\mathbf{w}), \gamma(\mathbf{v}))$ is larger than that of $\gamma(\mathbf{v})$ for all $\mathbf{v} \in \mathbf{V}^f$.

Example 4. We play a 3-regular game on $G(7)$ (recall that $G(p)$ was introduced in Example 1, and $G(4)$ is displayed in Fig. 1). Inspection shows that all collections of an even number of x_i are in $\mathbf{V}_0$; and $\mathbf{V}^f$ consists precisely of all collections of an even number of vertices with weight 1. Furthermore, $\gamma(x_j y_j) =$ smallest nonnegative integer not the Nim sum of at most three $\gamma(x_i y_i)$ for $i < j$. Also $\gamma(x_j y_j) = \gamma(x_i y_j)$ for all i. Thus $\{\gamma(x_1 y_i)\}_{i=1}^{7} = \{1, 2, 4, 8, 15, 16, 32\}$. Further, $\mathbf{W} = \mathsf{L}(x_1) = \{\Phi, x_1\}$ is the complement of $\mathbf{V}^f$, and $n - m - t =$

$1 = \dim(\mathbf{W})$ with basis $\beta^\infty = \{x_1\}$, where L denotes the linear span. Also $m = \dim \mathbf{V}_0 = 7$ since $\mathbf{V}_0$ contains the 2^6 subsets of an even number of x_i as well as $y_1y_2y_3y_4y_5$. Thus $t = 6$, which is consistent with $\gamma(x_7y_7) = 32$, since the value 64 is not attained. Moreover, we can put $\rho(x_i) = 64$ for all i, and $\{\rho(y_i)\}_{i=1}^7 = \{65, 66, 68, 72, 79, 80, 96\}$. Since $t = 6$ and the ρ-values on the singletons all have a nonzero bit in a position $> 2^t - 1$, the γ-value of any collection with an odd (even) number of vertices from G has value ∞ ($< \infty$). For example, $\rho(x_7y_1 \dots y_7) = \gamma(x_7y_1 \dots y_7) = 1 \oplus 2 \oplus 4 \oplus 8 \oplus 15 \oplus 16 \oplus 32 = 48$. So firing y_7 and complementing y_6 and any two of the x_i ($i < 7$) is a winning move. Incidentally, the sequence $\{1, 2, 4, 8, 15, 16, 32, 51, \dots\}$ has been used in [1] for a special case of the game "Turning Turtles".

This a posteriori verification is not very satisfactory, but at this stage the example nevertheless illustrates nicely Theorem 2 and Lemma 2. In § 6, where we construct ρ by embedding its computation in an algorithm for computing $\mathbf{V}_0$, $\mathbf{V}^f$ and $\mathbf{W}$, we will see how to compute a priori results such as these. The problem right now is that despite the precise information about the structure of $\mathbf{V}^f$, $\mathbf{V}_0$ and $\mathbf{V}^\infty$, the computation of say, $\mathbf{V}_0$, is exponential, as we may have to scan the 2^n vectors of $\mathbf{V}$ for membership in $\mathbf{V}_0$. Actually only a polynomial fragment of the 2^n vectors has to be examined, as we will see in the next section.

6. Sparse Vectors Suffice

A vector $\mathbf{u} \in \mathrm{GF}(2)^n$ has *weight* i, if it has precisely i 1-bits, i.e., $\sum_{k=1}^n u^k = i$. We write $w(\mathbf{u}) = i$ if $\mathbf{u}$ has weight i. This terminology is standard in coding theory. There should be no confusion between the weights $w(u^i)$, $w(\mathbf{u})$ and the vector $\mathbf{w} \in \mathbf{V}$.

The second key observation conducive to producing a strategy for cellular automata games is that for s-regular games, to which we now confine ourselves, $\mathbf{V}_0$, $\mathbf{V}^f$ and ρ can be computed by restricting attention to the linear span of vectors of weight at most $2(s+1)$.

We begin with some notation.

$$Z_i = \bigcup_{h=1}^{i} \{\mathbf{u} \in \mathbf{V}\colon w(\mathbf{u}) = h\} \qquad \text{(vectors of weight } \le i\text{)}$$

$$Z_i^f = Z_i \cap \mathbf{V}^f$$

$$S = Z_1 \cap \{\mathbf{u} \in \mathbf{V}\colon F(\mathbf{u}) = \mathbf{?}\,\} \cup \{\Phi\} \qquad \text{(leaves of weight } \le 1\text{)}$$

$$L = \{\mathbf{u} \in \mathbf{V}\colon F(\mathbf{u}) = \mathbf{?}\,\} \qquad \text{(set of all leaves)}$$

$$Q = (Z_{2(s+1)} \cap \mathbf{V}_0) \cup S$$

$$g(n,s) = (s+1)\binom{n-1}{s} - s$$

$$\phi(s) = \max(3s+2, 2s+4) = \begin{cases} 6 & \text{if } s=1, \\ 3s+2 & \text{if } s>1. \end{cases}$$

Theorem 3. *Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be the cellular automata graph of the finite digraph $G = (V, E)$. Then*

(i) $L = \mathsf{L}(S)$ (*linear span of* S *over* GF (2))
(ii) $\mathbf{V}_0 = \mathsf{L}(Q)$
(iii) $\mathbf{V}^f = \mathsf{L}(Q \cup Z^f_{s+1})$
(iv) $\{\gamma(\mathbf{u}) : \mathbf{u} \in \mathbf{V}^f\} = \{\gamma(\mathbf{u}) : \mathbf{u} \in (Z^f_{s+1} \cup \{\Phi\})\}$
(v) $t \leq \lfloor \log_2(1 + g(n, s)) \rfloor$ $(n \geq s+1)$*, where* $2^t - 1$ *is the maximum value of* γ *on* $\mathbf{V}^f$*.*

Comment. Since every vector in S, Q or $Q \cup Z^f_{s+1}$ has weight at most $2(s+1)$, it follows that $L, \mathbf{V}_0, \mathbf{V}^f$ can all be computed from a set of vectors of weight at most $2(s+1)$.

Proof. (i) Follows from the definition of a leaf.

(ii) Clearly $\mathsf{L}(Q) \subseteq \mathbf{V}_0$. If the result is false, let $\mathbf{u}$ with $c(\mathbf{u})$ minimal satisfy $\mathbf{u} \in \mathbf{V}_0$, $\mathbf{u} \notin \mathsf{L}(Q)$. In particular, $\mathbf{u} \notin \mathsf{L}(S)$, and so by (i), $\mathbf{u}$ is not a leaf. Hence there is $\mathbf{v} \in F(\mathbf{u})$. By **A** of Definition 1, $\gamma(\mathbf{v}) > 0$, and by **B**, there exists $\mathbf{w} \in F(\mathbf{v}) \cap \mathbf{V}_0$ with $c(\mathbf{w}) < c(\mathbf{u})$. By the minimality of $c(\mathbf{u})$, we have $\mathbf{w} \in \mathsf{L}(Q)$. Now $\mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus \sum'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell$ and $\mathbf{w} = \mathbf{v} \oplus \mathbf{z}_r \oplus \sum'_{z_h \in F^q(z_r)} \mathbf{z}_h$ say, so letting $\mathbf{y} = \mathbf{w} \oplus \mathbf{u}$ we have $\mathbf{y} = \mathbf{z}_k \oplus \sum'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell \oplus \mathbf{z}_r \oplus \sum'_{z_h \in F^q(z_r)} \mathbf{z}_h$, thus $w(\mathbf{y}) \in \{0, \ldots, 2(s+1)\}$. Hence $\mathbf{y} \in Q \subseteq \mathsf{L}(Q)$. Then $\mathbf{u} = \mathbf{w} \oplus \mathbf{y} \in \mathsf{L}(Q)$, since $\mathsf{L}(Q)$ is a subspace, which is a contradiction.

(iii) Clearly $\mathsf{L}(Q \cup Z^f_{s+1}) \subseteq \mathbf{V}^f$. Let $\mathbf{u} \in \mathbf{V}^f$. If $\mathbf{u} \in \mathbf{V}_0$, then by (ii), $\mathbf{u} \in \mathsf{L}(Q) \subseteq \mathsf{L}(Q \cup Z^f_{s+1})$. Otherwise, let $\mathbf{v} \in F(\mathbf{u}) \cap \mathbf{V}_0$, $\mathbf{w} = \mathbf{u} \oplus \mathbf{v} = \mathbf{z}_k \oplus \sum'_{z_\ell \in F^s(z_k)} \mathbf{z}_\ell$. Since $\mathbf{w} \neq \Phi$, we have $\mathbf{w} \in Z^f_{s+1}$. By (ii), $\mathbf{v} \in \mathsf{L}(Q)$. Hence $\mathbf{u} = \mathbf{v} \oplus \mathbf{w} \in \mathsf{L}(Q \cup Z^f_{s+1})$.

(iv) Denote the left hand set by S_ℓ and the right hand set by S_r. Clearly $S_r \subseteq S_\ell$. Let $j \in S_\ell$. If $j = 0$, then $j = \gamma(\Phi) \in S_r$. Otherwise, pick $\mathbf{u} \in \mathbf{V}_j$. There exists $\mathbf{v} \in F(\mathbf{u}) \cap \mathbf{V}_0$. Let $\mathbf{w} = \mathbf{u} \oplus \mathbf{v}$. Then $\gamma(\mathbf{w}) = j$, and $\mathbf{w} \in Z^f_{s+1} \in S_r$.

(v) Let $\mathbf{u} \in \mathbf{V}^f$ have maximum γ-value. By (iv) we may assume that $\mathbf{u} = \mathbf{z}_k \oplus \sum'_{z_\ell \in F^s(z_k)} \mathbf{z}_\ell \in Z^f_{s+1}$. Any of the $s+1$ tokens can fire into at most s followers in $n-1$ locations. At least s of the followers are identical. Thus the outdegree of $\mathbf{u}$ is at most $g(n, s)$. Then $2^t - 1 \leq g(n, s)$, which implies the result.

For the special case of 1-regular games $(s = q = 1)$, we can do a little better. This was done in [17]. For the sake of completeness, we reproduce it here, in a more transparent form. Define

$$Y_i = \{\mathbf{u} \in \mathbf{V} : w(\mathbf{u}) = i\} \qquad \text{(vectors of weight } i\text{)}$$
$$Y_i^f = Y_i \cap \mathbf{V}^f$$

$$\begin{aligned} S &= Y_1 \cap \{\mathbf{u} \in \mathbf{V}: F(\mathbf{u}) = \mathbf{?}\} \cup \{\Phi\} \qquad \text{(leaves of weight } \le 1) \\ L &= \{\mathbf{u} \in \mathbf{V}: F(\mathbf{u}) = \mathbf{?}\} \qquad \text{(set of all leaves)} \\ Q_i &= Y_i \cap \mathbf{V}_0 \\ Q &= Q_2 \cup Q_4 \cup S. \end{aligned}$$

Theorem 4. *Let* $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ *be the cellular automata graph of the finite digraph* $G = (V, E)$. *Then*

(i) $L = \mathsf{L}(S)$ (*linear span of* S *over* GF (2))
(ii) $\mathbf{V}_0 = \mathsf{L}(Q)$
(iii) $\mathbf{V}^f = \mathsf{L}(Q \cup Y_2^f)$
(iv) $\{\gamma(\mathbf{u}): \mathbf{u} \in \mathbf{V}^f\} = \{\gamma(\mathbf{u}): \mathbf{u} \in (Y_2^f \cup \{\Phi\})\}$
(v) $t \le \lceil \log_2 n \rceil$ $(n \ge 2)$, *where* $2^t - 1$ *is the maximum value of* γ *on* $\mathbf{V}^f$.

Proof. The proof is very similar to that of Theorem 3 and is therefore omitted.

7. An $O(n^6)$ Algorithm for γ for the Case $q = 1$

We shall now consolidate the results of the previous sections into a polynomial algorithm for computing γ for the cellular automata graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ of a digraph $G = (V, E)$ in polynomial time for the special case $q = 1$. We begin with some further notation.

$\beta_0 = \{B_0, \dots, B_{m-1}\}$ is a basis for $\mathbf{V}_0$, where $B_i \in Q$ $(0 \le i < m)$.
$\beta^f = \{B_m, \dots, B_{m+t-1}\} \cup \beta_0$ is a basis for $\mathbf{V}^f$, $B_{m+i} \in Y_{q+1}^f$ $(0 \le i < t)$.
$\beta^\infty = \{B_{m+t}, \dots, B_{n-1}\}$ is a basis for the complement $\mathbf{W}$ of $\mathbf{V}^f$ in $\mathbf{V}$, so $\beta = \beta^f \cup \beta^\infty$ is a basis for $\mathbf{V}$.
$\mathbf{G}^{[2]} = (\mathbf{V}^{[2]}, \mathbf{E}^{[2]})$ is the subgraph of $\mathbf{G}$ induced by $Y_2 \cup S$.
$\mathbf{G}^{[4]} = (\mathbf{V}^{[4]}, \mathbf{E}^{[4]})$ is the subgraph of $\mathbf{G}$ induced by $Y_4 \cup Y_2 \cup S$.

Algorithm CEL (cellular) for computing γ on $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ for $q = 1$ (without constructing the exponentially large $\mathbf{G}$!).
Input: Digraph $G = (V, E)$ with $|V| = n$.
Output: An $n \times n$ matrix $B = (\beta^f, \beta^\infty)$, and m, t, B^{-1}. The n columns of the bottom $n - m$ rows of B^{-1} constitute the homomorphism (see Lemma 2, § 5) $\rho(\mathbf{z}_i) = (\varepsilon_i^m, \dots, \varepsilon_i^{n-1}) \in \mathrm{GF}(2)^{n-m}$ $(0 \le i < n)$.

Notes.

1. For any vector $\mathbf{v} = (\delta^0, \dots, \delta^{n-1}) \in \mathbf{V}$, we have $\mathbf{v} = \sum_{i=0}^{n-1} \delta^i \mathbf{z}_i$, so $\rho(\mathbf{v}) = \sum_{i=0}^{\prime n-1} \delta^i \rho(\mathbf{z}_i)$ can thus be computed polynomially. Using Lemma 2 we then have either $\gamma(\mathbf{v}) = \rho(\mathbf{v})$, or $\gamma(\mathbf{v}) = \infty$. This polynomial computation is the main thrust of Algorithm CEL. It permits computing γ on all of $\mathbf{V}$.

2. The equation $B\mathbf{x} = \mathbf{v}$ has the solution $\mathbf{x} = B^{-1}\mathbf{v} = (\varepsilon^0, \ldots, \varepsilon^{n-1}) \in \mathrm{GF}(2)^n$, so $\mathbf{v}$ can be represented as a linear combination of the basis consisting of the columns of B:

$$\mathbf{v} = \varepsilon^0 B_0 \oplus \varepsilon^1 B_1 \oplus \cdots \oplus \varepsilon^{n-1} B_{n-1}.$$

This fact is used in the next section for forcing a win for $q = 1$ in polynomial time.

3. Notice that both $\mathbf{z}_0, \ldots, \mathbf{z}_{n-1}$ and $B_0, \ldots, B_{n-1}$ are bases of $\mathbf{V}$. The former is more convenient for the ρ-computation, and the latter for forcing a win.

Procedure: (i) Construct $\mathbf{G}^{[2]}$ and $\mathbf{G}^{[4]}$ ($\mathbf{G}^{[2]}$ has $O(n^2)$ vertices and $O(n^3)$ edges; $\mathbf{G}^{[4]}$ has $O(n^4)$ vertices and $O(n^5)$ edges).

(ii) Apply the first iteration ($i = 0$) of Algorithm GSG (§ 2) to $\mathbf{G}^{[4]}$. Store the resulting set $Q = \{\mathbf{q}_1, \ldots, \mathbf{q}_p\}$ of vectors in $\mathbf{V}_0$ together with their counter values c ($O(n^5)$ steps). (We may omit from Q the vector Φ and any other vectors which obviously depend linearly on the rest.)

(iii) Apply Algorithm GSG to $\mathbf{G}^{[2]}$. The largest γ-value will be $2^t - 1$ for some $t \in \mathsf{Z}^0$. Store vectors $\mathbf{v}_1, \ldots, \mathbf{v}_t$ with $\gamma(\mathbf{v}_i) = 2^{i-1}$, $w(\mathbf{v}_i) = 2$ $(1 \le i \le t)$ and t, together with their monotonic counter values c, such that

$$\min\bigl\{c(\mathbf{v}) : \mathbf{v} \in \mathbf{V}^{[2]}\bigr\} > \max\bigl\{c(\mathbf{u}) : \mathbf{u} \in \mathbf{V}^{[4]}\bigr\}$$

$\bigl(O(n^5)\text{steps}\bigr)$.

(iv) Construct the matrix $A = (\mathbf{q}_1, \ldots, \mathbf{q}_p, \mathbf{v}_1, \ldots, \mathbf{v}_t, \mathbf{z}_0, \ldots, \mathbf{z}_{n-1})$, where the $\mathbf{z}_i$ are the unit vectors ($O(n^5)$ steps, since $p = O(n^4)$, so A has order $n \times O(n^4)$).

(v) Transform A into a row-echelon matrix E, using elementary row operations ($O(n^6)$ steps; the $\mathbf{z}_0, \ldots, \mathbf{z}_{n-1}$ of A then become B^{-1} — see e.g., [22], Ch. 7, § 47).

(vi) Let $1 \le i_1 < \cdots < i_n \le p + t + n$ be the indices of the unit vectors of E. Then m is the largest subscript s such that $i_s \le p$. Let $B = (B_0, \ldots, B_{n-1})$ be the matrix consisting of the columns $A_{i_1}, \ldots, A_{i_n}$ of A. Store

$$\begin{aligned} \beta_0 &= \{B_0, \ldots, B_{m-1}\}, \\ \beta^f &= \{B_m, \ldots, B_{m+t-1}\} \cup \beta_0, \\ \beta &= \{B_{m+t}, \ldots, B_{n-1}\} \cup \beta^f, \end{aligned}$$

and the matrix $B^{-1} = (E_{p+t}, \ldots, E_{p+t+n-1})$ (the last n columns of E) $\bigl(O(n^4)\bigr)$. Compute $B^{-1}\mathbf{z}_i = (\varepsilon_i^0, \ldots, \varepsilon_i^{n-1})$; store $\rho(\mathbf{z}_i) = (\varepsilon_i^m, \ldots, \varepsilon_i^{n-1})$ $(0 \le i < n)$; the numerical values of the columns consisting of the last $n - m$ rows of B^{-1}; ($O(n^3)$ steps.) End.

Example 5. Play a 1-regular game on the digraph depicted in Fig. 3. We apply Algorithm CEL to it. The output of step (ii) is $Q = (0, 5, 10, 15)$, and step (iii) yields $\mathbf{v}_1 = \{3\}$ $(t = 1)$. The following constitutes steps (iv) and (v), where $\sim$ denotes equivalence under elementary row operations; we omitted the 0-vector

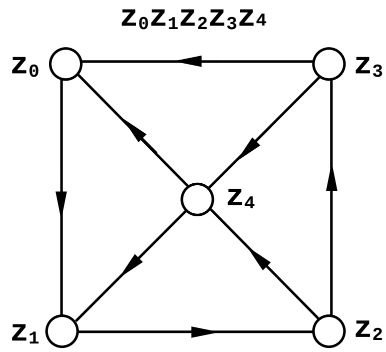

Figure 3. Play a 1-regular game. © 2000

from A.

$$A = \begin{pmatrix} 1&0&1&1&1&0&0&0&0\\ 0&1&1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1&0&0\\ 0&1&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1&1&1&0&0&0&0\\ 0&1&1&1&0&1&0&0&0\\ 0&0&0&1&1&0&1&0&0\\ 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 1&0&1&0&0&0&1&0&0\\ 0&1&1&0&1&1&1&0&0\\ 0&0&0&1&1&0&1&0&0\\ 0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} \sim \begin{pmatrix} 1&0&1&0&0&0&1&0&0\\ 0&1&1&0&0&0&0&1&0\\ 0&0&0&1&0&1&0&1&0\\ 0&0&0&0&1&1&1&1&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}$$
$$= E.$$

The last $n = 5$ columns of E constitute B^{-1} e.g., by [22], Ch. 7, § 47. The unit vectors are in columns 1, 2, 4, 5, 9. Since $p = 3$, we then have $m = 2$. Columns 1, 2, 4, 5, 9 of A constitute B. Thus,

$$B = \begin{pmatrix} 1&0&1&1&0\\ 0&1&1&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&1&0\\ 1&1&1&1&0\\ 0&0&0&0&1 \end{pmatrix},$$

and $\big(\rho(\mathbf{z}_0), \rho(\mathbf{z}_1), \rho(\mathbf{z}_2), \rho(\mathbf{z}_3), \rho(\mathbf{z}_4)\big) = (2,3,2,3,4)$, which are the bottom $n - m = 3$ rows of B^{-1}. Already in Example 4 we saw that these values determine the ρ-values of all positions of the game. Since the 5th and 9th columns of A contain 1 and 16 respectively, we have $\mathbf{W} = \{0, 1, 16, 17\}$.

Furthermore, for say $\mathbf{v} = \mathbf{z}_3 = 8$, the equation $B\mathbf{x} = \mathbf{v}$ has the solution $\mathbf{x} = B^{-1}\mathbf{v} = 14$, hence $\mathbf{v}$ is the following linear combination of the B-column vectors: $\mathbf{v} = 10 \oplus 3 \oplus 1 = 8$, and $\rho(\mathbf{v}) = \rho(10) \oplus \rho(3) \oplus \rho(1) = 0 \oplus 1 \oplus 2 = 3$. Since $\rho(\mathbf{v}) > 2^t - 1 = 1$, $\gamma(\mathbf{v}) = \infty$. Similarly, for $\mathbf{v} = \mathbf{z}_2 \oplus \mathbf{z}_3 = 12$, the equation $B\mathbf{x} = \mathbf{v}$ has the solution $x = 7$, hence $\mathbf{v} = 5 \oplus 10 \oplus 3 = 12$, and

$\rho(\mathbf{v}) = \rho(5)\oplus\rho(10)\oplus\rho(3) = 0\oplus 0\oplus 1 = 1$, so $\gamma(\mathbf{v}) = 1$. But a simpler computation of $\rho(\mathbf{v})$ in these two examples is to express $\mathbf{v}$ in terms of the unit vectors $\mathbf{z}_i$ and then apply the homomorphism ρ on them, as was done in Example 5.2: $\rho(8) = 3$, so $\gamma(\mathbf{v}) = \infty$; $\rho(12) = \rho(4 \oplus 8) = \rho(4) \oplus \rho(8) = 2 \oplus 3 = 1$, so $\gamma(12) = 1$.

Example 6. Suppose we put $w(z_4) = w(z_1) = 1$ for the digraph of Fig. 3, $w(z_1) = 1$ for the digraph of Fig. 2, and all other weights 0 on both digraphs. Play the sum of the 1-regular game on the digraph of Fig. 3 and the 2-regular game on the digraph of Fig. 2. We see that the γ-value of the former is $\infty(0,1)$ and of the latter 1, and $1 \oplus \infty(0,1) = \infty\,(1 \oplus (0,1)) = \infty(1,0)$. By (2) this is an N-position. The unique winning move is to to fire z_4 and complement $w(z_0)$.

For proving validity of Algorithm CEL, we need an auxiliary result. We first define a special type of subgraph which is closed under the operation of taking followers, i.e., it contains the followers of all of its vertices.

Definition 3. Let $G = (V,E)$ be any digraph. A subset $U \subseteq V$ is a *restriction* of V, if $F(U) \subseteq U$, where $F(U) = \{v \in V\colon (u,v) \in E,\ u \in U\}$.

Lemma 3. (Restriction Principle). *Let $G = (V,E)$ be a digraph, U a restriction of V, G_1 the subgraph of G induced by U. Then the γ-function computed on G_1 alone* (without considering G) is identical with the γ-function on G, restricted to U.

Proof. Let γ_1 be the γ-function of G restricted to G_1. Since $u \in U$ implies $F_G(u) \subseteq U$, Definition 1 implies that γ_1 is a Generalized Sprague–Grundy function on G_1. Since also γ computed on G_1 is, we have $\gamma_1 = \gamma$ by the uniqueness of γ.

Validity Proof of Algorithm CEL. The vertex set $\mathbf{V}^{[2]}$ of $\mathbf{G}^{[2]}$ is clearly a restriction of $\mathbf{V}$ $(q = 1)$; and $\mathbf{V}^{[4]}$ of $\mathbf{G}^{[4]}$ is also a restriction of $\mathbf{V}$. By Lemma 3, all the γ-values computed in steps (ii) and (iii) are γ-values of $\mathbf{G}$. By Theorem 4, these computed values generate γ on all of $\mathbf{G}$.

Note that the n unit vectors of A guarantee that A has rank n. The matrix E constructed in step (v) has the claimed properties by linear algebra properties over GF(2). In particular, the product of the elementary row operation matrices operated on the unit matrix $I = (\mathbf{z}_0, \dots, \mathbf{z}_{n-1})$ — which is the tail end of A — is B^{-1} as claimed. Since $\mathbf{V}_0 = \mathsf{L}\,(Q)$, the value of m is as stated in step (vi). Since the vectors $\mathbf{v}_1, \dots, \mathbf{v}_t$ are linearly independent, $\dim \mathbf{V}^f = m + t$.

Finally, we show that for any $\mathbf{u} \in \mathbf{V}$, if $B^{-1}\mathbf{u} = \varepsilon = (\varepsilon^0, \dots, \varepsilon^{n-1}) \in \mathrm{GF}(2)^n$, then $\rho(\mathbf{u}) = (\varepsilon^m, \dots, \varepsilon^{n-1}) \in \mathrm{GF}(2)^{n-m}$ is a homomorphism from $\mathbf{V}$ onto $\mathrm{GF}(2)^{n-m}$ with kernel $\mathbf{V}_0$ such that $\mathbf{u} \in \mathbf{V}^f$ if and only if $\rho(\mathbf{u}) \le 2^t - 1$, whence $\rho(\mathbf{u}) = \gamma(\mathbf{u})$. Let $\mathbf{v} \in \mathbf{V}$ and $B^{-1}\mathbf{v} = \delta = (\delta^0, \dots, \delta^{n-1}) \in \mathrm{GF}(2)^n$.

$$\rho(\mathbf{u}) \oplus \rho(\mathbf{v}) = (\varepsilon^m, \dots, \varepsilon^{n-1}) \oplus (\delta^m, \dots, \delta^{n-1}) = (\varepsilon^m \oplus \delta^m, \dots, \varepsilon^{n-1} \oplus \delta^{n-1}).$$

Now

$$\varepsilon \oplus \delta = (\varepsilon^0, \dots, \varepsilon^{n-1}) \oplus (\delta^0, \dots, \delta^{n-1}) = B^{-1}\mathbf{u} \oplus B^{-1}\mathbf{v} = B^{-1}(\mathbf{u} \oplus \mathbf{v})$$
$$= (\varepsilon^0 \oplus \delta^0, \dots, \varepsilon^{n-1} \oplus \delta^{n-1}).$$

Hence $\rho(\mathbf{u} \oplus \mathbf{v}) = (\varepsilon^m \oplus \delta^m, \dots, \varepsilon^{n-1} \oplus \delta^{n-1}) = \rho(\mathbf{u}) \oplus \rho(\mathbf{v})$, so ρ is a homomorphism $\mathbf{V} \to \mathrm{GF}(2)^{n-m}$. It is onto, since for any $\varepsilon = (\varepsilon^0, \dots, \varepsilon^{n-1}) \in \mathbf{V}$, the equation $B\varepsilon = \mathbf{u} \in \mathbf{V}$ has the solution $\varepsilon = B^{-1}\mathbf{u}$, and $\rho(\mathbf{u}) = (\varepsilon^m, \dots, \varepsilon^{n-1})$. By linear algebra, $B^{-1}\mathbf{u} = (\varepsilon^0, \dots, \varepsilon^{n-1})$ implies $\mathbf{u} = \varepsilon^0 B_0 \oplus \cdots \oplus \varepsilon^{n-1} B_{n-1}$. Since $\beta_0 = (B_0, \dots, B_{m-1})$ is a basis of $\mathbf{V}_0$, we see that $\varepsilon^m = \cdots = \varepsilon^{n-1} = 0$ if and only if $\mathbf{u} \in \mathbf{V}_0$. Hence the kernel is $\mathbf{V}_0$. The same argument shows that $\mathbf{u} \in \mathbf{V}^f$ if and only if $\varepsilon^{m+t} = \cdots = \varepsilon^{n-1} = 0$ if and only if $\rho(\mathbf{u}) = \gamma(\mathbf{u})$.

8. Forcing a Win in Cellular Automata Games for 1-Regular Games

By using ρ, computed in the previous section in $O(n^6)$ steps, we can compute the γ-function for every $\mathbf{u} \in \mathbf{V}$ for any cellular automata game-graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ played on a digraph $G = (V, E)$ with $|V| = n$, which determines the P-, N- and D-membership for every $\mathbf{u} \in \mathbf{V}$ by (1) and (2). However, our method only constructed an $O(n^4)$ fragment of the exponentially-large game-graph. In particular, we don't have a counter function for all $\mathbf{u} \in \mathbf{V}^f$, so the question arises, given any N-position in $\mathbf{G}$, how can we insure a win in a finite number of moves.

If $\mathbf{u} \in \mathsf{D}$, then $\mathbf{v} \in F(\mathbf{u}) \cap \mathsf{D}$ can be found polynomially by scanning $F(\mathbf{u})$ ($|F(\mathbf{u})| < n^2$): compute $\rho(\mathbf{v})$ for $\mathbf{v} \in F(\mathbf{u})$. If $\gamma(\mathbf{v}) = \infty$, then compute $\rho(\mathbf{w})$ for $\mathbf{w} \in F(\mathbf{v})$, to determine K such that $\gamma(\mathbf{v}) = \infty(K)$. Similarly, if $\mathbf{u} \in \mathsf{N}$, then $\mathbf{v} \in F(\mathbf{u}) \cap \mathsf{P}$ can be found polynomially. This, however, does not guarantee a win because of possible cycling and never reaching a leaf. The strategy of moving from $\mathbf{u} \in \mathsf{N}$ to any $F(\mathbf{u}) \cap \mathsf{P}$ guarantees a nonlosing outcome, nevertheless.

In this section we show how to force a win from a position $\mathbf{u} \in \mathsf{N}$ in polynomial time, including the case when the cellular automata game is a component in a sum of finitely many games. Let

$$\mathbf{R}_j = Q \bigcup_{i=1}^{j} (Y_2 \cap \mathbf{V}_{2^i}) \ (0 \le j < t), \text{ so } \mathbf{R}_0 = Q.$$

Informally, this third key idea is this. Given any $\mathbf{u} \in \mathbf{V}^f$, we can write $\mathbf{u} = \sum_{i=1}^{\prime h} \mathbf{u}_i$ with $\mathbf{u}_i \in \mathbf{R}_{t-1}$ $(1 \le i \le h)$, where the $\mathbf{u}_i$ can be computed polynomially using the matrix B produced by Algorithm CEL. We then say that $\widetilde{u} = \{\mathbf{u}_1, \dots, \mathbf{u}_h\}$ *represents* $\mathbf{u}$. Moreover, we can define a counter function $\widetilde{c}$ on representations $\widetilde{u}$ and arrange that the winner moves to a sequence of positions $\mathbf{u}^0, \mathbf{u}^1, \dots$ with $\widetilde{c}(\widetilde{u}^0) > \widetilde{c}(\widetilde{u}^1) > \cdots$ $\big(\widetilde{u}^i = (\mathbf{u}^i_1, \dots, \mathbf{u}^i_{h_i})\big)$, leading to a win. In doing this we will be confronted, analogously to Lemma 1 (§ 3) and the proof of Theorem 1, by the possibility of encountering a predecessor of a representation instead of the descendant we are seeking. That is, $\widetilde{c}(\widetilde{u}^i) > \widetilde{c}(\widetilde{u}^{i+1})$ with $\widetilde{u}^{i+1} \in \widetilde{F}(\widetilde{u}^i)$ implies $\mathbf{u}^{i+1} \in F(\mathbf{u}^i)) \cup F^{-1}(\mathbf{u}^i))$, where $\widetilde{F}$ is a follower function for representations, defined below, and $\mathbf{u}^i = \sum_{j=1}^{\prime h_i} \mathbf{u}_j^i$.

Definition 4. (i) Let $\mathbf{R} \subseteq \mathbf{V}$. A *representation* $\widetilde{u}$ of $\mathbf{u} \in \mathbf{V}$ over $\mathbf{R}$ is a subset $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\} \subseteq \mathbf{R}$ of distinct elements $\mathbf{u}_i$. If $\mathbf{R}$ is either indicated by the context or irrelevant, we may say simply that $\widetilde{u}$ is a *representation* of $\mathbf{u} = \sum_{j=1}^{\prime h} \mathbf{u}_j$, omitting over $\mathbf{R}$. The empty representation is denoted by $\widetilde{\mathbf{?}}$.

(ii) For $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\} \subseteq \mathbf{R}$, a follower function for representations is given by

$$\widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j) = \big(\widetilde{u} \ominus \{\mathbf{u}_j, \mathbf{v}_j\}\big),$$

where $\mathbf{u}_j \in \widetilde{u}$, $\mathbf{v}_j \in F_k^q(\mathbf{u}_j)$ for any $1 \le j \le h$, and where $\ominus$ denotes the symmetric difference: $\widetilde{x}_1 \ominus \widetilde{x}_2 = (\widetilde{x}_1 \cup \widetilde{x}_2) - (\widetilde{x}_1 \cap \widetilde{x}_2)$ $\big(\widetilde{x}_1, \widetilde{x}_2 \in \mathbf{R}\big)$. Thus,

$$\widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j) = \begin{cases} \{\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}, \mathbf{u}_{j+1}, \ldots, \mathbf{u}_h, \mathbf{v}_j\} \text{ if } \mathbf{v}_j \notin \widetilde{u} \\ \{\mathbf{u}_1, \ldots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_{j-1}, \mathbf{u}_{j+1}, \ldots, \mathbf{u}_h\} \text{ if } \mathbf{v}_j = \mathbf{u}_i \in \widetilde{u}. \end{cases}$$

(iii) We also define the set of all representation followers of $\widetilde{u}$ by

$$\widetilde{F}(\widetilde{u}) = \bigcup_{j=1}^{h} \bigcup_{\mathbf{v}_j \in F(\mathbf{u}_j)} \widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j).$$

Notation. Let $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\} \subseteq \mathbf{R}$, $\mathbf{v}_j \in F_k^k(\mathbf{u}_j)$. We put

$$\mu(\widetilde{u}) = \sum_{i=1}^{h}{}' \mathbf{u}_i, \quad \mu\big(\widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j)\big) = \mathbf{v}_j \oplus \sum_{i \ne j}{}' \mathbf{u}_i \in \mu\big(\widetilde{F}(\widetilde{u})\big),$$

where

$$\mu\big(\widetilde{F}(\widetilde{u})\big) = \bigcup_{j=1}^{h} \big(F(\mathbf{u}_j) \oplus \sum_{i \ne j}{}' \mathbf{u}_i\big), \quad \mu\big(\widetilde{F}(\widetilde{u})\big) = \bigcup_{j=1}^{h} \big(F(\mathbf{u}_j) \oplus \sum_{i \ne j}{}' \mathbf{u}_i\big).$$

Notes. (i) We see that $\widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j)$ is a representation, namely the representation of $\mu(\widetilde{u}) \oplus \mathbf{u}_j \oplus \mathbf{v}_j = \mathbf{v}_j \oplus \sum_{i \ne j}' \mathbf{u}_i$

(ii) Let $0 \le k < 2^t$. Every $\mathbf{u} \in \mathbf{V}_k$ has a representation over $\mathbf{R}_j$ where $j = \lceil \log_2(k+1) \rceil$. Such a representation can be constructed polynomially by computing $B^{-1}\mathbf{u}$ (see the Note at the beginning of Algorithm CEL). The elements of this representation have γ-values which are either 0 or nonnegative powers of 2.

(iii) If $\widetilde{u} = \widetilde{\mathbf{?}}$, then $\mu(\widetilde{u}) = \Phi$.

Lemma 1 implies directly:

(a) $F\big(\mu(\widetilde{u})\big) \subseteq \mu\big(\widetilde{F}(\widetilde{u})\big)$.

(b) $\mu\big(\widetilde{F}(\widetilde{u})\big) \subseteq F\big(\mu(\widetilde{u})\big) \cup F^{-1}\big(\mu(\widetilde{u})\big)$.

(c) Let $\widetilde{v} = \widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j)$, where $\mathbf{v}_j = F_k^q(\mathbf{u}_j)$. Then $\mu(\widetilde{u}) \in F\big(\mu(\widetilde{v})\big)$ if and only if either: (a) $u^k = 0$, or (b) for some $s \neq k$, $u^s = 0$ and $\mathbf{z}_k \oplus \sum'_{z_\ell \in F^q(z_k)} \mathbf{z}_\ell = \mathbf{z}_s \oplus \sum'_{z_t \in F^q(z_s)} \mathbf{z}_t$. (Note that $\mu(\widetilde{u}) = (u^0, \ldots, u^{n-1})$, $\mu(\widetilde{v}) = \mathbf{v}_j \oplus \sum'_{i \neq j} \mathbf{u}_i$.)

Example 7. We refer back to Example 5 (Fig. 3), § 7, and let $\mathbf{u} = 15 \in \mathbf{V}_0$. The equation $B\mathbf{x} = \mathbf{u}$ has then the solution $\mathbf{x} = B^{-1}\mathbf{u} = (11000)$, so $\mathbf{u}$ has a representation $\widetilde{u} = \{5, 10\}$ over Q (with $\mathbf{u} = 15$). Now

$$
\begin{aligned}
\widetilde{F}(\widetilde{u}) &= \bigcup_{j=1}^{h} \bigcup_{\mathbf{v}_j \in F(\mathbf{u}_j)} \widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j) \\
&= \widetilde{F}(\widetilde{u}; 5, 6) \cup \widetilde{F}(\widetilde{u}; 5, 9) \cup \widetilde{F}(\widetilde{u}; 5, 17) \cup \widetilde{F}(\widetilde{u}; 10, 3) \cup \widetilde{F}(\widetilde{u}; 10, 12) \cup \widetilde{F}(\widetilde{u}; 10, 18) \\
&= \{6, 10\} \cup \{9, 10\} \cup \{10, 17\} \cup \{3, 5\} \cup \{5, 12\} \cup \{5, 18\}.
\end{aligned}
$$

Thus, $\mu\big(\widetilde{F}(\widetilde{u})\big) = \{12, 3, 27, 6, 9, 23\}$. As can be seen directly from Fig. 3, $F\big(\mu(\widetilde{u})\big) = F(15) = \{3, 6, 9, 12, 23, 27\}$ so $F\big(\mu(\widetilde{u})\big) = \mu\big(\widetilde{F}(\widetilde{u})\big)$ in this case.

Recall that in steps (ii) and (iii) of Algorithm CEL, a counter function c on $\mathbf{G}^{[4]}$ and on $\mathbf{G}^{[2]}$ was computed. Since the vertex sets $\mathbf{V}^{[4]}$ of $\mathbf{G}^{[4]}$ and $\mathbf{V}^{[2]}$ of $\mathbf{G}^{[2]}$ have sizes $O(n^4)$ and $O(n^2)$ respectively, c has values bounded by $O(n^4)$ and $O(n^2)$ for these two cases. For $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\} \subseteq \mathbf{R}_{t-1}$, define $\widetilde{c}(\widetilde{u}) = \sum_{i=1}^{h} c(\mathbf{u}_i)$. We have $h \leq n$, since the $\mathbf{u}_i$ are computed using the $n \times n$ matrix B (Note 2, § 7). Thus, if $\widetilde{u} \subseteq \beta_0$, then the values of $\widetilde{c}$ are bounded by $O(n^5)$, and if $\widetilde{u} \subseteq \beta^f \setminus \beta_0$ then the values of $\widetilde{c}$ are bounded by $O(n^3)$.

Suppose now that we are given a sum of r games, one of which is a cellular automata game played on a finite cyclic digraph $G(V, E)$. It follows from (1) and (2) that for winning the sum by means of a move in G, it suffices if this winning move is of one of the following two types:

(i) Given $\mathbf{u} \in \mathbf{V}_p$ and $\mathbf{v} \in F(\mathbf{u})$ with $\gamma(\mathbf{v}) > p$, move to $\mathbf{w} \in F(\mathbf{v}) \cap \mathbf{V}_p$ with $\widetilde{c}(\widetilde{w}) < \widetilde{c}(\widetilde{u})$.

(ii) Given $\mathbf{u} \in \mathbf{V}_\ell$ and $p < \ell$; or $\gamma(\mathbf{u}) = \infty(K)$ with $p \in K$. Move to $\mathbf{v} \in F(\mathbf{u}) \cap \mathbf{V}_p$ such that $\widetilde{c}(\widetilde{v}) < \widetilde{c}(\widetilde{u})$, where we put $\widetilde{c}(\widetilde{u}) = \infty$ if $\gamma(\mathbf{u}) = \infty$.

These moves in G lead to a win when the *other* $r - 1$ component games in the sum have generalized Nim sum value p.

What's the complexity of computing these moves? The more complicated of these cases is (i). We deal with it in Theorem 5, and then summarize it, together with case (ii), in an overall strategy described in the proof of Theorem 6.

Theorem 5. *For any integer $0 \leq p \leq 2^t - 1$, letting $j = \lceil \log_2(p+1) \rceil$, a function $\Psi\colon (\mathbf{R}_j, \mathbf{V}) \to \mathbf{R}_j$ can be computed in polynomial time, such that if $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\} \subseteq \mathbf{R}_j$ is a representation of $\mu(\widetilde{u}) \in \mathbf{V}_p$ and $\mu(\widetilde{v}) \in F\big(\mu(\widetilde{u})\big)$ with $\gamma\big(\mu(\widetilde{v})\big) > p$, then $\Psi\big(\widetilde{u}, \mu(\widetilde{v})\big) = \widetilde{w} \subseteq \mathbf{R}_j$, where $\mu(\widetilde{w}) \in F\big(\mu(\widetilde{v})\big) \cap \mathbf{V}_p$ and $\widetilde{c}(\widetilde{w}) < \widetilde{c}(\widetilde{u})$.*

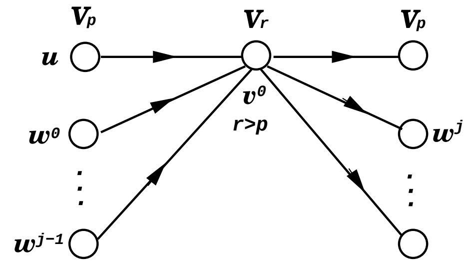

Figure 4. Illustrating Theorem 5.

Proof. Let $\mathbf{v}^0 \in F\big(\mu(\widetilde{u})\big)$ with $\gamma(\mathbf{v}^0) > \gamma(\mathbf{u})$. By (**a**), $\mathbf{v}^0 \in \mu\big(\widetilde{F}(\widetilde{u})\big)$, say $\mathbf{v}^0 = \mu\big(\widetilde{F}(\widetilde{u}; \mathbf{u}_j, \mathbf{v}_j)\big)$. Since $w(\mathbf{u}_j) \le 4$, the computation of j, k such that $\mathbf{v}_j = F_k^q(\mathbf{u}_j) = \mathbf{v}^0 \oplus \sum'_{i \ne j} \mathbf{u}_i$ takes $O(n^2)$ steps. For simplicity of notation, assume that $j = 1$. Since $\gamma(\mathbf{v}^0) > \gamma(\mathbf{u})$ and $\{\mathbf{u}_1, \ldots, \mathbf{u}_h\}$ is a representation over $\mathbf{R}_j$, i.e., its elements have γ-values 0 or distinct powers of 2, it follows that $\gamma(\mathbf{v}_1) > \gamma(\mathbf{u}_1)$ and $\mathbf{v}_1 \ne \mathbf{u}_i$ $(1 \le i \le h)$, so $\widetilde{v}^0 = \{\mathbf{v}_1, \mathbf{u}_2, \ldots, \mathbf{u}_h\}$ is a representation of $\mathbf{v}^0 = \mathbf{v}_1 \oplus \sum'^{h}_{i=2} \mathbf{u}_i$ over $\mathbf{V}$. Also $\mathbf{v}_1 \in \mathbf{V}^{[4]}$, hence $\mathbf{v}_1$ has only $O(n)$ followers, so we can compute $\mathbf{w}_1 \in F(\mathbf{v}_1) \cap \mathbf{V}^{[4]}$ with $\gamma(\mathbf{w}_1) = \gamma(\mathbf{u}_1)$ and $c(\mathbf{w}_1) < c(\mathbf{u}_1)$ in $O(n)$ steps. Hence $\widetilde{c}(\widetilde{w}^0) < \widetilde{c}(\widetilde{u})$, where $\widetilde{w}^0 = \{\mathbf{w}_1, \mathbf{u}_2, \ldots, \mathbf{u}_h\}$ if $\mathbf{w}_1 \ne \mathbf{u}_i$ $(2 \le i \le h)$, or $\widetilde{w}^0 = \{\mathbf{u}_2, \ldots, \mathbf{u}_{i-1}, \mathbf{u}_{i+1}, \ldots, \mathbf{u}_h\}$ otherwise, and in any case $\widetilde{w}^0 \subseteq \widetilde{F}(\widetilde{v}^0) \cap \mathbf{R}_j$, so $\mu(\widetilde{w}^0) \in \mu\big(\widetilde{F}(\widetilde{v}^0)\big)$. Hence by (**b**), $\mu(\widetilde{w}^0) \in F\big(\mu(\widetilde{v}^0)\big) \cup F^{-1}\big(\mu(\widetilde{v}^0)\big)$.

If $\mu(\widetilde{w}^0) \in F\big(\mu(\widetilde{v}^0)\big)$, we let $\Psi\big(\widetilde{u}, \mu(\widetilde{v}^0)\big) = \widetilde{w}^0$, which satisfies the desired requirements. If $\mu(\widetilde{v}^0) \in F\big(\mu(\widetilde{w}^0)\big)$, we replace the ancestor $\mu(\widetilde{u})$ of $\mu(\widetilde{v}^0)$ by its ancestor $\mu(\widetilde{w}^0)$ with representation $\widetilde{w}^0$. A representation $\widetilde{v}_1$ of $\mu(\widetilde{v}^0) = \mu(\widetilde{v}^1)$ can be obtained from $\widetilde{w}^0$ based on (**a**), as at the beginning of this proof. As before we get $\widetilde{w}^1 \subseteq \widetilde{F}(\widetilde{v}^1) \cap \mathbf{R}_j$ with $\widetilde{c}(\widetilde{w}^1) < \widetilde{c}(\widetilde{w}^0)$ and $\mu(\widetilde{w}^1) \in F\big(\mu(\widetilde{v}^1)\big) \cup F^{-1}\big(\mu(\widetilde{v}^1)\big)$.

This process thus leads to the formation of two sequences $\widetilde{v}^0, \widetilde{v}^1, \ldots; \widetilde{w}^0, \widetilde{w}^1, \ldots$, where $\mu(\widetilde{v}^0) = \mu(\widetilde{v}^i)$ $(i = 1, 2, \ldots)$, $\widetilde{w}^i \subseteq \mathbf{R}_j$, $\mu(\widetilde{w}^i) \in F\big(\mu(\widetilde{v}^i)\big) \cup F^{-1}\big(\mu(\widetilde{v}^i)\big)$ $(i = 1, 2, \ldots)$. Since $\widetilde{c}(\widetilde{w}^0) > \widetilde{c}(\widetilde{w}_1) > \cdots$, these sequences must be finite. In fact, each sequence has at most $O(n^5)$ terms. Since this process keeps producing a new sequence term if $\mu(\widetilde{w}^i) \in F^{-1}\big(\mu(\widetilde{v}^i)\big)$, there exists $j = O(n^5)$ such that $\mu(\widetilde{w}^j) \in F\big(\mu(\widetilde{v}^j)\big)$. We then define $\Psi\big(\widetilde{u}, \mu(\widetilde{v})\big) = \widetilde{w}^j$, which satisfies the desired requirements. The process is indicated schematically in Fig. 4.

Finally, it can be decided in $O(n)$ steps whether $\mu(\widetilde{w}^i) \in F\big(\mu(\widetilde{v}^i)\big)$ or $\mu(\widetilde{v}^i) \in F\big(\mu(\widetilde{w}^i)\big)$ by using (**c**).

Example 8. Continuing Example 6, suppose player I moves from $\mathbf{u} = 15 \in \mathbf{V}_0$ with $\widetilde{u} = \{5, 10\}$ $(\mathbf{u}_1 = 5,\ \mathbf{u}_2 = 10)$ to $\mathbf{v}^0 = 6 \in \mu\big(\widetilde{F}(\widetilde{u})\big)$. Then $\widetilde{v}^0 = \widetilde{F}(\widetilde{u}; 10, 3) = \{5, 3\} = \{\mathbf{u}_1, \mathbf{v}_2\}$ $\big(\mathbf{v}_2 = 3,\ \gamma(\mathbf{v}_2) > \gamma(\mathbf{u}_2)\big)$, and $\mu(\widetilde{v}^0) = \mathbf{v}^0$. Now $\mathbf{w}_2 \in F(\mathbf{v}_2)$ with $\gamma(\mathbf{w}_2) = \gamma(\mathbf{u}_2)$ and $c(\mathbf{w}_2) < c(\mathbf{u}_2)$ is clearly satisfied by $\mathbf{w}_2 = \Phi$. Thus $\widetilde{w}^0 = \{\mathbf{u}_1\}$, $\mu(\widetilde{w}^0) = 5$. Since $5 \in F^{-1}(6)$, we replace the ancestor $\mathbf{u}$ of $\mathbf{v}^0$ by $\mu(\widetilde{w}^0) = 5$ with representation $\widetilde{w}^0 = \{\mathbf{u}_1\}$, pretending that play began from $\mu(\widetilde{w}^0) = 5$, rather than from $\mathbf{u} = 15$. Then $\widetilde{v}^1 = \widetilde{F}(\widetilde{w}^0; 5, 6) = \{6\}$. Now $\widetilde{w}^1 = \widetilde{\mathbf{?}} \in F\big(\mu(\widetilde{v}^1)\big)$ with $\mu(\widetilde{w}^1) = \mathbf{?}$. Thus $\Psi\big(\widetilde{u}^0, \mu(\widetilde{v}^0)\big) = \widetilde{w}^1$, and so player II moves to $\mu(\widetilde{w}^1) \in \mathbf{V}_0$.

Given a sum of r games, one of which is a two-player cellular automata game played on a finite cyclic digraph $G = (V, E)$. Given an N-position of the sum, consider the subset M of moves in the cellular automata game which realize a win. How large is M ? What's the complexity of computing it?

Theorem 6. *Given an N-position in a sum of r games containing a two-player cellular automata game played on a finite cyclic digraph $G = (V, E)$ with $|V| = n$. The subset M of moves on G leading to a win has size $O(n^5)$, and its computation needs $O(n^6)$ steps.*

Proof. Apply Algorithm CEL to G $\big(O(n^6)\big)$. Given an N-position of the sum, we may assume that a winning move is of type (ii). So we have to move from a vertex u in G, which corresponds to $\mathbf{u} \in \mathbf{V}_\ell$ or to $\gamma(\mathbf{u}) = \infty(K)$, $p \in K$ in the cellular automata game-graph, to $\mathbf{v} \in F(\mathbf{u}) \cap \mathbf{V}_p$.

Assume first $\mathbf{u} \in \mathbf{V}_\ell$. Compute $B^{-1}\mathbf{u}$ to get a representation $\widetilde{u} = \{\mathbf{u}_1, \dots, \mathbf{u}_h\} \subseteq \mathbf{R}_s$ with $\mu(\widetilde{u}) = \sum'^{h}_{i=1} \mathbf{u}_i$, where $s = \lceil \log_2(\ell + 1) \rceil$ $\big(O(n^2)\big)$. For a move of type (ii), let $\mathbf{v} \in F\big(\mu(\widetilde{u})\big) \cap \mathbf{V}_p$. By **(a)**, $\mathbf{v} \in \mu\big(\widetilde{F}(\widetilde{u})\big)$, say $\mathbf{v} = \mu\big(\widetilde{F}(\widetilde{u}; \mathbf{u}_1, \mathbf{v}_1)\big)$. As we saw at the beginning of the proof of Theorem 5, the computation of $\mathbf{v}_1 = F^1_k(\mathbf{u}_1)$ takes $O(n^2)$ steps. It can always be arranged so that $\gamma(\mathbf{v}_1) < \gamma(\mathbf{u}_1)$. Also $w(\mathbf{v}_1) \le 4$. Thus $\widetilde{v}_1 = \{\mathbf{v}_1\}$ is a representation. Replacing $\mathbf{u}_1$ by $\mathbf{v}_1$ in $\widetilde{u}$ and deleting $\mathbf{v}_1$ if $\mathbf{v}_1 = \mathbf{u}_i$ for some i, we get a representation $\widetilde{v}$ of $\mathbf{v}$ over $\mathbf{R}_j$, where $j = \lceil \log_2(p + 1) \rceil$ $\big(O(n)\big)$. Since $p < \ell$ and c is monotonic we have $\widetilde{c}(\widetilde{v}) < \widetilde{c}(\widetilde{u})$.

Secondly assume $\gamma(\mathbf{u}) = \infty(K)$. Scan the $O(n^2)$ followers of $\mathbf{u}$, to locate one, say $\mathbf{v}$, which is in $\mathbf{V}_p$. Compute $B^{-1}\mathbf{v}$ to yield a representation $\widetilde{v} = \{\mathbf{v}_1, \dots, \mathbf{v}_h\} \subseteq \mathbf{R}_j$ with $\mathbf{v} = \mu(\widetilde{v}) = \sum'^{h}_{i=1} \mathbf{v}_i$, where $j = \lceil \log_2(p + 1) \rceil$ $\big(O(n^4)\big)$. By definition, $\widetilde{c}(\widetilde{v}) < \widetilde{c}(\widetilde{u})$.

In any subsequent move of type (ii) we compute the new representation from the previous one as was done above for the case $\mathbf{u} \in \mathbf{V}_\ell$, where $\widetilde{v}$ was computed from $\widetilde{u}$ in $O(n)$ steps.

For a move of type (i), assume that player II moves from $\mathbf{u}^i = \mu(\widetilde{u}^i) \in \mathbf{V}_p$ with $\widetilde{u}^i \subseteq \mathbf{R}_j$ $\big(j = \lceil \log_2(p + 1) \rceil\big)$ to $\mathbf{v}^i \in F(\mathbf{u}^i)$ with $\gamma(\mathbf{v}^i) > \gamma(\mathbf{u}^i)$. Then player I computes $\widetilde{u}^{i+1} = \Psi(\widetilde{u}_i, \mathbf{v}^i)$ $\big(O(n)\big)$ and moves to $\mathbf{u}^{i+1} = \mu(\widetilde{u}^{i+1}) \in F(\mathbf{v}^i) \cap \mathbf{V}_p$ such that $\widetilde{c}(\widetilde{u}^{i+1}) < \widetilde{c}(\widetilde{u}^i)$. This can be done as we saw in Theorem 5.

Thus $\widetilde{c}$ decreases strictly for both a move of type (i) and of type (ii). Since $\widetilde{c}(\widetilde{u}) = O(n^5)$, player I can win in $O(n^5)$ moves made in the cellular automata game, for whatever sequence of moves of type (i) and (ii) is taken. This is in addition to any other moves in the other sum components. Since each computation of one move of type (ii) and of Ψ requires $O(n)$ steps, the entire computation time for player I in the cellular automata game is $O(n^6)$ steps.

9. Epilogue

A collection of two-player cellular automata games with only a minimum of the underlying theory can be found in [12], [13].

An obvious remaining question is whether two-player cellular automata games have a polynomial strategy also for every $q > 1$. We have, in fact, provided a polynomial infrastructure for the general case, in the sense that everything up to the end of § 6 is consistent with a polynomial strategy for all $q \geq 1$. But $\mathbf{V}^{[q+1]}$ and $\mathbf{V}^{[2(q+1)]}$ are not restrictions of $\mathbf{V}$ when $q > 1$, so we cannot apply Lemma 3. Therefore we cannot prove polynomiality for $q > 1$ in the same way we used for 1-regular games.

The special case of 1-regular games are the so-called *annihilation games*, analyzed in [9], [15], [17]. The present paper is a generalization of [17], and § 7 and § 8 of the present paper are simplifications and clarifications of the corresponding parts there. For annihilation games it is natural to consider a vertex with weight 1 to be occupied by a token, and one with weight 0 to be unoccupied. Two tokens are then mutually annihilated on impact, hence the name. Misère play (in which the player making the last move loses, and the opponent wins) of annihilation games was investigated in [8]. Our motivation for examining annihilation games, suggested to us by John Conway, was to create games which exhibit some interaction between tokens, yet still have a polynomial strategy.

Annihilation games are "barely" polynomial, in several senses. Their complexity is $O(|V|^6)$, and just about any perturbation of them yields Pspace-hard games (see [16], [14], [19]). Moreover, the polynomial computation of a winning move may require a "strategy in the broad sense" (see [11], § 4).

Kalmár [21] and Smith [30] defined a *strategy in the wide sense* to be a strategy which depends on the present position and on all its antecedents, from the beginning of play. Having defined this notion, both authors concluded that it seems logical that it suffices to consider a *strategy in the narrow sense*, which is a strategy that depends only on the present position (the terminology *Markoff strategy* suggests itself here). They then promptly restricted attention to strategies in the narrow sense.

Let us define a *strategy in the broad sense* to be a strategy that depends on the present position v and on *all* its predecessors $u \in F^{-1}(v)$, whether or not such u is a position in the play of the game. This notion, if anything, seems to be even less needed than a strategy in the wide sense.

Yet, in § 8, we did employ a strategy in the broad sense, for computing a winning move in polynomial time. It was needed, since the counter function associated with γ was computed only for a small subgraph of size $O(n^4)$ of the game-graph of size $O(2^n)$, in order to preserve polynomiality. This suggests the possibility that a polynomial strategy in the narrow sense may not exist; but we have not proved anything like this. We only report that we didn't find a polynomial strategy in the narrow sense, and that perhaps the polynomial strategy in the broad sense used here suggested itself precisely because the game is "barely" polynomial, so to speak.

Annihilation games can lead to linear error correcting codes [10], but their Hamming distance is ≤ 4. The current work was motivated by the desire to create games which naturally induce codes of Hamming distance > 4. Cellular automata games may provide such codes, a topic to be taken up elsewhere. In practice, the computation of $\mathbf{V}_0$, which is all that's needed for the codes, can often be done by inspection. The best codes may be derived from a digraph which is a simplification of that of Figure 1, where $\mathbf{V}_0$ can also be computed easily. The lexicodes method [6], [5], [28] for deriving codes related to games is plainly exponential.

Another motivation was to further explore polynomial games with "token interactions". Last but not least was the desire to create two-player cellular automata games, which traditionally have been solitaire and 0-player only.

In conclusion, we have have extended cellular automata games to two-player games in a natural way and given a strategy for them. Four key ideas were used for doing this: (I). Showing that γ is essentially additive: $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)$ if $\mathbf{u}_1 \in \mathbf{V}^f$ or $\mathbf{u}_2 \in \mathbf{V}^f$. (II). Showing that $\mathbf{V}_0$, $\mathbf{V}^f$ and γ can be computed by restricting attention to the linear span of "sparse" vectors of polynomial size. (III). Providing an algorithm to compute the sparse vector space. (IV). Computing a winning move. Whereas (I) and (II) and (IV) are polynomial for all two-player cellular automata games, this has been shown for (III) only for the special case $q = 1$ (annihilation games). So the main open question is the complexity status of (III) for $q > 1$. Another question is what happens when loops are permitted in the groundgraph. This question and the polynomiality of (III) have been settled for the case of annihilation games in [17].

Finally, we point out that our strategy also solves a two-player cellular automata game with a modified move-rule: Instead of firing a neighborhood of z_k of size $q(k)$ (see the beginning of § 3), we fire a neighborhood of z_k of size *at most* $q(k)$. Indeed, we can reduce our original game to the modified one by adjoining to every z_k with $q(k) > 0$, $q(k) - 1$ edges to leaves and playing a regular two-player cellular automata game on the modified game. See also [10], Remark 4.2.

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Note added in proof The present work has indeed led to a much improved algorithm for producing *lexicodes*. Previously known algorithms had complexity exponential in n. The new algorithm has complexity $O(n^{d-1})$, where n is the size of the code, and d its distance. See A. S. Fraenkel and O. Rahat, "Complexity of error-correcting codes derived from combinatorial games"; preprint available from my homepage.

References

[1] E. R. Berlekamp, J. H. Conway and R. K. Guy [1982], *Winning Ways for Your Mathematical Plays* (two volumes), Academic Press, London.

[2] N. L. Biggs [1999], Chip-firing and the critical group of a graph, *J. Algebr. Comb.* **9**, 25–45.

[3] A. Björner and L. Lovász [1992], Chip-firing on directed graphs, *J. Algebr. Comb.* **1**, 305–328.

[4] J. H. Conway [1976], *On Numbers and Games*, Academic Press, London.

[5] J. H. Conway [1990], Integral lexicographic codes, *Discrete Math.* **83**, 219–235.

[6] J. H. Conway and N. J. A. Sloane [1986], Lexicographic codes: error-correcting codes from game theory, *IEEE Trans. Inform. Theory* **IT-32**, 337–348.

[7] Y. Dodis and P. Winkler [2001], Universal configurations in light-flipping games, *Proc. 12th Annual ACM-SIAM Sympos. on Discrete Algorithms* (Washington, DC, 2001), ACM, New York, pp. 926–927.

[8] T. S. Ferguson [1984], Misère annihilation games, *J. Combin. Theory* (Ser. A) **37**, 205–230.

[9] A. S. Fraenkel [1974], Combinatorial games with an annihilation rule, in: The Influence of Computing on Mathematical Research and Education, *Proc. Symp. Appl. Math.* (J. P. LaSalle, ed.), Vol. 20, Amer. Math. Soc., Providence, RI, pp. 87–91.

[10] A. S. Fraenkel [1996], Error-correcting codes derived from combinatorial games, in: *Games of No Chance* (R. J. Nowakowski, ed.), MSRI Publications **29**, Cambridge University Press, Cambridge, pp. 417–431.

[11] A. S. Fraenkel [1996], Scenic trails ascending from sea-level Nim to alpine chess, in: *Games of No Chance* (R. J. Nowakowski, ed.), MSRI Publications **29**, Cambridge University Press, Cambridge, pp. 13–42.

[12] A. S. Fraenkel [2001], Virus versus mankind, *Proc. 2nd Intern. Conference on Computers and Games CG'2000* (T. Marsland and I. Frank, eds.), Vol. 2063, Hamamatsu, Japan, Oct. 2000, Lecture Notes in Computer Science, Springer, pp. 204–213.

[13] A. S. Fraenkel [2002], Mathematical chats between two physicists, in: *Puzzler's Tribute: a Feast for the Mind* (D. Wolfe and T. Rodgers, eds.), honoring Martin Gardner, A K Peters, Natick, MA., pp. 383–386.

[14] A. S. Fraenkel and E. Goldschmidt [1987], Pspace-hardness of some combinatorial games, *J. Combin. Theory* (Ser. A) **46**, 21–38.

[15] A. S. Fraenkel and Y. Yesha [1976], Theory of annihilation games, *Bull. Amer. Math. Soc.* **82**, 775–777.

[16] A. S. Fraenkel and Y. Yesha [1979], Complexity of problems in games, graphs and algebraic equations, *Discrete Appl. Math.* **1**, 15–30.

[17] A. S. Fraenkel and Y. Yesha [1982], Theory of annihilation games — I, *J. Combin. Theory* (Ser. B) **33**, 60–86.

[18] A. S. Fraenkel and Y. Yesha [1986], The generalized Sprague–Grundy function and its invariance under certain mappings, *J. Combin. Theory* (Ser. A) **43**, 165–177.

[19] A. S. Goldstein and E. M. Reingold [1995], The complexity of pursuit on a graph, *Theoret. Comput. Sci.* (Math Games) **143**, 93–112.

[20] E. Goles [1991], Sand piles, combinatorial games and cellular automata, *Math. Appl.* **64**, 101–121.

[21] L. Kalmár [1928], Zur Theorie der abstrakten Spiele, *Acta Sci. Math. Univ. Szeged* **4**, 65–85.

[22] J. L. Kelley [1965], Algebra, A Modern Introduction, Van Nostrand, Princeton, NJ.

[23] J. Lambek and L. Moser [1959], On some two way classifications of the integers, *Can. Math. Bull.* **2**, 85–89.

[24] C. M. López [1997], Chip firing and the Tutte polynomial, *Ann. of Comb.* **1**, 253–259.

[25] Ó. Martín-Sánchez and C. Pareja-Flores [2001], Two reflected analyses of lights out, *Math. Mag.* **74**, 295–304.

[26] J. Missigman and R. Weida [2001], An easy solution to mini lights out, *Math. Mag.* **74**, 57-59.

[27] D. H. Pelletier [1987], Merlin's magic square, *Amer. Math. Monthly* **94**, 143–150.

[28] V. Pless [1991], Games and codes, in: *Combinatorial Games,* Proc. Symp. Appl. Math. (R. K. Guy, ed.), Vol. 43, Amer. Math. Soc., Providence, RI, pp. 101–110.

[29] A. Shen [2000], Lights out, *Math. Intelligencer* **22**, 20-21.

[30] C. A. B. Smith [1966], Graphs and composite games, *J. Combin. Theory* **1**, 51–81. Reprinted in slightly modified form in: *A Seminar on Graph Theory* (F. Harary, ed.), Holt, Rinehart and Winston, New York, NY, 1967.

[31] D. L. Stock [1989], Merlin's magic square revisited, *Amer. Math. Monthly* **96**, 608–610.

[32] K. Sutner [1988], On σ-automata, *Complex Systems* **2**, 1–28.

[33] K. Sutner [1989], Linear cellular automata and the Garden-of-Eden, *Math. Intelligencer* **11**, 49–53.

[34] K. Sutner [1990], The σ-game and cellular automata, *Amer. Math. Monthly* **97**, 24–34.

[35] K. Sutner [1995], On the computational complexity of finite cellular automata, *J. Comput. System Sci.* **50**, 87–97.

[36] G. P. Tollisen and T. Lengyel [2000], Color switching games, *Ars Combin.* **56**, 223-234.

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