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On the Classification of Finite-Dimensional Triangular Hopf Algebras

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Abstract. A fundamental problem in the theory of Hopf algebras is the classification and construction of finite-dimensional quasitriangular Hopf algebras over $\mathbb{C}$. Quasitriangular Hopf algebras constitute a very important class of Hopf algebras, introduced by Drinfeld. They are the Hopf algebras whose representations form a braided tensor category. However, this intriguing problem is extremely hard and is still widely open. Triangular Hopf algebras are the quasitriangular Hopf algebras whose representations form a symmetric tensor category. In that sense they are the closest to group algebras. The structure of triangular Hopf algebras is far from trivial, and yet is more tractable than that of general Hopf algebras, due to their proximity to groups. This makes triangular Hopf algebras an excellent testing ground for general Hopf algebraic ideas, methods and conjectures. A general classification of triangular Hopf algebras is not known yet. However, the problem was solved in the semisimple case, in the minimal triangular pointed case, and more generally for triangular Hopf algebras with the Chevalley property. In this paper we report on all of this, and explain in full details the mathematics and ideas involved in this theory. The classification in the semisimple case relies on Deligne's theorem on Tannakian categories and on Movshev's theory in an essential way. We explain Movshev's theory in details, and refer to [G5] for a detailed discussion of the first aspect. We also discuss the existence of grouplike elements in quasitriangular semisimple Hopf algebras, and the representation theory of cotriangular semisimple Hopf algebras. We conclude the paper with a list of open problems; in particular with the question whether any finite-dimensional triangular Hopf algebra over $\mathbb{C}$ has the Chevalley property.

1. Introduction

A fundamental problem in the theory of Hopf algebras is the classification and construction of finite-dimensional quasitriangular Hopf algebras (A, R) over an algebraically closed field k. Quasitriangular Hopf algebras constitute a very important class of Hopf algebras, which were introduced by Drinfeld [Dr1] in order to supply solutions to the quantum Yang-Baxter equation that arises in mathematical physics. Quasitriangular Hopf algebras are the Hopf algebras whose finite-dimensional representations form a braided rigid tensor category, which naturally relates them to low dimensional topology. Furthermore, Drinfeld showed that *any* finite-dimensional Hopf algebra can be embedded in a finite-dimensional quasitriangular Hopf algebra, known now as its Drinfeld double or quantum double. However, this intriguing problem turns out to be extremely hard and it is still widely open. One can hope that resolving this problem first in the *triangular* case would contribute to the understanding of the general problem.

Triangular Hopf algebras are the Hopf algebras whose representations form a symmetric tensor category. In that sense, they are the class of Hopf algebras closest to group algebras. The structure of triangular Hopf algebras is far from trivial, and yet is more tractable than that of general Hopf algebras, due to their proximity to groups and Lie algebras. This makes triangular Hopf algebras an excellent testing ground for general Hopf algebraic ideas, methods and conjectures. A general classification of triangular Hopf algebras is not known yet. However, there are two classes that are relatively well understood. One of them is semisimple triangular Hopf algebras over k (and cosemisimple if the characteristic of k is positive) for which a complete classification is given in [EG1, EG4]. The key theorem about such Hopf algebras states that each of them is obtained by twisting a group algebra of a finite group [EG1, Theorem 2.1] (see also [G5]).

Another important class of Hopf algebras is that of *pointed* ones. These are Hopf algebras whose all simple comodules are 1-dimensional. Theorem 5.1 in [G4] (together with [AEG, Theorem 6.1]) gives a classification of *minimal* triangular pointed Hopf algebras.

Recall that a finite-dimensional algebra is called *basic* if all of its simple modules are 1-dimensional (i.e., if its dual is a pointed coalgebra). The same Theorem 5.1 of [G4] gives a classification of minimal triangular basic Hopf algebras, since the dual of a minimal triangular Hopf algebra is again minimal triangular.

Basic and semisimple Hopf algebras share a common property. Namely, the Jacobson radical $\mathrm{Rad}(A)$ of such a Hopf algebra A is a Hopf ideal, and hence the quotient $A/\mathrm{Rad}(A)$ (the semisimple part) is itself a Hopf algebra. The representation-theoretic formulation of this property is: The tensor product of two simple A-modules is semisimple. A remarkable classical theorem of Chevalley [C, p. 88] states that, in characteristic 0, this property holds for the group algebra of any (not necessarily finite) group. So we called this property of A **the Chevalley property** [AEG].

In [AEG] it was proved that any finite-dimensional triangular Hopf algebra with the Chevalley property is obtained by twisting a finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2, and that any finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2 is a suitable modification of a finite-dimensional cocommutative Hopf superalgebra (i.e., the group algebra of a finite supergroup). On the other hand, by a theorem of Kostant [Ko], a finite supergroup is a semidirect product of a finite group with an odd vector space on which this group acts. Moreover, the converse result that any such Hopf algebra does have the Chevalley property is also proved in [AEG]. As a corollary, we proved that any finite-dimensional triangular Hopf algebra whose coradical is a Hopf subalgebra (e.g., pointed) is obtained by twisting a triangular Hopf algebra with R-matrix of rank ≤ 2.

The purpose of this paper is to present all that is currently known to us about the classification and construction of finite-dimensional triangular Hopf algebras, and to explain the mathematics and ideas involved in this theory.

The paper is organized as follows. In Section 2 we review some necessary material from the theory of Hopf algebras. In particular the important notion of a twist for Hopf algebras, which was introduced by Drinfeld [Dr1].

In Section 3 we explain in details the theory of Movshev on twisting in group algebras of finite groups [Mov]. The results of [EG4, EG5] (described in Sections 4 and 5 below) rely, among other things, on this theory in an essential way.

In Section 4 we concentrate on the theory of triangular semisimple and cosemisimple Hopf algebras. We first describe the classification and construction of triangular semisimple and cosemisimple Hopf algebras over *any* algebraically closed field k, and then describe some of the consequences of the classification theorem, in particular the one concerning the existence of grouplike elements in triangular semisimple and cosemisimple Hopf algebras over k [EG4]. The classification uses, among other things, Deligne's theorem on Tannakian categories [De1] in an essential way. We refer the reader to [G5] for a detailed discussion of this aspect. The proof of the existence of grouplike elements relies on a theorem from [HI] on central type groups being solvable, which is proved using the classification of finite simple groups. The classification in positive characteristic relies also on the lifting functor from [EG5].

In Section 5 we concentrate on the dual objects of Section 4; namely, on semisimple and cosemisimple *cotriangular* Hopf algebras over k, studied in [EG3]. We describe the representation theory of such Hopf algebras, and in particular obtain that Kaplansky's 6th conjecture [Kap] holds for them (i.e., they are of Frobenius type).

In Section 6 we concentrate on the pointed case, studied in [G4] and [AEG, Theorem 6.1]. The main result in this case is the classification of minimal triangular pointed Hopf algebras.

In Section 7 we generalize and concentrate on the classification of finite-dimensional triangular Hopf algebras with the Chevalley property, given in [AEG]. We note that similarly to the case of semisimple Hopf algebras, the proof of the main result of [AEG] is based on Deligne's theorem [De1]. In fact, we used Theorem 2.1 of [EG1] to prove the main result of this paper.

In Section 8 we conclude the paper with a list of relevant questions raised in [AEG] and [G4].

Throughout the paper the ground field k is assumed to be algebraically closed. The symbol $\mathbb{C}$ will always denote the field of complex numbers. For a Hopf (super)algebra A, $\mathbf{G}(A)$ will denote its group of grouplike elements.

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2. Preliminaries

In this Section we recall the necessary background needed for this paper. We refer the reader to the books [ES, Kass, Mon, Sw] for the general theory of Hopf algebras and quantum groups.

2.1. Quasitriangular Hopf algebras. We recall Drinfeld's notion of a (quasi)triangular Hopf algebra [Dr1]. Let $(A,m,1,\Delta,\varepsilon,S)$ be a finite-dimensional Hopf algebra over k, and let $R=\sum_i a_i\otimes b_i\in A\otimes A$ be an invertible element. Define a linear map $f_R:A^*\to A$ by $f_R(p)=\sum_i\langle p,a_i\rangle b_i$ for $p\in A^*$. The tuple $(A,m,1,\Delta,\varepsilon,S,R)$ is said to be a *quasitriangular* Hopf algebra if the following axioms hold:

$$(\Delta\otimes\mathrm{Id})(R)=R_{13}R_{23},\ (\mathrm{Id}\otimes\Delta)(R)=R_{13}R_{12} \tag{2–1}$$

where Id is the identity map of A, and

$$\Delta^{\mathrm{cop}}(a)R=R\Delta(a)\ \text{forany}\ a\in A; \tag{2–2}$$

or, equivalently, if $f_R:A^*\to A^{\mathrm{cop}}$ is a Hopf algebra map and (2–2) is satisfied. The element R is called an R-matrix. Observe that using Sweedler's notation for the comultiplication [Sw], (2–2) is equivalent to

$$\sum\langle p_{(1)},a_{(2)}\rangle a_{(1)}f_{_R}(p_{(2)})=\sum\langle p_{(2)},a_{(1)}\rangle f_{_R}(p_{(1)})a_{(2)} \tag{2–3}$$

for any $p\in A^*$ and $a\in A$.

A quasitriangular Hopf algebra (A,R) is called *triangular* if $R^{-1}=R_{21}$; or equivalently, if $f_R*f_{R_{21}}=\varepsilon$ in the convolution algebra $\mathrm{Hom}_k(A^*,A)$, i.e. (using Sweedler's notation again)

$$\sum f_R(p_{(1)})f_{R_{21}}(p_{(2)})=\langle p,1\rangle 1\ \text{for any}\ p\in A^*. \tag{2–4}$$

Let

$$u:=\sum_i S(b_i)a_i \tag{2–5}$$

be the *Drinfeld element* of (A,R). Drinfeld showed [Dr2] that u is invertible and that

$$S^2(a)=uau^{-1}\ \text{for any}\ a\in A. \tag{2–6}$$

He also showed that (A,R) is triangular if and only if u is a grouplike element [Dr2].

Suppose also that $(A,m,1,\Delta,\varepsilon,S,R)$ is *semisimple and cosemisimple* over k.

LEMMA 2.1.1. *The Drinfeld element u is central, and*

$$u = S(u). \tag{2–7}$$

PROOF. By [LR1] in characteristic 0, and by [EG5, Theorem 3.1] in positive characteristic, $S^2 = I$. Hence by (2–6), u is central. Now, we have $(S\otimes S)(R) = R$ [Dr2], so $S(u) = \sum_i S(a_i)S^2(b_i) = \sum_i a_i S(b_i)$. This shows that $\mathrm{tr}(u) = \mathrm{tr}(S(u))$ in every irreducible representation of A. But u and $S(u)$ are central, so they act as scalars in this representation, which proves (2–7). □

LEMMA 2.1.2. *In particular*,

$$u^2 = 1. \tag{2–8}$$

PROOF. Since $S(u) = u^{-1}$, the result follows from (2–7). □

Let us demonstrate that it is always possible to replace R with a new R-matrix $\tilde{R}$ so that the new Drinfeld element $\tilde{u}$ equals 1. Indeed, if k does not have characteristic 2, set

$$R_u := \tfrac{1}{2}(1\otimes 1 + 1\otimes u + u\otimes 1 - u\otimes u). \tag{2–9}$$

If k is of characteristic 2 (in which case $u = 1$ by semisimplicity), set $R_u := 1$. Set $\tilde{R} := RR_u$.

LEMMA 2.1.3. $(A,\tilde{R})$ *is a triangular semisimple and cosemisimple Hopf algebra with Drinfeld element* 1.

PROOF. Straightforward. □

This observation allows to reduce questions about triangular semisimple and cosemisimple Hopf algebras over k to the case when the Drinfeld element is 1.

Let (A,R) be *any* triangular Hopf algebra over k. Write $R = \sum_{i=1}^{n} a_i \otimes b_i$ in the shortest possible way, and let A_m be the Hopf subalgebra of A generated by the a_i's and b_i's. Following [R2], we will call A_m the *minimal part* of A. We will call $n = \dim(A_m)$ the *rank* of the R-matrix R. It is straightforward to verify that the corresponding map $f_R : A_m^{*\mathrm{cop}} \to A_m$ defined by $f_R(p) = (p\otimes I)(R)$ is a Hopf algebra isomorphism. This property of minimal triangular Hopf algebras will play a central role in our study of the pointed case (see Section 6 below). It implies in particular that $\mathbf{G}(A_m) \cong \mathbf{G}((A_m)^*)$, and hence that the group $\mathbf{G}(A_m)$ is *abelian* (see e.g., [G2]). Thus, $\mathbf{G}(A_m) \cong \mathbf{G}(A_m)^\vee$ (where $\mathbf{G}(A_m)^\vee$ denotes the character group of $\mathbf{G}(A_m)$), and we can identify the Hopf algebras $k[\mathbf{G}(A_m)^\vee]$ and $k[\mathbf{G}(A_m)]^*$. Also, if (A,R) is minimal triangular and pointed then f_R being an isomorphism implies that A^* is pointed as well.

Note that if (A,R) is (quasi)triangular and $\pi : A \to A'$ is a surjective map of Hopf algebras, then (A',R') is (quasi)triangular as well, where $R' := (\pi\otimes\pi)(R)$.

2.2. Hopf superalgebras.

2.2.1. Supervector spaces. We start by recalling the definition of the category of supervector spaces. A Hopf algebraic way to define this category is as follows. Let us assume that $k = \mathbb{C}$.

Let u be the generator of the order-two group $\mathbb{Z}_2$, and let $R_u \in \mathbb{C}[\mathbb{Z}_2]\otimes\mathbb{C}[\mathbb{Z}_2]$ be as in (2–9). Then $(\mathbb{C}[\mathbb{Z}_2], R_u)$ is a minimal triangular Hopf algebra.

Definition 2.2.1.1. The category of supervector spaces over $\mathbb{C}$ is the symmetric tensor category $\mathrm{Rep}(\mathbb{C}[\mathbb{Z}_2], R_u)$ of representations of the triangular Hopf algebra $(\mathbb{C}[\mathbb{Z}_2], R_u)$. This category will be denoted by SuperVect.

For $V \in$ SuperVect and $v \in V$, we say that v is even if $uv = v$ and odd if $uv = -v$. The set of even vectors in V is denoted by V_0 and the set of odd vectors by V_1, so $V = V_0 \oplus V_1$. We define the parity of a vector v to be $p(v) = 0$ if v is even and $p(v) = 1$ if v is odd (if v is neither odd nor even, $p(v)$ is not defined).

Thus, as an ordinary tensor category, SuperVect is equivalent to the category of representations of $\mathbb{Z}_2$, but the commutativity constraint is different from that of $\mathrm{Rep}(\mathbb{Z}_2)$ and equals $\beta := R_u P$, where P is the permutation of components. In other words, we have

$$\beta(v \otimes w) = (-1)^{p(v)p(w)} w \otimes v, \tag{2–10}$$

where both v, w are either even or odd.

2.2.2. Hopf superalgebras. Recall that in any symmetric (more generally, braided) tensor category, one can define an algebra, coalgebra, bialgebra, Hopf algebra, triangular Hopf algebra, etc, to be an object of this category equipped with the usual structure maps (morphisms in this category), subject to the same axioms as in the usual case. In particular, any of these algebraic structures in the category SuperVect is usually identified by the prefix "super". For example:

Definition 2.2.2.1. A Hopf superalgebra is a Hopf algebra in SuperVect.

More specifically, a Hopf superalgebra $\mathcal{A}$ is an ordinary $\mathbb{Z}_2$-graded associative unital algebra with multiplication m, equipped with a coassociative map

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

(a morphism in SuperVect) which is multiplicative in the super-sense, and with a counit and antipode satisfying the standard axioms. Here multiplicativity in the super-sense means that Δ satisfies the relation

$$\Delta(ab) = \sum (-1)^{p(a_2)p(b_1)} a_1 b_1 \otimes a_2 b_2 \tag{2–11}$$

for all $a, b \in \mathcal{A}$ (where $\Delta(a) = \sum a_1 \otimes a_2$, $\Delta(b) = \sum b_1 \otimes b_2$). This is because the tensor product of two algebras A, B in SuperVect is defined to be $A \otimes B$ as a vector space, with multiplication

$$(a \otimes b)(a' \otimes b') := (-1)^{p(a')p(b)} aa' \otimes bb'. \tag{2–12}$$

REMARK 2.2.2.2. Hopf superalgebras appear in [Ko], under the name of "graded Hopf algebras".

Similarly, a (quasi)triangular Hopf superalgebra $(\mathcal{A}, \mathcal{R})$ is a Hopf superalgebra with an R-matrix (an *even* element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$) satisfying the usual axioms. As in the even case, an important role is played by the Drinfeld element u of $(\mathcal{A}, \mathcal{R})$:

$$u := m \circ \beta \circ (\mathrm{Id} \otimes S)(\mathcal{R}). \tag{2–13}$$

For instance, $(\mathcal{A}, \mathcal{R})$ is triangular if and only if u is a grouplike element of $\mathcal{A}$.

As in the even case, the tensorands of the R-matrix of a (quasi)triangular Hopf superalgebra $\mathcal{A}$ generate a finite-dimensional sub Hopf superalgebra $\mathcal{A}_m$, called the *minimal part of* $\mathcal{A}$ (the proof does not differ essentially from the proof of the analogous fact for Hopf algebras). A (quasi)triangular Hopf superalgebra is said to be minimal if it coincides with its minimal part. The dimension of the minimal part is the *rank* of the R-matrix.

2.2.3. Cocommutative Hopf superalgebras.

DEFINITION 2.2.3.1. We will say that a Hopf superalgebra $\mathcal{A}$ is commutative (resp. cocommutative) if $m = m \circ \beta$ (resp. $\Delta = \beta \circ \Delta$).

EXAMPLE 2.2.3.2 [Ko]. Let G be a group, and $\mathbf{g}$ a Lie superalgebra with an action of G by automorphisms of Lie superalgebras. Let $\mathcal{A} := \mathbb{C}[G] \ltimes \mathrm{U}(\mathbf{g})$, where $\mathrm{U}(\mathbf{g})$ denotes the universal enveloping algebra of $\mathbf{g}$. Then $\mathcal{A}$ is a cocommutative Hopf superalgebra, with $\Delta(x) = x \otimes 1 + 1 \otimes x$, $x \in \mathbf{g}$, and $\Delta(g) = g \otimes g$, $g \in G$. In this Hopf superalgebra, we have $S(g) = g^{-1}$, $S(x) = -x$, and in particular $S^2 = \mathrm{Id}$.

The Hopf superalgebra $\mathcal{A}$ is finite-dimensional if and only if G is finite, and $\mathbf{g}$ is finite-dimensional and purely odd (and hence commutative). Then $\mathcal{A} = \mathbb{C}[G] \ltimes \Lambda V$, where $V = \mathbf{g}$ is an odd vector space with a G-action. In this case, $\mathcal{A}^*$ is a commutative Hopf superalgebra.

REMARK 2.2.3.3. We note that as in the even case, it is convenient to think about $\mathcal{A}$ and $\mathcal{A}^*$ in geometric terms. Consider, for instance, the finite-dimensional case. In this case, it is useful to think of the "affine algebraic supergroup" $\tilde{G} := G \ltimes V$. Then one can regard $\mathcal{A}$ as the group algebra $\mathbb{C}[\tilde{G}]$ of this supergroup, and $\mathcal{A}^*$ as its function algebra $F(\tilde{G})$. Having this in mind, we will call the algebra $\mathcal{A}$ **a supergroup algebra**.

It turns out that like in the even case, any cocommutative Hopf superalgebra is of the type described in Example 2.2.3.2. Namely, we have the following theorem.

THEOREM 2.2.3.4. ([**Ko**], **Theorem** 3.3) *Let $\mathcal{A}$ be a cocommutative Hopf superalgebra over $\mathbb{C}$. Then $\mathcal{A} = \mathbb{C}[\mathbf{G}(\mathcal{A})] \ltimes U(P(\mathcal{A}))$, where $U(P(\mathcal{A}))$ is the universal* *enveloping algebra of the Lie superalgebra of primitive elements of* $\mathcal{A}$, *and* $\mathbf{G}(\mathcal{A})$ *is the group of grouplike elements of* $\mathcal{A}$.

In particular, in the finite-dimensional case we get:

COROLLARY 2.2.3.5. *Let* $\mathcal{A}$ *be a finite-dimensional cocommutative Hopf superalgebra over* $\mathbb{C}$. *Then* $\mathcal{A} = \mathbb{C}[\mathbf{G}(\mathcal{A})] \ltimes \Lambda V$, *where* V *is the space of primitive elements of* $\mathcal{A}$ (*regarded as an odd vector space*) *and* $\mathbf{G}(\mathcal{A})$ *is the finite group of grouplikes of* $\mathcal{A}$. *In other words*, $\mathcal{A}$ *is a supergroup algebra*.

2.3. Twists. Let $(A, m, 1, \Delta, \varepsilon, S)$ be a Hopf algebra over a field k. We recall Drinfeld's notion of a *twist* for A [Dr1].

DEFINITION 2.3.1. A quasitwist for A is an element $J \in A \otimes A$ which satisfies

$$\left.\begin{aligned}(\Delta \otimes \mathrm{Id})(J)(J \otimes 1) = (\mathrm{Id} \otimes \Delta)(J)(1 \otimes J),\\ (\varepsilon \otimes \mathrm{Id})(J) = (\mathrm{Id} \otimes \varepsilon)(J) = 1.\end{aligned}\right\} \tag{2–14}$$

An invertible quasitwist for A is called a *twist*.

Given a twist J for A, one can define a new Hopf algebra structure

$$(A^J, m, 1, \Delta^J, \varepsilon, S^J)$$

on the algebra $(A, m, 1)$ as follows. The coproduct is determined by

$$\Delta^J(a) = J^{-1}\Delta(a)J \text{ for any } a \in A, \tag{2–15}$$

and the antipode is determined by

$$S^J(a) = Q^{-1}S(a)Q \text{ for any } a \in A, \tag{2–16}$$

where $Q := m \circ (S \otimes \mathrm{Id})(J)$. If A is (quasi)triangular with the universal R-matrix R, then so is A^J, with the universal R-matrix $R^J := J_{21}^{-1}RJ$.

EXAMPLE 2.3.2. Let G be a finite *abelian* group, and $G^\vee$ its character group. Then the set of twists for $A := k[G]$ is in one to one correspondence with the set of 2-cocycles c of $G^\vee$ with coefficients in k^*, such that $c(0,0) = 1$. Indeed, let J be a twist for A, and define $c : G^\vee \times G^\vee \to k^*$ via $c(\chi, \psi) := (\chi \otimes \psi)(J)$. Then it is straightforward to verify that c is a 2-cocycle of $G^\vee$ (see e.g., [Mov, Proposition 3]), and that $c(0,0) = 1$.

Conversely, let $c : G^\vee \times G^\vee \to k^*$ be a 2-cocycle of $G^\vee$ with coefficients in k^*, such that $c(0,0) = 1$. Note that the 2-cocycle condition implies that $c(0,\chi) = 1 = c(\chi, 0)$ for all $\chi \in G^\vee$. For $\chi \in G^\vee$, let $E_\chi := |G|^{-1}\sum_{g\in G}\chi(g)g$ be the associated idempotent of A. Then it is straightforward to verify that $J := \sum_{\chi,\psi\in G^\vee} c(\chi,\psi)E_\chi \otimes E_\psi$ is a twist for A (see e.g., [Mov, Proposition 3]). Moreover it is easy to check that the above two assignments are inverse to each other.

REMARK 2.3.3. Unlike for finite abelian groups, the study of twists for finite non-abelian groups is much more involved. This was done in [EG2, EG4, Mov] (see Section 4 below).

If J is a (quasi)twist for A and x is an invertible element of A such that $\varepsilon(x) = 1$, then

$$J^x := \Delta(x)J(x^{-1} \otimes x^{-1}) \tag{2–17}$$

is also a (quasi)twist for A. We will call the (quasi)twists J and J^x *gauge equivalent*. Observe that if (A, R) is a (quasi)triangular Hopf algebra, then the map $(A^J, R^J) \to (A^{J^x}, R^{J^x})$ determined by $a \mapsto xax^{-1}$ is an isomorphism of (quasi)triangular Hopf algebras.

Let A be a group algebra of a finite group. We will say that a twist J for A is *minimal* if the right (and left) components of the R-matrix $R^J := J_{21}^{-1}J$ span A, i.e., if the corresponding triangular Hopf algebra $(A^J, J_{21}^{-1}J)$ is minimal.

A twist for a Hopf algebra in *any symmetric tensor category* is defined in the same way as in the usual case. For instance, if $\mathcal{A}$ is a Hopf superalgebra then a twist for $\mathcal{A}$ is an invertible *even* element $\mathcal{J} \in \mathcal{A} \otimes \mathcal{A}$ satisfying (2–14).

2.4. Projective representations and central extensions. Here we recall some basic facts about projective representations and central extensions. They can be found in textbooks, e.g. [CR, Section 11E].

A projective representation over k of a group H is a vector space V together with a homomorphism of groups $\pi_V : H \to \mathrm{PGL}(V)$, where $\mathrm{PGL}(V) \cong \mathrm{GL}(V)/k^*$ is the projective linear group.

A linearization of a projective representation V of H is a central extension $\hat{H}$ of H by a central subgroup ζ together with a linear representation $\tilde{\pi}_V : \hat{H} \to \mathrm{GL}(V)$ which descends to π_V. If V is a finite-dimensional projective representation of H then there exists a linearization of V such that ζ is finite (in fact, one can make $\zeta = \mathbb{Z}/(\dim(V))\mathbb{Z}$).

Any projective representation V of H canonically defines a cohomology class $[V] \in H^2(H, k^*)$. The representation V can be lifted to a linear representation of H if and only if $[V] = 0$.

2.5. Pointed Hopf algebras. The Hopf algebras which are studied in Section 6 are pointed. Recall that a Hopf algebra A is *pointed* if its simple subcoalgebras are all 1-dimensional or equivalently (when A is finite-dimensional) if the irreducible representations of A^* are all 1-dimensional (i.e., A^* is basic). For any $g, h \in \mathbf{G}(A)$, we denote the vector space of $g : h$ *skew primitives* of A by $P_{g,h}(A) := \{x \in A \mid \Delta(x) = x \otimes g + h \otimes x\}$. Thus the classical *primitive* elements of A are $P(A) := P_{1,1}(A)$. The element $g - h$ is always $g : h$ skew primitive. Let $P'_{g,h}(A)$ denote a complement of $sp_k\{g - h\}$ in $P_{g,h}(A)$. Taft-Wilson theorem

[TW] states that the first term A_1 of the coradical filtration of A is given by:

$$A_1 = k[\mathbf{G}(A)] \bigoplus \biggl(\bigoplus_{g,h \in \mathbf{G}(A)} P'_{g,h}(A) \biggr). \tag{2–18}$$

In particular, if A is *not* cosemisimple then there exists $g \in \mathbf{G}(A)$ such that $P'_{1,g}(A) \neq 0$.

If A is a Hopf algebra over the field k, which is generated as an algebra by a subset S of $\mathbf{G}(A)$ and by $g : g'$ skew primitive elements, where g, g' run over S, then A is pointed and $\mathbf{G}(A)$ is generated as a group by S (see e.g., [R4, Lemma 1]).

3. Movshev's Theory on the Algebra Associated with a Twist

In this section we describe Movshev's theory on twisting in group algebras of finite groups [Mov]. Our classification theory of triangular semisimple and cosemisimple Hopf algebras [EG4] (see Section 4 below), and our study of the representation theory of cotriangular semisimple and cosemisimple Hopf algebras [EG3] (see Section 5 below) rely, among other things, on this theory in an essential way.

Let k be an algebraically closed field whose characteristic is relatively prime to $|G|$. Let $A := k[G]$ be the group algebra of a finite group G, equipped with the usual multiplication, unit, comultiplication, counit and antipode, denoted by m, 1, Δ, ε and S respectively. Let $J \in A \otimes A$. Movshev had the following nice idea of characterizing quasitwists [Mov]. Let $(A_J, \Delta_J, \varepsilon)$ where $A_J = A$ as vector spaces, and Δ_J is the map

$$\Delta_J : A \to A \otimes A,\ a \mapsto \Delta(a)J. \tag{3–1}$$

Proposition 3.1. *$(A_J, \Delta_J, \varepsilon)$ is a coalgebra if and only if J is a quasitwist for A.*

Proof. Straightforward. □

Regard A as the left regular representation of G. Then $(A_J, \Delta_J, \varepsilon)$ is a G-coalgebra (i.e., $\Delta_J(ga) = (g \otimes g)\Delta_J(a)$ and $\varepsilon(ga) = \varepsilon(a)$ for all $g \in G$, $a \in A$). In fact, we have the following important result.

Proposition 3.2 [Mov, Proposition 5]. *Suppose that $(C, \tilde{\Delta}, \tilde{\varepsilon})$ is a G-coalgebra which is isomorphic to the regular representation of G as a G-module. Then there exists a quasitwist $J \in A \otimes A$ such that $(C, \tilde{\Delta}, \tilde{\varepsilon})$ and $(A, \Delta_J, \varepsilon)$ are isomorphic as G-coalgebras. Moreover, J is unique up to gauge equivalence.*

Proof. We can choose an element $\lambda \in C$ such that the set $\{g \cdot \lambda \mid g \in G\}$ forms a basis of C, and $\tilde{\varepsilon}(\lambda) = 1$. Now, write $\tilde{\Delta}(\lambda) = \sum_{a,b \in G} \gamma(a,b) a \cdot \lambda \otimes b \cdot \lambda$, and set

$$J := \sum_{a,b \in G} \gamma(a,b) a \otimes b \in A \otimes A. \tag{3–2}$$

We have to show that J is a quasitwist for A. Indeed, let $f : A \to C$ be determined by $f(a) = a \cdot \lambda$. Clearly, f is an isomorphism of G-modules which satisfies $\tilde{\Delta}(f(a)) = (f \otimes f)\Delta_J(a)$, $a \in A$. Therefore $(A_J, \Delta_J, \varepsilon)$ is a coalgebra, which is equivalent to saying that J is a quasitwist by Proposition 3.1. This proves the first claim.

Suppose that $(A_{J'}, \Delta_{J'}, \varepsilon)$ and $(A_J, \Delta_J, \varepsilon)$ are isomorphic as G-coalgebras via $\phi : A \to A$. We have to show that J, J' are gauge equivalent. Indeed, ϕ is given by right multiplication by an invertible element $x \in A$, $\phi(a) = ax$. On one hand, $(\phi \otimes \phi)(\Delta_J(1)) = J(x \otimes x)$, and on the other hand, $\Delta_{J'}(\phi(1)) = \Delta(x)J'$. The equality between the two right hand sides implies the desired result. □

We now focus on the dual algebra $(A_J)^*$ of the coalgebra $(A_J, \Delta_J, \varepsilon)$, and summarize Movshev's results about it [Mov]. Note that $(A_J)^*$ is a G-algebra which is isomorphic to the regular representation of G as a G-module.

Proposition 3.3 [Mov, Propositions 6 and 7]. 1. *The algebra* $(A_J)^*$ *is semisimple.*

2. *There exists a subgroup* St *of* G (*the stabilizer of a maximal two sided ideal* I *of* $(A_J)^*$) *such that* $(A_J)^*$ *is isomorphic to the algebra of functions from the set* G/St *to the matrix algebra* $M_{|\mathrm{St}|^{1/2}}(k)$.

Note that, in particular, the group St acts on the matrix algebra $(A_J)^*/I \cong M_{|\mathrm{St}|^{1/2}}(k)$. Hence this algebra defines a projective representation $T : \mathrm{St} \to \mathrm{PGL}(|\mathrm{St}|^{1/2}, k)$ (since $\mathrm{Aut}(M_{|\mathrm{St}|^{1/2}}(k)) = \mathrm{PGL}(|\mathrm{St}|^{1/2}, k)$).

Proposition 3.4 [Mov, Propositions 8 and 9]. *T is irreducible, and the associated 2-cocycle* $c : \mathrm{St} \times \mathrm{St} \to k^*$ *is nontrivial.*

Consider the twisted group algebra $k[\mathrm{St}]^c$. This algebra has a basis $\{X_g \mid g \in \mathrm{St}\}$ with relations $X_g X_h = c(g,h)X_{gh}$, and a natural structure as a St-algebra given by $a \cdot X_g := X_a X_g (X_a)^{-1}$ for all $a \in \mathrm{St}$ (see also [Mov, Proposition 10]). Recall that c is called *nondegenerate* if for all $1 \neq g \in \mathrm{St}$, the map $C_{\mathrm{St}}(g) \to k^*$, $m \mapsto c(m,g)/c(g,m)$ is a nontrivial homomorphism of the centralizer of g in St to k^*. In [Mov, Propositions 11,12] Movshev reproduces the following well known criterion for $k[\mathrm{St}]^c$ to be a simple algebra (i.e., isomorphic to the matrix algebra $M_{|\mathrm{St}|^{1/2}}(k)$).

Proposition 3.5. *The twisted group algebra* $k[\mathrm{St}]^c$ *is simple if and only if* c *is nondegenerate. Furthermore, if this is the case, then* $k[\mathrm{St}]^c$ *is isomorphic to the regular representation of* St *as a* St*-module.*

Assume c is nondegenerate. By Proposition 3.2, the simple St-coalgebra $(k[\mathrm{St}]^c)^*$ is isomorphic to the St-coalgebra $(k[\mathrm{St}]_{\tilde{J}}, \Delta_{\tilde{J}})$ for some unique (up to gauge equivalence) quasitwist $\tilde{J} \in k[\mathrm{St}] \otimes k[\mathrm{St}]$.

Proposition 3.6 [Mov, Propositions 13 and 14]. *$\tilde{J}$ is in fact a twist for* $k[\mathrm{St}]$ (*i.e., it is invertible*). *Furthermore,* J *is the image of* $\tilde{J}$ *under the coalgebra embedding* $(k[\mathrm{St}]^c)^* \hookrightarrow A_J$.

PROOF. We only reproduce here the proof of the invertibility of $\tilde{J}$ (in a slightly expanded form). Set $C := k[\mathrm{St}]_{\tilde{J}}$. Suppose on the contrary that $\tilde{J}$ is not invertible. Then there exists $0 \neq L \in C^* \otimes C^*$ such that $\tilde{J}L = 0$. Let $F : C \otimes C \to C \otimes C$ be defined by $F(x) = xL$. Clearly, F is a morphism of $\mathrm{St} \times \mathrm{St}$-representations, and $F \circ \Delta_{\tilde{J}} = 0$. Thus the image $\mathrm{Im}(F^*)$ of the morphism of $\mathrm{St} \times \mathrm{St}$-representations $F^* : C^* \otimes C^* \to C^* \otimes C^*$ is contained in the kernel of the multiplication map $m := (\Delta_{\tilde{J}})^*$. Let $U := (C^* \otimes 1)\mathrm{Im}(F^*)(1 \otimes C^*)$. Clearly, U is contained in the kernel of m too. But, for any $x \in U$ and $g \in \mathrm{St}$, $(1 \otimes X_g)x(1 \otimes X_g)^{-1} \in U$. Thus, U is a left $C^* \otimes C^*$-module under left multiplication. Similarly, it is a right module over this algebra under right multiplication. So, it is a bimodule over $C^* \otimes C^*$. Since $U \neq 0$, this implies that $U = C^* \otimes C^*$. This is a contradiction, since we get that $m = 0$. Hence $\tilde{J}$ is invertible as desired. □

REMARK 3.7. In the paper [Mov] it is assumed that the characteristic of k is equal to 0, but all the results generalize in a straightforward way to the case when the characteristic of k is positive and relatively prime to the order of the group G.

4. The Classification of Triangular Semisimple and Cosemisimple Hopf Algebras

In this section we describe the classification of triangular semisimple and cosemisimple Hopf algebras over *any* algebraically closed field k, given in [EG4].

4.1. Construction of triangular semisimple and cosemisimple Hopf algebras from group-theoretical data. Let H be a finite group such that $|H|$ is not divisible by the characteristic of k. Suppose that V is an irreducible projective representation of H over k satisfying $\dim(V) = |H|^{1/2}$. Let $\pi : H \to \mathrm{PGL}(V)$ be the projective action of H on V, and let $\tilde{\pi} : H \to \mathrm{SL}(V)$ be any lifting of this action ($\tilde{\pi}$ need not be a homomorphism). We have $\tilde{\pi}(x)\tilde{\pi}(y) = c(x,y)\tilde{\pi}(xy)$, where c is a 2-cocycle of H with coefficients in k^*. This cocycle is nondegenerate (see Section 3) and hence the representation of H on $\mathrm{End}_k(V)$ is isomorphic to the regular representation of H (see e.g., Proposition 3.5). By Propositions 3.2 and 3.6, this gives rise to a twist $J(V)$ for $k[H]$, whose equivalence class is canonically associated to (H, V).

Now, for any group $G \supseteq H$, whose order is relatively prime to the characteristic of k, define a triangular semisimple Hopf algebra

$$F(G,H,V) := \big(k[G]^{J(V)}, J(V)_{21}^{-1}J(V)\big). \tag{4–1}$$

We wish to show that it is also cosemisimple.

LEMMA 4.1.1. *The Drinfeld element of the triangular semisimple Hopf algebra* $(A,R) := F(G,H,V)$ *equals* 1.

Proof. The Drinfeld element u is a grouplike element of A, and for any finite-dimensional A-module V one has $\mathrm{tr}|_V(u) = \dim_{\mathrm{Rep}_k(A)}(V) = \dim(V)$ (since $\mathrm{Rep}_k(A)$ is equivalent to $\mathrm{Rep}_k(G)$, see e.g., [G5]). In particular, we can set V to be the regular representation, and find that $\mathrm{tr}|_A(u) = \dim(A) \neq 0$ in k. But it is clear that if g is a nontrivial grouplike element in any finite-dimensional Hopf algebra A, then $\mathrm{tr}|_A(g) = 0$. Thus, $u = 1$. □

Remark 4.1.2. Lemma 4.1.1 fails for infinite-dimensional *cotriangular* Hopf algebras, which shows that this lemma can not be proved by an explicit computation. We refer the reader to [EG6] for the study of infinite-dimensional cotriangular Hopf algebras which are obtained from twisting in function algebras of affine proalgebraic groups.

Corollary 4.1.3. *The triangular semisimple Hopf algebra* $(A, R) := F(G, H, V)$ *is also cosemisimple.*

Proof. Since $u = 1$, one has $S^2 = \mathrm{Id}$ and so A is cosemisimple (as $\dim A \neq 0$). □

Thus we have assigned a triangular semisimple and cosemisimple Hopf algebra with Drinfeld element $u = 1$ to any triple (G, H, V) as above.

4.2. The classification in characteristic 0**.** In this subsection we assume that k is of characteristic 0. We first recall Theorem 2.1 from [EG1] and Theorem 3.1 from [EG4], and state them in a single theorem which is the key structure theorem for triangular semisimple Hopf algebras over k.

Theorem 4.2.1. *Let* (A, R) *be a triangular semisimple Hopf algebra over an algebraically closed field* k *of characteristic* 0, *with Drinfeld element* u. *Set* $\tilde{R} := RR_u$. *Then there exist a finite group* G, *a subgroup* $H \subseteq G$ *and a minimal twist* $J \in k[H] \otimes k[H]$ *such that* $(A, \tilde{R})$ *and* $(k[G]^J, J_{21}^{-1}J)$ *are isomorphic as triangular Hopf algebras. Moreover, the data* (G, H, J) *is unique up to isomorphism of groups and gauge equivalence of twists. That is, if there exist a finite group* G', *a subgroup* $H' \subseteq G'$ *and a minimal twist* $J' \in k[H'] \otimes k[H']$ *such that* $(A, \tilde{R})$ *and* $(k[G']^{J'}, J_{21}^{\prime -1}J')$ *are isomorphic as triangular Hopf algebras, then there exists an isomorphism of groups* $\phi : G \to G'$ *such that* $\phi(H) = H'$ *and* $(\phi \otimes \phi)(J)$ *and* J' *are gauge equivalent as twists for* $k[H']$.

The proof of this theorem relies, among other things, on the following (special case of a) deep theorem of Deligne on Tannakian categories [De1] in an essential way.

Theorem 4.2.2. *Let* k *be an algebraically closed field of characteristic* 0, *and* $(\mathcal{C}, \otimes, \mathbf{1}, a, l, r, c)$ *a* k*-linear abelian symmetric rigid category with* $End(\mathbf{1}) = k$, *which is semisimple with finitely many irreducible objects. If categorical dimensions of objects are nonnegative integers, then there exist a finite group* G *and an equivalence of symmetric rigid categories* $F : \mathcal{C} \to Rep_k(G)$.

We refer the reader to [G5, Theorems 5.3, 6.1, Corollary 6.3] for a complete and detailed proof of [EG1, Theorem 2.1] and [EG4, Theorem 3.1], along with a discussion of Tannakian categories.

Let (A, R) be a triangular semisimple Hopf algebra over k whose Drinfeld element u is 1, and let (G, H, J) be the associated group-theoretic data given in Theorem 4.2.1.

PROPOSITION 4.2.3. *The H-coalgebra $(k[H]_J, \Delta_J)$ (see (3–1)) is simple, and is isomorphic to the regular representation of H as an H-module.*

PROOF. By Proposition 3.6, J is the image of $\tilde{J}$ under the embedding $k[\mathrm{St}]_{\tilde{J}} \hookrightarrow k[H]_J$. Since J is minimal and $\mathrm{St} \subseteq H$, it follows that $\mathrm{St} = H$. Hence the result follows from the discussion preceding Proposition 3.6. □

We are now ready to prove our first classification result.

THEOREM 4.2.4. *The assignment $F : (G, H, V) \mapsto (A, R)$ is a bijection between*

1. *isomorphism classes of triples (G, H, V) where G is a finite group, H is a subgroup of G, and V is an irreducible projective representation of H over k satisfying $\dim(V) = |H|^{1/2}$, and*
2. *isomorphism classes of triangular semisimple Hopf algebras over k with Drinfeld element $u = 1$.*

PROOF. We need to construct an assignment F' in the other direction, and check that both $F' \circ F$ and $F \circ F'$ are the identity assignments.

Let (A, R) be a triangular semisimple Hopf algebra over k whose Drinfeld element u is 1, and let (G, H, J) be the associated group-theoretic data given in Theorem 4.2.1. By Proposition 4.2.3, the H-algebra $(k[H]_J)^*$ is simple. So we see that $(k[H]_J)^*$ is isomorphic to $\mathrm{End}_k(V)$ for some vector space V, and we have a homomorphism $\pi : H \to \mathrm{PGL}(V)$. Thus V is a projective representation of H. By Proposition 3.4, this representation is irreducible, and it is obvious that $\dim(V) = |H|^{1/2}$.

It is clear that the isomorphism class of the representation V does not change if J is replaced by a twist J' which is gauge equivalent to J as twists for $k[H]$. Thus, to any isomorphism class of triangular semisimple Hopf algebras (A, R) over k, with Drinfeld element 1, we have assigned an isomorphism class of triples (G, H, V). Let us write this as

$$F'(A, R) := (G, H, V). \tag{4–2}$$

The identity $F \circ F' = id$ follows from Proposition 3.2. Indeed, start with $(A, R) \cong (k[G]^J, J_{21}^{-1}J)$, where J is a minimal twist for $k[H]$, H a subgroup of G. Then by Proposition 4.2.3, we have that $(k[H]_J, \Delta_J)$ is a simple H-coalgebra which is isomorphic to the regular representation of H. Now let V be the associated irreducible projective representation of H, and $J(V)$ the associated twist as in (4–1). Then $(k[H]_J, \Delta_J)$ and $(k[H]_{J(V)}, \Delta_{J(V)})$ are isomorphic as H-coalgebras, and the claim follows from Proposition 3.2.

The identity $F' \circ F = id$ follows from the uniqueness part of Theorem 4.2.1. □

Remark 4.2.5. Observe that it follows from Theorem 4.2.4 that the twist $J(V)$ associated to (H, V) is *minimal*.

Now let (G, H, V, u) be a quadruple, in which (G, H, V) is as above, and u is a central element of G of order ≤ 2. We extend the map F to quadruples by setting

$$F(G, H, V, u) := (A, RR_u) \text{ where } (A, R) := F(G, H, V). \tag{4–3}$$

Theorem 4.2.6. *The assignment F given in* (4–3) *is a bijection between*

1. *isomorphism classes of quadruples (G, H, V, u) where G is a finite group, H is a subgroup of G, V is an irreducible projective representation of H over k satisfying* $\dim(V) = |H|^{1/2}$, *and $u \in G$ is a central element of order ≤ 2, and*
2. *isomorphism classes of triangular semisimple Hopf algebras over k.*

Proof. Define F' by $F'(A, R) := (F'(A, RR_u), u)$, where $F'(A, RR_u)$ is defined in (4–2). Using Theorem 4.2.4, it is straightforward to see that both $F' \circ F$ and $F \circ F'$ are the identity assignments. □

Theorem 4.2.6 implies the following classification result for *minimal* triangular semisimple Hopf algebras over k.

Proposition 4.2.7. *$F(G, H, V, u)$ is minimal if and only if G is generated by H and u.*

Proof. As we have already pointed out in Remark 4.2.5, if $(A, R) := F(G, H, V)$ then the sub Hopf algebra $k[H]^J \subseteq A$ is minimal triangular. Therefore, if $u = 1$ then $F(G, H, V)$ is minimal if and only if $G = H$. This obviously remains true for $F(G, H, V, u)$ if $u \neq 1$ but $u \in H$. If $u \notin H$ then it is clear that the R-matrix of $F(G, H, V, u)$ generates $k[H']$, where $H' = H \cup uH$. This proves the proposition. □

Remark 4.2.8. As was pointed out already by Movshev, the theory developed in [Mov] and extended in [EG4] is an analogue, for finite groups, of the theory of quantization of skew-symmetric solutions of the classical Yang-Baxter equation, developed by Drinfeld [Dr3]. In particular, the operation F is the analogue of the operation of quantization in [Dr3].

4.3. The classification in positive characteristic. In this subsection we assume that k is of positive characteristic p, and prove an analogue of Theorem 4.2.6 by using this theorem itself and the lifting techniques from [EG5].

We first recall some notation from [EG5]. Let $\mathcal{O} := W(k)$ be the ring of Witt vectors of k (see e.g., [Se, Sections 2.5, 2.6]), and K the field of fractions of $\mathcal{O}$. Recall that $\mathcal{O}$ is a local complete discrete valuation ring, and that the characteristic of K is zero. Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}$, which is generated by p. One has $\mathfrak{m}^n/\mathfrak{m}^{n+1} = k$ for any $n \geq 0$ (here $\mathfrak{m}^0 := \mathcal{O}$).

Let F be the assignment defined in (4–3). We now have the following classification result.

THEOREM 4.3.1. *The assignment F is a bijection between*

1. *isomorphism classes of quadruples (G, H, V, u) where G is a finite group of order prime to p, H is a subgroup of G, V is an irreducible projective representation of H over k satisfying $\dim(V) = |H|^{1/2}$, and $u \in G$ is a central element of order ≤ 2, and*
2. *isomorphism classes of triangular semisimple and cosemisimple Hopf algebras over k.*

PROOF. As in the proof of Theorem 4.2.6 we need to construct the assignment F'.

Let (A, R) be a triangular semisimple and cosemisimple Hopf algebra over k. Lift it (see [EG5]) to a triangular semisimple Hopf algebra $(\bar{A}, \bar{R})$ over K. By Theorem 4.2.6, we have that $(\bar{A} \otimes_K \bar{K}, \bar{R}) = F(G, H, V, u)$. We can now reduce V "*mod p*" to get V_p which is an irreducible projective representation of H over the field k. This can be done since V is defined by a nondegenerate 2-cocycle c (see Section 3) with values in roots of unity of degree $|H|^{1/2}$ (as the only irreducible representation of the simple H-algebra with basis $\{X_h \mid h \in H\}$, and relations $X_g X_h = c(g, h) X_{gh}$). This cocycle can be reduced *mod p* and remains nondegenerate (since the groups of roots of unity of order $|H|^{1/2}$ in k and K are naturally isomorphic), so it defines an irreducible projective representation V_p. Define $F'(A, R) := (G, H, V_p, u)$. It is shown like in characteristic 0 that $F \circ F'$ and $F' \circ F$ are the identity assignments. □

The following is the analogue of Theorem 4.2.1 in positive characteristic.

COROLLARY 4.3.2. *Let (A, R) be a triangular semisimple and cosemisimple Hopf algebra over any algebraically closed field k, with Drinfeld element u. Set $\tilde{R} := RR_u$. Then there exist a finite group G, a subgroup $H \subseteq G$ and a minimal twist $J \in k[H] \otimes k[H]$ such that $(A, \tilde{R})$ and $(k[G]^J, J_{21}^{-1} J)$ are isomorphic as triangular Hopf algebras. Moreover, the data (G, H, J) is unique up to isomorphism of groups and gauge equivalence of twists.*

PROPOSITION 4.3.3. *Proposition* 4.2.7 *holds in positive characteristic as well.*

PROOF. As before, if $(A, R) := F(G, H, V)$, the sub Hopf algebra $k[H]^J \subseteq A$ is minimal triangular. This follows from the facts that it is true in characteristic 0, and that the rank of a triangular structure does not change under lifting. Thus, Proposition 4.2.7 holds in characteristic p. □

REMARK. The class of finite-dimensional triangular cosemisimple Hopf algebras over k is invariant under twisting (see Remark 3.7 in [AEGN]). Using this and

Theorem 4.2.6, we were able in Theorem 6.3 of [AEGN] to classify the isomorphism classes of finite-dimensional triangular cosemisimple Hopf algebras over k. Namely, we proved that the classification is the same as in characteristic 0 (see Theorem 4.2.6) except that the subgroup H has to be of order coprime to the characteristic of k.

4.4. The solvability of the group underlying a minimal triangular semisimple Hopf algebra. In this subsection we consider finite groups which admit a minimal twist as studied in [EG4]. We also consider the existence of nontrivial grouplike elements in triangular semisimple and cosemisimple Hopf algebras, following [EG4].

A classical fact about complex representations of finite groups is that the dimension of any irreducible representation of a finite group K does not exceed $|K : Z(K)|^{1/2}$, where $Z(K)$ is the center of K. Groups of central type are those groups for which this inequality is in fact an equality. More precisely, a finite group K is said to be of *central type* if it has an irreducible representation V such that $\dim(V)^2 = |K : Z(K)|$ (see e.g., [HI]). We shall need the following theorem (conjectured by Iwahori and Matsumoto in 1964) whose proof uses the classification of finite simple groups.

THEOREM 4.4.1 [HI, Theorem 7.3]. *Any group of central type is solvable.*

As corollaries, we have the following results.

COROLLARY 4.4.2. *Let H be a finite group which admits a minimal twist. Then H is solvable.*

PROOF. We may assume that k has characteristic 0 (otherwise we can lift to characteristic 0). As we showed in the proof of Theorem 4.2.4, H has an irreducible projective representation V with $\dim(V) = |H|^{1/2}$. Let K be a finite central extension of H with central subgroup Z, such that V lifts to a linear representation of K. We have $\dim(V)^2 = |K : Z|$. Since $\dim(V)^2 \leq |K : Z(K)|$ we get that $Z = Z(K)$ and hence that K is a group of central type. But by Theorem 4.4.1, K is solvable and hence $H \cong K/Z(K)$ is solvable as well. □

REMARK 4.4.3. Movshev conjectures in the introduction to [Mov] that any finite group with a nondegenerate 2-cocycle is solvable. As explained in the Proof of Corollary 4.4.2, this result follows from Theorem 4.4.1.

COROLLARY 4.4.4. *Let A be a triangular semisimple and cosemisimple Hopf algebra over k of dimension bigger than* 1. *Then A has a nontrivial grouplike element.*

PROOF. We can assume that the Drinfeld element u is equal to 1 and that A is not cocommutative. Let A_m be the minimal part of A. By Corollary 4.4.2, $A_m = k[H]^J$ for a solvable group H, $|H| > 1$. Therefore, A_m has nontrivial 1-dimensional representations. Since $A_m \cong A_m^{*\mathrm{op}}$ as Hopf algebras, we get that A_m, and hence A, has nontrivial grouplike elements. □

4.5. Biperfect quasitriangular semisimple Hopf algebras. Corollary 4.4.4 motivates the following question. Let (A, R) be a *quasitriangular* semisimple Hopf algebra over k with characteristic 0 (e.g., the quantum double of a semisimple Hopf algebra), and let $\dim(A) > 1$. Is it true that A possesses a nontrivial grouplike element? We now follow [EGGS] and show that the answer to this question is *negative*.

Let G be a finite group. If G_1 and G_2 are subgroups of G such that $G = G_1G_2$ and $G_1 \cap G_2 = 1$, we say that $G = G_1G_2$ is an *exact factorization*. In this case G_1 can be identified with G/G_2, and G_2 can be identified with G/G_1 as sets, so G_1 is a G_2-set and G_2 is a G_1-set. Note that if $G = G_1G_2$ is an exact factorization, then $G = G_2G_1$ is also an exact factorization by taking the inverse elements.

Following Kac [KaG] and Takeuchi [T], one constructs a semisimple Hopf algebra from these data, as follows: Take the vector space $H := \mathbb{C}[G_2]^* \otimes \mathbb{C}[G_1]$. Introduce a product on H by:

$$(\varphi \otimes a)(\psi \otimes b) = \varphi(a \cdot \psi) \otimes ab \qquad (4\text{–}4)$$

for all $\varphi, \psi \in \mathbb{C}[G_2]^*$ and $a, b \in G_1$. Here $\cdot$ denotes the associated action of G_1 on the algebra $\mathbb{C}[G_2]^*$, and $\varphi(a \cdot \psi)$ is the multiplication of φ and $a \cdot \psi$ in the algebra $\mathbb{C}[G_2]^*$.

Identify the vector spaces

$$\begin{aligned} H \otimes H &= (\mathbb{C}[G_2] \otimes \mathbb{C}[G_2])^* \otimes (\mathbb{C}[G_1] \otimes \mathbb{C}[G_1]) \\ &= \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G_2] \otimes \mathbb{C}[G_2], \mathbb{C}[G_1] \otimes \mathbb{C}[G_1]) \end{aligned}$$

in the usual way, and introduce a coproduct on H by:

$$(\Delta(\varphi \otimes a))(b \otimes c) = \varphi(bc) a \otimes b^{-1} \cdot a \qquad (4\text{–}5)$$

for all $\varphi \in \mathbb{C}[G_2]^*$, $a \in G_1$ and $b, c \in G_2$. Here $\cdot$ denotes the action of G_2 on G_1.

Introduce a counit on H by:

$$\varepsilon(\varphi \otimes a) = \varphi(1_G) \qquad (4\text{–}6)$$

for all $\varphi \in \mathbb{C}[G_2]^*$ and $a \in G_1$.

Finally, identify the vector spaces $H = \mathbb{C}[G_2]^* \otimes \mathbb{C}[G_1] = \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G_2], \mathbb{C}[G_1])$ in the usual way, and introduce an antipode on H by

$$S\left(\sum_{a \in G_1} \varphi_a \otimes a\right)(x) = \sum_{a \in G_1} \varphi_{(x^{-1} \cdot a)^{-1}} \left((a^{-1} \cdot x)^{-1}\right) a \qquad (4\text{–}7)$$

for all $\sum_{a \in G_1} \varphi_a \otimes a \in H$ and $x \in G_2$, where the first $\cdot$ denotes the action of G_2 on G_1, and the second one denotes the action of G_1 on G_2.

THEOREM 4.5.1 [KaG, T]. *The multiplication, comultiplication, counit and antipode described in* (4–4)-(4–7) *determine a semisimple Hopf algebra structure on the vector space* $H := \mathbb{C}[G_2]^* \otimes \mathbb{C}[G_1]$.

The Hopf algebra H is called the *bicrossproduct* Hopf algebra associated with G, G_1, G_2, and is denoted by $H(G, G_1, G_2)$.

THEOREM 4.5.2 [Ma2]. *$H(G, G_2, G_1) \cong H(G, G_1, G_2)^*$ as Hopf algebras.*

Let us call a Hopf algebra H *biperfect* if the groups $G(H)$, $G(H^*)$ are both trivial. We are ready now to prove:

THEOREM 4.5.3. *$H(G, G_1, G_2)$ is biperfect if and only if G_1, G_2 are self normalizing perfect subgroups of G.*

PROOF. It is well known that the category of finite-dimensional representations of $H(G, G_1, G_2)$ is equivalent to the category of G_1-equivariant vector bundles on G_2, and hence that the irreducible representations of $H(G, G_1, G_2)$ are indexed by pairs (V, x) where x is a representative of a G_1-orbit in G_2, and V is an irreducible representation of $(G_1)_x$, where $(G_1)_x$ is the isotropy subgroup of x. Moreover, the dimension of the corresponding irreducible representation is

$$\frac{\dim(V)|G_1|}{|(G_1)_x|}.$$

Thus, the 1-dimensional representations of $H(G, G_1, G_2)$ are indexed by pairs (V, x) where x is a fixed point of G_1 on $G_2 = G/G_1$ (i.e., $x \in N_G(G_1)/G_1$), and V is a 1-dimensional representation of G_1. The result follows now using Theorem 4.5.2. □

By Theorem 4.5.3, in order to construct an example of a biperfect semisimple Hopf algebra, it remains to find a finite group G which admits an exact factorization $G = G_1G_2$, where G_1, G_2 are self normalizing perfect subgroups of G. Amazingly the Mathieu simple group $G := M_{24}$ of degree 24 provides such an example!

THEOREM 4.5.4. *The group G contains a subgroup $G_1 \cong \mathrm{PSL}(2, 23)$, and a subgroup $G_2 \cong A_7 \ltimes (\mathbb{Z}_2)^4$ where A_7 acts on $(\mathbb{Z}_2)^4$ via the embedding $A_7 \subset A_8 = \mathrm{SL}(4, 2) = \mathrm{Aut}((\mathbb{Z}_2)^4)$ (see [AT]). These subgroups are perfect, self normalizing and G admits an exact factorization $G = G_1G_2$. In particular, $H(G, G_1, G_2)$ is biperfect.*

We suspect that not only is M_{24} the smallest example but it may be the only finite simple group with a factorization with all the needed properties.

Clearly, the Drinfeld double $D(H(G, G_1, G_2))$ is an example of a biperfect quasitriangular semisimple Hopf algebra.

4.6. Minimal triangular Hopf algebras constructed from a bijective 1-cocycle. In this subsection we describe an explicit way of constructing minimal twists for certain solvable groups (hence of central type groups) given in [EG2]. For simplicity we let $k := \mathbb{C}$.

Definition 4.6.1. Let G, A be finite groups and $\rho : G \to \mathrm{Aut}(A)$ a homomorphism. By a 1-cocycle of G with coefficients in A we mean a map $\pi : G \to A$ which satisfies the equation

$$\pi(gg') = \pi(g)(g \cdot \pi(g')),\ \ g, g' \in G, \tag{4–8}$$

where $\rho(g)(x) = g \cdot x$ for $g \in G, x \in A$.

We will be interested in the case when π is *a bijection* (so in particular, $|G| = |A|$), because of the following proposition.

Proposition 4.6.2. *Let G, A be finite groups, $\pi : G \to A$ a bijective 1-cocycle, and J a twist for $\mathbb{C}[A]$ which is G-invariant. Then $\bar{J} := (\pi^{-1} \otimes \pi^{-1})(J)$ is a quasitwist for $\mathbb{C}[G]$.*

Proof. It is obvious that the second equation of (2–14) is satisfied for $\bar{J}$. So we only have to prove the first equation of (2–14) for $\bar{J}$. Write $J = \sum a_{xy} x \otimes y$. Then

$$\begin{aligned} &(\pi \otimes \pi \otimes \pi)((\Delta \otimes \mathrm{Id})(\bar{J})(\bar{J} \otimes 1)) \\ &= \sum_{x,y,z,t \in A} a_{xy} a_{zt} \pi(\pi^{-1}(x)\pi^{-1}(z)) \otimes \pi(\pi^{-1}(x)\pi^{-1}(t)) \otimes \pi(\pi^{-1}(y)) \\ &= \sum_{x,y,z,t \in A} a_{xy} a_{zt} x(\pi^{-1}(x)z) \otimes x(\pi^{-1}(x)t) \otimes y. \end{aligned}$$

Using the G-invariance of J, we can remove the $\pi^{-1}(x)$ in the last expression and get

$$(\pi \otimes \pi \otimes \pi)((\Delta \otimes \mathrm{Id})(\bar{J})(\bar{J} \otimes 1)) = (\Delta \otimes \mathrm{Id})(J)(J \otimes 1). \tag{4–9}$$

Similarly,

$$(\pi \otimes \pi \otimes \pi)((\mathrm{Id} \otimes \Delta)(\bar{J})(1 \otimes \bar{J})) = (\mathrm{Id} \otimes \Delta)(J)(1 \otimes J). \tag{4–10}$$

But J is a twist, so the right hand sides of (4–9) and (4–10) are equal. Since π is bijective, this implies equation (2–14) for $\bar{J}$. □

Now, given a quadruple (G, A, ρ, π) as above such that A is *abelian*, define $\tilde{G} := G \ltimes A^\vee$, $\tilde{A} := A \times A^\vee$, $\tilde{\rho} : \tilde{G} \to \mathrm{Aut}(\tilde{A})$ by $\tilde{\rho}(g) = \rho(g) \times \rho^*(g)^{-1}$, and $\tilde{\pi} : \tilde{G} \to \tilde{A}$ by $\tilde{\pi}(a^* g) = \pi(g) a^*$ for $a^* \in A^\vee$, $g \in G$. It is straightforward to check that $\tilde{\pi}$ is a bijective 1-cocycle. We call the quadruple $(\tilde{G}, \tilde{A}, \tilde{\rho}, \tilde{\pi})$ the *double* of (G, A, ρ, π).

Consider the element $J \in \mathbb{C}[\tilde{A}] \otimes \mathbb{C}[\tilde{A}]$ given by

$$J := |A|^{-1} \sum_{x \in A, y^* \in A^\vee} e^{(x,y^*)} x \otimes y^*,$$

where $(,)$ is the duality pairing between A and $A^\vee$. It is straightforward to check that J is a twist for $\mathbb{C}[\tilde{A}]$, and that it is G-invariant. This allows to construct the corresponding element

$$\bar{J} := |A|^{-1} \sum_{x \in A, y^* \in A^\vee} e^{(x,y^*)} \pi^{-1}(x) \otimes y^* \in \mathbb{C}[\tilde{G}] \otimes \mathbb{C}[\tilde{G}]. \tag{4–11}$$

PROPOSITION 4.6.3. *$\bar{J}$ is a twist for $\mathbb{C}[\tilde{G}]$, and*

$$\bar{J}^{-1} = |A|^{-1} \sum_{z \in A, t^* \in A^\vee} e^{-(z,t^*)} \pi^{-1}(T(z)) \otimes t^*, \tag{4–12}$$

where $T : A \to A$ is a bijective map (not a homomorphism, in general) defined by

$$\pi^{-1}(x^{-1})\pi^{-1}(T(x)) = 1.$$

PROOF. Denote the right hand side of (4–12) by J'. We need to check that $J' = \bar{J}^{-1}$. It is enough to check it after evaluating any $a \in A$ on the second component of both sides. We have

$$(1 \otimes a)(\bar{J}) = |A|^{-1} \sum_{x,y^*} e^{(xa,y^*)} \pi^{-1}(x) = \pi^{-1}(a^{-1})(1 \otimes a)(J')$$
$$= |A|^{-1} \sum_{x,y^*} e^{-(xa^{-1},y^*)} \pi^{-1}(T(x)) = \pi^{-1}(T(a)).$$

This concludes the proof of the proposition. □

We can now prove:

THEOREM 4.6.4. *Let $\bar{J}$ be as in (4–11). Then $\bar{J}$ is a minimal twist for $\mathbb{C}[\tilde{G}]$, and it gives rise to a minimal triangular semisimple Hopf algebra $(\mathbb{C}[\tilde{G}]^{\bar{J}}, R^{\bar{J}})$, with universal R-matrix*

$$R^{\bar{J}} = |A|^{-2} \sum_{\substack{x,y \in A \\ x^*, y^* \in A^\vee}} e^{(x,y^*)-(y,x^*)} x^* \pi^{-1}(x) \otimes \pi^{-1}(T(y))y^*.$$

PROOF. Minimality of $\bar{J}$ follows from the fact that $\{x^*\pi^{-1}(x) \mid x^* \in A^\vee, x \in A\}$ and $\{\pi^{-1}(T(y))y^* \mid y \in A, y^* \in A^\vee\}$ are bases of $\mathbb{C}[\tilde{G}]$, and the fact that the matrix $c_{xx^*,yy^*} = e^{(x,y^*)-(y,x^*)}$ is invertible (because it is proportional to the matrix of Fourier transform on $A \times A^\vee$). □

REMARK 4.6.5. By Theorem 4.6.4, every bijective 1-cocycle $\pi : G \to A$ gives rise to a minimal triangular structure on $\mathbb{C}[G \ltimes A^\vee]$. So it remains to construct a supply of bijective 1-cocycles. This was done in [ESS]. The theory of bijective 1-cocycles was developed in [ESS], because it was found that they correspond to set-theoretical solutions of the quantum Yang-Baxter equation. In particular, many constructions of these 1-cocycles were found. We refer the reader to [ESS] for further details.

We now give two examples of nontrivial minimal triangular semisimple Hopf algebras. The first one has the least possible dimension; namely, dimension 16, and the second one has dimension 36.

EXAMPLE 4.6.6. Let $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators x, y, and $A := \mathbb{Z}_4$ with generator a. Define an action of G on A by letting x act trivially, and y act as an automorphism via $y \cdot a = a^{-1}$. Eli Aljadeff pointed out to us that the group $\tilde{G} := (\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_4$ has a 2-cocycle c with coefficients in $\mathbb{C}^*$, such that the twisted group algebra $\mathbb{C}[\tilde{G}]^c$ is simple. This implies that $\tilde{G}$ has a minimal twist (see Subsection 4.1). We now use Theorem 4.6.4 to explicitly construct our example.

Define a bijective 1-cocycle $\pi : G \to A$ as follows: $\pi(1) = 1$, $\pi(x) = a^2$, $\pi(y) = a$ and $\pi(xy) = a^3$. Then by Theorem 4.6.4, $\mathbf{C}[\tilde{G}]^{\bar{J}}$ is a non-commutative and non-cocommutative minimal triangular semisimple Hopf algebra of dimension 16.

We remark that it follows from the classification of semisimple Hopf algebras of dimension 16 [Kash], that the Hopf algebra $\mathbf{C}[\tilde{G}]^{\bar{J}}$ constructed above, appeared first in [Kash]. However, our triangular structure on this Hopf algebra is new. Indeed, Kashina's triangular structure on this Hopf algebra is not minimal, since it arises from a twist of a subgroup of $\tilde{G}$ which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 4.6.7. Let $G := S_3$ be the permutation group of three letters, and $A := \mathbb{Z}_2 \times \mathbb{Z}_3$. Define an action of G on A by $s(a, b) = (a, (-1)^{sign(s)}b)$ for $s \in G$, $a \in \mathbb{Z}_2$ and $b \in \mathbb{Z}_3$. Define a bijective 1-cocycle $\pi = (\pi_1, \pi_2) : G \to A$ as follows: $\pi_1(s) = 0$ if s is even and $\pi_1(s) = 1$ if s is odd, and $\pi_2(id) = 0$, $\pi_2((123)) = 1$, $\pi_2((132)) = 2$, $\pi_2((12)) = 2$, $\pi_2((13)) = 0$ and $\pi_2((23)) = 1$. Then by Theorem 4.6.4, $\mathbf{C}[\tilde{G}]^{\bar{J}}$ is a noncommutative and noncocommutative minimal triangular semisimple Hopf algebra of dimension 36.

We now wish to determine the group-theoretical data corresponding, under the bijection of the classification given in Theorem 4.2.6, to the minimal triangular semisimple Hopf algebras constructed in Theorem 4.6.4.

Let $H := G \ltimes A^\vee$. Following Theorem 4.6.4, we can associate to this data the element

$$J := |A|^{-1} \sum_{g \in G, b \in A^\vee} e^{(\pi(g), b)} b \otimes g$$

(for convenience we use the opposite element to the one we used before). We proved that this element is a minimal twist for $\mathbb{C}[H]$, so $\mathbb{C}[H]^J$ is a minimal triangular semisimple Hopf algebra with Drinfeld element $u = 1$. Now we wish to find the irreducible projective representation V of H which corresponds to $\mathbb{C}[H]^J$ under the correspondence of Theorem 4.2.6.

Let $V := \mathrm{Fun}(A, \mathbb{C})$ be the space of $\mathbb{C}$-valued functions on A. It has a basis $\{\delta_a \mid a \in A\}$ of characteristic functions of points. Define a projective action ϕ of H on V by

$$\phi(b)\delta_a = e^{-(a,b)}\delta_a,\ \phi(g)\delta_a = \delta_{g \cdot a + \pi(g)} \text{ and } \phi(bg) = \phi(b)\phi(g) \qquad (4\text{–}13)$$

for $g \in G$ and $b \in A^\vee$. It is straightforward to verify that this is indeed a projective representation.

Proposition 4.6.8. *The representation V is irreducible, and corresponds to $\mathbb{C}[H]^J$ under the bijection of the classification given in Theorem* 4.2.6.

PROOF. It is enough to show that the H-algebras $(\mathbb{C}[H]_J)^*$ and $\mathrm{End}_{\mathbb{C}}(V)$ are isomorphic.

Let us compute the multiplication in the algebra $(\mathbb{C}[H]_J)^*$. We have

$$\Delta_J(bg) = |A|^{-1} \sum_{g' \in G, b' \in A^\vee} e^{(\pi(g'),b')} b(g \cdot b')g \otimes bgg'. \tag{4–14}$$

Let $\{Y_{bg}\}$ be the dual basis of $(\mathbb{C}[H]_J)^*$ to the basis $\{bg\}$ of $\mathbb{C}[H]_J$. Let $*$ denote the multiplication law dual to the coproduct Δ_J. Then, dualizing equation (4–14), we have

$$Y_{b_2g_2} * Y_{b_1g_1} = e^{(\pi(g_1)-\pi(g_2),b_2-b_1)} Y_{b_1g_2} \tag{4–15}$$

for g_1, $g_2 \in G$ and b_1, $b_2 \in A^\vee$ (here for convenience we write the operations in A and $A^\vee$ additively). Define $Z_{bg} := e^{(\pi(g),b)} Y_{bg}$ for $g \in G$ and $b \in A^\vee$. In the basis $\{Z_{bg}\}$ the multiplication law in $(\mathbb{C}[H]_J)^*$ is given by

$$Z_{b_2g_2} * Z_{b_1g_1} = e^{(\pi(g_1),b_2)} Z_{b_1g_2}. \tag{4–16}$$

Now let us introduce a left action of $(\mathbb{C}[H]_J)^*$ on V. Set

$$Z_{bg}\delta_a := e^{(a,b)} \delta_{\pi(g)}. \tag{4–17}$$

It is straightforward to check, using (4–16), that (4–17) is indeed a left action. It is also straightforward to compute that this action is H-invariant. Thus, (4–17) defines an isomorphism $(\mathbb{C}[H]_J)^* \to \mathrm{End}_k(V)$ as H-algebras, which proves the proposition. □

5. The Representation Theory of Cotriangular Semisimple and Cosemisimple Hopf Algebras

If (A, R) is a minimal triangular Hopf algebra then A and $A^{*\mathrm{op}}$ are isomorphic as Hopf algebras. But any nontrivial triangular semisimple and cosemisimple Hopf algebra A, over any algebraically closed field k, which is *not* minimal, gives rise to a new Hopf algebra A^*, which is also semisimple and cosemisimple. These are very interesting semisimple and cosemisimple Hopf algebras which arise from finite groups, and they are abundant by the constructions given in [EG2, EG4] (see Section 4). Generally, the dual Hopf algebra of a triangular Hopf algebra is called *cotriangular* in the literature.

In this section we explicitly describe the representation theory of cotriangular semisimple and cosemisimple Hopf algebras $A^* = (k[G]^J)^*$ studied in [EG3], in terms of representations of some associated groups. As a corollary we prove that Kaplansky's 6th conjecture [Kap] holds for A^*; that is, that the dimension of any irreducible representation of A^* divides the dimension of A.

5.1. The algebras associated with a twist. Let $A := k[H]$ be the group algebra of a finite group H whose order is relatively prime to the characteristic of k. Let $J \in A \otimes A$ be a *minimal* twist, and $A_1 := (A_J, \Delta_J, \varepsilon)$ be as in (3–1). Similarly, we define the coalgebra $A_2 := ({}_JA, {}_J\Delta, \varepsilon)$, where ${}_JA = k[H]$ as vector spaces, and

$$ {}_J\Delta : A \to A \otimes A,\ {}_J\Delta(a) = J^{-1}\Delta(a) \tag{5–1}$$

for all $a \in A$. Note that since J is a twist, ${}_J\Delta$ is indeed coassociative. For $h \in H$, let $\delta_h : k[H] \to k$ be the linear map determined by $\delta_h(h) = 1$ and $\delta_h(h') = 0$ for $h \neq h' \in H$. Clearly the set $\{\delta_h \mid h \in H\}$ forms a linear basis of the dual algebras $(A_1)^*$ and $(A_2)^*$.

Theorem 5.1.1. 1. *$(A_1)^*$ and $(A_2)^*$ are H-algebras via*

$$\rho_1(h)\delta_y = \delta_{hy},\ \ \rho_2(h)\delta_y = \delta_{yh^{-1}}$$

respectively.

2. *$(A_1)^* \cong (A_2)^{*\mathrm{op}}$ as H-algebras (where H acts on $(A_2)^{*\mathrm{op}}$ as it does on $(A_2)^*$).*
3. *The algebras $(A_1)^*$ and $(A_2)^*$ are simple, and are isomorphic as H-modules to the regular representation R_H of H.*

Proof. The proof of part 1 is straightforward.

The proof of part 3 follows from Proposition 4.2.3 and Part 2.

Let us prove part 2. It is straightforward to verify that $(S \otimes S)(J) = (Q \otimes Q)J_{21}^{-1}\Delta(Q)^{-1}$ where Q is as in (2–16) (see e.g., (2.17) in [Ma1, Section 2.3]). Hence the map $(A_2)^* \to (A_1)^{*\mathrm{op}}$, $\delta_x \mapsto \delta_{S(x)Q^{-1}}$ determines an H-algebra isomorphism. □

Since the algebras $(A_1)^*$, $(A_2)^*$ are simple, the actions of H on $(A_1)^*$, $(A_2)^*$ give rise to projective representations $H \to \mathrm{PGL}(|H|^{1/2}, k)$. We will denote these projective representations by V_1, V_2 (they can be thought of as the simple modules over $(A_1)^*$, $(A_2)^*$, with the induced projective action of H). Note that Part 2 of Theorem 5.1.1 implies that V_1, V_2 are dual to each other, hence that $[V_1] = -[V_2]$.

5.2. The main result. Let (A, R) be a triangular semisimple and cosemisimple Hopf algebra over k, with Drinfeld element $u = 1$, and let H, G and J be as before. Consider the dual Hopf algebra A^*. It has a basis of δ-functions δ_g. The first simple but important fact about the structure of A^* as an algebra is the following:

Proposition 5.2.1. *Let Z be a double coset of H in G, and $(A^*)_Z := \oplus_{g\in Z} k\delta_g \subset A^*$. Then $(A^*)_Z$ is a subalgebra of A^*, and $A^* = \oplus_Z (A^*)_Z$ as algebras.*

Proof. Straightforward. □

Thus, to study the representation theory of A^*, it is sufficient to describe the representations of $(A^*)_Z$ for any Z.

Let Z be a double coset of H in G, and let $g \in Z$. Let $K_g := H \cap gHg^{-1}$, and define the embeddings $\theta_1, \theta_2 : K_g \to H$ given by $\theta_1(a) = g^{-1}ag$, $\theta_2(a) = a$. Denote by W_i the pullback of the projective H-representation V_i to K_g by means of θ_i, $i = 1, 2$.

Our main result is the following theorem, which is proved in the next subsection.

THEOREM 5.2.2. *Let W_1, W_2 be as above, and let $(\hat{K}_g, \tilde{\pi}_W)$ be any linearization of the projective representation $W := W_1 \otimes W_2$ of K_g. Let ζ be the kernel of the projection $\hat{K}_g \to K_g$, and $\chi : \zeta \to k^*$ be the character by which ζ acts in W. Then there exists a 1–1 correspondence between*

1. *isomorphism classes of irreducible representations of $(A^*)_Z$ and*
2. *isomorphism classes of irreducible representations of $\hat{K}_g$ with ζ acting by χ.*

Moreover, if a representation Y of $(A^*)_Z$ corresponds to a representation X of $\hat{K}_g$, then

$$\dim(Y) = \frac{|H|}{|K_g|} \dim(X).$$

As a corollary we get Kaplansky's 6th conjecture [Kap] for cotriangular semisimple and cosemisimple Hopf algebras.

COROLLARY 5.2.3. *The dimension of any irreducible representation of a cotriangular semisimple and cosemisimple Hopf algebra over k divides the dimension of the Hopf algebra.*

PROOF. Since $\dim(X)$ divides $|K_g|$ (see e.g., [CR, Proposition 11.44]), we have

$$\frac{|G|}{\frac{|H|}{|K_g|}\dim(X)} = \frac{|G|}{|H|}\frac{|K_g|}{\dim(X)}$$

and the result follows. □

In some cases the classification of representations of $(A^*)_Z$ is even simpler. Namely, let $\overline{g} \in \mathrm{Aut}(K_g)$ be given by $\overline{g}(a) = g^{-1}ag$. Then we have:

COROLLARY 5.2.4. *If the cohomology class $[W_1]$ is $\overline{g}$-invariant then irreducible representations of $(A^*)_Z$ correspond in a 1–1 manner to irreducible representations of K_g, and if Y corresponds to X, then*

$$\dim(Y) = \frac{|H|}{|K_g|} \dim(X).$$

PROOF. plus 3mu For any $\alpha \in \mathrm{Aut} K_g$ and $f \in \mathrm{Hom}((K_g)^n, k^*)$, let $\alpha \circ f \in \mathrm{Hom}((K_g)^n, k^*)$ be given by $(\alpha \circ f)(h_1, \ldots, h_n) = f(\alpha(h_1), \ldots, \alpha(h_n))$ (which determines the action of α on $H^i(K_g, k^*)$). Then it follows from the identity $[V_1] = -[V_2]$ that $[W_1] = -\overline{g} \circ [W_2]$. Thus, in our situation $[W] = 0$, hence W comes from a linear representation of K_g. Thus, we can set $\hat{K}_g = K_g$ in the theorem, and the result follows. □

Example 5.2.5. Let $k := \mathbb{C}$. Let $p > 2$ be a prime number, and $H := (\mathbb{Z}/p\mathbb{Z})^2$ with the standard symplectic form $(\cdot,\cdot) : H \times H \to k^*$ given by $\big((x,y),(x',y')\big) = e^{2\pi i(xy'-yx')/p}$. Then the element $J := p^{-2}\sum_{a,b\in H}(a,b)a\otimes b$ is a minimal twist for $\mathbb{C}[H]$. Let $g \in \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ be an automorphism of H, and G_0 be the cyclic group generated by g. Construct the group $G := G_0 \ltimes H$. It is easy to see that in this case, the double cosets are ordinary cosets g^kH, and $K_{g^k} = H$. Moreover, one can show either explicitly or using Proposition 3.4, that $[W_1]$ is a generator of $H^2(H,\mathbb{C}^*)$ which is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. The element g^k acts on $[W_1]$ by multiplication by $\det(g^k)$. Therefore, by Corollary 5.2.4, the algebra $(A^*)_{g^kH}$ has p^2 1-dimensional representations (corresponding to linear representations of H) if $\det(g^k) = 1$.

However, if $\det(g^k) \neq 1$, then $[W]$ generates $H^2(H,\mathbb{C}^*)$. Thus, W comes from a linear representation of the Heisenberg group $\hat{H}$ (a central extension of H by $\mathbb{Z}/p\mathbb{Z}$) with some central character χ. Thus, $(A^*)_{g^kH}$ has one p-dimensional irreducible representation, corresponding to the unique irreducible representation of $\hat{H}$ with central character χ (which is W).

5.3. Proof of Theorem 5.2.2. Let $Z \subset G$ be a double coset of H in G. For any $g \in Z$ define the linear map

$$F_g : (A^*)_Z \to (A_2)^* \otimes (A_1)^*, \quad \delta_y \mapsto \sum_{h,h'\in H: y=hgh'} \delta_h \otimes \delta_{h'}.$$

Proposition 5.3.1. *Let ρ_1,ρ_2 be as in Theorem* 5.1.1. *Then*:

1. *The map F_g is an injective homomorphism of algebras.*
2. $F_{aga'}(\varphi) = (\rho_2(a)\otimes\rho_1(a')^{-1})F_g(\varphi)$ *for any* $a,a' \in H$, $\varphi \in (A^*)_Z$.

Proof. 1. It is straightforward to verify that the map $(F_g)^* : A_2 \otimes A_1 \to A_Z$ is determined by $h \otimes h' \mapsto hgh'$, and that it is a surjective homomorphism of coalgebras. Hence the result follows.

2. Straightforward. □

For any $a \in K_g$ define $\rho(a) \in \mathrm{Aut}((A_2)^* \otimes (A_1)^*)$ by $\rho(a) = \rho_2(a) \otimes \rho_1(a^g)$, where $a^g := g^{-1}ag$ and ρ_1,ρ_2 are as in Theorem 5.1.1. Then ρ is an action of K_g on $(A_2)^* \otimes (A_1)^*$.

Proposition 5.3.2. *Let $U_g := ((A_2)^* \otimes (A_1)^*)^{\rho(K_g)}$ be the algebra of invariants. Then* $\mathrm{Im}(F_g) = U_g$, *so* $(A^*)_Z \cong U_g$ *as algebras.*

Proof. It follows from Proposition 5.3.1 that $\mathrm{Im}(F_g) \subseteq U_g$, and $\mathrm{rk}(F_g) = \dim((A^*)_Z) = |H|^2/|K_g|$. On the other hand, by Theorem 5.1.1, $(A_1)^*, (A_2)^*$ are isomorphic to the regular representation R_H of H. Thus, $(A_1)^*$ and $(A_2)^*$ are isomorphic to $(|H|/|K_g|)R_{K_g}$ as representations of K_g, via $\rho_1(a)$ and $\rho_2(a^g)$. Thus,

$$(A_2)^* \otimes (A_1)^* \cong \frac{|H|^2}{|K_g|^2}(R_{K_g} \otimes R_{K_g}) \cong \frac{|H|^2}{|K_g|}R_{K_g}.$$

So U_g has dimension $|H|^2/|K_g|$, and the result follows. □

Now we are in a position to prove Theorem 5.2.2. Since $W_1 \otimes W_1^* \cong (A_1)^*$ and $W_2 \otimes W_2^* \cong (A_2)^*$, it follows from Theorem 5.1.1 that $W_1 \otimes W_2 \otimes W_1^* \otimes W_2^* \cong (|H|^2/|K_g|)R_{K_g}$ as $\hat{K}_g$ modules. Thus, if χ_W is the character of $W := W_1 \otimes W_2$ as a $\hat{K}_g$ module then

$$|\chi_W(x)|^2 = 0,\, x \notin \zeta \;\; \textit{and} \;\; |\chi_W(x)|^2 = |H|^2,\, x \in \zeta.$$

Therefore,

$$\chi_W(x) = 0,\, x \notin \zeta \;\; \textit{and} \;\; \chi_W(x) = |H| \cdot x_W,\, x \in \zeta,$$

where x_W is the root of unity by which x acts in W. Now, it is clear from the definition of U_g (see Proposition 5.3.2) that $U_g = \mathrm{End}_{\hat{K}_g}(W)$. Thus if $W = \bigoplus_{M \in \mathrm{Irr}(\hat{K}_g)} W(M) \otimes M$, where $W(M) := \mathrm{Hom}_{\hat{K}_g}(M, W)$ is the multiplicity space, then

$$U_g = \bigoplus_{M : W(M) \neq 0} \mathrm{End}_k(W(M)).$$

So $\{W(M) \mid W(M) \neq 0\}$ is the set of irreducible representations of U_g. Thus the following result implies the theorem:

LEMMA. 1. $W(M) \neq 0$ *if and only if for all* $x \in \zeta$, $x_{|M} = x_{|W}$.
2. *If* $W(M) \neq 0$ *then* $\dim(W(M)) = \dfrac{|H|}{|K_g|} \dim(M)$.

PROOF. The "only if" part of 1 is clear. For the "if" part compute $\dim(W(M))$ as the inner product (χ_W, χ_M). We have

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|\hat{K}_g|} x_{|W} \cdot \dim(M) \cdot \bar{x}_{|M}.$$

If $x_{|M} = x_{|W}$ then

$$(\chi_W, \chi_M) = \sum_{x \in \zeta} \frac{|H|}{|\hat{K}_g|} \dim(M) = \frac{|H||\zeta|}{|\hat{K}_g|} \dim(M) = \frac{|H|}{|K_g|} \dim(M).$$

This proves part 2 as well. □

This concludes the proof of the theorem. □

6. The Pointed Case

In this section we consider finite-dimensional triangular pointed Hopf algebras over an algebraically closed field k of characteristic 0, and in particular describe the classification and explicit construction of minimal triangular pointed Hopf algebras, given in [G4]. Throughout the section, unless otherwise specified, the ground field k will be assumed to be algebraically closed with characteristic 0.

6.1. The antipode of triangular pointed Hopf algebras. In this subsection we prove that the fourth power of the antipode of any triangular pointed Hopf algebra (A, R) is the identity. Along the way we prove that the group algebra of the group of grouplike elements of A_R (which must be abelian) admits a minimal triangular structure and consequently that A has the structure of a biproduct [R1].

THEOREM 6.1.1. *Let (A, R) be a minimal triangular pointed Hopf algebra over k with Drinfeld element u, and set $K := k[\mathbf{G}(A)]$. Then there exists a projection of Hopf algebras $\pi : A \to K$, and consequently $A = B \times K$ is a biproduct where $B := \{x \in A \mid (I \otimes \pi)\Delta(x) = x \otimes 1\} \subseteq A$. Moreover, K admits a minimal triangular structure with Drinfeld element $u_K = u$.*

PROOF. Since $\mathbf{G}(A)$ is abelian, $K^* \cong K$ and $K \cong k[\mathbf{G}(A^{*\mathrm{cop}})]$ as Hopf algebras. Hence, $\dim(K^*) = \dim(k[\mathbf{G}(A^*)])$. Consider the series of Hopf algebra homomorphisms

$$K \xhookrightarrow{i} A^{\mathrm{cop}} \xrightarrow{(f_R)^{-1}} A^* \xrightarrow{i^*} K^*,$$

where i is the inclusion map. Since A^* is pointed it follows from the above remarks that $i^*_{|k[\mathbf{G}(A^*)]} : k[\mathbf{G}(A^*)] \to K^*$ is an isomorphism of Hopf algebras (see e.g., [Mon, 5.3.5]), and hence that $i^* \circ (f_R)^{-1} \circ i$ determines a minimal quasitriangular structure on K^*. This structure is in fact triangular since $(f_R)^{-1}$ determines a triangular structure on A^*. Clearly, $(i^* \circ (f_R)^{-1} \circ i)(u) = (u_{K^*})^{-1} = u_{K^*}$ is the Drinfeld element of K^*. Since K and K^* are isomorphic as Hopf algebras we conclude that K admits a minimal triangular structure with Drinfeld element $u_K = u$.

Finally, set $\varphi := i^* \circ (f_R)^{-1} \circ i$ and $\pi := \varphi^{-1} \circ i^* \circ (f_R)^{-1}$. Then $\pi : A \to K$ is onto, and moreover $\pi \circ i = \varphi^{-1} \circ i^* \circ (f_R)^{-1} \circ i = \varphi^{-1} \circ \varphi = id_K$. Hence π is a projection of Hopf algebras and by [R1], $A = B \times K$ is a biproduct where $B := \{x \in A \mid (I \otimes \pi)\Delta(x) = x \otimes 1\}$ as desired. This concludes the proof of the theorem. ☐

THEOREM 6.1.2. *Let (A, R) be any triangular pointed Hopf algebra with antipode S and Drinfeld element u over any field k of characteristic* 0. *Then* $S^4 = \mathrm{Id}$. *If in addition A_m is not semisimple and A is finite-dimensional then* $\dim(A)$ *is divisible by* 4.

PROOF. We may assume that k is algebraically closed. By (2–6), $S^2(a) = uau^{-1}$ for all $a \in A$. Let $K := k[\mathbf{G}(A_m)]$. Since $u \in A_m$, and by Theorem 6.1.1, $u = u_K$ and $(u_K)^2 = 1$, we have that $S^4 = \mathrm{Id}$.

In order to prove the second claim, we may assume that (A, R) is minimal (since by [NZ], $\dim(A_m)$ divides $\dim(A)$). Since A is not semisimple it follows from [LR1] that $S^2 \neq \mathrm{Id}$, and hence that $u \neq 1$. In particular, $|\mathbf{G}(A)|$ is even. Now, let B be as in Theorem 6.1.1. Since $S^2(B) = B$, B has a basis $\{a_i, b_j \mid S^2(a_i) = a_i, S^2(b_j) = -b_j, 1 \leq i \leq n, 1 \leq j \leq m\}$. Hence by Theorem

6.1.1,

$$\{a_i g, b_j g \mid g \in \mathbf{G}(A), 1 \le i \le n, 1 \le j \le m\}$$

is a basis of A. Since by [R3], $\mathrm{tr}(S^2) = 0$, we have $0 = \mathrm{tr}(S^2) = |\mathbf{G}(A)|(n - m)$, which implies that $n = m$, and hence that $\dim(B)$ is even as well. □

In fact, the first part of Theorem 6.1.2 can be generalized.

THEOREM 6.1.3. *Let (A, R) be a finite-dimensional quasitriangular Hopf algebra with antipode S over any field k of characteristic* 0, *and suppose that the Drinfeld element u of A acts as a scalar in any irreducible representation of A* (*e.g.*, *when A^* is pointed*). *Then $u = S(u)$ and in particular* $S^4 = \mathrm{Id}$.

PROOF. We may assume that k is algebraically closed. In any irreducible representation V of A, $\mathrm{tr}_{|V}(u) = \mathrm{tr}_{|V}(S(u))$ (see Subsection 2.1). Since $S(u)$ also acts as a scalar in V (the dual of $S(u)_{|V}$ equals $u_{|V^*}$) it follows that $u = S(u)$ in any irreducible representation of A. Therefore, there exists a basis of A in which the operators of left multiplication by u and $S(u)$ are represented by upper triangular matrices with the same main diagonal. Hence the special grouplike element $uS(u)^{-1}$ is unipotent. Since it has a finite order we conclude that $uS(u)^{-1} = 1$, and hence that $S^4 = \mathrm{Id}$. □

REMARK 6.1.4. If (A, R) is a minimal triangular pointed Hopf algebra then all its irreducible representations are 1-dimensional. Hence Theorem 6.1.3 is applicable, and the first part of Theorem 6.1.2 follows.

EXAMPLE 6.1.5. Let A be Sweedler's 4-dimensional Hopf algebra [Sw]. It is generated as an algebra by a grouplike element g and a $1 : g$ skew primitive element x satisfying the relations $g^2 = 1$, $x^2 = 0$ and $gx = -xg$. It is known that A admits minimal triangular structures all of which with g as the Drinfeld element [R2]. In this example, $K = k[\langle g \rangle]$ and $B = sp\{1, x\}$. Note that g is central in K but is not central in A, so $(S_{|K})^2 = \mathrm{Id}$ but $S^2 \ne \mathrm{Id}$ in A. However, $S^4 = \mathrm{Id}$.

6.2. Construction of minimal triangular pointed Hopf algebras. In this section we give a method for the construction of minimal triangular pointed Hopf algebras which are *not* necessarily semisimple.

Let G be a finite abelian group, and $F : G \times G \to k^*$ be a non-degenerate skew symmetric bilinear form on G. That is, $F(xy, z) = F(x, z)F(y, z)$, $F(x, yz) = F(x, y)F(x, z)$, $F(1, x) = F(x, 1) = 1$, $F(x, y) = F(y, x)^{-1}$ for all $x, y, z \in G$, and the map $f : G \to G^\vee$ defined by $\langle f(x), y\rangle = F(x, y)$ for all $x, y \in G$ is an isomorphism. Let $U_F : G \to \{-1, 1\}$ be defined by $U_F(g) = F(g, g)$. Then U_F is a homomorphism of groups. Denote $U_F^{-1}(-1)$ by I_F.

DEFINITION 6.2.1. Let k be an algebraically closed field of characteristic zero. A datum $\mathcal{D} = (G, F, n)$ is a triple where G is a finite abelian group, $F : G \times G \to k^*$ is a non-degenerate skew symmetric bilinear form on G, and n is a non-negative integer function $I_F \to \mathbb{Z}^+$, $g \mapsto n_g$.

REMARK 6.2.2. (1) The map $f : k[G] \to k[G^\vee]$ determined by $\langle f(g), h\rangle = F(g,h)$ for all g, $h \in G$ determines a minimal triangular structure on $k[G^\vee]$.
(2) If I_F is not empty then G has an even order.

To each datum $\mathcal{D}$ we associate a Hopf algebra $H(\mathcal{D})$ in the following way. For each $g \in I_F$, let V_g be a vector space of dimension n_g, and let $\mathcal{B} = \bigoplus_{g\in I_F} V_g$. Then $H(\mathcal{D})$ is generated as an algebra by $G \cup \mathcal{B}$ with the following additional relations (to those of the group G and the vector spaces V_g's):

$$xy = F(h,g)yx \quad \text{and} \quad xa = F(a,g)ax \tag{6–1}$$

for all $g, h \in I_F$, $x \in V_g$, $y \in V_h$ and $a \in G$.

The coalgebra structure of $H(\mathcal{D})$ is determined by letting $a \in G$ be a grouplike element and $x \in V_g$ be a $1 : g$ skew primitive element for all $g \in I_F$. In particular, $\varepsilon(a) = 1$ and $\varepsilon(x) = 0$ for all $a \in G$ and $x \in V_g$.

In the special case where $G = \mathbb{Z}_2 = \{1, g\}$, $F(g,g) = -1$ and $n := n_g$, the associated Hopf algebra will be denoted by $H(n)$. Clearly, $H(0) = k\mathbb{Z}_2$, $H(1)$ is Sweedler's 4-dimensional Hopf algebra, and $H(2)$ is the 8-dimensional Hopf algebra studied in [G1, Section 2.2] in connection with KRH invariants of knots and 3-manifolds. We remark that the Hopf algebras $H(n)$ are studied in [PO1, PO2] where they are denoted by $E(n)$.

For a finite-dimensional vector space V we let $\bigwedge V$ denote the exterior algebra of V. Set $B := \bigotimes_{g\in I_F} \bigwedge V_g$.

PROPOSITION 6.2.3. 1. *The Hopf algebra $H(\mathcal{D})$ is pointed and $\mathbf{G}(H(\mathcal{D})) = G$.*
2. *$H(\mathcal{D}) = B \times k[G]$ is a biproduct.*
3. *$H(\mathcal{D})_1 = k[G] \bigoplus (k[G]\mathcal{B})$, and $P_{a,b}(H(\mathcal{D})) = sp\{a-b\} \bigoplus aV_{a^{-1}b}$ for all $a, b \in G$ (here we agree that $V_{a^{-1}b} = 0$ if $a^{-1}b \notin I_F$).*

PROOF. Part 1 follows since (by definition) $H(\mathcal{D})$ is generated as an algebra by grouplike elements and skew primitive elements. Now, it is straightforward to verify that the map $\pi : H(\mathcal{D}) \to k[G]$ determined by $\pi(a) = a$ and $\pi(x) = 0$ for all $a \in G$ and $x \in \mathcal{B}$, is a projection of Hopf algebras. Since $B = \{x \in H(\mathcal{D}) \mid (I \otimes \pi)\Delta(x) = x \otimes 1\}$, Part 2 follows from [R1]. Finally, by Part 2, B is a braided graded Hopf algebra in the Yetter-Drinfeld category ${}^{k[G]}_{k[G]}\mathcal{YD}$ (see e.g., [AS]) with respect to the grading where the elements of $\mathcal{B}$ are homogeneous of degree 1. Write $B = \bigoplus_{n\geq 0} B(n)$, where $B(n)$ denotes the homogeneous component of degree n. Then, $B(0) = k1 = B_1$ (since $B \cong H(\mathcal{D})/H(\mathcal{D})k[G]^+$ as coalgebras, it is connected). Furthermore, by similar arguments used in the proof of [AS, Lemma 3.4], $P(B) = B(1) = \mathcal{B}$. But then by [AS, Lemma 2.5], $H(\mathcal{D})$ is coradically graded (where the nth component $H(\mathcal{D})(n)$ is just $B(n) \times k[G]$) which means by definition that $H(\mathcal{D})_1 = H(\mathcal{D})(0) \bigoplus H(\mathcal{D})(1) = k[G] \bigoplus (k[G]\mathcal{B})$ as desired. The second statement of Part 3 follows now, using (1), by counting dimensions. □

In the following we determine *all* the minimal triangular structures on $H(\mathcal{D})$. Let $f : k[G] \to k[G^\vee]$ be the isomorphism from Remark 6.2.2(1), and set $I'_F := \{g \in I_F \mid n_g \neq 0\}$. Let Φ be the set of all isomorphisms $\varphi : G^\vee \to G$ satisfying $\varphi^*(\alpha) = \varphi(\alpha^{-1})$ for all $\alpha \in G^\vee$, and $(\varphi \circ f)(g) = g$ for all $g \in I'_F$ (here we identify G with $G^{\vee\vee}$).

Extend any $\alpha \in G^\vee$ to an algebra homomorphism $H(\mathcal{D}) \to k$ by setting $\alpha(z) = 0$ for all $z \in \mathcal{B}$. Extend any $x \in V_g^*$ to $P_x \in H(\mathcal{D})^*$ by setting $\langle P_x, ay\rangle = 0$ for all $a \in G$ and $y \in \bigotimes_{g\in I_F} \bigwedge V_g$ of degree different from 1, and $< P_x, ay >= \delta_{g,h}\langle x, y\rangle$ for all $a \in G$ and $y \in V_h$. We shall identify the vector spaces V_g^* and $\{P_x \mid x \in V_g^*\}$ via the map $x \mapsto P_x$.

For $g \in I'_F$, let $S_g(k)$ be the set of all isomorphisms $M_g : V_g^* \to V_{g^{-1}}$. Let $S(k) \subseteq \times_{g\in I'_F} S_g(k)$ be the set of all tuples (M_g) satisfying $M_g^* = M_{g^{-1}}$ for all $g \in I'_F$.

THEOREM 6.2.4. (1) *For each $T := (\varphi, (M_g)) \in \Phi \times S(k)$, there exists a unique Hopf algebra isomorphism $f_T : H(\mathcal{D})^{*\mathrm{cop}} \to H(\mathcal{D})$ determined by $\alpha \mapsto \varphi(\alpha)$ and $P_x \mapsto M_g(x)$ for $\alpha \in G^\vee$ and $x \in V_g^*$.*
(2) *There is a one to one correspondence between $\Phi\times S(k)$ and the set of minimal triangular structures on $H(\mathcal{D})$ given by $T \mapsto f_T$.*

PROOF. We first show that f_T is a well defined isomorphism of Hopf algebras. Using Proposition 6.2.3(2), it is straightforward to verify that

$$\Delta(P_x) = \varepsilon \otimes P_x + P_x \otimes f(g^{-1}),$$
$$P_x\alpha = \langle \alpha, g\rangle \alpha P_x, \text{ and}$$
$$P_xP_y = F(h, g)P_yP_x$$

for all $\alpha \in G^\vee$, $g, h \in I_F$, $x \in V_g^*$ and $y \in V_h^*$. Let $\mathcal{B}^* := \{P_x \mid x \in V_g^*,\ g \in I_F\}$, and H be the sub Hopf algebra of $H(\mathcal{D})^{*\mathrm{cop}}$ generated as an algebra by $G^\vee \cup \mathcal{B}^*$. Then, using (4) and our assumptions on T, it is straightforward to verify that the map $f_T^{-1} : H(\mathcal{D}) \to H$ determined by $a \mapsto \varphi^{-1}(a)$ and $z \mapsto M_g^{-1}(z)$ for $a \in G$ and $z \in V_{g^{-1}}$, is a surjective homomorphism of Hopf algebras. Let us verify for instance that $f_T^{-1}(za) = F(a, g)f_T^{-1}(az)$. Indeed, this is equivalent to $\langle\varphi^{-1}(a), g\rangle = \langle f(a), g\rangle$ which in turn holds by our assumptions on φ. Now, using Proposition 4.3(3), it is straightforward to verify that f_T^{-1} is injective on $P_{a,b}(H(\mathcal{D}))$ for all $a, b \in G$. Since $H(\mathcal{D})$ is pointed, f_T^{-1} is also injective (see e.g., [Mon, Corollary 5.4.7]). This implies that $H = H(\mathcal{D})^{*\mathrm{cop}}$, and that $f_T : H(\mathcal{D})^{*\mathrm{cop}} \to H(\mathcal{D})$ is an isomorphism of Hopf algebras as desired. Note that in particular, $G^\vee = G(H(\mathcal{D})^*)$.

The fact that f_T satisfies (2–3) follows from a straightforward computation (using (6–1)) since it is enough to verify it for algebra generators $p \in G^\vee \cup \mathcal{B}^*$ of $H(\mathcal{D})^{*\mathrm{cop}}$, and $a \in G \cup \mathcal{B}$ of $H(\mathcal{D})$.

We have to show that f_T satisfies (2.3.1). Indeed, it is straightforward to verify that $f_T^* : H(\mathcal{D})^{*\mathrm{op}} \to H(\mathcal{D})$ is determined by $\alpha \mapsto \varphi(\alpha^{-1})$ and $P_x \mapsto gM_g(x)$

for $\alpha \in G^\vee$ and $x \in V_g^*$. Hence, $f_T^* = f_T \circ S$, where S is the antipode of $H(\mathcal{D})^*$, as desired.

We now have to show that *any* minimal triangular structure on $H(\mathcal{D})$ comes from f_T for some T. Indeed, let $\mathbf{f} : H(\mathcal{D})^{*\mathrm{cop}} \to H(\mathcal{D})$ be any Hopf isomorphism. Then $\mathbf{f}$ must map $G^\vee$ onto G, $\{f(g^{-1}) \mid g \in I'_F\}$ onto I'_F, and $P_{f(g^{-1}),\varepsilon}(H(\mathcal{D})^{*\mathrm{cop}})$ bijectively onto $P_{1,\varphi(f(g^{-1}))}(H(\mathcal{D}))$. Therefore there exists an invertible operator $M_g : V_g^* \to V_{\varphi(f(g^{-1}))}$ such that $\mathbf{f}$ is determined by $\alpha \mapsto \varphi(\alpha)$ and $P_x \mapsto M_g(x)$. Suppose $\mathbf{f}$ satisfies (2–3). Then letting $p = P_x$ and $a \in G$ in (2–3) yields $a\mathbf{f}(P_x) = F(a, g)\mathbf{f}(P_x)a$ for all $a \in G$. But by (6–1), this is equivalent to $(\varphi \circ f)(g) = g$ for all $g \in I'_F$. Since by Theorem 6.1.1, $\varphi : k[G^\vee] \to k[G]$ determines a minimal triangular structure on $k[G]$ it follows that $\varphi \in \Phi$. Since $\mathbf{f} : H(\mathcal{D})^{*\mathrm{cop}} \to H(\mathcal{D})$ satisfies (2.3.1), $(M_g) \in S(k)$, and hence $\mathbf{f}$ is of the form f_T for some T as desired. □

For a triangular structure on $H(\mathcal{D})$ corresponding to the map f_T, we let R_T denote the corresponding R-matrix.

Remark 6.2.5. Note that if $n_{g^{-1}} \neq n_g$ for some $g \in I'_F$, then $S(k)$ is empty and $H(\mathcal{D})$ does *not* have a minimal triangular structure.

6.3. The classification of minimal triangular pointed Hopf algebras. In this subsection we use Theorems 6.1.1, 6.1.2 and [AEG, Theorem 6.1] to classify minimal triangular pointed Hopf algebras. Namely, we prove:

Theorem 6.3.1. *Let (A, R) be a minimal triangular pointed Hopf algebra over an algebraically closed field k of characteristic* 0. *There exist a datum $\mathcal{D}$ and $T \in \Phi \times S(k)$ such that $(A, R) \cong (H(\mathcal{D}), R_T)$ as triangular Hopf algebras.*

Before we prove Theorem 6.3.1 we need to fix some notation and prove a few lemmas.

In what follows, (A, R) will always be a minimal triangular pointed Hopf algebra over k, $G := \mathbf{G}(A)$ and $K := k[\mathbf{G}(A)]$. For any $g \in G$, $P_{1,g}(A)$ is a $\langle g\rangle$-module under conjugation, and $sp\{1 - g\}$ is a submodule of $P_{1,g}(A)$. Let $V_g \subset P_{1,g}(A)$ be its complement, and set $n_g := \dim(V_g)$.

By Theorem 6.1.1, $A = B \times K$ where $B = \{x \in A \mid (I \otimes \pi)\Delta(x) = x \otimes 1\} \subseteq A$ is a left coideal subalgebra of A (equivalently, B is an object in the Yetter-Drinfeld category ${}^{k[G]}_{k[G]}\mathcal{YD}$). Note that $B \cap K = k1$. Let $\rho : B \to K \otimes B$ be the associated comodule structure and write $\rho(x) = \sum x^1 \otimes x^2$, $x \in B$. By [R1], $B \cong A/AK^+$ as coalgebras, hence B is a connected pointed coalgebra. Let $P(B) := \{x \in B \mid \Delta_B(x) = x \otimes 1 + 1 \otimes x\}$ be the space of primitive elements of B.

Lemma 6.3.2. *For any $g \in G$, $V_g = \{x \in P(B) \mid \rho(x) = g \otimes x\}$.*

Proof. Let $x \in V_g$. Since g acts on V_g by conjugation we may assume by [G1, Lemma 0.2], that $gx = \omega xg$ for some $1 \neq \omega \in k$. Since $\pi(x)$ and $\pi(g) = g$ commute we must have that $\pi(x) = 0$. But then $(I \otimes \pi)\Delta(x) = x \otimes 1$ and hence

$x \in B$. Since $\Delta(x) = \sum x_1 \times x_2^1 \otimes x_2^2 \times 1$, applying the maps $\varepsilon \otimes I \otimes I \otimes \varepsilon$ and $I \otimes \varepsilon \otimes I \otimes \varepsilon$ to both sides of the equation $\sum x_1 \times x_2^1 \otimes x_2^2 \times 1 = x \times 1 \otimes 1 \times 1 + 1 \times g \otimes x \times 1$, yields that $x \in P(B)$ and $\rho(x) = g \otimes x$ as desired.

Suppose that $x \in P(B)$ satisfies $\rho(x) = g \otimes x$. Since $\Delta(x) = x \otimes 1 + \rho(x)$, it follows that $x \in V_g$ as desired. □

LEMMA 6.3.3. *For every* $x \in V_g$, $x^2 = 0$ *and* $gx = -xg$.

PROOF. Suppose $V_g \neq 0$ and let $0 \neq x \in V_g$. Then $S^2(x) = g^{-1}xg$, $g^{-1}xg \neq x$ by [G1, Lemma 0.2], and $g^{-1}xg \in V_g$. Since by Theorem 6.1.2, $S^4 = \mathrm{Id}$ it follows that g^2 and x commute, and hence that $gx = -xg$ for every $x \in V_g$.

Second we wish to show that $x^2 = 0$. By Lemma 6.3.2, $x \in B$ and hence $x^2 \in B$ (B is a subalgebra of A). Since $\Delta(x^2) = x^2 \otimes 1 + g^2 \otimes x^2$, and x^2 and g^2 commute, it follows from [G1, Lemma 0.2] that $x^2 = \alpha(1 - g^2) \in K$ for some $\alpha \in k$. We thus conclude that $x^2 = 0$, as desired. □

Recall that the map $f_R : A^{*\mathrm{cop}} \to A$ is an isomorphism of Hopf algebras, and let $F : G \times G \to k^*$ be the associated non-degenerate skew symmetric bilinear form on G defined by $F(g, h) := \langle f_R^{-1}(g), h\rangle$ for every $g, h \in G$.

LEMMA 6.3.4. *For any* $x \in V_g$ *and* $y \in V_h$, $xy = F(h, g)yx$.

PROOF. If either $V_g = 0$ or $V_h = 0$, there is nothing to prove. Suppose $V_g, V_h \neq 0$, and let $0 \neq x \in V_g$ and $0 \neq y \in V_h$. Set $P := f_R^{-1}(x)$. Then $P \in P_{f_R^{-1}(g),\varepsilon}(A^{*\mathrm{cop}})$. Substituting $p := P$ and $a := y$ in equation (2–3) yields $yx - F(g, h)xy = \langle P, y\rangle(1-gh)$. Since $\langle P, y\rangle(1 - gh) \in B \cap K$, it is equal to 0, and hence $yx = F(g, h)xy$. □

LEMMA 6.3.5. *For any* $a \in G$ *and* $x \in V_g$, $xa = F(a, g)ax$.

PROOF. Set $P := f_R^{-1}(x)$. Then the result follows by letting $p := P$ and $a \in G$ in (2–3), and noting that $P \in P_{f_R^{-1}(g),\varepsilon}(A^{*\mathrm{cop}})$. □

We can now prove Theorem 6.3.1.

PROOF OF THEOREM 6.3.1. Let $n : I_F \to \mathbb{Z}^+$ be the nonnegative integer function defined by $n(g) = n_g$, and let $\mathcal{D} := (G, F, n)$. By [AEG, Theorem 6.1], A is generated as an algebra by $G \cup (\bigoplus_{g \in I_F} V_g)$. By Lemmas 6.3.3-6.3.5, relations (6–1) are satisfied. Therefore there exists a surjection of Hopf algebras $\varphi : H(\mathcal{D}) \to A$. Using Proposition 4.3(3), it is straightforward to verify that φ is injective on $P_{a,b}(H(\mathcal{D}))$ for all $a, b \in G$. Since $H(\mathcal{D})$ is pointed, φ is also injective (see e.g., [Mon, Corollary 5.4.7]). Hence φ is an isomorphism of Hopf algebras. The rest of the theorem follows now from Theorem 6.2.4. □

REMARK 6.3.6. Theorem 6.1 in [AEG] states that a finite-dimensional cotriangular pointed Hopf algebra is generated by its grouplike and skewprimitive elements. This confirms the conjecture that this is the case for *any* finite-dimensional pointed Hopf algebra over $\mathbb{C}$ [AS2], in the cotriangular case. The proof uses a categorical point of view (or, alternatively, the Lifting method; see [AS1, AS2]).

7. Triangular Hopf Algebras with the Chevalley Property

As we said in the introduction, semisimple cosemisimple triangular Hopf algebras and minimal triangular pointed Hopf algebras share in common the Chevalley property. In this section we describe the classification of finite-dimensional triangular Hopf algebras with the Chevalley property, given in [AEG].

7.1. Triangular Hopf algebras with Drinfeld element of order ≤ 2. We start by classifying triangular Hopf algebras with R-matrix of rank ≤ 2. We show that such a Hopf algebra is a suitable modification of a cocommutative Hopf superalgebra (i.e., the group algebra of a supergroup). On the other hand, by Corollary 2.2.3.5, a finite supergroup is a semidirect product of a finite group with an odd vector space on which this group acts.

7.1.1. The correspondence between Hopf algebras and superalgebras. We start with a correspondence theorem between Hopf algebras and Hopf superalgebras.

THEOREM 7.1.1.1. *There is a one to one correspondence between*

1. *isomorphism classes of pairs (A, u) where A is an ordinary Hopf algebra, and u is a grouplike element in A such that $u^2 = 1$, and*
2. *isomorphism classes of pairs $(\mathcal{A}, g)$ where $\mathcal{A}$ is a Hopf superalgebra, and g is a grouplike element in $\mathcal{A}$ such that $g^2 = 1$ and $gxg^{-1} = (-1)^{p(x)}x$ (i.e., g acts on x by its parity),*

such that the tensor categories of representations of A and $\mathcal{A}$ are equivalent.

PROOF. Let (A, u) be an ordinary Hopf algebra with comultiplication Δ, counit ε, antipode S, and a grouplike element u such that $u^2 = 1$. Let $\mathcal{A} = A$ regarded as a superalgebra, where the $\mathbb{Z}_2$-grading is given by the adjoint action of u. For $a \in A$, let us define Δ_0, Δ_1 by writing $\Delta(a) = \Delta_0(a) + \Delta_1(a)$, where $\Delta_0(a) \in A \otimes A_0$ and $\Delta_1(a) \in A \otimes A_1$. Define a map $\tilde{\Delta} : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by $\tilde{\Delta}(a) := \Delta_0(a) - (-1)^{p(a)}(u \otimes 1)\Delta_1(a)$. Define $\tilde{S}(a) := u^{p(a)}S(a)$, $a \in A$. Then it is straightforward to verify that $(\mathcal{A}, \tilde{\Delta}, \varepsilon, \tilde{S})$ is a Hopf superalgebra.

The element u remains grouplike in the new Hopf superalgebra, and acts by parity, so we can set $g := u$.

Conversely, suppose that $(\mathcal{A}, g)$ is a pair where $\mathcal{A}$ is a Hopf superalgebra with comultiplication $\tilde{\Delta}$, counit ε, antipode $\tilde{S}$, and a grouplike element g, with $g^2 = 1$, acting by parity. For $a \in \mathcal{A}$, let us define $\tilde{\Delta}_0, \tilde{\Delta}_1$ by writing $\tilde{\Delta}(a) = \tilde{\Delta}_0(a) + \tilde{\Delta}_1(a)$, where $\tilde{\Delta}_0(a) \in \mathcal{A} \otimes \mathcal{A}_0$ and $\tilde{\Delta}_1(a) \in \mathcal{A} \otimes \mathcal{A}_1$. Let $A = \mathcal{A}$ as algebras, and define a map $\Delta : A \to A \otimes A$ by $\Delta(a) := \tilde{\Delta}_0(a) - (-1)^{p(a)}(g \otimes 1)\tilde{\Delta}_1(a)$. Define $S(a) := g^{p(a)}\tilde{S}(a)$, $a \in A$. Then it is straightforward to verify that $(A, \Delta, \varepsilon, S)$ is an ordinary Hopf algebra, and we can set $u := g$.

It is obvious that the two assignments constructed above are inverse to each other. The equivalence of tensor categories is straightforward to verify. The theorem is proved. □

Theorem 7.1.1.1 implies the following. Let $\mathcal{A}$ be *any* Hopf superalgebra, and $\mathbb{C}[\mathbb{Z}_2] \ltimes \mathcal{A}$ be the semidirect product, where the generator g of $\mathbb{Z}_2$ acts on $\mathcal{A}$ by $gxg^{-1} = (-1)^{p(x)}x$. Then we can define an ordinary Hopf algebra $\overline{\mathcal{A}}$, which is the one corresponding to $(\mathbb{C}[\mathbb{Z}_2] \ltimes \mathcal{A}, g)$ under the correspondence of Theorem 7.1.1.1.

The constructions of this subsection have the following explanation in terms of Radford's biproduct construction [R1]. Namely $\mathcal{A}$ is a Hopf algebra in the Yetter-Drinfeld category of $\mathbb{C}[\mathbb{Z}_2]$, so Radford's biproduct construction yields a Hopf algebra structure on $\mathbb{C}[\mathbb{Z}_2] \otimes \mathcal{A}$, and it is straightforward to see that this Hopf algebra is exactly $\overline{\mathcal{A}}$. Moreover, it is clear that for any pair (A, u) as in Theorem 7.1.1.1, gu is central in $\overline{\mathcal{A}}$ and $A = \overline{\mathcal{A}}/(gu - 1)$.

7.1.2. Correspondence of twists. Let us say that a twist J for a Hopf algebra A with an involutive grouplike element g is *even* if it is invariant under $\mathrm{Ad}(g)$.

PROPOSITION 7.1.2.1. *Let $(\mathcal{A}, g)$ be a pair as in Theorem 7.1.1.1, and let A be the associated ordinary Hopf algebra. Let $\mathcal{J} \in \mathcal{A} \otimes \mathcal{A}$ be an even element. Write $\mathcal{J} = \mathcal{J}_0 + \mathcal{J}_1$, where $\mathcal{J}_0 \in \mathcal{A}_0 \otimes \mathcal{A}_0$ and $\mathcal{J}_1 \in \mathcal{A}_1 \otimes \mathcal{A}_1$. Define $J := \mathcal{J}_0 - (g \otimes 1)\mathcal{J}_1$. Then J is an even twist for A if and only if $\mathcal{J}$ is a twist for $\mathcal{A}$. Moreover, $\mathcal{A}^{\mathcal{J}}$ corresponds to A^J under the correspondence in Theorem 7.1.1.1. Thus, there is a one to one correspondence between even twists for A and twists for $\mathcal{A}$, given by $J \to \mathcal{J}$.*

PROOF. Straightforward. □

7.1.3. The Correspondence between triangular Hopf algebras and superalgebras. Let us now return to our main subject, which is triangular Hopf algebras and superalgebras. For triangular Hopf algebras whose Drinfeld element u is involutive, we will make the natural choice of the element u in Theorem 7.1.1.1, namely define it to be the Drinfeld element of A.

THEOREM 7.1.3.1. *The correspondence of Theorem 7.1.1.1 extends to a one to one correspondence between*

1. *isomorphism classes of ordinary triangular Hopf algebras A with Drinfeld element u such that $u^2 = 1$, and*
2. *isomorphism classes of pairs $(\mathcal{A}, g)$ where $\mathcal{A}$ is a triangular Hopf superalgebra with Drinfeld element 1 and g is an element of $\mathbf{G}(\mathcal{A})$ such that $g^2 = 1$ and $gxg^{-1} = (-1)^{p(x)}x$.*

PROOF. Let (A, R) be a triangular Hopf algebra with $u^2 = 1$. Since $(S \otimes S)(R) = R$ and $S^2 = \mathrm{Ad}(u)$ [Dr2], $u \otimes u$ and R commute. Hence we can write $R = R_0 + R_1$, where $R_0 \in A_0 \otimes A_0$ and $R_1 \in A_1 \otimes A_1$. Let $\mathcal{R} := (R_0 + (1 \otimes u)R_1)R_u$. Then $\mathcal{R}$

is even. Indeed, since

$$R_0 = 1/2(R + (u \otimes 1)R(u \otimes 1)) \text{ and}$$
$$R_1 = 1/2(R - (u \otimes 1)R(u \otimes 1)),$$

$u \otimes u$ and $\mathcal{R}$ commute.

It is now straightforward to show that $(\mathcal{A}, \mathcal{R})$ is triangular with Drinfeld element 1. Let us show for instance that $\mathcal{R}^{-1} = \mathcal{R}_{21}$. Let us use the notation $a * b, X^{21}$ for multiplication and opposition in the tensor square of a superalgebra, and the notation ab, X^{op} for usual algebras. Then,

$$\mathcal{R} * \mathcal{R}_{21} = (R_0 + (1 \otimes u)R_1)R_u * (R_0^{\text{op}} - (u \otimes 1)R_1^{\text{op}})R_u.$$

Since, $R_u R_0 = R_0 R_u$, $R_u R_1 = -(u \otimes u)R_1 R_u$, we get that the RHS equals

$$(R_0 + (1 \otimes u)R_1) * (R_0^{\text{op}} + (1 \otimes u)R_1^{\text{op}}) = R_0 R_0^{\text{op}} + R_1 R_1^{\text{op}} + (1 \otimes u)(R_1 R_0^{\text{op}} + R_0 R_1^{\text{op}}).$$

But, $R_0 R_0^{\text{op}} + R_1 R_1^{\text{op}} = 1$ and $(1 \otimes u)(R_1 R_0^{\text{op}} + R_0 R_1^{\text{op}}) = 0$, since $RR^{\text{op}} = 1$, so we are done.

Conversely, suppose that $(\mathcal{A}, g)$ is a pair where $\mathcal{A}$ is a triangular Hopf superalgebra with R-matrix $\mathcal{R}$ and Drinfeld element 1. Let $\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_1$, where $\mathcal{R}_0$ has even components, and $\mathcal{R}_1$ has odd components. Let $R := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g$. Then it is straightforward to show that (A, R) is triangular with Drinfeld element $u = g$. The theorem is proved. □

Corollary 7.1.3.2. *If $(\mathcal{A}, \mathcal{R})$ is a triangular Hopf superalgebra with Drinfeld element* 1, *then the Hopf algebra $\overline{\mathcal{A}}$ is also triangular*, *with the R-matrix*

$$\overline{R} := (\mathcal{R}_0 + (1 \otimes g)\mathcal{R}_1)R_g, \tag{7–1}$$

where g is the grouplike element adjoined to $\mathcal{A}$ to obtain $\overline{\mathcal{A}}$. Moreover, *$\mathcal{A}$ is minimal if and only if so is $\overline{\mathcal{A}}$.*

Proof. Clear. □

The following corollary, combined with Kostant's theorem, gives a classification of triangular Hopf algebras with R-matrix of rank ≤ 2 (i.e., of the form R_u as in (2–9), where u is a grouplike of order ≤ 2).

Corollary 7.1.3.3. *The correspondence of Theorem* 7.1.3.1 *restricts to a one to one correspondence between*

1. *isomorphism classes of ordinary triangular Hopf algebras with R-matrix of rank ≤ 2, and*
2. *isomorphism classes of pairs $(\mathcal{A}, g)$ where $\mathcal{A}$ is a cocommutative Hopf superalgebra and g is an element of $\mathbf{G}(\mathcal{A})$ such that $g^2 = 1$ and $gxg^{-1} = (-1)^{p(x)}x$.*

Proof. Let (A, R) be an ordinary triangular Hopf algebra with $R-$ matrix of rank ≤ 2. In particular, the Drinfeld element u of A satisfies $u^2 = 1$, and $R = R_u$. Hence by Theorem 7.1.3.1, $(\mathcal{A}, \tilde{\Delta}, \mathcal{R})$ is a triangular Hopf superalgebra. Moreover, it is cocommutative since $\mathcal{R} = R_u R_u = 1$.

Conversely, for any $(\mathcal{A}, g)$, by Theorem 7.1.3.1, the pair (A, R_g) is an ordinary triangular Hopf algebra, and clearly the rank of R_g is ≤ 2. □

In particular, Corollaries 2.2.3.5 and 7.1.3.3 imply that finite-dimensional triangular Hopf algebras with R-matrix of rank ≤ 2 correspond to supergroup algebras. In view of this, we make the following definition.

Definition 7.1.3.4. A finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2 is called a modified supergroup algebra.

7.1.4. Construction of twists for supergroup algebras.

Proposition 7.1.4.1. *Let $\mathcal{A} = \mathbb{C}[G] \ltimes \Lambda V$ be a supergroup algebra. Let $r \in S^2V$. Then $\mathcal{J} := e^{r/2}$ is a twist for $\mathcal{A}$. Moreover, $((\Lambda V)^{\mathcal{J}}, \mathcal{J}_{21}^{-1}\mathcal{J})$ is minimal triangular if and only if r is nondegenerate.*

Proof. Straightforward. □

Example 7.1.4.2. Let G be the group of order 2 with generator g. Let $V := \mathbb{C}$ be the nontrivial 1-dimensional representation of G, and write $\Lambda V = sp\{1, x\}$. Then the associated ordinary triangular Hopf algebra to $(\mathcal{A}, g) := (\mathbb{C}[G] \ltimes \Lambda V, g)$ is Sweedler's 4-dimensional Hopf algebra A [Sw] (see Example 6.1.5) with the triangular structure R_g. It is known [R2] that the set of triangular structures on A is parameterized by $\mathbb{C}$; namely, R is a triangular structure on A if and only if

$$R = R_\lambda := R_g - \frac{\lambda}{2}(x \otimes x - gx \otimes x + x \otimes gx + gx \otimes gx),\ \lambda \in \mathbb{C}.$$

Clearly, (A, R_λ) is minimal if and only if $\lambda \neq 0$.

Let $r \in S^2V$ be defined by $r := \lambda x \otimes x$, $\lambda \in \mathbb{C}$. Set $\mathcal{J}_\lambda := e^{r/2} = 1 + \frac{1}{2}\lambda x \otimes x$; it is a twist for $\mathcal{A}$. Hence, $J_\lambda := 1 - \frac{1}{2}\lambda gx \otimes x$ is a twist for A. It is easy to check that $R_\lambda = (J_\lambda)_{21}^{-1} R_g J_\lambda$. Thus, $(A, R_\lambda) = (A, R_0)^{J_\lambda}$.

Remark 7.1.4.3. In fact, Radford's classification of triangular structures on A can be easily deduced from Lemma 7.3.2.6 below.

7.2. The Chevalley property. Recall that in the introduction we made the following definition.

Definition 7.2.1. A Hopf algebra A over $\mathbb{C}$ is said to have the Chevalley property if the tensor product of any two simple A-modules is semisimple. More generally, let us say that a tensor category has the Chevalley property if the tensor product of two simple objects is semisimple.

Let us give some equivalent formulations of the Chevalley property.

Proposition 7.2.2. *Let A be a finite-dimensional Hopf algebra over $\mathbb{C}$. The following conditions are equivalent*:

1. *A has the Chevalley property.*
2. *The category of (right) A^*-comodules has the Chevalley property.*
3. $\mathrm{Corad}(A^*)$ *is a Hopf subalgebra of* A^*.
4. $\mathrm{Rad}(A)$ *is a Hopf ideal and thus* $A/\mathrm{Rad}(A)$ *is a Hopf algebra.*
5. $S^2 = \mathrm{Id}$ *on* $A/\mathrm{Rad}(A)$, *or equivalently on* $\mathrm{Corad}(A^*)$.

Proof. ($1 \Leftrightarrow 2$) Clear, since the categories of left A-modules and right A^*-comodules are equivalent.

($2 \Rightarrow 3$) Recall the definition of a matrix coefficient of a comodule V over A^*. If $\rho : V \to V \otimes A^*$ is the coaction, $v \in V$, $\alpha \in V^*$, then

$$\phi^V_{v,\alpha} := (\alpha \otimes \mathrm{Id})\rho(v) \in A^*.$$

It is well-known that:

(a) The coradical of A^* is the linear span of the matrix coefficients of all simple A^*-comodules.

(b) The product in A^* of two matrix coefficients is a matrix coefficient of the tensor product. Specifically,

$$\phi^V_{v,\alpha}\phi^W_{w,\beta} = \phi^{V\otimes W}_{v\otimes w,\alpha\otimes\beta}.$$

It follows at once from (a) and (b) that $\mathrm{Corad}(A^*)$ is a subalgebra of A^*. Since the coradical is stable under the antipode, the claim follows.

($3 \Leftrightarrow 4$) To say that $\mathrm{Rad}(A)$ is a Hopf ideal is equivalent to saying that $\mathrm{Corad}(A^*)$ is a Hopf algebra, since $\mathrm{Corad}(A^*) = (A/\mathrm{Rad}(A))^*$.

($4 \Rightarrow 1$) If V, W are simple A-modules then they factor through $A/\mathrm{Rad}(A)$. But $A/\mathrm{Rad}(A)$ is a Hopf algebra, so $V \otimes W$ also factors through $A/\mathrm{Rad}(A)$, so it is semisimple.

($3 \Rightarrow 5$) Clear, since a cosemisimple Hopf algebra is involutory.

($5 \Rightarrow 3$) Consider the subalgebra B of A^* generated by $\mathrm{Corad}(A^*)$. This is a Hopf algebra, and $S^2 = \mathrm{Id}$ on it. Thus, B is cosemisimple and hence $B = \mathrm{Corad}(A^*)$ is a Hopf subalgebra of A^*. □

Remark 7.2.3. The assumption that the base field has characteristic 0 is needed only in the proof of ($5 \Leftrightarrow 3$)

7.3. The classification of triangular Hopf algebras with the Chevalley property.

7.3.1. The main theorem. The main result of Section 7 is the following theorem.

Theorem 7.3.1.1. *Let A be a finite-dimensional triangular Hopf algebra over $\mathbb{C}$. Then the following are equivalent*:

1. *A is a twist of a finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2 (i.e., of a modified supergroup algebra).*

2. *A has the Chevalley property.*

7.3.2. Proof of the main theorem. The proof will take the remainder of this subsection. We shall need the following result, whose proof is given in [AEG].

THEOREM 7.3.2.1. *Let $\mathcal{A}$ be a local finite-dimensional Hopf superalgebra (not necessarily supercommutative). Then $\mathcal{A} = \Lambda V^*$ for a finite-dimensional vector space V. In other words, $\mathcal{A}$ is the function algebra of an odd vector space V.*

REMARK 7.3.2.2. Note that in the commutative case Theorem 7.3.2.1 is a special case of Proposition 3.2 of [Ko].

We start by giving a super-analogue of Theorem 3.1 in [G4].

LEMMA 7.3.2.3. *Let $\mathcal{A}$ be a minimal triangular pointed Hopf superalgebra. Then $\mathrm{Rad}(\mathcal{A})$ is a Hopf ideal, and $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is minimal triangular.*

PROOF. The proof is a tautological generalization of the proof of Theorem 3.1 in [G4] to the super case.

First of all, it is clear that $\mathrm{Rad}(\mathcal{A})$ is a Hopf ideal, since its orthogonal complement (the coradical of $\mathcal{A}^*$) is a sub Hopf superalgebra (as $\mathcal{A}^*$ is isomorphic to $\mathcal{A}^{\mathrm{cop}}$ as a coalgebra, and hence is pointed). Thus, it remains to show that the triangular structure on $\mathcal{A}$ descends to a minimal triangular structure on $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$. For this, it suffices to prove that the composition of the Hopf superalgebra maps

$$\mathrm{Corad}(\mathcal{A}^{*\mathrm{cop}}) \hookrightarrow \mathcal{A}^{*\mathrm{cop}} \to \mathcal{A} \to \mathcal{A}/\mathrm{Rad}(\mathcal{A})$$

(where the middle map is given by the R-matrix) is an isomorphism. But this follows from the fact that for any surjective coalgebra map $\eta : C_1 \to C_2$, the image of the coradical of C_1 contains the coradical of C_2 (see e.g., [Mon, Corollary 5.3.5]): One needs to apply this statement to the map $\mathcal{A}^{*\mathrm{cop}} \to \mathcal{A}/\mathrm{Rad}(\mathcal{A})$. □

LEMMA 7.3.2.4. *Let $\mathcal{A}$ be a minimal triangular pointed Hopf superalgebra, such that the R-matrix $\mathcal{R}$ of $\mathcal{A}$ is unipotent (which is to say, $\mathcal{R} - 1 \otimes 1$ is 0 in $\mathcal{A}/\mathrm{Rad}(\mathcal{A}) \otimes \mathcal{A}/\mathrm{Rad}(\mathcal{A})$). Then $\mathcal{A} = \Lambda V$ as a Hopf superalgebra, and $\mathcal{R} = e^r$, where $r \in S^2V$ is a nondegenerate symmetric (in the usual sense) bilinear form on V^*.*

PROOF. By Lemma 7.3.2.3, $\mathrm{Rad}(\mathcal{A})$ is a Hopf ideal, and $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is minimal triangular. But the R-matrix of $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ must be $1 \otimes 1$, so $\mathcal{A}/\mathrm{Rad}(\mathcal{A})$ is 1-dimensional. Hence $\mathcal{A}$ is local, so by Theorem 7.3.2.1, $\mathcal{A} = \Lambda V$. If $\mathcal{R}$ is a triangular structure on $\mathcal{A}$ then it comes from an isomorphism $\Lambda V^* \to \Lambda V$ of Hopf superalgebras, which is induced by a linear isomorphism $r : V^* \to V$. So $\mathcal{R} = e^r$, where r is regarded as an element of $V \otimes V$. Since $\mathcal{R}\mathcal{R}_{21} = 1$, we have $r + r^{21} = 0$ (where $r^{21} = -r^{\mathrm{op}}$ is the opposite of r in the supersense), so $r \in S^2V$. □

REMARK 7.3.2.5. The classification of pointed finite-dimensional Hopf algebras with coradical of dimension 2 is known [CD, N]. In [AEG] we used the Lifting method [AS1, AS2] to give an alternative proof. Below we shall need the following more precise version of this result in the triangular case.

LEMMA 7.3.2.6. *Let A be a minimal triangular pointed Hopf algebra, whose coradical is $\mathbb{C}[\mathbb{Z}_2] = sp\{1,u\}$, where u is the Drinfeld element of A. Then $A = \overline{(\Lambda V)^{\mathcal{J}}}$ with the triangular structure of Corollary 7.1.3.2, where $\mathcal{J} = e^{r/2}$, with $r \in S^2V$ a nondegenerate element. In particular, A is a twist of a modified supergroup algebra.*

PROOF. Let $\mathcal{A}$ be the associated triangular Hopf superalgebra to A, as described in Theorem 7.1.3.1. Then the R-matrix of $\mathcal{A}$ is unipotent, because it turns into $1 \otimes 1$ after killing the radical.

Let $\mathcal{A}_m$ be the minimal part of $\mathcal{A}$. By Lemma 7.3.2.4, $\mathcal{A}_m = \Lambda V$ and $\mathcal{R} = e^r$, $r \in S^2V$. So if $\mathcal{J} := e^{r/2}$ then $\mathcal{A}^{\mathcal{J}^{-1}}$ has R-matrix equal to $1 \otimes 1$. Thus, $\mathcal{A}^{\mathcal{J}^{-1}}$ is cocommutative, so by Corollary 2.2.3.5, it equals $\mathbb{C}[\mathbb{Z}_2] \ltimes \Lambda V$. Hence $\mathcal{A} = \mathbb{C}[\mathbb{Z}_2] \ltimes (\Lambda V)^{\mathcal{J}}$, and the result follows from Proposition 7.1.2.1. $\square$

We shall need the following lemma.

LEMMA 7.3.2.7. *Let $B \subseteq A$ be finite-dimensional associative unital algebras. Then any simple B-module is a constituent (in the Jordan-Holder series) of some simple A-module.*

PROOF. Since A, considered as a B-module, contains B as a B-module, any simple B-module is a constituent of A.

Decompose A (in the Grothendieck group of A) into simple A-modules: $A = \sum V_i$. Further decomposing as B-modules, we get $V_i = \sum W_{ij}$, and hence $A = \sum_i \sum_j W_{ij}$. Now, by Jordan-Holder theorem, since A (as a B-module) contains all simple B-modules, any simple B-module X is in $\{W_{ij}\}$. Thus, X is a constituent of some V_i, as desired. $\square$

PROPOSITION 7.3.2.8. *Any minimal triangular Hopf algebra A with the Chevalley property is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2.*

PROOF. By Proposition 7.2.2, the coradical A_0 of A is a Hopf subalgebra, since $A \cong A^{*\mathrm{cop}}$, being minimal triangular. Consider the Hopf algebra map $\varphi : A_0 \to A^{*\mathrm{cop}}/\mathrm{Rad}(A^{*\mathrm{cop}})$, given by the composition of the following maps:

$$A_0 \hookrightarrow A \cong A^{*\mathrm{cop}} \to A^{*\mathrm{cop}}/\mathrm{Rad}(A^{*\mathrm{cop}}),$$

where the second map is given by the R-matrix. We claim that φ is an isomorphism. Indeed, A_0 and $A^{*\mathrm{cop}}/\mathrm{Rad}(A^{*\mathrm{cop}})$ have the same dimension, since $\mathrm{Rad}(A^{*\mathrm{cop}}) = (A_0)^{\perp}$, and φ is injective, since A_0 is semisimple by [LR]. Let $\pi : A \to A_0$ be the associated projection.

We see, arguing exactly as in [G4, Theorem 3.1], that A_0 is also minimal triangular, say with R-matrix R_0.

Now, by [EG1, Theorem 2.1], we can find a twist J in $A_0 \otimes A_0$ such that $(A_0)^J$ is isomorphic to a group algebra and has R-matrix $(R_0)^J$ of rank ≤ 2. Notice that here we are relying on Deligne's theorem, as mentioned in the introduction.

Let us now consider J as an element of $A_0 \otimes A_0$ and the twisted Hopf algebra A^J, which is again triangular.

The projection $\pi : A^J \to (A_0)^J$ is still a Hopf algebra map, and sends R^J to $(R_0)^J$. It induces a projection $(A^J)_m \to \mathbb{C}[\mathbb{Z}_2]$, whose kernel K_m is contained in the kernel of π. Because any simple $(A^J)_m$-module is contained as a constituent in a simple A-module (see Lemma 7.3.2.7), $K_m = \text{Rad}((A^J)_m)$. Hence, $(A^J)_m$ is minimal triangular and $(A^J)_m/\text{Rad}((A^J)_m) = (\mathbb{C}[\mathbb{Z}_2], R_u)$. It follows, again by minimality, that $(A^J)_m$ is also pointed with coradical isomorphic to $\mathbb{C}[\mathbb{Z}_2]$. So by Lemma 7.3.2.6, $(A^J)_m$, and hence A^J, can be further twisted into a triangular Hopf algebra with R-matrix of rank ≤ 2, as desired. □

Now we can prove the main theorem.

Proof of Theorem 7.3.1.1. ($2 \Rightarrow 1$) By Proposition 7.2.2, $A/\text{Rad}(A)$ is a semisimple Hopf algebra. Let A_m be the minimal part of A, and A'_m be the image of A_m in $A/\text{Rad}(A)$. Then A'_m is a semisimple Hopf algebra.

Consider the kernel K of the projection $A_m \to A'_m$. Then $K = \text{Rad}(A) \cap A_m$. This means that any element $k \in K$ is zero in any simple A-module. This implies that k acts by zero in any simple A_m-module, since by Lemma 7.3.2.7, any simple A_m-module occurs as a constituent of some simple A-module. Thus, K is contained in $\text{Rad}(A_m)$. On the other hand, A_m/K is semisimple, so $K = \text{Rad}(A_m)$. This shows that $\text{Rad}(A_m)$ is a Hopf ideal. Thus, A_m is minimal triangular satisfying the conditions of Proposition 7.3.2.8. By Proposition 7.3.2.8, A_m is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2. Hence A is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2 (by the same twist), as desired.

($1 \Rightarrow 2$) By assumption, $\text{Rep}(A)$ is equivalent to $\text{Rep}(\tilde{G})$ for some supergroup $\tilde{G}$ (as a tensor category without braiding). But we know that supergroup algebras have the Chevalley property, since, modulo their radicals, they are group algebras. This concludes the proof of the main theorem. □

Remark 7.3.2.9. Notice that it follows from the proof of the main theorem that any triangular Hopf algebra with the Chevalley property can be obtained by twisting of a triangular Hopf algebra with R-matrix of rank ≤ 2 by an *even* twist.

Definition 7.3.2.10. If a triangular Hopf algebra A over $\mathbb{C}$ satisfies condition 1. or 2. of Theorem 7.3.1.1, we will say that H is of supergroup type.

7.3.3. Corollaries of the main theorem.

Corollary 7.3.3.1. *A finite-dimensional triangular Hopf algebra A is of supergroup type if and only if so is its minimal part A_m.*

Proof. If A is of supergroup type then $\mathrm{Rad}(A)$ is a Hopf ideal, so like in the proof of Theorem 7.3.1.1 ($2 \Rightarrow 1$) we conclude that $\mathrm{Rad}(A_m)$ is a Hopf ideal, i.e., A_m is of supergroup type.

Conversely, if A_m is of supergroup type then A_m is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2. Hence A is a twist of a triangular Hopf algebra with R-matrix of rank ≤ 2 (by the same twist), so A is of supergroup type. □

Corollary 7.3.3.2. *A finite-dimensional triangular Hopf algebra whose coradical is a Hopf subalgebra is of supergroup type. In particular, this is the case for finite-dimensional triangular pointed Hopf algebras.*

Proof. This follows from Corollary 7.3.3.1. □

Corollary 7.3.3.3. *Any finite-dimensional triangular basic Hopf algebra is of supergroup type.*

Proof. A basic Hopf algebra automatically has the Chevalley property since all its irreducible modules are 1-dimensional. Hence the result follows from the main theorem. □

Remark 7.3.3.4. The classification in Theorem 7.3.1.1 can be made more effective and explicit. Indeed, Theorem 2.2 in [EG7] states a bijection between the set of isomorphism classes of finite-dimensional triangular Hopf algebras with the Chevalley property and the set of isomorphism classes of septuples (G, W, H, Y, B, V, u) where G is a finite group, W is a finite-dimensional representation of G, H is a subgroup of G, Y is an $H-$invariant subspace of W, B is an $H-$invariant non-degenerate element in S^2Y, V is an irreducible projective representation of H of dimension $|H|^{1/2}$, and $u \in G$ is a central element of order ≤ 2 acting by -1 on W. In the semisimple case, the septuples reduce to the quadruples of Theorem 4.2.6. In the minimal pointed case, we recover Theorem 6.3.1.

7.4. Categorical dimensions in symmetric categories with finitely many irreducibles are integers. In [AEG] we classified finite-dimensional triangular Hopf algebras with the Chevalley property. We also gave one result that is valid in a greater generality for any finite-dimensional triangular Hopf algebra, and even for any symmetric rigid category with finitely many irreducible objects.

Let $\mathcal{C}$ be a $\mathbb{C}$-linear abelian symmetric rigid category with $\mathbf{1}$ as its unit object, and suppose that $\mathrm{End}(\mathbf{1}) = \mathbb{C}$. Recall that there is a natural notion of dimension in $\mathcal{C}$, generalizing the ordinary dimension of an object in Vect, and having the properties of being additive and multiplicative with respect to the tensor product. Let β denote the commutativity constraint in $\mathcal{C}$, and for an object V, let ev_V, $coev_V$ denote the associated evaluation and coevaluation morphisms.

DEFINITION 7.4.1 [DM]. The categorical dimension $\dim_c(V) \in \mathbb{C}$ of $V \in \mathcal{C}$ is the morphism

$$\dim_c(V) : \mathbf{1} \xrightarrow{ev_V} V \otimes V^* \xrightarrow{\beta_{V,V^*}} V^* \otimes V \xrightarrow{coev_V} \mathbf{1}. \tag{7–2}$$

The main result of this subsection is the following:

THEOREM 7.4.2. *In any $\mathbb{C}$-linear abelian symmetric rigid tensor category $\mathcal{C}$ with finitely many irreducible objects, the categorical dimensions of objects are integers.*

PROOF. First note that the categorical dimension of any object V of $\mathcal{C}$ is an algebraic integer. Indeed, let $V_1 \ldots, V_n$ be the irreducible objects of $\mathcal{C}$. Then $\{V_1 \ldots, V_n\}$ is a basis of the Grothendieck ring of $\mathcal{C}$. Write $V \otimes V_i = \sum_j N_{ij}(V) V_j$ in the Grothendieck ring. Then $N_{ij}(V)$ is a matrix with integer entries, and $\dim_c(V)$ is an eigenvalue of this matrix. Thus, $\dim_c(V)$ is an algebraic integer.

Now, if $\dim_c(V) = d$ then it is easy to show (see e.g. [De1]) that

$$\dim_c(S^k V) = d(d+1)\cdots(d+k-1)/k!,$$

and

$$\dim_c(\Lambda^k V) = d(d-1)\cdots(d-k+1)/k!,$$

hence they are also algebraic integers. So the theorem follows from:

LEMMA. *Suppose d is an algebraic integer such that $d(d+1)\cdots(d+k-1)/k!$ and $d(d-1)\cdots(d-k+1)/k!$ are algebraic integers for all k. Then d is an integer.*

PROOF. Let Q be the minimal monic polynomial of d over $\mathbb{Z}$. Since

$$d(d-1)\cdots(d-k+1)/k!$$

is an algebraic integer, so are the numbers $d'(d'-1)\cdots(d'-k+1)/k!$, where d' is any algebraic conjugate of d. Taking the product over all conjugates, we get that

$$N(d)N(d-1)\cdots N(d-k+1)/(k!)^n$$

is an integer, where n is the degree of Q. But $N(d-x) = (-1)^n Q(x)$. So we get that $Q(0)Q(1)\cdots Q(k-1)/(k!)^n$ is an integer. Similarly from the identity for S^kV, it follows that $Q(0)Q(-1)\cdots Q(1-k)/(k!)^n$ is an integer. Now, without loss of generality, we can assume that $Q(x) = x^n + ax^{n-1} + \cdots$, where $a \le 0$ (otherwise replace $Q(x)$ by $Q(-x)$; we can do it since our condition is symmetric under this change). Then for large k, we have $Q(k-1) < k^n$, so the sequence $b_k := Q(0)Q(1)\cdots Q(k-1)/k!^n$ is decreasing in absolute value or zero starting from some place. But a sequence of integers cannot be strictly decreasing in absolute value forever. So $b_k = 0$ for some k, hence Q has an integer root. This means that d is an integer (i.e., Q is linear), since Q must be irreducible over the rationals. This concludes the proof of the lemma, and hence of the theorem. □

□

Corollary 7.4.3. *For any triangular Hopf algebra A (not necessarily finite-dimensional), the categorical dimensions of its finite-dimensional representations are integers.*

Proof. It is enough to consider the minimal part A_m of A which is finite-dimensional, since $\dim_c(V) = \mathrm{tr}(u_{|V})$ for any module V (where u is the Drinfeld element of A), and $u \in A_m$. Hence the result follows from Theorem 7.4.2. □

Remark 7.4.4. Theorem 7.4.2 is false without the finiteness conditions. In fact, in this case any complex number can be a dimension, as is demonstrated in examples constructed by Deligne [De2, p. 324–325]. Also, it is well known that the theorem is false for ribbon, nonsymmetric categories (e.g., for fusion categories of semisimple representations of finite-dimensional quantum groups at roots of unity [L], where dimensions can be irrational algebraic integers).

Remark 7.4.5. In any rigid braided tensor category with finitely many irreducible objects, one can define the Frobenius-Perron dimension of an object V, $\mathrm{FPdim}(V)$, to be the largest positive eigenvalue of the matrix of multiplication by V in the Grothendieck ring. This dimension is well defined by the Frobenius-Perron theorem, and has the usual additivity and multiplicativity properties. For example, for the category of representations of a quasi-Hopf algebra, it is just the usual dimension of the underlying vector space. If the answer to Question 8.7 is positive then $\mathrm{FPdim}(V)$ for symmetric categories is always an integer, which is equal to $\dim_c(V)$ modulo 2. It would be interesting to check this, at least in the case of modules over a triangular Hopf algebras, when the integrality of FPdim is automatic (so only the mod 2 congruence has to be checked).

8. Questions

We conclude the paper with some natural questions motivated by the above results [AEG, G4].

Question 8.1. Let (A, R) be *any* finite-dimensional triangular Hopf algebra with Drinfeld element u. Is it true that $S^4 = \mathrm{Id}$? Does u satisfy $u^2 = 1$? Is it true that $S^4 = \mathrm{Id}$ implies $u^2 = 1$?

Remark 8.2. A positive answer to the second question in Question 8.1 will imply that an odd-dimensional triangular Hopf algebra must be semisimple.

Note that if A is of supergroup type, then the answer to Question 8.1 is positive. Indeed, since $S^2 = \mathrm{Id}$ on the semisimple part of A, u acts by a scalar in any irreducible representation of A. In fact, since $\mathrm{tr}(u) = \mathrm{tr}(u^{-1})$, we have that $u = 1$ or $u = -1$ on any irreducible representation of A, and hence $u^2 = 1$ on any irreducible representation of A. Thus, u^2 is unipotent. But it is of finite order (as it is a grouplike element), so it is equal to 1 as desired.

QUESTION 8.3. Does any finite-dimensional triangular Hopf algebra over $\mathbb{C}$ have the Chevalley property (i.e., is of supergroup type)? Is it true under the assumption that $S^4 = \mathrm{Id}$ or at least under the assumption that $u^2 = 1$?

REMARK 8.4. Note that the answer to question 8.3 is negative in the infinite dimensional case. Namely, although the answer is positive in the cocommutative case (by [C]), it is negative already for triangular Hopf algebras with R-matrix of rank 2, which correspond to cocommutative Hopf superalgebras. Indeed, let us take the cocommutative Hopf superalgebra $\mathcal{A} := \mathrm{U}(\mathrm{gl}(n|n))$ (for the definition of the Lie superalgebra $\mathrm{gl}(n|n)$, see [KaV, p. 29]). The associated triangular Hopf algebra $\overline{\mathcal{A}}$ does not have the Chevalley property, since it is well known that Chevalley theorem fails for Lie superalgebras (e.g., $\mathrm{gl}(n|n)$); more precisely, already the product of the vector and covector representations for this Lie superalgebra is not semisimple.

REMARK 8.5. It follows from Corollary 7.3.3.1 that a positive answer to Question 8.3 in the minimal case would imply the general positive answer.

Here is a generalization of Question 8.3.

QUESTION 8.6. Does any $\mathbb{C}$-linear abelian symmetric rigid tensor category, with $\mathrm{End}(\mathbf{1}) = \mathbb{C}$ and finitely many simple objects, have the Chevalley property?

Even a more ambitious question:

QUESTION 8.7. Is such a category equivalent to the category of representations of a finite-dimensional triangular Hopf algebra with R-matrix of rank ≤ 2? In particular, is it equivalent to the category of representations of a supergroup, as a category without braiding? Are these statements valid at least for categories with Chevalley property? For semisimple categories?

REMARK 8.8. Note that Theorem 7.4.2 can be regarded as a piece of supporting evidence for a positive answer to Question 8.7.

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