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The Brauer Group of a Hopf Algebra

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Abstract. Let H be a Hopf algebra with a bijective antipode over a commutative ring k with unit. The Brauer group of H is defined as the Brauer group of Yetter–Drinfel'd H-module algebras, which generalizes the Brauer–Long group of a commutative and cocommutative Hopf algebra and those known Brauer groups of structured algebras.

Introduction

The Brauer group is something like a mathematical chameleon, it assumes the characteristics of its environment. For example, if you look at it from the point of view of representation theory you seem to be dealing with classes of noncommutative algebras appearing in the representation theory of finite groups, a purely group theoretical point of view presents it as the second Galois-cohomology group, over number fields it becomes an arithmetical tool related to the local theory via complete fields, over an algebraic function field or some coordinate rings it gets a distinctive geometric meaning and category theoretical aspects are put in evidence when relating the Brauer group to K-theory, in particular the K_2-group. When looking at the vast body of theory existing for the Brauer group one cannot escape to note the central role very often played by group actions and group gradings. This is most evident for example in the appearance of crossed products or generalizations of these.

Another typical case is presented by Clifford algebras and the $\mathbb{Z}_2$ (i.e., $\mathbb{Z}/2\mathbb{Z}$) graded theory contained in the study of the well-known Brauer–Wall group [57], as well as the generalized Clifford algebras in the Brauer–Long group for an abelian group [29]. At that point the theory was ripe for an approach via Hopf algebras where certain actions and co-actions (like the grading by a group) may be adequately combined in one unifying theory [30], but for commutative cocommutative Hopf algebras only. However, the cohomological interpretation for such

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Brauer–Long groups presented some technical problems that probably slowed down the development of a general theory. The cohomological description was obtained years later by S. Caenepeel a.o. [6; 7; 8; 9] prompted by new interest in the matter stemming from earlier work of Van Oystaeyen and Caenepeel, Van Oystaeyen on another type of graded Brauer group. The problem of considering noncommutative noncocommutative Hopf algebras remained and became more fascinating because of the growing interest in quantum groups. The present authors then defined and studied the Brauer group of a quantum group first in terms of the category of Yetter–Drinfel'd modules, but quickly generalized it to the Brauer group of a braided category [53], thus arriving at the final generality one would hope for after [38]. The Brauer group of a quantum group or even of a general Hopf algebra presents us with an interesting new invariant but a warning is in place. Not only is this group non-abelian, it is even non-torsion in general! Even restriction to cohomology describable or split parts does not reduce the complexity much. On the other hand, at least for finite dimensional Hopf algebras explicit calculations should be possible. Note that even the case of the Brauer group of the group ring of a non-abelian group is a very new and interesting object. Recently concrete calculations have been finalized for Sweedler's four dimensional Hopf algebra, group rings of dihedral groups and a few more low dimensional examples [16; 54; 56].

The arrangement of this paper is as follows:

(i) Basic notions and conventions
(ii) Quaternion algebras
(iii) The definition of the Brauer group
(iv) An exact sequence for the Brauer group $\mathrm{BC}(k, H, R)$
(v) The Hopf automorphism group
(vi) The second Brauer group

We do not repeat here a survey of main results because the paper is itself an expository paper albeit somewhat enriched by new results at places. We have adopted a very constructive approach starting with a concrete treatment of actions and coactions on quaternion algebras (Section 2), so that the abstractness of the definition in Section 3 is well-motivated and is made look natural. We shall not include the Brauer–Long group theory in this paper as the reader may find a comprehensive introduction in the book [6].

1. Basic Notions and Conventions

Throughout k is a commutative ring with unit unless it is specified and $(H, \Delta, \varepsilon, S)$ or simply H is a Hopf algebra over k where (H, Δ, ε) is the underlying coalgebra and S is a bijective antipode. Since the antipode S is bijective, the opposite H^{op} and the co-opposite H^{cop} are again Hopf algebras with antipode

S^{-1}. We will often use Sweedler's sigma notation; for example, we will write, for $h \in H$,

i. $\Delta h = \sum h_{(1)} \otimes h_{(2)}$,

ii. $(1 \otimes \Delta)\Delta h = (\Delta \otimes 1)\Delta h = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$

etc. For more detail concerning the theory of Hopf algebras we refer to [1; 35; 47].

1.1. Dimodules and Yetter–Drinfel'd modules. A k-module is said to be of *finite type* if it is finitely generated projective. If a k-module M of finite type is faithful, then M is said to be *faithfully projective*

Let H be a Hopf algebra. We will take from [47] the theory of H-modules and H-comodules for granted. Sigma notations such as $\sum m_{(0)} \otimes m_{(1)}$ for the comodule structure $\chi(m)$ of an element m of a left H-comodule M will be adapted from [47]. In this paper we will use χ for comodule structures over Hopf algebras and use ρ for comodule structures over coalgebras in order to distinguish two comodule structures when they happen to be together.

Write ${}_H\mathbf{M}$ for the category of left H-modules and H-module morphisms. If M and N are H-modules the diagonal H-module structure on $M \otimes N$ and the adjoint H-module structure on $\mathrm{Hom}(M, N)$ are given by:

i. $h \cdot (m \otimes n) = \sum h_{(1)} \cdot m \otimes h_{(2)} \cdot n$,

ii. $(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m)$,

for $h \in H, n \in N, f : M \longrightarrow N$. The category ${}_H\mathbf{M}$ together with the tensor product and the trivial H-module k forms a *monoidal category* (see [31]). If M is left H-module, we have a k-module of invariants

$$M^H = \{m \in M \mid h \cdot m = \varepsilon(h)m.\}$$

In a dual way, we have a monoidal category of right H-comodules, denoted $(\mathbf{M}^H, \otimes, k)$ or simply $\mathbf{M}^H$. For instance, if M and N are two right H-comodules, the codiagonal H-comodule structure on $M \otimes N$ is given by

$$\chi(m \otimes n) = \sum m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}$$

for $m \in M$ and $n \in N$. If an H-comodule M is of finite type, then $\mathrm{Hom}(M, N) \cong N \otimes M^*$ has a comodule structure:

$$\chi(f)(m) = \sum f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)} S(m_{(1)})$$

for $f \in \mathrm{Hom}(M, N)$ and $m \in M$. For a right H-comodule M the k-module

$$M^{coH} = \{m \in M \mid \chi(m) = m \otimes 1\}$$

is called the coinvariant submodule of M.

A k-module M which is both an H-module and an H-comodule is called an H-*dimodule* if the action and the coaction of H commute, that is, for all $m \in M, h \in H$,

$$\sum (h \cdot m)_{(0)} \otimes (h \cdot m)_{(1)} = \sum h \cdot m_{(0)} \otimes m_{(1)}.$$

Write $\mathbf{D}^H$ for the category of H-dimodules. When the Hopf algebra H is finite, we obtain equivalences of categories

$$\mathbf{M}^{H\otimes H^*} \cong \mathbf{D}^H \cong_{H^*\otimes H} \mathbf{M};$$

here $H\otimes H^*$ and $H^*\otimes H$ are tensor Hopf algebras. For more details on dimodules, we refer to [29; 30].

Recall that a Yetter–Drinfel'd H-module (simply a YD H-module) M is a left crossed H-bimodule [58]. That is, M is a k-module which is a left H-module and a right H-comodule satisfying the following equivalent compatibility conditions [27, 5.1.1]:

i. $\sum h_{(1)} \cdot m_{(0)} \otimes h_{(2)} m_{(1)} = \sum (h_{(2)} \cdot m)_{(0)} \otimes (h_{(2)} \cdot m)_{(1)} h_{(1)}$

ii. $\chi(h \cdot m) = \sum (h_{(2)} \cdot m_{(0)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})$.

Denote by $\mathcal{Q}^H$ (${}_H YD^H$ in several references) the category of YD H-modules and YD H-module morphisms. For two YD H-modules M and N, the diagonal H-module structure and the codiagonal H^{op}-comodule structure on tensor product $M \otimes N$ satisfy the compatibility conditions of a YD H-module. So $M \otimes N$ is a YD H-module, denoted $M\tilde{\otimes}N$. It is easy to see that the natural map

$$\Gamma : (X\tilde{\otimes}Y)\tilde{\otimes}Z \longrightarrow X\tilde{\otimes}(Y\tilde{\otimes}Z)$$

is a YD H-module isomorphism, and the trivial YD H-module k is a unit with respect to $\tilde{\otimes}$. Therefore $(\mathcal{Q}^H, \tilde{\otimes}, k)$ forms a monoidal category (for details concerning monoidal categories we refer to [31; 58]).

Let M and N be YD H-modules. Then there exists a YD H-module isomorphism Ψ between $M\tilde{\otimes}N$ and $N\tilde{\otimes}M$:

$$\Psi : M\tilde{\otimes}N \longrightarrow N\tilde{\otimes}M, m\tilde{\otimes}n \mapsto \sum n_{(0)}\tilde{\otimes}n_{(1)} \cdot m$$

with inverse $\Psi^{-1}(n\tilde{\otimes}m) = \sum S(n_{(1)}) \cdot m\tilde{\otimes}n_{(0)}$. It is not hard to check that $(\mathcal{Q}^H, \tilde{\otimes}, \Gamma, \Psi, k)$ is a braided monoidal (or quasitensor) category (see [31; 58]). If in addition, H is a finite Hopf algebra, then there is a category equivalence:

$${}_{D(H)}\mathbf{M} \sim \mathcal{Q}^H$$

where $D(H)$ is the Drinfel'd double $(H^{\text{op}})^* \bowtie H$ which is a finite quasitriangular Hopf algebra over k as described in [23; 32; 40].

1.2. H-dimodule and YD H-module algebras. An algebra A is a (left) H-module algebra if there is a measuring action of H on A, i.e., for $h \in H, a, b \in A$,

i. A is a left H-module,

ii. $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,

iii. $h \cdot 1 = \varepsilon(h)1$.

Similarly, an algebra is called a (right) H-comodule algebra if A is a right H-comodule with the comodule structure $\chi : A \longrightarrow A \otimes H$ being an algebra map, i.e., for $a, b \in A$,

i. $\chi(ab) = \sum a_{(0)}b_{(0)} \otimes a_{(1)}b_{(1)}$,
ii. $\chi(1) = 1 \otimes 1$.

An *H-dimodule algebra* A is an H-dimodule and a k-algebra which is both an H-module algebra and an H-comodule algebra. Suppose that H is both commutative and cocommutative. Let A and B be two H-dimodule algebras. The smash product $A\#B$ is defined as follows: $A\#B = A \otimes B$ as a k-module and the multiplication is given by

$$(a\#b)(c\#d) = \sum a(b_{(1)} \cdot c)\#b_{(0)}d.$$

Then $A\#B$ furnished with the diagonal H-module structure and codiagonal comodule structure $A \otimes B$ is again an H-dimodule algebra.

The *H-opposite* $\overline{A}$ of an H-dimodule algebra A is equal to A as an H-dimodule, but with multiplication given by

$$\overline{a} \cdot \overline{b} = \overline{\sum (a_{(1)} \cdot b)a_{(0)}}$$

which is again an H-dimodule algebra.

A Yetter–Drinfel'd H-module algebra A is a YD H-module and a k-algebra which is a left H-module algebra and a right H^{op}-comodule algebra. Note that here we replace H by H^{op} when we deal with *comodule algebra structures*.

As examples pointed out in [11], $(H^{\mathrm{op}}, \Delta, \mathrm{ad}')$ and (H, χ, ad) are regular Yetter–Drinfel'd H-module algebras with H-structures defined as follows:

$$h \ \mathrm{ad}' \ x = \sum h_{(2)}xS^{-1}(h_{(1)})$$
$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$$
$$h \ \mathrm{ad} \ x = \sum h_{(1)}xS(h_{(2)})$$
$$\chi(x) = \sum x_{(2)} \otimes S^{-1}(x_{(1)}).$$

Let A and B be two YD H-module algebras. We may define a *braided product*, still denoted $\#$, on the YD H-module $A\tilde{\otimes}B$:

$$(a\#b)(c\#d) = \sum ac_{(0)}\#(c_{(1)} \cdot b)d \tag{1–1}$$

for $a, c \in A$ and $b, d \in B$. The braided product $\#$ makes $A\#B$ a left H-module algebra and a right H^{op}-comodule algebra so that $A\#B$ is a YD H-module algebra. Note that the braided product $\#$ is associative.

Now let A be a YD H-module algebra. The *H-opposite algebra* $\overline{A}$ of A is the YD H-module algebra defined as follows: $\overline{A}$ equals A as a YD H-module, with multiplication given by the formula

$$\overline{a} \circ \overline{b} = \sum \overline{b_{(0)}(b_{(1)} \cdot a)}$$

for all $\overline{a}, \overline{b} \in \overline{A}$. In case the antipode of H is of order two, $\overline{\overline{A}}$ is equal to A as a YD H-module algebra.

Let M be a YD H-module such that M is of finite type. The endomorphism algebra $\mathrm{End}_k(M)$ is a YD H-module algebra with the H-structures induced by those of M, i.e., for $h \in H$, $f \in \mathrm{End}_k(M)$ and $m \in M$,

$$\begin{aligned} (h \cdot f)(m) &= \textstyle\sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m), \\ \chi(f)(m) &= \textstyle\sum f(m_{(0)})_{(0)} \otimes S^{-1}(m_{(1)}) f(m_{(0)})_{(1)}. \end{aligned} \tag{1–2}$$

Recall from [11, 4.2] that the H-opposite of $\mathrm{End}_k(M)$ is isomorphic as an YD H-algebra to $\mathrm{End}_k(M)^{\mathrm{op}}$, where the latter has YD H-module structure given by

$$\begin{aligned} (h \cdot f)(m) &= \textstyle\sum h_{(2)} \cdot f(S^{-1}(h_{(1)}) \cdot m), \\ \chi(f)(m) &= \textstyle\sum f(m_{(0)})_{(0)} \otimes f(m_{(0)})_{(1)} S(m_{(0)}) \end{aligned} \tag{1–3}$$

for $m \in M,\ h \in H$ and $f \in \mathrm{End}_k(M)$.

1.3. Quasitriangular and coquasitriangular Hopf algebras. A *quasitriangular Hopf algebra* is a pair $(H, \mathcal{R})$, where H is a Hopf algebra with an invertible element $\mathcal{R} = \sum R^{(1)} \otimes R^{(2)} \in H \otimes H$ satisfying the following axioms ($r = \mathcal{R}$):

(QT1) $\sum \Delta(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$,
(QT2) $\sum \varepsilon(R^{(1)}) R^{(2)} = 1$,
(QT3) $\sum R^{(1)} \otimes \Delta(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
(QT4) $\sum R^{(1)} \varepsilon(R^{(2)}) = 1$,
(QT5) $\Delta^{\mathrm{cop}}(h)\mathcal{R} = \mathcal{R}\Delta(h)$,

where $\Delta^{\mathrm{cop}} = \tau\Delta$ is the comultiplication of the Hopf algebra H^{cop} and τ is the switch map.

Now let M be a left H-module. It is well-known that there is an induced H-comodule structure on M as follows:

$$\chi(m) = \sum R^{(2)} \cdot m \otimes R^{(1)} \tag{1–4}$$

for $m \in M$ such that the left H-module M together with (1–4) is a YD H-module. When M is a left H-module algebra, then (1–4) makes M into a right H^{op}-comodule algebra and hence a YD H-module algebra. It is easy to see that $\mathrm{Hom}_H(M, N) = \mathrm{Hom}_H^H(M, N)$ for any two YD H-modules M, N with comodule structures (1–4) stemming from the left module structures. Thus the category ${}_H\mathbf{M}$ of left H-modules and H-morphisms can be embedded into the category $\mathcal{Q}^H$ as a full subcategory, which we denote by ${}_H\mathbf{M}^{\mathcal{R}}$. Moreover ${}_H\mathbf{M}^{\mathcal{R}}$ is a braided monoidal subcategory of $\mathcal{Q}^H$ since the tensor product is closed in ${}_H\mathbf{M}$ and the braiding Ψ of $\mathcal{Q}^H$ restricts to the braiding of ${}_H\mathbf{M}^{\mathcal{R}}$ which is nothing but $\Psi_{\mathcal{R}}$ induced by the $\mathcal{R}$-matrix:

$$\Psi_{\mathcal{R}}(m \otimes n) = \sum R^{(2)} \cdot n \otimes R^{(1)} \cdot m$$

where $m \in M, n \in N$ and $M, N \in {}_H\mathbf{M}^{\mathcal{R}}$.

A *coquasitriangular Hopf algebra* is a pair (H, R), where H is a Hopf algebra and $R \in (H \otimes H)^*$ is a convolution invertible element and satisfies the following axioms:

(CQT1) $R(h \otimes 1) = R(1 \otimes h) = \varepsilon(h)1_H$,
(CQT2) $R(ab \otimes c) = \sum R(a \otimes c_{(1)})R(b \otimes c_{(2)})$,
(CQT3) $R(a \otimes bc) = \sum R(a_{(1)} \otimes c)R(a_{(2)} \otimes b)$,
(CQT4) $\sum b_{(1)}a_{(1)}R(a_{(2)} \otimes b_{(2)}) = \sum R(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)}$.

Let M be a right H-comodule. There is an induced left H-module structure on M given by

$$h \triangleright_1 a = \sum a_{(0)}R(h \otimes a_{(1)}) \tag{1–5}$$

for all $a \in A, h \in H$, such that M is a YD H-module. The right H-comodule category $\mathbf{M}^H$ can be embedded into $\mathcal{Q}^H$ as a full braided monoidal subcategory. We denote by $\mathbf{M}^H_R$ the braided subcategory of $\mathcal{Q}^H$.

It is easy to check that an H^{op}-comodule algebra A with the H-module structure described in (1–5) is a YD H-module algebra.

2. Quaternion Algebras

Let k be a field. Quaternion algebras play a very important role in the study of the Brauer group $\text{Br}(k)$ of k. On the other hand, quaternion algebras also represent elements in the Brauer–Wall group $\text{BW}(k)$ of $\mathbb{Z}_2$-graded algebras. The natural $\mathbb{Z}_2$-gradings of quaternion algebras are obtained from certain involutions related to the canonical quadratic forms of quaternion algebras. However, one may find that the same quaternion algebra will represent two different elements in $\text{BW}(k)$. When one turns to the Brauer–Long group $\text{BD}(k, \mathbb{Z}_2)$ of $\mathbb{Z}_2$-dimodule algebras where actions of $\mathbb{Z}_2$ commute with the $\mathbb{Z}_2$-gradings, the quaternion algebras now represent four different elements of order two. Now if we add a differential on the $\mathbb{Z}_2$-graded algebras such that they become differential superalgebras, we may form the Brauer group of differential superalgebras, and the quaternion algebras are now differential superalgebras. If we mimic the process used by C.T.C. Wall, we obtain a Brauer group $\text{BDS}(k)$ of differential superalgebras. A new interesting fact now shows, i.e., a quaternion algebra may represent an element of infinite order in $\text{BDS}(k)$. As a consequence, the Brauer group $\text{BDS}(k)$ is a non-torsion infinite group if k has characteristic zero.

Recall the definition of a quaternion algebra. For $\alpha, \beta \in k^\bullet = k\backslash 0$, define a 4-dimensional algebra with basis $\{1, u, v, w\}$ by the following multiplication table:

$$uv = w, \quad u^2 = \alpha 1, \quad v^2 = \beta 1, \quad vu = -w.$$

Here 1 denotes the unit. We denote this algebra by $\left(\frac{\alpha,\beta}{k}\right)$. The elements in the subspace $ku + kv + kw$ are called *pure quaternions*. The subspace of pure quaternions is independent of the choice of standard basis and is determined by the algebra structure of $\left(\frac{\alpha,\beta}{k}\right)$.

There exists a canonical linear involution given by

$$ {}^{-}: \left(\frac{\alpha,\beta}{k}\right) \longrightarrow \left(\frac{\alpha,\beta}{k}\right), \quad \overline{x} = \overline{x_0 + x_1} = x_0 - x_1 $$

where $x_0 \in k$ and $x_1 \in ku + kv + kw$. It follows that $\left(\frac{\alpha,\beta}{k}\right)$ is isomorphic to its opposite algebra $\left(\frac{\alpha,\beta}{k}\right)^{\mathrm{op}}$. One may easily calculate that the center of $\left(\frac{\alpha,\beta}{k}\right)$ is k and that $\left(\frac{\alpha,\beta}{k}\right)$ has no proper ideals except $\{0\}$. An algebra is called a *central simple algebra* if its center is canonically isomorphic to k and it has no proper non-zero ideals. Any $n\times n$-matrix algebra $M_n(k)$ is a central simple algebra. The opposite algebra of a central simple algebra is obviously a central simple algebra. The tensor product of two central simple algebras is still a central simple algebra. There are several characterizations of a central simple algebra [19; 39]:

Proposition 2.1. *Let A be a finite dimensional algebra over a field k. The following are equivalent*:

(1) *A is a central simple algebra.*
(2) *A is a central separable algebra* (*here A is separable if mult* : $A \otimes A \longrightarrow A$ *splits as an A-bimodule map*).
(3) *A is isomorphic to a matrix algebra $M_n(D)$ over a skew field D where the center of D is k.*
(4) *The canonical linear algebra map can* : $A \otimes A^{\mathrm{op}} \longrightarrow \mathrm{End}(A)$ *given by can*$(a \otimes b)(c) = acb$ *for $a, b, c \in A$ is an isomorphism.*

A finite dimensional algebra satisfying one of the above equivalent conditions is called an *Azumaya algebra*. Let $B(k)$ be the set of all isomorphism classes of Azumaya algebras. Then $B(k)$ is a semigroup with the multiplication induced by the tensor product and with the unit represented by the one dimensional algebra k.

Define an equivalence relation $\sim$ on $B(k)$ as follows: Two central simple algebras A and B are equivalent, denoted $A \sim B$, if there are two positive integers m and n such that

$$ A \otimes M_n(k) \cong B \otimes M_m(k) $$

as algebras. Then the quotient set of $B(k)$ modulo the equivalence relation $\sim$ is a group and is called the *Brauer group* of k, denoted $\mathrm{Br}(k)$.

The Brauer group $\mathrm{Br}(k)$ can be defined more intuitively as the quotient $B(k)/M(k)$, where $M(k)$ is a sub-semigroup generated by the isomorphism classes of matrix algebras over k. If $[A]$ is an element in $\mathrm{Br}(k)$ represented by a central simple algebra A, then the inverse $[A]^{-1}$ is represented by the opposite algebra A^{op} because $A \otimes A^{\mathrm{op}}$ is isomorphic to a matrix algebra. The Brauer group $\mathrm{Br}(k)$ can be generalized to the Brauer group of a commutative ring by making use of the equivalent condition (2) or (4) of Proposition 2.1. That is, an Azumaya algebra A over a commutative ring is a faithfully projective algebra such that the condition (2) or (4) of Proposition.2.1 holds. One may refer to [2; 19] for the details on the Brauer group of a commutative ring. However, in this section we restrict our attention to the case where k is a field.

Let us return to the consideration of quaternion algebras. We know that a quaternion algebra is a central simple algebra and it is isomorphic to its opposite algebra due to the canonical involution map. Thus $[(\frac{\alpha,\beta}{k})]$ is an element of order not greater than two. Actually any element of order two in $\mathrm{Br}(k)$ can be represented by a tensor product of quaternion algebras (see [39]).

The quaternion algebra $(\frac{\alpha,\beta}{k})$ has a canonical $\mathbb{Z}_2$-grading defined as follows:

$$\left(\frac{\alpha,\beta}{k}\right) = A_0 + A_1, \quad A_0 = k + kw, \quad A_1 = ku + kv. \tag{2–1}$$

In [57], Wall introduced the notion of a $\mathbb{Z}_2$*-graded Azumaya algebra* which is a graded central and a graded separable algebra A in the following sense:

i. the graded center $Z_g(A) = \{a \in A \mid ab = ba_0 + b_0a_1 - b_1a_1, \forall b \in A\} = k.$

ii. A is a simple graded algebra, i.e., A has no proper non-zero graded ideals.

As in Proposition 2.1, we may replace 'graded simplicity' by 'graded separability' if the characteristic of k is different from 2. That is, condition ii can be replaced by

iii. $A \otimes A \longrightarrow A$ splits as a $\mathbb{Z}_2$-graded A-bimodule map, where the grading on $A \otimes A$ is the diagonal one.

Given two graded algebras A and B. The product $A\hat{\otimes}B$ of two graded algebras A and B is defined as follows:

$$(a\hat{\otimes}b)(c\hat{\otimes}d) = (-1)^{\partial(b)\partial(c)}ac\hat{\otimes}bd \tag{2–2}$$

where b and c are homogeneous elements and $\partial(b), \partial(c)$ are the graded degrees of b and c respectively. If A and B are graded Azumaya algebras, then the product $A\hat{\otimes}B$ is a graded Azumaya algebra. Now one may repeat the definition of $\mathrm{Br}(k)$ by adding the term '($\mathbb{Z}_2-$) graded' to obtain the Brauer group of graded algebras which is referred to as *the Brauer–Wall group*, denoted $\mathrm{BW}(k)$. Notice that in the definition of the equivalence $\sim$, the grading of any matrix algebra $M_n(k)$ must be 'good', namely, $M_n(k) \cong \mathrm{End}(M)$ as graded algebras for some n-dimensional graded module M. The Brauer–Wall group $\mathrm{BW}(k)$ can be completely described in terms of the usual Brauer group $\mathrm{Br}(k)$ and the group of graded quadratic extensions:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BW}(k) \longrightarrow Q_2(k) \longrightarrow 1$$

where $Q_2(k) = \mathbb{Z}_2 \times k^\bullet/k^{\bullet 2}$ with multiplication given by

$$(e,d)(e',d') = (e+e', (-1)^{ee'}dd')$$

for $e, e' \in \mathbb{Z}_2$ and $d, d' \in k^\bullet/k^{\bullet 2}$. One may write down the multiplication rule for the product $\mathrm{Br}(k) \times Q_2(k)$ so that $\mathrm{BW}(k)$ is isomorphic to $\mathrm{Br}(k) \times Q_2(k)$ (for details see [18; 46]).

Again let us look at the quaternion algebras $\left(\frac{\alpha,\beta}{k}\right)$, $\alpha, \beta \in k^\bullet$. Let $k\langle\sqrt{\alpha}\rangle$ be the graded algebra $k \oplus ku$ and $k\langle\sqrt{\beta}\rangle$ be the graded algebra $k \oplus kv$. It is easy to show that $k\langle\sqrt{\alpha}\rangle$ is a graded Azumaya algebra and that $\left(\frac{\alpha,\beta}{k}\right)$ is isomorphic to the graded product $k\langle\sqrt{\alpha}\rangle \hat{\otimes} k\langle\sqrt{\beta}\rangle$. It follows that $\left(\frac{\alpha,\beta}{k}\right)$ is a graded Azumaya algebra. We denote by $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ the graded Azumaya algebra $\left(\frac{\alpha,\beta}{k}\right)$ in order to make the difference between $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ and $\left(\frac{\alpha,\beta}{k}\right)$. Since $[k\langle\sqrt{\alpha}\rangle] \in \mathrm{BW}(k)$ is an element of order two (or one) if $\alpha \notin k^{\bullet 2}$ (or $\alpha \in k^{\bullet 2}$), $[\left(\frac{\alpha,\beta}{k}\right)]$ is of order equal or less than two. Though $\left(\frac{\alpha,\beta}{k}\right)$ and $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ are the same algebra, they do represent two different elements of order two in $\mathrm{BW}(k)$ when $\left(\frac{\alpha,\beta}{k}\right)$ is a division algebra.

Furthermore, the quaternion algebras are no longer the 'smallest' nontrivial graded Azumaya algebras in terms of dimension. Here the smallest ones are quadratic extensions of k. This prompts the idea that the more extra structures you put on algebras, the more classes of such structured Azumaya algebras you will get, and the richer the corresponding Brauer group will be. So we look again at quadratic extensions and quaternion algebras. For a quadratic extension $k\langle\sqrt{\alpha}\rangle$, there is a natural k-linear $\mathbb{Z}_2$-action on it:

$$\sigma(1) = 1, \quad \sigma(u) = -u \tag{2–3}$$

where σ is the generator of the group $\mathbb{Z}_2$ and u is the generator of the field $k\langle\sqrt{\alpha}\rangle$. It is easy to see that the action (2–3) commutes with the canonical grading on $k\langle\sqrt{\alpha}\rangle$. The action (2–3) extends to any quaternion algebra $\left(\frac{\alpha,\beta}{k}\right)$ in the way of diagonal group action. In fact, to any graded algebra $A = A_0 \oplus A_1$, one may associate a natural $\mathbb{Z}_2$-action on A as follows:

$$\sigma(a_i) = (-1)^i a_i \tag{2–4}$$

where $a_i \in A_i$ is a homogeneous element of A.

A $\mathbb{Z}_2$-graded algebra with a $\mathbb{Z}_2$-action that commutes with the grading is a $\mathbb{Z}_2$-dimodule algebra. The notion of a dimodule algebra for a finite abelian group was introduced by F. W. Long in 1972 [29], it is extended for a commutative and cocommutative Hopf algebra in [30]. Let A be a $\mathbb{Z}_2$-graded algebra. Having the canonical $\mathbb{Z}_2$-action (2–4), A is a $\mathbb{Z}_2$-dimodule algebra. The product (2–2) respects the action (2–4). In this case, we may forget the action (2–4). However, if we take any two $\mathbb{Z}_2$-dimodule algebras A and B, the graded product (2–2) may not respect actions of $\mathbb{Z}_2$. For instance, A is a graded Azumaya algebra with the action (2–4) and B is a graded Azumaya algebra with the trivial $\mathbb{Z}_2$-action (i.e., σ acts as the identity map). Both A and B are dimodule algebras, but $A \hat{\otimes} B$ is not a dimodule algebra.

In order to have a product for dimodule algebras, we have to modify the product (2–2) such that the action of $\mathbb{Z}_2$ is involved. This is the situation dealt with by F.W. Long. Let A and B be two dimodule algebras. Long defined a product $\#$ on $A \otimes B$ as follows:

$$(a\#b)(c\#d) = ac\#\sigma^{\partial(c)}(b)d \tag{2–5}$$

where c is a homogeneous element. The product (2–5) preserves the dimodule structures, and restricts to the product (2–2) when the dimodule algebras have the canonical action (2–4). With this product (2–5) Long was able to define the Brauer group of dimodule algebras which is now referred to as the Brauer–Long group of $\mathbb{Z}_2$ and is denoted $\mathrm{BD}(k, \mathbb{Z}_2)$. The definition of an Azumaya $\mathbb{Z}_2$-dimodule algebra is similar to the definition of a graded Azumaya algebra.

Suppose that the characteristic of the field k is different from two. A $\mathbb{Z}_2$-dimodule algebra is called an Azumaya dimodule algebra if A satisfies the following two conditions:

i. A is $\mathbb{Z}_2$-central, namely, $\{a \in A \mid ab = b\sigma^i(a), \forall b \in A_i\} = \{a \in A \mid ba = a_0b + a_1\sigma(b), \forall b \in A\} = k$.

ii. the multiplication map $A\#A \longrightarrow A$ splits as A-bimodule and $\mathbb{Z}_2$-dimodule map.

Note that the foregoing definition is not the original definition given by Long, but it is equivalent to that if the characteristic of k is different from two. The equivalence relation $\sim$ is defined as follows: for two Azumaya dimodule algebras A and B, $A \sim B$ if and only if there exists two finite dimensional dimodules M and N such that

$$A\#\mathrm{End}(M) \cong B\#\mathrm{End}(N)$$

as dimodule algebras. The Brauer–Long group $\mathrm{BD}(k, \mathbb{Z}_2)$ contains the Brauer Wall-group $\mathrm{BW}(k)$ as a subgroup.

Let us investigate the role played by quaternion algebras in $\mathrm{BD}(k, \mathbb{Z}_2)$. If $\left(\frac{\alpha,\beta}{k}\right)$ is a quaternion algebra, then there are eight types of dimodule structures on $\left(\frac{\alpha,\beta}{k}\right)$:

(1) the trivial action and the trivial grading,

(2) the trivial action and the canonical grading (2–1),

(3) the canonical action (2–4) and the trivial grading,

(4) the action (2–4) and the grading (2–1), i.e., the dimodule structure of $\left\langle\frac{\alpha,\beta}{k}\right\rangle$,

(5) the action (2–4) and the grading $A_0 = k \oplus ku, A_1 = kv \oplus kw$,

(6) the grading (2–1) and the action given by $\sigma(u) = u, \sigma(v) = -v$.

If we switch the roles of u and v in (5) and (6), we will obtain two more dimodule structures on $\left(\frac{\alpha,\beta}{k}\right)$. One may take a while to check that the first four types of dimodule structures make $\left(\frac{\alpha,\beta}{k}\right)$ into $\mathbb{Z}_2$-Azumaya dimodule algebras. However, though $\left(\frac{\alpha,\beta}{k}\right)$ is an Azumaya algebra the dimodule algebra $\left(\frac{\alpha,\beta}{k}\right)$ of type five or six is not a $\mathbb{Z}_2$-Azumaya algebra because it is not $\mathbb{Z}_2$-central. For instance, the left center

$$\{a \in A \mid ab = b\sigma^i(a), \forall b \in A_i\} = k \oplus ku.$$

is not trivial in the case of type five.

Nevertheless, the $\left(\frac{\alpha,\beta}{k}\right)$ of type (1)–(4) represent four different elements of order two in $\mathrm{BD}(k,\mathbb{Z}_2)$ when $\left(\frac{\alpha,\beta}{k}\right)$ is a division algebra. Let $\left(\frac{\alpha,\beta}{k}\right)_i$ be the $\mathbb{Z}_2$-Azumaya dimodule algebra of type (i), where $i=1,2,3,4$. Since the multiplication of the group is induced by the braided product (2–5), $[\left(\frac{\alpha,\beta}{k}\right)_1]$ commutes with $[\left(\frac{\alpha,\beta}{k}\right)_i]$, but in general $[\left(\frac{\alpha,\beta}{k}\right)_2][\left(\frac{\alpha,\beta}{k}\right)_3]\neq[\left(\frac{\alpha,\beta}{k}\right)_3][\left(\frac{\alpha,\beta}{k}\right)_2]$ (pending on k, see [18, Thm]). For example, when $k=\mathbb{R}$, the real number field, we have

$$[\mathbb{H}_2][\mathbb{H}_3]=(1,-1,1,-1)(1,1,-1,-1)=(1,-1,-1,1),$$
$$[\mathbb{H}_3][\mathbb{H}_2]=(1,1,-1,-1)(1,-1,1,-1)=(1,-1,-1,-1),$$

where $\mathbb{H}=\left(\frac{-1,-1}{k}\right)$ and $\mathrm{BD}(\mathbb{R},\mathbb{Z}_2)=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathrm{Br}(\mathbb{R})$ ($\mathrm{Br}(\mathbb{R})=\mathbb{Z}_2$) with multiplication rules given by [18, Thm] or [7, 13.12.14]. In fact, $\mathrm{BD}(\mathbb{R},\mathbb{Z}_2)\cong D_8$, the dihedral group of 16 elements and $\mathrm{BW}(\mathbb{R})\cong\mathbb{Z}_8$, the cyclic group of 8 elements (see [29] or [7, 13.12.15]). Thus $\mathrm{BD}(k,\mathbb{Z}_2)$ may not be an abelian group though $\mathrm{BW}(k)$ is an abelian group. Nonetheless, both $\mathrm{BW}(k)$ and $\mathrm{BD}(k,\mathbb{Z}_2)$ are torsion groups (see [18]). This will not be the case for the Brauer group of differential superalgebras introduced hereafter.

For convenience we call a $\mathbb{Z}_2$-graded algebra a *super* algebra. Let $A=A_0\oplus A_1$ be a superalgebra. A linear endomorphism δ of A is called a *super-derivation* of A if δ is a degree one graded endomorphism and satisfies the following condition:

$$\delta(ab)=a\delta(b)+(-1)^{\partial(b)}\delta(a)b$$

where b is homogeneous and a is arbitrary. A super-derivation δ is called a *differential* if $\delta^2=0$.

A graded algebra A with a differential δ is called a *differential superalgebra* (simply DS algebra), denoted (A,δ) or just A if there is no confusion. Two DS algebras (A,δ_A) and (B,δ_B) can be multiplied by means of the graded product (2–2). So we obtain a new DS algebra $(A\hat{\otimes}B,\delta_{A\hat{\otimes}B})$, where $\delta_{A\hat{\otimes}B}$ is given by

$$\delta_{A\hat{\otimes}B}(a\hat{\otimes}b)=a\hat{\otimes}\delta_B(b)+(-1)^i\delta_A(a)\hat{\otimes}b$$

for $a\in A$ and $b\in B_i$.

Let M be a graded module. M is called a *differential graded module* if there exists a degree one graded linear endomorphism δ_M of M such that $\delta_M^2=0$ (in the sequel δ_M will be called a *differential on* M). The endomorphism ring $\mathrm{End}(M)$ is a DS algebra. The grading on $\mathrm{End}(M)$ is the induced grading and the differential δ on $\mathrm{End}(M)$ is induced by δ_M, namely,

$$\delta(f)(m)=f(\delta_M(m))+(-1)^{\partial(m)}\delta(f(m)) \tag{2–6}$$

for any homogeneous element $m\in M$ and $f\in\mathrm{End}(M)$.

Now let A be the graded Azumaya algebra $\left\langle\frac{\alpha,\beta}{k}\right\rangle$. There is a natural differential on A given by Doi and Takeuchi [22]:

$$\delta(1)=\delta(u)=0,\quad \delta(v)=1,\quad \delta(w)=u. \tag{2–7}$$

So any quaternion algebra is a DS algebra. As mentioned before, the graded Azumaya algebra $\langle\frac{\alpha,\beta}{k}\rangle$ represents an element of order not greater than two in $\mathrm{BW}(k)$. This means that the product graded Azumaya algebra $A\hat{\otimes}A$ is a graded 4×4-matrix algebra. So there is a 4-dimensional graded module M such that $A\hat{\otimes}A\cong\mathrm{End}(M)$ as graded algebras. Is this the same when we add the canonical differential (2–7) to $\langle\frac{\alpha,\beta}{k}\rangle$? In other words, does there exist a graded module M with a differential δ_M such that $A\hat{\otimes}A$ is isomorphic to $\mathrm{End}(M)$ as a DS algebra? The answer is even negative for the graded matrix algebra $\langle\frac{1,-1}{k}\rangle$. Before we answer the question let us first define the Brauer group of DS algebras.

Definition 2.2. A DS algebra A is called a DS Azumaya algebra if A is a graded Azumaya algebra. Two DS Azumaya algebras A and B are said to be equivalent, denoted $A\sim B$, if there are two differential graded module M and N such that $A\hat{\otimes}\mathrm{End}(M)\cong B\hat{\otimes}\mathrm{End}(N)$ as DS algebras.

Let $B(k)$ be the set of isomorphism classes of DS Azumaya algebras. It is a routine verification that the quotient set of $B(k)$ modulo the equivalence relation $\sim$ is a group and is called the *Brauer group of DS algebras*, denoted $\mathrm{BDS}(k)$. If A represents an element $[A]$ of $\mathrm{BDS}(k)$, then the graded opposite $\overline{A}$ represents the inverse of $[A]$ in $\mathrm{BDS}(k)$, where A and $\overline{A}$ share the same differential. The unit of the group $\mathrm{BDS}(k)$ is represented by matrix DS algebras which are the endomorphism algebras of some finite dimensional differential graded modules. In other words, if A is a DS Azumaya algebra such that $[A]=1$, then A is isomorphic to $\mathrm{End}(M)$ as a DS algebra for some finite dimensional differential graded module M. This follows from the fact that the Brauer equivalence $\sim$ is the same as the differential graded Morita equivalence which can be done straightforward by adding the 'differential' to graded Morita equivalence. We would rather wait till next section to see a far more general H-Morita theory.

Following Definition 2.2 all quaternion algebras $\langle\frac{\alpha,\beta}{k}\rangle$, $\alpha,\beta\in k^\bullet$, are DS Azumaya algebras. We show first that the DS matrix algebra $\langle\frac{\alpha,-\alpha}{k}\rangle$ does not represents the unit of $\mathrm{BDS}(k)$. In order to prove this, we need to consider the associated automorphism σ given by (2–4) of a differential graded module M. Since the differential δ of M is a degree one graded endomorphism, it follows that δ anti-commutes with σ, namely, $\sigma\delta+\delta\sigma=0$.

Lemma 2.3. *For any $\alpha\in k^\bullet$, the DS algebra $[\langle\frac{\alpha,-\alpha}{k}\rangle]\neq 1$ in* $\mathrm{BDS}(k)$.

Proof. Let A be $\langle\frac{\alpha,-\alpha}{k}\rangle$ and assume that $[\langle\frac{\alpha,-\alpha}{k}\rangle]=1$ in $\mathrm{BDS}(k)$. Then there exists a two dimensional differential graded module M such that $\langle\frac{\alpha,-\alpha}{k}\rangle\cong\mathrm{End}(M)$. Since the differential δ_E and the automorphism σ_E of $\mathrm{End}(M)$ are induced by the differential δ_M and automorphism σ_M of M respectively (see (2–6) for the differential), $A\cong\mathrm{End}(M)$ implies that there exist two elements ν and ω in A such that the canonical differential δ given by (2–7) is induced by ν and the automorphism σ_A given by (2–4) is the inner automorphism induced by

ω, i.e.,

$$\delta(a) = a\nu - (-1)^{\partial(a)}\nu a, \quad \sigma(b) = \omega b \omega^{-1}$$

where $a, b \in A$ and a is homogeneous. Furthermore, ν and ω satisfy the relations that δ_M and σ_M obey, i.e.,

$$\nu^2 = 0, \omega^2 = 1, \quad \nu\omega + \omega\nu = 0.$$

Let u, v be the two generators of $\left\langle \frac{\alpha,-\alpha}{k} \right\rangle$. Then we have the following relations:

$$\begin{aligned}\delta(u) &= u\nu + \nu u = 0,\\ \delta(v) &= v\nu + \nu v = 1,\\ \sigma(u) &= \omega u \omega^{-1} = -u,\\ \sigma(v) &= \omega v \omega^{-1} = -v.\end{aligned}$$

It follows that $\nu = -\frac{\alpha^{-1}}{2}v + suv$ for some $s \in k$ and $\omega = \alpha^{-1}uv$. Since $\nu^2 = 0$, we have

$$0 = (-\frac{\alpha^{-1}}{2}v + suv)^2 = -\frac{\alpha^{-1}}{4} + s^2\alpha^2$$

So s cannot be zero. However, the anti-commutativity of ν with ω implies that

$$\begin{aligned}0 &= \nu\omega + \omega\nu\\ &= (-\tfrac{1}{2}\alpha^{-1}v + suv)\alpha^{-1}uv + \alpha^{-1}uv(-\tfrac{1}{2}\frac{\alpha^{-1}}{v} + suv)\\ &= 2s\alpha.\end{aligned}$$

So s must be zero. Contradiction! Thus we have proved that it is impossible to have $\left\langle \frac{\alpha,-\alpha}{k} \right\rangle \cong \mathrm{End}(M)$ for some 2-dimensional differential graded module M, and hence $[\left\langle \frac{\alpha,-\alpha}{k} \right\rangle] \neq 1$. □

From Lemma 2.3 we see that a DS matrix algebra $\left\langle \frac{\alpha,-\alpha}{k} \right\rangle$ ($\alpha \in k^{\bullet}$) representing the unit in $\mathrm{BW}(k)$ now represents a non-unit element in $\mathrm{BDS}(k)$. In the following we show that $\left\langle \frac{\alpha,-\alpha}{k} \right\rangle$ represents an element of infinite order in $\mathrm{BDS}(k)$ if the characteristic of k is zero. In fact:

Proposition 2.4 [54, Prop. 7]. *Let $(k, +)$ be the additive group of k. Then*

$$\tau : (k, +) \longrightarrow \mathrm{BDS}(k), \quad \alpha \mapsto \left[\left\langle \frac{\alpha^{-1}, -\alpha^{-1}}{k} \right\rangle\right], \quad \alpha \neq 0, 0 \mapsto 1$$

is a group monomorphism.

Proof. By Lemma 2.3 it is sufficient to show that τ is a group homomorphism. Consider the product $\left\langle \frac{\alpha^{-1},-\alpha^{-1},0}{k} \right\rangle \hat{\otimes} \left\langle \frac{\beta^{-1},-\beta^{-1},0}{k} \right\rangle$. If $\alpha + \beta = 0$, then $\left\langle \frac{\beta^{-1},-\beta^{-1},0}{k} \right\rangle = \overline{\left\langle \frac{\alpha^{-1},-\alpha^{-1},0}{k} \right\rangle}$ and

$$\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle \hat{\otimes} \left\langle \frac{\beta^{-1}, -\beta^{-1}, 0}{k} \right\rangle \cong \mathrm{End}\left\langle \frac{\alpha^{-1}, -\alpha^{-1}, 0}{k} \right\rangle,$$

which represents the unit in BDS(k).

Assume that $\alpha+\beta \neq 0$. Let $\{u, v\}$ and $\{u', v'\}$ be the generators of $\left\langle\frac{\alpha^{-1},-\alpha^{-1}}{k}\right\rangle$ and of $\left\langle\frac{\beta^{-1},-\beta^{-1}}{k}\right\rangle$ respectively. Let

$$U = \frac{\alpha\beta}{\alpha+\beta}(u\hat{\otimes}w' + w\hat{\otimes}u'), V = \frac{\alpha}{\alpha+\beta}v\hat{\otimes}1 + \frac{\beta}{\alpha+\beta}1\hat{\otimes}v'\,.$$

Then

$$U^2 = (\alpha+\beta)^{-1}, \quad V^2 = -(\alpha+\beta)^{-1}, \quad UV + VU = 0.$$

Thus U and V generate the matrix algebra $\left(\frac{\sigma^{-1},-\sigma^{-1}}{k}\right)$, where $\sigma = \alpha+\beta$. One may further check that the induced $\mathbb{Z}_2$-grading and the induced differential on $\left(\frac{\sigma^{-1},-\sigma^{-1}}{k}\right)$ are given by (2–1) and (2–7). Thus U and V generate a DS quaternion subalgebra $\left\langle\frac{(\alpha+\beta)^{-1},-(\alpha+\beta)^{-1}}{k}\right\rangle$ in $\left\langle\frac{\alpha^{-1},-\alpha^{-1}}{k}\right\rangle\hat{\otimes}\left\langle\frac{\beta^{-1},-\beta^{-1}}{k}\right\rangle$. Applying the commutator theorem for Azumaya algebras (see [19]), we obtain

$$\left\langle\frac{\alpha^{-1},-\alpha^{-1}}{k}\right\rangle\hat{\otimes}\left\langle\frac{\beta^{-1},-\beta^{-1}}{k}\right\rangle = \left\langle\frac{\sigma^{-1},-\sigma^{-1}}{k}\right\rangle \otimes M_2(k)$$

as algebras, where $\sigma = \alpha+\beta$. We leave it to readers to check that they are equal as DS algebras (or see [55, Coro.2]). It follows that

$$\begin{aligned}\tau(\alpha)\tau(\beta) &= \left[\left\langle\frac{\alpha^{-1},-\alpha^{-1}}{k}\right\rangle\hat{\otimes}\left\langle\frac{\beta^{-1},-\beta^{-1}}{k}\right\rangle\right] = \left[\left\langle\frac{(\alpha+\beta)^{-1},-(\alpha+\beta)^{-1}}{k}\right\rangle\right]\\ &= \tau(\alpha+\beta)\end{aligned}$$

in the Brauer group BDS(k). So we have proved that τ is a group homomorphism. □

Note that when the characteristic of k is 0, $(k,+)$ is not a torsion group. The element represented by the matrix algebra $\left\langle\frac{1,-1}{k}\right\rangle$ in BDS(k) generates a subgroup which is isomorphic to $\mathbb{Z}$. In this case any quaternion algebra $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ with the canonical grading and the canonical differential represents an element of infinite order in the Brauer group BDS(k). If the characteristic of k is $p \neq 2$, then $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ represents an element of order not greater than p in BDS(k). The group $(k,+)$ indicates the substantial difference between the Brauer–Wall group BW(k) and the Brauer group BDS(k) of DS algebras. Actually, this subgroup comes only from extra differentials added to graded Azumaya algebras.

Theorem 2.5 [54, Thm. 8]. BDS(k) = BW(k) $\times$ $(k,+)$.

Proof. By definition of a DS Azumaya algebra, we have a well-defined group homomorphism

$$\gamma : \mathrm{BDS}(k) \longrightarrow \mathrm{BW}(k), \quad [A] \longrightarrow [A]$$

by forgetting the differential on the latter A. It is clear that γ is a surjective map as a graded Azumaya algebra with a trivial differential is a DS Azumaya algebra. Since the graded Azumaya algebra $\left\langle\frac{\alpha^{-1},-\alpha^{-1}}{k}\right\rangle$ represents the unit in BW(k), we have $\tau(k,+) \subseteq \mathrm{Ker}(\gamma)$. To prove that $\mathrm{Ker}(\gamma) \subseteq \tau(k,+)$, we need to use the associated automorphism σ given by (2–4) of a graded algebra. Let A be a DS Azumaya algebra representing a non-trivial element in $\mathrm{Ker}(\gamma)$. Since $[A]=1$ in $\mathrm{BW}(k)$, A is a graded matrix algebra. Since A is Azumaya, the associated automorphism σ is an inner automorphism induced by some invertible element $u\in A$ such that $u^2=1$. Similarly, the differential δ is an inner super-derivation induced by some element $v\in A$ in the sense that

$$\delta(a)=va-(-1)^{\partial(a)}va$$

for any homogeneous element $a\in A$. Note that $v^2\neq 0$ by the proof of Lemma 2.3. Now one may apply the properties that $\delta^2=0$, $\sigma\delta+\delta\sigma=0$ and $\sigma^2=1$ to obtain that u and v generate a quaternion subalgebra $\left(\frac{\alpha,\beta}{k}\right)$ for some $\alpha,\beta\in k^\bullet$ with α being a square number. Here u,v are not necessarily the canonical generators of $\left(\frac{\alpha,\beta}{k}\right)$ (see [54, Thm. 8] for more detail). Thus $\left(\frac{\alpha,\beta}{k}\right)$ is a matrix algebra and A is a tensor product $\left(\frac{\alpha,\beta}{k}\right)\otimes M_n(k)$ of two matrix algebras for some integer n. Since u and v generate $\left(\frac{\alpha,\beta}{k}\right)$ and $M_n(k)$ commutes with $\left(\frac{\alpha,\beta}{k}\right)$, σ and δ act on $M_n(k)$ trivially and $\left(\frac{\alpha,\beta}{k}\right)$ is a DS subalgebra of A. It follows that $A=\left(\frac{\alpha,\beta}{k}\right)\hat{\otimes}M_n(k)$. Finally one may take a while to check that there is a pair of new generators u',v' of $\left(\frac{\alpha,\beta}{k}\right)$ such that the DS algebra $\left(\frac{\alpha,\beta}{k}\right)$ can be written as $\left\langle\frac{\alpha,\beta}{k}\right\rangle$ with u',v' being the canonical generators. So $[A]=[\left\langle\frac{\alpha,\beta}{k}\right\rangle\hat{\otimes}M_n(k)]=[\left\langle\frac{\alpha,\beta}{k}\right\rangle]\in\tau(k,+)$. Finally, since γ is split by the inclusion map, the Brauer group $\mathrm{BDS}(k)$ is a direct product of $\mathrm{BW}(k)$ with $(k,+)$. □

DS algebras may be generalized to differential $\mathbb{Z}_2$-dimodule algebras adding one differential to a dimodule algebra such that the action of the differential anti-commutes with the action of the non-unit element of $\mathbb{Z}_2$. The Brauer group $\mathrm{BDD}(k,\mathbb{Z}_2)$ of differential dimodule algebras can be defined and computed. Once again quaternion algebras play the same roles as they do in the Brauer group of DS algebras. As an exercise for readers, the Brauer group $\mathrm{BDD}(k,\mathbb{Z}_2)$ is isomorphic to the group $(k,+)\times\mathrm{BD}(k,\mathbb{Z}_2)$ [56]. Other exercises include adding more differentials, say n differentials $\delta_1,\cdots,\delta_n$, to graded Azumaya algebras or dimodule Azumaya algebras. For instance, one may obtain the Brauer group $\mathrm{BDS}_n(k)$ of n-differential superalgebras which is isomorphic to the group $(k,+)^n\times\mathrm{BW}(k)$.

From the proofs of Lemma 2.3 and Theorem 2.5 one may find that the argument there is actually involved with actions of an automorphism and a differential which satisfy the relations:

$$\sigma^2=1,\quad \delta^2=0,\quad \sigma\delta+\delta\sigma=0$$

where σ is the non-unit element of $\mathbb{Z}_2$. In fact, the four dimensional algebra generated by σ and δ is a Hopf algebra with comultiplication given by

$$\Delta(\sigma)=\sigma\otimes\sigma,\quad \Delta(\delta)=1\otimes\delta+\delta\otimes\sigma$$

and counit given by $\varepsilon(\sigma) = 1$ and $\varepsilon(\delta) = 0$. This Hopf algebra is called Sweedler Hopf algebra, denoted H_4. The two generators σ and δ are usually replaced by g and h. Thus a DS algebra is nothing else but an H_4-module algebra.

Conversely if A is an H_4-module algebra, there is a natural $\mathbb{Z}_2$-grading on A given by

$$A_0 = \{x \in A \mid g(x) = x\}, \quad A_1 = \{x \in A \mid g(x) = -x\}. \qquad (2\text{–}8)$$

With respect to the grading (2–8), the action of h is a differential on A so that A is a DS algebra. Moreover, the H_4-module algebra is a YD H_4-module algebra with the coaction given by the grading (2–8).

In this way we may identify DS algebras with YD H_4-module algebras with coactions given by the grading (2–8). This is the reason why the Brauer group of DS algebras can be defined. The elements in the Brauer group of DS algebras are eventually represented by those so called YD H_4-Azumaya algebras which will be introduced in the next section. In particular, quaternion algebras are YD H_4-Azumaya algebras.

3. The Definition of the Brauer Group

Throughout this section H is a flat k-Hopf algebra with a bijective antipode S, and all k-modules (except H) are faithfully projective over k. Let A be a YD H-module algebra. The two YD H-module algebras $A\#\overline{A}$ and $\overline{A}\#A$ are called the left and right *H-enveloping algebras* of A (see (1–1) for definition of #). We are now able to define the concept of an H-Azumaya algebra, and construct the Brauer group of the Hopf algebra H.

Definition 3.1. A YD H-module algebra A is called an *H-Azumaya* algebra if it is faithfully projective as a k-module and if the following YD H-module algebra maps are isomorphisms:

$$F : A\#\overline{A} \longrightarrow \operatorname{End}(A), \qquad F(a\hat{\otimes}\overline{b})(x) = \sum a x_{(0)}(x_{(1)} \cdot b),$$
$$G : \overline{A}\#A \longrightarrow \operatorname{End}(A)^{\mathrm{op}}, \quad G(\overline{a}\#b)(x) = \sum a_{(0)}(a_{(1)} \cdot x)b.$$

where the YD H-structures of $\operatorname{End}(A)$ and $\operatorname{End}(A)^{\mathrm{op}}$ are given by (1–2) and (1–3).

It follows from the definition that a usual Azumaya algebra with trivial H-structures is an H-Azumaya algebra. One may take a while to check (or see [11]) that the H-opposite algebra $\overline{A}$ of an H-Azumaya algebra A and the braided product $A\#B$ of two H-Azumaya algebras A and B are H-Azumaya algebras. In particular, the YD H-module algebra $\operatorname{End}(M)$ of any faithfully projective YD H-module M is an H-Azumaya algebra. An H-Azumaya algebra of the form $\operatorname{End}(M)$ is called an *elementary H-Azumaya algebra*. As usual we may define an equivalence relation on the set $B(k, H)$ of isomorphism classes of H-Azumaya algebras.

Definition 3.2. Let A and B be two H-Azumaya algebras. A and B are said to be Brauer equivalent, denoted $A \sim B$, if there exist two faithfully projective YD H-modules M and N such that $A\#\mathrm{End}(M) \cong B\#\mathrm{End}(N)$ as YD H-module algebras.

As expected the quotient set of $B(k,H)$ modulo the Brauer equivalence is a group with multiplication induced by the braided product # and with inverse operator induced by the H-opposite $^{-}$. Denote by $\mathrm{BQ}(k,H)$ the group $B(k,H)/\sim$ and call it the *Brauer group of the Hopf algebra H* or the *Brauer group of Yetter–Drinfel'd H-module algebras*. Since a usual Azumaya algebra with trivial YD H-module structures is H-Azumaya and the Brauer equivalence restricts to the usual Brauer equivalence, the classical Brauer group $\mathrm{Br}(k)$ of k is a subgroup of $\mathrm{BQ}(k,H)$ sitting in the center of $\mathrm{BQ}(k,H)$.

Let E be a commutative ring with unit. Suppose that we have a ring homomorphism $f : k \longrightarrow E$. By usual base change $H_E = H \otimes_k E$ is a E-Hopf algebra. Now in a way similar to [30, 4.7, 4.8] we obtain an induced group homomorphism on the Brauer group level.

Proposition 3.3. *The functor $M \mapsto M \otimes_k E$ induces a group homomorphism $\mathrm{BQ}(k,H) \longrightarrow \mathrm{BQ}(E,H_E)$, mapping the class of A to the class of A_E.*

The kernel of the foregoing homomorphism, denoted by $\mathrm{BQ}(E/k,H)$, is called *the relative Brauer group of H w.r.t. the extension E/k*. Denote by $\mathrm{BQ}^s(k,H)$ the union of relative Brauer groups $\mathrm{BQ}(E/k,H)$ of all faithfully flat extensions E of k. $\mathrm{BQ}^s(k,H)$ is called the *split part* of $\mathrm{BQ}(k,H)$. In [12], $\mathrm{BQ}^s(k,H)$ was described in a complex:

$$1 \longrightarrow \mathrm{BQ}^s(k,H) \longrightarrow \mathrm{BQ}(k,H) \longrightarrow \mathrm{O}(E(H))$$

where $\mathrm{O}(E(H))$ is a subgroup of the automorphism group $\mathrm{Aut}(E(H))$ and $E(H)$ is the group of group-like elements of the dual Drinfel'd double $D(H)^*$ of H (see [12, 3.11-3.14] for details).

Now let H be a commutative and cocommutative Hopf algebra. In this situation, a YD H-module (algebra) is an H-dimodule (algebra). But an H-Azumaya algebra in the sense of Definition 3.1 is not an Azumaya H-dimodule algebra in the sense of Long (see [30] for detail on the Brauer group of dimodule algebras we refer to [6; 30]). The reason for this is that the braided product we choose in $\mathcal{Q}^H$ is the inverse product of $\mathbf{D}^H$ when H is commutative and cocommutative. However we have the following:

Proposition 3.4 [12, Prop.5.8]. *Let H be a commutative and cocommutative Hopf algebra. If A is an H-Azumaya algebra, then A^{op} is an H-Azumaya dimodule algebra. Moreover, $\mathrm{BQ}(k,H)$ is isomorphic to $\mathrm{BD}(k,H)$. The isomorphism is given by $[A] \mapsto [A^{\mathrm{op}}]^{-1}$.*

When H is a commutative and cocommutative Hopf algebra, the Brauer–Long group $\mathrm{BD}(k,H)$ has two subgroups $\mathrm{BM}(k,H)$ and $\mathrm{BC}(k,H)$ (see [30, 1.10, 2.13]). The subgroup $\mathrm{BM}(k,H)$ consists of isomorphism classes represented by H-Azumaya dimodule algebras with trivial H-comodule structures, and the subgroup $\mathrm{BC}(k,H)$ consists of isomorphism classes represented by H-Azumaya dimodule algebras with trivial H-module structures. These two subgroups were calculated by M. Beattie in [3] which are completely determined by the groups of Galois objects of H and H^* respectively (e.g., see Corollary 4.3.5).

If H is not cocommutative (or not commutative) an H-module (or comodule) algebra with the trivial H^{op}-comodule (or the trivial H-module) structure does not need to be YD H-module algebra. In general, we do not have subgroups like $\mathrm{BM}(k,H)$ or $\mathrm{BC}(k,H)$ in the Brauer group $\mathrm{BQ}(k,H)$ when H is non-commutative or non-cocommutative. However, when H is quasitriangular or coquasitriangular, we have subgroups similar to $\mathrm{BM}(k,H)$ or $\mathrm{BC}(k,H)$.

It is well known that the notion of a quasitriangular Hopf algebra is the generalization of the notion of a cocommutative Hopf algebra. If $(H,\mathcal{R})$ is a quasitriangular Hopf algebra, an H-module algebra is automatically a YD H-module algebra with the freely granted H^{op}-comodule structure (1–4). Since ${}_H\mathbf{M}^{\mathcal{R}}$ is a braided subcategory of $\mathcal{Q}^H$ and the braided product $\#$ given by (1–1) commutes (or is compatible) with the H^{op}-coaction (1–4), the canonical H-linear map F and G in Definition 3.1 are automatically H^{op}-colinear. Thus the subset of $\mathrm{BQ}(k,H)$ consisting of isomorphism classes represented by H-Azumaya algebras with H^{op}-coactions of the form (1–4) stemming from H-actions is a subgroup, denoted $\mathrm{BM}(k,H,\mathcal{R})$. We now have the following inclusions for a QT Hopf algebra.

$$\mathrm{Br}(k) \subseteq \mathrm{BM}(k,H,\mathcal{R}) \subseteq \mathrm{BQ}(k,H).$$

Similarly if (H,R) is a coquasitriangular Hopf algebra, the Brauer group $\mathrm{BQ}(k,H)$ possesses a subgroup $\mathrm{BC}(k,H,R)$ consisting of isomorphism classes represented by H-Azumaya algebras with H-actions (1–5) stemming from the H^{op}-coactions. In this case we have the following inclusions of groups for a CQT Hopf algebra:

$$\mathrm{Br}(k) \subseteq \mathrm{BC}(k,H,R) \subseteq \mathrm{BQ}(k,H)$$

When H is a finite commutative and cocommutative Hopf algebra, a CQT structure can be interpreted by a Hopf algebra map from H into H^*. As a matter of fact, there is one-to-one correspondence between the CQT structures on H and the Hopf algebra maps from H to H^* which form a group $\mathrm{Hopf}(H,H^*)$ with the convolution product. The correspondence is given by

$$\{\text{CQT structures on } H\} \longrightarrow \mathrm{Hopf}(H,H^*), \quad R \mapsto \theta_R,\ \theta_R(h)(l) = R(h \otimes l)$$

for any $h,l \in H$. In this case the Brauer group $\mathrm{BC}(k,H,R)$ is Orzech's Brauer group $\mathrm{B}_\theta(k,H)$ of $\mathrm{BD}(k,H)$ consisting of classes of θ-dimodule algebras. For θ-dimodule algebras one may refer to [36] in the case that H is a group Hopf algebra with a finite abelian group and to [6, § 12.4] in the general case. In a special case that $H = kG$ is a finite abelian group algebra, $\gamma : G \times G \longrightarrow k^\bullet$ a bilinear map, we may view γ as a coquasitriangular structure on H. Then the Brauer group $\mathrm{B}_\gamma(k, G)$ of graded algebras investigated by Childs, Garfinkel and Orzech (see [14; 15]) is isomorphic to $\mathrm{BC}(k, H, \gamma)$.

Note that a cocommutative Hopf algebra with a coquasitriangular structure is necessarily commutative. Similarly, a commutative quasitriangular Hopf algebra is cocommutative.

Now let us consider the finite case. Suppose that H is a faithfully projective Hopf algebra. Then the Drinfel'd double $\mathrm{D}(H)$ is a quasitriangular Hopf algebra with the canonical QT structure $\mathcal{R}$ represented by a pair of dual bases of H and H^*, e.g., [23; 40], and there is a one-to-one correspondence between left $\mathrm{D}(H)$-module algebras and Yetter–Drinfel'd H-module algebras, [32]. It follows that $\mathrm{BQ}(k, H) = \mathrm{BM}(k, \mathrm{D}(H), \mathcal{R})$. So

$$\mathrm{BQ}(k, H) \subseteq \mathrm{BQ}(k, \mathrm{D}(H)).$$

Now write $\mathrm{D}^n(H)$ for $\mathrm{D}(\mathrm{D}^{n-1}(H))$, the n-*th* Drinfel'd double. Then we have the following chain of inclusions:

$$\mathrm{BQ}(k, H) \subseteq \mathrm{BQ}(k, \mathrm{D}(H)) \subseteq \mathrm{BQ}(k, \mathrm{D}^2(H)) \subseteq \cdots \subseteq \mathrm{BQ}(k, \mathrm{D}^n(H)) \subseteq \cdots$$

A natural question arises: when is the foregoing ascending chain finite?

To end this section, let us look once again at Definition 3.1 and the definition of the Brauer equivalence. It is not surprising that these definitions are essentially categorical in nature. This means that an H-Azumaya algebra can be characterized in terms of monoidal category equivalences. The Brauer equivalence is in essence the Morita equivalence. In particular, the unit in the Brauer group $\mathrm{BQ}(k, H)$ is only represented by elementary H-Azumaya algebras.

Let A be a YD H-module algebra. A *left A-module M in $\mathcal{Q}^H$* is both a left A-module and a YD H-module satisfying the compatibility conditions:

i. $h \cdot (am) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot m)$,

ii. $\chi(am) = \sum a_{(0)} m_{(0)} \otimes m_{(1)} a_{(1)}$.

That is, M is a left $A\#H$-module and a right Hopf module in ${}_A\mathbf{M}^{H^{\mathrm{op}}}$. Here $A\#H$ is the usual smash product rather than the braided product. Denote by ${}_A\mathcal{Q}^H$ the category of left A-modules in $\mathcal{Q}^H$ and A-module morphisms in $\mathcal{Q}^H$. Similarly, we may define a *right A-module M in $\mathcal{Q}^H$* as a right A-module and a YD H-module such that the following two compatibility conditions hold:

i. $h \cdot (ma) = \sum (h_{(1)} \cdot m)(h_{(2)} \cdot a)$,

ii. $\chi(ma) = \sum m_{(0)} a_{(0)} \otimes a_{(1)} m_{(1)}$.

Denote by $\mathcal{Q}^H_A$ the category of right A-modules in $\mathcal{Q}^H$ and their morphisms. Now let A and B be two YD H-module algebras. An *$(A$-$B)$-bimodule M in $\mathcal{Q}^H$* is an $(A$-$B)$-bimodule which belongs to ${}_A\mathcal{Q}^H$ and $\mathcal{Q}^H_B$. Denote by ${}_A\mathcal{Q}^H_B$ the category of $(A$-$B)$-bimodules. View k as a trivial YD H-module algebra. Then ${}_A\mathcal{Q}^H$ is just the category of (A-k)-bimodules in $\mathcal{Q}^H$. Similarly $\mathcal{Q}^H_A$ is the category of (k-A)-bimodules in $\mathcal{Q}^H$.

Definition 3.5. A (strict) Morita context (A,B,P,Q,φ,ψ) is called a (strict) H-Morita context in $\mathcal{Q}^H$ if the following conditions hold:

(1) A and B are YD H-module algebras,
(2) P is an (A-B)-bimodule in $\mathcal{Q}^H$ and Q is an (B-A)-bimodule in $\mathcal{Q}^H$,
(3) φ and ψ are (surjective) YD H-module algebra maps: $\varphi: P\tilde{\otimes}_B Q \longrightarrow A$ and $\psi: Q\tilde{\otimes}_A P \longrightarrow B$.

An H-Morita context in $\mathcal{Q}^H$ is a usual Morita context if one forgets the H-structures. When one works with the base category $\mathcal{Q}^H$, the usual Morita theory applies fully and one obtains an H-Morita theory in $\mathcal{Q}^H$. Here are few basic properties of H-Morita contexts.

Proposition 3.6. (1) *If P is a faithfully projective YD H-module*, *then*

$$(\mathrm{End}(P), k, P, P^*, \varphi, \psi)$$

is a strict H-Morita context in $\mathcal{Q}^H$. Here φ and ψ are given by $\varphi(p\otimes f)(x) = pf(x)$ and $\psi(f\otimes p) = f(p)$.

(2) *Let B be a YD H-module algebra. If $P \in \mathcal{Q}^H_B$ is a B-progenerator*, *then $(A = \mathrm{End}_B(P), B, P, Q = \mathrm{Hom}_B(P,B), \varphi, \psi)$ is a strict H-Morita context in $\mathcal{Q}^H$. Here φ and ψ are given by $\varphi(p\otimes f)(x) = pf(x)$ and $\psi(f\otimes p) = f(p)$*,

where $\mathrm{End}_B(P)$ *is a YD H-module algebra with adjoint H-structures given in Subsection* 1.1. *Like usual Morita theory*, *if (A,B,P,Q,φ,ψ) is a strict H-Morita context*, *then the pairs of functors*

$$Q\tilde{\otimes}_A - :\ {}_A\mathcal{Q}^H \longrightarrow {}_B\mathcal{Q}^H \quad \textit{and} \quad P\tilde{\otimes}_B - :\ {}_B\mathcal{Q}^H \longrightarrow {}_A\mathcal{Q}^H,$$
$$-\tilde{\otimes}_A P :\ \mathcal{Q}^H_A \longrightarrow \mathcal{Q}^H_B \quad \textit{and} \quad -\tilde{\otimes}_B Q :\ \mathcal{Q}^H_B \longrightarrow \mathcal{Q}^H_A$$

define equivalences between the categories of bimodules in $\mathcal{Q}^H$.

Let A be a YD H-module algebra, $\overline{A}$ is the H-opposite of A. Write A^e for $A\#\overline{A}$ and ${}^eA$ for $\overline{A}\#A$. Then A may be regarded as a left A^e-module and a right ${}^eA$-module as follows:

$$(a\#\bar{b})\cdot x = \sum a x_{(0)}(x_{(1)}\cdot b), \text{ and } x\cdot(\bar{a}\#b) = \sum a_{(0)}(a_{(1)}\cdot x)b. \qquad (3\text{–}1)$$

It is clear that A with foregoing A^e and ${}^eA$-module structures is in ${}_{A^e}\mathcal{Q}^H$ and $\mathcal{Q}^H_{{}^eA}$ respectively. Now consider the categories ${}_{A^e}\mathcal{Q}^H$ and $\mathcal{Q}^H_{{}^eA}$. To a left A^e-module M in ${}_{A^e}\mathcal{Q}^H$ we associate a YD H-submodule

$$M^A = \{m \in M \mid (a\#1)m = (1\#\bar{a})m, \forall a \in A\}.$$

This correspondence gives rise to a functor $(-)^A$ from ${}_{A^e}\mathcal{Q}^H$ to $\mathcal{Q}^H$. On the other hand, we have an *induction functor* $A\tilde{\otimes}-$ from $\mathcal{Q}^H$ to ${}_{A^e}\mathcal{Q}^H$. It is easy to see that

$$\begin{aligned} A\tilde{\otimes}- : \mathcal{Q}^H &\longrightarrow {}_{A^e}\mathcal{Q}^H,\ N \mapsto A\tilde{\otimes}N, \\ (-)^A : {}_{A^e}\mathcal{Q}^H &\longrightarrow \mathcal{Q}^H, \quad M \mapsto M^A. \end{aligned} \tag{3–2}$$

is an adjoint pair of functors. Similarly we have an adjoint pair of functors between categories $\mathcal{Q}^H$ and $\mathcal{Q}^H_{{}^eA}$:

$$\begin{aligned} -\tilde{\otimes}A : \mathcal{Q}^H &\longrightarrow \mathcal{Q}^H_{{}^eA}, \quad N \mapsto N\tilde{\otimes}A, \\ {}^A(-) : \mathcal{Q}^H_{{}^eA} &\longrightarrow \mathcal{Q}^H, \quad M \mapsto {}^AM \end{aligned} \tag{3–3}$$

where ${}^AM = \{m \in M \mid m(1\#a) = m(\overline{a}\#1), \forall a \in A\}$.

Proposition 3.7 [12, Prop.2.6]. *Let A be a YD H-module algebra. Then A is H-Azumaya if and only if* (3–2) *and* (3–3) *define equivalences of categories.*

In fact, (3–2) and (3–3) define the equivalences between the braided monoidal categories if A is H-Azumaya (see [12]).

With the previous preparation one is able to show that the 'Brauer equivalence' is equivalent to the 'H-Morita equivalence' in $\mathcal{Q}^H$. We denote by $A\overset{m}{\sim}B$ that A is H-Morita equivalent to B, and denote by $A\overset{b}{\sim}B$ that A is Brauer equivalent to B, i.e., $[A] = [B] \in \mathrm{BQ}(k, H)$.

Theorem 3.8 [12, Thm. 2.10]. *Let A, B be H-Azumaya algebras. $A\overset{b}{\sim}B$ if and only if $A\overset{m}{\sim}B$.*

As a direct consequence, we have that if $[A] = 1$ in $\mathrm{BQ}(k, H)$, then $A \cong \mathrm{End}(P)$ for some faithfully projective YD H-module P.

4. An Exact Sequence for the Brauer Group $\mathrm{BC}(k, H, R)$

As we explained in Section 3, when a Hopf algebra H is finite, the Brauer group $\mathrm{BQ}(k, H)$ of H is equal to the Brauer group $\mathrm{BC}(k, D(H)^*, R)$. So in the finite case, it is sufficient to consider the Brauer group $\mathrm{BC}(k, H, R)$ of a finite coquasitriangular Hopf algebra. We present a general approach to the calculation of the Brauer group $\mathrm{BC}(k, H, R)$. The idea of this approach is basically the one of Wall in [57] where he introduced the first Brauer group of structured algebras, i.e., the Brauer group of super (or $\mathbb{Z}_2$-graded) algebras which is now called the Brauer–Wall group, denoted $\mathrm{BW}(k)$. He proved that the Brauer group of super algebras over a field k is an extension of the Brauer group $\mathrm{Br}(k)$ by the group $Q_2(k)$ of $\mathbb{Z}_2$-graded quadratic extensions of k, i.e., there is an exact sequence of group homomorphisms:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BW}(k) \longrightarrow Q_2(k) \longrightarrow 1$$

In 1972, Childs, Garfinkel and Orzech studied the Brauer group of algebras graded by a finite abelian group [14], and Childs (in [15]) generalized Wall's sequence by constructing a non-abelian group $\mathrm{Galz}(G)$ of bigraded Galois objects replacing the graded quadratic group of k in Wall's sequence. They obtained an exact group sequence:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BC}(k, G, \phi) \longrightarrow \mathrm{Galz}(G)$$

where $\mathrm{Galz}(G)$ may be described by another exact sequence when G is a p-group.

The complexity of the group $\mathrm{Galz}(G)$ is evident. An object in $\mathrm{Galz}(G)$ involves both two-sided G-gradings and two-sided G-actions such that the actions and gradings commute. In 1992, K. Ulbrich extended the exact sequences of Childs to the case of the Brauer–Long group of a commutative and cocommutative Hopf algebra (see [48]). The technique involved is essentially the Hopf Galois theory of a finite Hopf algebra. However, when the Hopf algebra is not commutative, a similar group of Hopf Galois objects does not exists although one still obtains a group of Hopf bigalois objects with respect to the cotensor product (see [42; 55]). The idea of this section is to apply the Hopf quotient Galois theory. This requires the deformation of the Hopf algebra H. Throughout, (H, R) is a finite CQT Hopf algebra.

4.1. The algebra $\mathcal{H}_R$. We start with the definition of the new product $\star$ on the k-module H:

$$\begin{aligned} h \star l &= \sum l_{(2)} h_{(2)} R(S^{-1}(l_{(3)}) l_{(1)} \otimes h_{(1)}) \\ &= \sum h_{(2)} l_{(1)} R(l_{(2)} \otimes S(h_{(1)}) h_{(3)}) \end{aligned}$$

where h and l are in H. $(H, \star)$ is an algebra with unit 1. We denote by $\mathcal{H}_R$ the algebra $(H, \star)$. It is easy to see that the counit map ε of H is still an augmentation map from $\mathcal{H}_R$ to k.

There is a double Hopf algebra for a CQT Hopf algebra (H, R) (not necessarily finite). This double Hopf algebra, denoted $D[H]$ due to Doi and Takeuchi (see [21]), is equal to $H \otimes H$ as a coalgebra with the multiplication given by

$$(h \otimes l)(h' \otimes l') = \sum h h'_{(2)} \otimes l_{(2)} l' R(h'_{(1)} \otimes l_{(1)}) R(S(h'_{(3)}) \otimes l_{(3)})$$

for h, l, h' and $l' \in H$. The antipode of $D[H]$ is given by

$$S(h \otimes l) = (1 \otimes S(l))(S(h) \otimes 1)$$

for all $h, l \in H$. The counit of $D[H]$ is $\varepsilon \otimes \varepsilon$.

Since H is finite, the canonical Hopf algebra homomorphism $\Theta_l : H \longrightarrow H^{*\mathrm{op}}$ given by $\Theta_l(h)(l) = R(h \otimes l)$ induces an Hopf algebra homomorphism from $D[H]$ to $D(H)$, the Drinfel'd quantum double $H^{*\mathrm{op}} \bowtie H$.

$$\Phi : D[H] \longrightarrow D(H), \quad \Phi(h \bowtie l) = \Theta_l(h) \bowtie l.$$

When Θ_l is an isomorphism, we may identify $D[H]$ with $D(H)$. Thus a YD H-module is automatically a left $D[H]$-module. Moreover, the following algebra monomorphism ϕ shows that $\mathcal{H}_R$ can be embedded into $D[H]$.

$$\phi : \mathcal{H}_R \longrightarrow D[H], \quad \phi(h) = \sum S^{-1}(h_{(2)}) \bowtie h_{(1)}.$$

Thus we may view $\mathcal{H}_R$ as a subalgebra of the double $D[H]$. Moreover, one may check that the image of ϕ in $D[H]$ is a left coideal of $D[H]$. We obtain the following:

Proposition 4.1.1. *$\mathcal{H}_R$ is a left $D[H]$-comodule algebra with the comodule structure given by*

$$\chi : \mathcal{H}_R \longrightarrow D[H] \otimes \mathcal{H}_R, \quad \chi(h) = \sum (S^{-1}(h_{(3)}) \bowtie h_{(1)}) \otimes h_{(2)}.$$

The left $D[H]$-comodule structure of $\mathcal{H}_R$ in Proposition 4.1.1 demonstrates that $\mathcal{H}_R$ can be embedded into $D[H]$ as a left coideal subalgebra. In fact, $\mathcal{H}_R$ can be further embedded into $D(H)$ as a left coideal subalgebra.

Corollary 4.1.2. *The composite algebra map*

$$\mathcal{H}_R \xrightarrow{\ \phi\ } D[H] \xrightarrow{\ \Phi\ } D(H)$$

is injective, and $\mathcal{H}_R$ is isomorphic to a left coideal subalgebra of $D(H)$.

Let us now consider Yetter–Drinfel'd H-modules and $\mathcal{H}_R$-bimodules. Let M be a Yetter–Drinfel'd module over H, or a left $D(H)$-module. The following composite map:

$$\mathcal{H}_R \otimes M \xrightarrow{\phi \otimes \iota} D[H] \otimes M \xrightarrow{\Phi \otimes \iota} D(H) \otimes M$$

makes M into a left $\mathcal{H}_R$-module. If we write $-\!\triangleright$ for the above left action, then we have the explicit formula:

$$h \multimap m = \sum (h_{(2)} \cdot m_{(0)}) R(S^{-1}(h_{(4)}) \otimes h_{(3)} m_{(1)} S^{-1}(h_{(1)})) \tag{4–1}$$

for $h \in \mathcal{H}_R$ and $m \in M$.

Since there is an augmentation map ε on $\mathcal{H}_R$, we may define the $\mathcal{H}_R$-invariants of a left $\mathcal{H}_R$-module M which is

$$M^{\mathcal{H}_R} = \{m \in M \mid h \multimap m = \varepsilon(h)m, \forall h \in \mathcal{H}_R\}.$$

When a left $\mathcal{H}_R$-module comes from a YD H-module we have

$$M^{\mathcal{H}_R} = \{m \in M \mid h \cdot m = h \triangleright_1 m = \sum m_{(0)} R(h \otimes m_{(1)}), \forall h \in H\}.$$

Now we define a right $\mathcal{H}_R$-module structure on a YD H-module M. Observe that the right H-comodule structure of M induces two left H-module structures. The first one is (1–5), and the second one is given by

$$h \triangleright_2 m = \sum m_{(0)} R(S(m_{(1)}) \otimes h) \tag{4–2}$$

for $h \in H$ and $m \in M$. With this second left H-action (4–2) on M, M can be made into a right $D[H]$-module.

LEMMA 4.1.3. *Let M be a Yetter–Drinfel'd H-module. Then M is a right $D[H]$-module defined by*

$$m \leftharpoondown (h \bowtie l) = S(l) \triangleright_2 (S(h) \cdot m)$$

for $h, l \in H$ and $m \in M$. Moreover, if A is a YD H-module algebra, then A is a right $D[H]^{\mathrm{cop}}$-module algebra.

The right $D[H]$-module structure in Lemma 4.1.3 does not match the canonical left $D[H]$-module structure induced by the Hopf algebra map Φ so as to yield a $D[H]$-bimodule structure on M. However the right $\mathcal{H}_R$-module structure on M given by

$$M \otimes \mathcal{H}_R \xrightarrow{\iota \otimes \phi} M \otimes D[H] \longrightarrow M,$$

more precisely:

$$m \mathrel{\triangleleft\!\!-} h = \sum (h_{(3)} \cdot m_{(0)}) R(h_{(4)} m_{(1)} S^{-1}(h_{(2)}) \otimes h_{(1)}) \tag{4–3}$$

for $m \in M$ and $h \in \mathcal{H}_R$, together with the left $\mathcal{H}_R$-module structure (4–1), defines an $\mathcal{H}_R$-bimodule structure on M.

PROPOSITION 4.1.4. *Let M be a YD H-module. Then M is an $\mathcal{H}_R$-bimodule via* (4–1) *and* (4–3).

If A is a YD H-module algebra, then Proposition 4.1.4 implies that A is an $\mathcal{H}_R$-bimodule algebra in the sense that:

$$\begin{array}{l} h \mathrel{-\!\!\triangleright} (ab) = \sum (h_{(-1)} \cdot a)(h_{(0)} \mathrel{-\!\!\triangleright} b) \\ (ab) \mathrel{\triangleleft\!\!-} h = \sum (a \cdot h_{(0)})(b \mathrel{\triangleleft\!\!-} h_{(-1)}) \end{array} \tag{4–4}$$

for $a, b \in A$ and $h \in \mathcal{H}_R$, where $\chi(h) = \sum h_{(-1)} \otimes h_{(0)} \in D[H] \otimes \mathcal{H}_R$.

To end this subsection we present the dual comodule version of (4–4) which is needed in the next subsection. Observe that the dual coalgebra $\mathcal{H}_R^*$ is a left $D[H]^*$-module quotient coalgebra of the dual Hopf algebra $D[H]^*$ in the sense that the following coalgebra map is a surjective $D[H]$-comodule map:

$$\phi^* : D[H]^* \longrightarrow \mathcal{H}_R^*, \quad p \bowtie q \mapsto q S^{-1}(p).$$

Thus a left (or right) $D[H]$-comodule M is a left (or right) $\mathcal{H}_R^*$-comodule in the natural way through ϕ^*. In order to distinguish $D[H]$ or $\mathcal{H}_R^*$-comodule structures from the H-comodule structures (e.g., a YD H-module has all three comodule structures) we use different uppercase Sweedler sigma notations:

i. $\sum x^{[-1]} \otimes x^{[0]}$, $\sum x^{[0]} \otimes x^{[1]}$ stand for left and right $D[H]^*$-comodule structures,

ii. $\sum x^{(-1)} \otimes x^{(0)}$, $\sum x^{(0)} \otimes x^{(1)}$ stand for left and right $\mathcal{H}_R^*$-comodule structures, where x is an element in a due comodule. Now let A be a YD H-module algebra. Then A is both a left and right $D[H]$-module algebra, and therefore an $\mathcal{H}_R$-bimodule algebra in the sense of (4–4). Thus the dual comodule versions of the formulas in (4–4) read as follows:

$$\begin{aligned}\textstyle\sum(ab)^{(0)}\otimes(ab)^{(1)}&=\textstyle\sum a^{[0]}b^{(0)}\otimes a^{[1]}\rightharpoondown b^{(1)},\\ \textstyle\sum(ab)^{(-1)}\otimes(ab)^{(0)}&=\textstyle\sum b^{[-1]}\rightharpoondown a^{(-1)}\otimes a^{(0)}b^{[0]}\end{aligned}\tag{4–5}$$

for $a,b\in A$, where $\rightharpoondown$ is the left action of $D[H]^*$ on $\mathcal{H}_R^*$. We will call A a *right (or left)* $\mathcal{H}_R^*$*-comodule algebra* in the sense of (4–5).

Finally, for a YD H-module M, we will write $M_\diamond$ (or ${}_\diamond M$) for the right (or left) $\mathcal{H}_R^*$-coinvariants. For instance,

$$M_\diamond=\{m\in M\mid\sum m^{(0)}\otimes m^{(1)}=m\otimes\varepsilon.\}$$

It is obvious that $M_\diamond=M^{\mathcal{H}_R}$.

4.2. The group $\mathrm{Gal}(\mathcal{H}_R)$**.** We are going to construct a group $\mathrm{Gal}(\mathcal{H}_R)$ of 'Galois' objects for the deformation $\mathcal{H}_R$. The group $\mathrm{Gal}(\mathcal{H}_R)$ plays the vital role in an exact sequence to be constructed.

Definition 4.2.1. Let A be a right $D[H]^*$-comodule algebra. $A/A_\diamond$ is said to be a right $\mathcal{H}_R^*$-Galois extension if the linear map

$$\beta^r:A\otimes_{A_\diamond}A\longrightarrow A\otimes\mathcal{H}_R^*,\quad \beta^r(a\otimes b)=\sum a^{(0)}b\otimes a^{(1)}$$

is an isomorphism. Similarly, if A is a left $D[H]^*$-comodule algebra, then $A/{}_\diamond A$ is said to be left Galois if the linear map

$$\beta^l:A\otimes_{{}_\diamond A}A\longrightarrow\mathcal{H}_R^*\otimes A,\quad \beta^l(a\otimes b)=\sum b^{(-1)}\otimes ab^{(0)}$$

is an isomorphism. If in addition the subalgebra ${}_\diamond A$ (or $A_\diamond$) is trivial, then A is called a *left* (or *right*) $\mathcal{H}_R^*$*-Galois object*. For more detail on Hopf quotient Galois theory, readers may refer to [33; 44; 45].

The objects we are interested in are those $\mathcal{H}_R^*$-bigalois objects which are both left and right $\mathcal{H}_R^*$-Galois such that the left and right $\mathcal{H}_R^*$-coactions commute. Denote by $\mathcal{E}(\mathcal{H}_R)$ the category of YD H-module algebras which are $\mathcal{H}_R^*$-bigalois objects. The morphisms in $\mathcal{E}(\mathcal{H}_R)$ are YD H-module algebra isomorphisms. We are going to define a product in the category $\mathcal{E}(\mathcal{H}_R)$. Let $\#_R$ be the braided product in the category $\mathbf{M}_R^H$ to differ from the braided product in $\mathcal{Q}^H$. This makes sense when a YD H-module algebra A can be treated as an algebra in $\mathbf{M}_R^H$ forgetting the H-module structure of A and endowing with the induced H-module structure (1–5).

Given two objects X and Y in $\mathcal{E}(\mathcal{H}_R)$, we define a generalized cotensor product $X\wedge Y$ (in terms of $\mathcal{H}_R^*$-bicomodules) as a subset of $X\#_RY$:

$$\left\{\sum x_i\#y_i\in X\#_RY\mid\sum x_i\triangleleft\!\!-\,h\#y_i=\sum x_i\#h\,-\!\!\triangleright\,y_i,\forall h\in\mathcal{H}_R.\right\}$$

In the foregoing formula we may change the actions $\triangleleft\!-$ and $-\!\triangleright$ of $\mathcal{H}_R$ into the actions of H which are easier to check.

$$X \wedge Y = \{\sum x_i \# y_i \in X \#_R Y \mid \sum h_{(1)} \cdot x_i \# h_{(2)} \triangleright_1 y_i \\ = \sum h_{(1)} \triangleright_2 x_i \# h_{(2)} \cdot y_i, \forall h \in H\}. \tag{4–6}$$

The formula (4–6) allow us to define a left H-action on $X \wedge Y$:

$$h \cdot \sum (x_i \# y_i) = \sum h_{(1)} \cdot x_i \# h_{(2)} \triangleright_1 y_i = \sum h_{(1)} \triangleright_2 x_i \# h_{(2)} \cdot y_i \tag{4–7}$$

whenever $\sum x_i \# y_i \in X \wedge Y$ and $h \in H$. The left H-action is YD compatible with the right diagonal H-coaction, so we obtain:

PROPOSITION 4.2.2. *If X, Y are two objects of $\mathcal{E}(\mathcal{H}_R)$, then $X \wedge Y$ with the H-action* (4–7) *and the H-coaction inherited from $X \#_R Y$ is a YD H-module algebra. Moreover*, $X \wedge Y$ *is an object of* $\mathcal{E}(\mathcal{H}_R)$.

Let H^* be the convolution algebra of H. There is a canonical YD H-module structure on H^* such that H^* is a YD H-module algebra:

$$\begin{aligned} h \cdot p &= \sum p_{(1)} < p_{(2)}, h >, \ H\text{-action} \\ h^* \cdot p &= \sum h^*_{(2)} p S^{-1}(h^*_{(1)}), \ H\text{-coaction} \end{aligned} \tag{4–8}$$

for $h^*, p \in H^*$ and $h \in H$. One may easily check that H^* with the YD H-module structure (4–8) is an object in $\mathcal{E}(\mathcal{H}_R)$, denoted I. Moreover I is the unit object of $\mathcal{E}(H)$ with respect to the product $\wedge$. It follows that the category $\mathcal{E}(\mathcal{H}_R)$ is a monoidal category.

Denote by $E(\mathcal{H}_R)$ the set of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$. The fact that $\mathcal{E}(\mathcal{H}_R)$ is a monoidal category implies that the set $E(\mathcal{H}_R)$ is a semigroup. In general, $E(\mathcal{H}_R)$ is not necessarily a group. However, it contains a subgroup of a nice type.

Recall that a YD H-module algebra A is said to be *quantum commutative* (q.c.) if

$$ab = \sum b_{(0)}(b_{(1)} \cdot a) \tag{4–9}$$

for any $a, b \in A$. That is, A is a commutative algebra in $\mathcal{Q}^H$.

Let X be a q.c. object in $\mathcal{E}(\mathcal{H}_R)$. Let $\overline{X}$ be the opposite algebra in $\mathbf{M}_R^H$. That is, $\overline{X} = X$ as a right H^{op}-comodule, but with the multiplication given by

$$\overline{x} \circ \overline{y} = \sum \overline{y_{(0)} x_{(0)}} R(y_{(1)} \otimes x_{(1)})$$

where $\overline{x}, \overline{y} \in \overline{X}$. Since the H-action on X does not define an H-module algebra structure on $\overline{X}$, we have to define a new H-action on $\overline{X}$ such that $\overline{X}$ together with the inherited H^{op}-comodule structure is a YD H-module algebra. Let H act on $\overline{X}$ as follows:

$$h \rightharpoonup \overline{x} = \sum \overline{h^u_{(3)} \cdot (h_{(2)} \triangleright_2 (h_{(5)} \triangleright_1 x))} R(S(h_{(4)}) \otimes h_{(1)}) \tag{4–10}$$

where $h \in \mathcal{H}_R$, $\overline{x} \in \overline{X}$, $h^u = \sum S(h_{(2)})u^{-1}(h_{(1)})$ and $u = \sum S(R^{(2)})R^{(1)} \in H^*$ is the Drinfel'd element of H^*.

Proposition 4.2.3. *Let X be an object in $\mathcal{E}(\mathcal{H}_R)$ such that X is q.c. Then*:

(1) $\overline{X}$ *together with the H-action* (4–10) *is a YD H-module algebra.*
(2) $\overline{X}$ *is a q.c. object in $\mathcal{E}(\mathcal{H}_R)$ and $X \wedge \overline{X} \cong I = \overline{X} \wedge X$.*

Since the proof is lengthy, we refer reader to [59] for the complete proof. Denote by $\mathrm{Gal}(\mathcal{H}_R)$ the subset of $E(\mathcal{H}_R)$ consisting of the isomorphism classes of objects in $\mathcal{E}(\mathcal{H}_R)$ such that the objects are quantum commutative in $\mathcal{Q}^H$. We have

Theorem 4.2.4 [59, Thm. 3.12]. *The set* $\mathrm{Gal}(\mathcal{H}_R)$ *is a group with product induced by $\wedge$ and inverse operator induced by H-opposite .*

4.3. The exact sequence. For convenience we will call an H-Azumaya algebra A an *R-Azumaya algebra* if the H-action on A is of form (1–5). That is, A represents an element of $\mathrm{BC}(k, H, R)$. In this subsection we investigate the R-Azumaya algebras which are Galois extensions of the coinvariants, and establish a group homomorphism from $\mathrm{BC}(k, H, R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$ constructed in the previous subsection.

In the sequel, we will write:

$$M_0 = \{m \in M \mid \sum m_{(0)} \otimes m_{(1)} = m \otimes 1\}$$

for the coinvariant k-submodule of a right H-comodule M, in order to make a difference between $\mathcal{H}_R^*$-coinvariants and H-coinvariants. We start with a special elementary R-Azumaya algebra.

Lemma 4.3.1. *Let $M = H^{\mathrm{op}}$ be the right H^{op}-comodule, and let A be the elementary R-Azumaya algebra* $\mathrm{End}(M)$. *Then $A \cong H^{*\mathrm{op}} \# H^{\mathrm{op}}$, where the left H^{op}-action on $H^{*\mathrm{op}}$ is given by $h \cdot p = \sum p_{(1)}\langle p_{(2)}, S^{-1}(h)\rangle = S^{-1}(h) \rightharpoonup p$, whenever $h \in H^{\mathrm{op}}$ and $p \in H^{*\mathrm{op}}$.*

Let A be an R-Azumaya algebra. We have $[A\#\mathrm{End}(H^{\mathrm{op}})] = [A]$ since $\mathrm{End}(H^{\mathrm{op}})$ represents the unit of $\mathrm{BC}(k, H, R)$. Now the composite algebra map

$$H^{\mathrm{op}} \xrightarrow{\lambda} \mathrm{End}(H^{\mathrm{op}}) \hookrightarrow A\#\mathrm{End}(H^{\mathrm{op}})$$

is H^{op}-colinear. It follows that $A\#\mathrm{End}(H^{\mathrm{op}})$ is a smash product algebra $B\#H^{\mathrm{op}}$ where $B = (A\#\mathrm{End}(H^{\mathrm{op}}))_0$. Thus we obtain that any element of $\mathrm{BC}(k, H, R)$ can be represented by an R-Azumaya algebra which is a smash product. Since any smash product algebra is a Galois extension of its coinvariants, we have that any element of $\mathrm{BC}(k, H, R)$ can be represented by an R-Azumaya algebra which is an H^{op}-Galois extension of its coinvariants. Moreover, one may easily prove that if A is an R-Azumaya algebra such that it is an H^{op}-Galois extension of A_0, then $\overline{A}$ is a H^{op}-Galois extension of $(A_0)^{\mathrm{op}}$.

An R-Azumaya algebra A is said to be *Galois* if it is a right H^{op}-Galois extension of its coinvariant subalgebra A_0. Let A be a Galois R-Azumaya algebra. Denote by $\pi(A)$ the centraliser subalgebra $C_A(A_0)$ of A_0 in A. It is clear that $\pi(A_0)$ is an H^{op}-comodule subalgebra of A. The *Miyashita–Ulbrich–Van Oystaeyen* (MUVO) *action* (see [34; 48; 50; 51]; the last author mentioned considered it first in the situation of purely inseparable splitting rings in [50]) of H on $\pi(A)$ is given by

$$h \rightharpoonup a = \sum X_i^h a Y_i^h \tag{4–11}$$

where $\sum X_i^h \otimes Y_i^h = \beta^{-1}(1 \otimes h)$, for $h \in H$ and β is the canonical Galois map given by $\beta(a \otimes b) = \sum ab_{(0)} \otimes b_{(1)}$. It is well-known (e.g., see [48; 11]) that $\pi(A)$ together with the MUVO action (4–11) is a new YD H-module algebra. Moreover, $\pi(A)$ is quantum commutative in the sense of (4–9) [48; 52].

Recall that when a Galois H^{op}-comodule algebra A is an Azumaya algebra, the centraliser $\pi(A)$ is a right H^*-Galois extension of k with respect to the MUVO action (4–11); compare [48]. This is not the case when A is an R-Azumaya algebra. However, $\pi(A)$ turns out to be an $\mathcal{H}_R^*$-Galois object, instead of an H^*-Galois object.

Proposition 4.3.2 [59, Prop. 4.5]. *Let A be a Galois R-Azumaya algebra. Then $\pi(A)/k$ is an $\mathcal{H}_R^*$-biextension and $\pi(A)$ is an object in* $\mathrm{Gal}(\mathcal{H}_R)$.

It is natural to expect the functor π to be a monoidal functor from the monoidal category of Galois R-Azumaya algebras to the monoidal category $\mathcal{E}(\mathcal{H}_R)$. This is indeed the case.

Proposition 4.3.3. *π is a monoidal functor. That is*:

(1) *If A and B are two Galois R-Azumaya algebras, then $\pi(A\#B) = \pi(A) \wedge \pi(B)$; and*
(2) *If M is a finite right H^{op}-comodule, and $A = \mathrm{End}(M)$ is the elementary R-Azumaya algebra such that A is a Galois R-Azumaya algebra, then $\pi(A) \cong I$.*

It follows that π induces a group homomorphism $\widetilde{\pi}$ from the Brauer group $\mathrm{BC}(k, H, R)$ to the group $\mathrm{Gal}(\mathcal{H}_R)$ sending element $[A]$ to element $[\pi(A)]$, where A is chosen as a Galois R-Azumaya algebra.

In order to describe the kernel of $\widetilde{\pi}$, one has to analyze the H-coactions on the Galois R-Azumaya algebras. We obtain that the kernel of $\widetilde{\pi}$ is isomorphic to the usual Brauer group $\mathrm{Br}(k)$. Thus we obtain the following exact sequence:

Theorem 4.3.4 [59, Thm. 4.11]. *We have an exact sequence of group homomorphisms*:

$$1 \longrightarrow \mathrm{Br}(k) \xrightarrow{\ \iota\ } \mathrm{BC}(k, H, R) \xrightarrow{\ \widetilde{\pi}\ } \mathrm{Gal}(\mathcal{H}_R). \tag{4–12}$$

Note that the exact sequence (4–12) indicates that the factor group

$$\mathrm{BC}(k, H, R)/\mathrm{Br}(k)$$

is completely determined by the $\mathcal{H}_R^*$-bigalois objects. In particular, when k is an algebraically closed field $\mathrm{BC}(k, H, R)$ is a subgroup of $\mathrm{Gal}(\mathcal{H}_R)$.

Now let us look at some special cases. First let H be a commutative Hopf algebra. H has a trivial coquasitriangular structure $R = \varepsilon \otimes \varepsilon$. In this case. $\mathcal{H}_R$ is equal to H as an algebra and $D[H] = H \otimes H$ is the tensor product algebra. An R-Azumaya algebra is an Azumaya algebra which is a right H-comodule algebra with the trivial left H-action. On the other hand, the $\mathcal{H}_R$-bimodule structures (4–1) and (4–3) of a YD H-module M coincide and are exactly the left H-module structure of M. So in this case an object in the category $\mathcal{E}(\mathcal{H}_R)$ is nothing but an H^*-Galois object which is automatically an H^*-bigalois object since H^* is cocommutative. So the group $\mathrm{Gal}(\mathcal{H}_R)$ is the group $E(H^*)$ of H^*-Galois objects with the cotensor product over H^*. So we obtain the following exact sequence due to Beattie.

Corollary 4.3.5 [3]. *Let H be a finite commutative Hopf algebra. Then the following group sequence is exact and split*:

$$1 \longrightarrow \mathrm{Br}(k) \xrightarrow{\iota} \mathrm{BC}(k, H) \xrightarrow{\widetilde{\pi}} E(H^*) \longrightarrow 1$$

where the group map $\widetilde{\pi}$ is surjective and split because any H^*-Galois object B is equal to $\pi(B\#H)$ and the smash product $B\#H$ is a right H-comodule Azumaya algebra which represents an element in $\mathrm{BC}(k, H)$.

Secondly we let R be a non-trivial coquasitriangular structure of H, but let H be a commutative and cocommutative finite Hopf algebra over k. In this case, $\mathcal{H}_R$ is isomorphic to H as an algebra and becomes a Hopf algebra. In this case, an object in $\mathrm{Gal}(\mathcal{H}_R)$ is an H^*-bigalois object. It is not difficult to check that YD H-module (or H-dimodule) structures commute with both H^*-Galois structures.

Let θ be the Hopf algebra homomorphism corresponding to the coquasitriangular structure R, that is,

$$\theta : H \longrightarrow H^*, \quad \theta(h)(l) = R(l \otimes h)$$

for $h, l \in H$. Let $\rightharpoonup$ be the induced H-action on a right H-comodule M:

$$h \rightharpoonup m = \sum m_{(0)} \theta(h)(m_{(1)}) = \sum m_{(0)} R(m_{(1)} \otimes h)$$

for $h \in H$ and $m \in M$. In [49], Ulbrich constructed a group $D(\theta, H^*)$ consisting of isomorphism classes of H^*-bigalois objects which are also H-dimodule algebras such that all H and H^* structures commute, and satisfy the following additional conditions interpreted by means of R [49, (14), (16)]:

$$\begin{aligned} &h \mathbin{-\!\!\triangleright} a = \textstyle\sum a_{(0)} \mathbin{\triangleleft\!\!-} h_{(1)} R(a_{(1)} \otimes S(h_{(2)})) R(S(h_{(3)}) \otimes a_{(2)}) \\ &\textstyle\sum x_{(0)} (a \mathbin{\triangleleft\!\!-} x_{(1)}) = \sum (x_{(1)} \rightharpoonup a) x_{(0)}, \end{aligned} \tag{4–13}$$

Let us check that any object A in the category $\mathcal{E}(\mathcal{H}_R)$ satisfies the conditions (4–13) so that A represents an element of $D(\theta, H^*)$. Indeed, since H is commutative and cocommutative, we have

$$\begin{aligned} h \multimap a &= \sum (h_{(2)} \cdot a_{(0)}) R(S^{-1}(h_{(4)}) \otimes h_{(3)} a_{(1)} S^{-1}(h_{(1)})) \\ &= \sum (h_{(1)} \cdot a_{(0)}) R(S(h_{(2)}) \otimes a_{(1)}) \\ &= \sum (h_{(2)} \cdot a_{(0)}) R(a_{(1)} \otimes h_{(1)}) R(a_{(2)} \otimes S(h_{(3)})) R(S(h_{(4)}) \otimes a_{(3)}) \\ &= \sum (a_{(0)} \triangleleft\!- h_{(1)}) R(a_{(1)} \otimes S(h_{(2)})) R(S(h_{(3)}) \otimes a_{(2)}), \end{aligned}$$

and

$$\begin{aligned} \sum x_{(0)}(a \triangleleft\!- x_{(1)}) &= \sum x_{(0)}(x_{(1)} \cdot a_{(0)}) R(a_{(1)} \otimes x_{(1)}) \\ &= \sum a_{(0)} x_{(0)} R(a_{(1)} \otimes x_{(1)}) \quad \textit{(by q.c.)} \\ &= \sum (x_{(1)} \rightharpoonup a) x_{(0)} \end{aligned}$$

for any $a, x \in A$ and $h \in H$. It follows that the group $\mathrm{Gal}(\mathcal{H}_R)$ is contained in $D(\theta, H^*)$. As a consequence, we obtain Ulbrich's exact sequence [49, 1.10]:

$$1 \longrightarrow \mathrm{Br}(k) \longrightarrow \mathrm{BD}(\theta, H^*) \xrightarrow{\pi_\theta} D(\theta, H^*)$$

for a commutative and cocommutative finite Hopf algebra with a Hopf algebra homomorphism θ from H to H^*.

4.4. An example. Let k be a field with characteristic different from two. Let H_4 be the Sweedler four dimensional Hopf algebra over k. That is, H_4 is generated by two elements g and h satisfying

$$g^2 = 1, h^2 = 0, gh + hg = 0.$$

The comultiplication, the counit and the antipode are as follows:

$$\begin{aligned} &\Delta(g) = g \otimes g, \Delta(h) = 1 \otimes h + h \otimes g, \\ &\varepsilon(g) = 1, \qquad \varepsilon(h) = 0, \\ &S(g) = g, \qquad S(h) = gh. \end{aligned}$$

There is a family of CQT structures R_t on H_4 parameterized by $t \in k$ as follows:

R_t	1	g	h	gh
1	1	1	0	0
g	1	-1	0	0
h	0	0	t	$-t$
gh	0	0	t	t

It is not hard to check that the Hopf algebra homomorphisms Θ_l and Θ_r induced by R_t are as follows:

$$\Theta_l : H_4^{\text{cop}} \longrightarrow H_4^*, \quad \Theta_l(g) =, \overline{1} - \overline{g}, \quad \Theta_l(h) = t(\overline{h} - \overline{gh}),$$
$$\Theta_r : H_4^{\text{op}} \longrightarrow H_4^*, \quad \Theta_r(g) = \overline{1} - \overline{g}, \quad \Theta_r(h) = t(\overline{h} + \overline{gh}).$$

When t is non-zero, Θ_l and Θ_r are isomorphisms, so that H_4 is a self-dual Hopf algebra.

We have that $\mathcal{H}_{R_t}$ is a 4-dimensional algebra generated by two elements u and v such that u and v satisfy the relations:

$$u^2 = 1, uv - vu = 0, v^2 = t(1-u),$$

which is isomorphic to the commutative algebra $k[y]/\langle y^4 - 2ty^2\rangle$ when t is not zero.

The double algebra $D[H_4]$ with respect to R_t is generated by four elements, g_1, g_2, h_1 and h_2 such that

$$g_i^2 = 1,\ h_i^2 = 0, g_ih_j + h_jg_i = 0,$$
$$g_1g_2 = g_2g_1, \qquad h_1h_2 + h_2h_1 = t(1 - g_1g_2).$$

The comultiplication of $D[H_4]$ is easy because the Hopf subalgebras generated by g_i, h_i, $i = 1, 2$, are isomorphic to H_4. Thus the algebra embedding ϕ reads as follows:

$$\mathcal{H}_{R_t} \longrightarrow D[H_4], \quad \phi(u) = g_1g_2, \phi(v) = g_1(h_2 - h_1).$$

Let us consider the triangular case (H_4, R), where $R = R_0$. In this case, an algebra A is an H_4-module algebra if and only if it a DS-algebra (see section 2), and A is R-Azumaya algebra if and only if A is a DS-Azumaya algebra. From Theorem 2.5 we know that the Brauer group $\mathrm{BC}(k, H_4, R)$ is isomorphic to $(k, +) \times \mathrm{BW}(k)$. Let us work out the group $\mathrm{Gal}(\mathcal{H}_R)$ and calculate the Brauer group $\mathrm{BC}(k, H_4, R)$ using the sequence (4–12).

First of all we have the following structure theorem of bigalois objects in $\mathcal{E}(\mathcal{H}_R)$ [59].

THEOREM 4.4.1 [59, Thm. 5.7]. *Let A be a bigalois object in $\mathcal{E}(\mathcal{H}_R)$. Then A is either of type (A) or of type (B)*:

Type (A): *A is a generalized quaternion algebra $\left(\frac{\alpha,\beta}{k}\right)$, $\alpha \neq 0$, with the following YD H-module structures*:

$$\begin{aligned} &g \cdot u = -u, && g \cdot v = -v,\\ &h \cdot u = 0, && h \cdot v = 1,\\ &\rho(u) = u \otimes 1 - 2uv \otimes gh, && \rho(v) = v \otimes g + 2\beta \otimes h.\end{aligned}$$

Type (B): *A is a commutative algebra $k(\sqrt{\alpha})\otimes k(\sqrt{\beta})$ with the following YD H-module structures*:

$$\begin{aligned} &g\cdot u = u, && g\cdot v=-v,\\ &h\cdot u=0, && h\cdot v=1,\\ &\rho(u)=u\otimes 1+2uv\otimes h, && \rho(v)=v\otimes g+2\beta\otimes h, \end{aligned}$$

where $k(\sqrt{\alpha})$ and $k(\sqrt{\beta})$ are generated by elements u and v respectively, and u, v satisfy the relations: $u^2=\alpha, uv=vu$ *and* $v^2=\beta$.

As a consequence of Theorem 4.4.1, we have the group structure of the group $\mathrm{Gal}(\mathcal{H}_R)$:

Proposition 4.4.2 [59, 5.8–5.9]. *The group* $\mathrm{Gal}(\mathcal{H}_R)$ *is equal to* $k\times(k^\bullet/k^{\bullet 2})\times\mathbb{Z}_2$ *as a set. The multiplication rule on the set is given by*

$$(\beta,\alpha,i)(\beta',\alpha',j)=(\beta+\beta',(-1)^{ij}\alpha\alpha',i+j).$$

The foregoing multiplication rule of $\mathrm{Gal}(\mathcal{H}_R)$ shows that $\mathrm{Gal}(\mathcal{H}_R)$ is a direct product of $(k,+)$ and the group $k^\bullet/k^{\bullet 2}\rtimes\mathbb{Z}_2$ which is isomorphic to the group $Q_2(k)$ of graded quadratic extensions of k (see [57]).

Notice that an object of type (A) in $\mathrm{Gal}(\mathcal{H}_R)$ is some generalized quaternion algebra $\left(\frac{\alpha,\beta}{k}\right)$ with the H_4-action and coaction given in Theorem 4.4.1, where $\alpha\in k^\bullet$ and $\beta\in k$. When $\beta\neq 0$, $\left(\frac{\alpha,\beta}{k}\right)$ is a Galois R-Azumaya algebra if we forget the left H_4-module structure. Since the coinvariant subalgebra of $\left(\frac{\alpha,\beta}{k}\right)$ is trivial, we have $\pi(\left(\frac{\alpha,\beta}{k}\right))=\left(\frac{\alpha,\beta}{k}\right)$ if $\beta\neq 0$. To get an object $\left(\frac{\alpha,0}{k}\right)$ in $\mathrm{Gal}(\mathcal{H}_R)$, where $\alpha\in k^\bullet$, we consider the Galois R-Azumaya algebra $\left(\frac{\alpha,1}{k}\right)\#\left(\frac{1,-1}{k}\right)$. Since π is monoidal, we have

$$\pi\left(\left(\frac{\alpha,1}{k}\right)\#\left(\frac{1,-1}{k}\right)\right)=\left(\frac{\alpha,1}{k}\right)\wedge\left(\frac{1,-1}{k}\right)=\left(\frac{\alpha,0}{k}\right)$$

for any $\alpha\in k^\bullet$.

For an object $k(\sqrt{\alpha})\otimes k(\sqrt{\beta})$ of type (B) in $\mathrm{Gal}(\mathcal{H}_R)$, we choose a Galois R-Azumaya algebra A such that $\pi(A)=\left(\frac{\alpha,\beta}{k}\right)$ (assured by the foregoing arguments). Then it is easy to check that

$$\pi(A\# k(\sqrt{1}))=k(\sqrt{\alpha})\otimes k(\sqrt{\beta})$$

for $\alpha\in k^\bullet$ and $\beta\in k$. Thus we have shown that the homomorphism $\tilde{\pi}$ is surjective and we have an exact sequence:

$$1\longrightarrow \mathrm{Br}(k)\longrightarrow \mathrm{BC}(k,H_4,R)\xrightarrow{\tilde{\pi}}\mathrm{Gal}(\mathcal{H}_R)\longrightarrow 1. \tag{4–14}$$

Recall that the Brauer–Wall group $\mathrm{BW}(k)$ is $\mathrm{BC}(k,k\mathbb{Z}_2,R')$, where $k\mathbb{Z}_2$ is the sub-Hopf algebra of H_4 generated by the group-like element $g\in H_4$, and R' is the restriction of R to $k\mathbb{Z}_2$. The following well-known exact sequence is a special case of (4–12):

$$1\longrightarrow \mathrm{Br}(k)\longrightarrow \mathrm{BW}(k)\xrightarrow{\tilde{\pi}} Q_2(k)\longrightarrow 1, \tag{4–15}$$

where $Q_2(k)$ is nothing but $\mathrm{Gal}(\mathcal{H}_{R'})$ and $\mathcal{H}_{R'} \cong k\mathbb{Z}_2$, here $H = k\mathbb{Z}_2$.

The sequence (4–15) can be also obtained if we restrict the homomorphism $\tilde{\pi}$ in (4–14) to the subgroup $\mathrm{BW}(k)$ of $\mathrm{BC}(k, H_4, R)$. The subgroup $\tilde{\pi}(\mathrm{BW}(k))$ of $\mathrm{Gal}(\mathcal{H}_R)$ consists of all objects of forms: $\left(\frac{\alpha,0}{k}\right)$ of type (A) and $k(\sqrt{\alpha}) \otimes k(\sqrt{0})$ of type (B), which is isomorphic to $Q_2(k)$ (see [59] for details). In fact we have the following commutative diagram:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Br}(k) \cap K & \longrightarrow & K & \xrightarrow{\tilde{\pi}} & (k,+) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \iota} & & \\
1 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{BC}(k, H_4, R) & \xrightarrow{\tilde{\pi}} & \mathrm{Gal}(\mathcal{H}_{R0}) & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle p} & & \\
1 & \longrightarrow & \mathrm{Br}(k) & \longrightarrow & \mathrm{BW}(k) & \xrightarrow{\tilde{\pi}} & Q_2(k) & \longrightarrow & 1,
\end{array}
$$

where γ is the canonical map defined in Theorem 2.5, K is the kernel of γ, ι is the inclusion map and p is the projection from $(k,+) \times Q_2(k)$ onto $Q_2(k)$. Here $\tilde{\pi}(K) = (k,+)$ because $\tilde{\pi} \circ \gamma = p \circ \tilde{\pi}$ (which can be easily checked on Galois R-Azumaya algebras $\left(\frac{\alpha,\beta}{k}\right)$ and $\left(\frac{\alpha,\beta}{k}\right)\#k(\sqrt{1})$). By definition of γ we have $\mathrm{Br}(k) \cap K = 1$. It follows that $K \cong (k,+)$. Since γ is split, we obtain that the Brauer group $\mathrm{BC}(k, H_4, R)$ is isomorphic to the direct product group $(k,+) \times \mathrm{BW}(k)$, which coincides with Theorem 2.5.

Recently, G. Carnovale proved in [13] that the Brauer group $\mathrm{BC}(k, H_4, R_t)$ is isomorphic to $\mathrm{BC}(k, H_4, R_0)$ for any $t \neq 0$ although (H_4, R_t) is not coquasitriangularly isomorphic to (H_4, R_0) when $t \neq 0$ [40].

5. The Hopf Automorphism Group

Let H be a faithfully projective Hopf algebra over a commutative ring k. As we have seen from the previous section, the Brauer group $\mathrm{BQ}(k, H)$ may be approximated by computing the group $\mathrm{Gal}(\mathcal{H}_R)$, where $\mathcal{H}_R$ is a deformation of the dual $D(H)^*$ of the quantum double $D(H)$. However, to compute explicitly the group $\mathrm{BQ}(k, H)$ is a hard task. On the other hand, there are some subgroups of $\mathrm{BQ}(k, H)$ which are (relatively) easier to calculate. For instance, when H is commutative and cocommutative, various subgroups of the Brauer–Long group could more easily be studied [3; 4; 7; 8; 10; 17]. One of these subgroups is Deegan's subgroup introduced in [17] which involves the Hopf algebra structure of H itself and in fact turns out to be isomorphic to the Hopf algebra automorphism group $\mathrm{Aut}(H)$ [17; 8]. The connection between $\mathrm{Aut}(H)$ and $\mathrm{BD}(k, H)$ for some particular commutative and cocommutative Hopf algebra H was probably first studied by M.Beattie in [3] where she established the existence of an exact sequence:

$$1 \longrightarrow \mathrm{BC}(k,G)/\mathrm{Br}(k) \times \mathrm{BM}(k,G)/\mathrm{Br}(k) \longrightarrow \mathrm{B}(k,G)/\mathrm{Br}(k) \xrightarrow{\boldsymbol{\beta}} \mathrm{Aut}(G) \longrightarrow 1$$

where $\mathrm{B}(k,G)$ is the subgroup of $\mathrm{BD}(k,G)$ consisting of the classes represented by G-dimodule Azumaya algebras whose underlying algebras are Azumaya, $\mathrm{Aut}(G)$ is the automorphism group of G, G is a finite abelian group and k is a connected ring. Based on Beattie's construction of the map $\boldsymbol{\beta}$, Deegan constructed his subgroup $\mathrm{BT}(k,G)$ which is then isomorphic to $\mathrm{Aut}(G)$; the resulting embedding of $\mathrm{Aut}(G)$ in the Brauer–Long group (group case) is known as Deegan's embedding theorem. In [8], S.Caenepeel looked at the Picard group of a Hopf algebra, and extended Deegan's embedding theorem from abelian groups to commutative and cocommutative Hopf algebras. But if H is a quantum group (i.e., a (co)quasitriangular Hopf algebra) or just any non-commutative non-cocommutative Hopf algebra then it seems that the method of Deegan and Caenepeel cannot be extended to obtain a group homomorphism from some subgroup of $\mathrm{BQ}(k,H)$ to the automorphism group $\mathrm{Aut}(H)$. In fact, $\mathrm{Aut}(H)$ can no longer be embedded into $\mathrm{BQ}(k,H)$. On the other hand, the idea of Deegan's construction can still be applied to our non-commutative and non-cocommutative case.

Let M be a faithfully projective Yetter–Drinfel'd H-module. Then $\mathrm{End}_k(M)$ is an H-Azumaya YD H-module algebra. However, if M is an H-bimodule, that is, a left H-module and a right H-comodule, but not a YD H-module, it may still happen that $\mathrm{End}_k(M)$ is a YD H-module algebra.

Take a non-trivial Hopf algebra isomorphism $\alpha \in \mathrm{Aut}(H)$ (for example, if the antipode S of H is not of order two, S^2 is a non-trivial Hopf automorphism). We define a left H-module and a right H-comodule H_α as follows: as a k-module $H_\alpha = H$; we equip H_α with the obvious H-comodule structure given by Δ, and an H-module structure given by

$$h \cdot x = \sum \alpha(h_{(2)}) x S^{-1}(h_{(1)})$$

for $h \in H$, $x \in H_\alpha$. Since α is nontrivial H_α is not a YD H-module. Let $A_\alpha = \mathrm{End}(H_\alpha)$ with H-structures induced by the H-structures of H_α, that is,

$$(h \cdot f)(x) = \sum h_{(1)} f(S(h_{(2)}) \cdot x)$$
$$\chi(f)(x) = \sum f(x_{(0)})_{(0)} \otimes S^{-1}(x_{(1)}) f(x_{(0)})_{(1)}$$

for $f \in A_\alpha$, $x \in H_\alpha$.

Lemma 5.1 [12, 4.6, 4.7]. *If H is a faithfully projective Hopf algebra and α is a Hopf algebra automorphism of H, then A_α is an Azumaya YD-module algebra and the following map defines a group homomorphism*:

$$\omega : \mathrm{Aut}(H) \longrightarrow \mathrm{BQ}(k,H), \quad \alpha \mapsto [A_{\alpha^{-1}}].$$

In the sequel, we will compute the kernel of the map ω. Let $D(H)$ denote the Drinfel'd double of H. Let A be an H-module algebra. Recall from [5] that the H-action on A is said to be *strongly inner* if there is an algebra map $f : H \longrightarrow A$ such that

$$h \cdot a = \sum f(h_{(1)}) a f(S(h_{(2)})), \quad a \in A,\ h \in H.$$

LEMMA 5.2. *Let M be a faithfully projective k-module. Suppose that* $\mathrm{End}(M)$ *is a $D(H)$-Azumaya algebra. Then* $[\mathrm{End}(M)] = 1$ *in* $\mathrm{BM}(k, D(H))$ *if and only if the $D(H)$-action on A is strongly inner.*

The proof has its own interest. Suppose that the $D(H)$-action on A is strongly inner. There is an algebra map $f : D(H) \to A$ such that $t\cdot a = \sum f(t_{(1)}) a f(S(t_{(2)}))$, $t \in D(H), a \in A$. This inner action yields a $D(H)$-module structure on M given by

$$t \rightharpoonup m = f(t)(m),\ t \in D(H), m \in M.$$

Since f is an algebra representation map the above action does define a module structure. Now it is straightforward to check that the $D(H)$-module structure on A is exactly induced by the $D(H)$-module structure on M defined above. By definition $[\mathrm{End}(M)] = 1$ in $\mathrm{BM}(k, D(H))$.

Conversely, if $[A] = 1$, then there exists a faithfully projective $D(H)$-module N such that $A \cong \mathrm{End}(N)$ as $D(H)$-module algebras by [12, 2.11]. Now $D(H)$ acts strongly innerly on $\mathrm{End}(N)$. Let $u : D(H) \longrightarrow \mathrm{End}(N)$ be the algebra representation map. Now one may easily verify that the strongly inner action induced by the composite algebra map:

$$\mu : D(H) \xrightarrow{u} \mathrm{End}(N) \cong A$$

exactly defines the $D(H)$-module structure on A.

LEMMA 5.3. *For a faithfully projective k-module M, let $u, v : H \longrightarrow \mathrm{End}(M)$ define H-module structures on M, call them M_u and M_v. If* $\mathrm{End}(M_u) = \mathrm{End}(M_v)$ *as left H-modules via* (1–2), *then $(v \circ S) * u$ is an algebra map from H to k, i.e., a grouplike element in H^*. Similarly, if M admits two H-comodule structures ρ, χ such that the induced H-comodule structures on* $\mathrm{End}(M)$ *given by* (1–2) *coincide, then there is a grouplike element $g \in G(H)$ such that $\chi = (1 \otimes g)\rho$, i.e., $\chi(x) = \sum x_{(0)} \otimes g x_{(1)}$ if $\rho(x) = \sum x_{(0)} \otimes x_{(1)}$ for $x \in M$.*

PROOF. For any $m \in M, h \in H, \phi \in \mathrm{End}(M_u) = \mathrm{End}(M_v)$,

$$\sum u(h_{(1)})[\phi[u(S(h_{(2)}))(m)]] = \sum v(h_{(1)})[\phi[v(S(h_{(2)}))(m)]],$$

or equivalently,

$$\sum v(S(h_{(1)}))[u(h_{(2)})(\phi[u(S(h_{(3)}))(m)])] = \phi(v(S(h))(m)).$$

Let $\lambda = (v \circ S) * u : H \longrightarrow \mathrm{End}(M)$ with convolution inverse $(u \circ S) * v$. Letting $m = u(h_{(4)})(x)$ for any $x \in H$ in the equation above, we obtain $\lambda(h) \in$

$Z(\mathrm{End}(M)) = k$ for all $h \in H$. Since u, v are algebra maps, it is easy to see that λ is an algebra map from H to k. □

Given a group-like element $g \in G(H)$, g induces an inner Hopf automorphism of H denoted $\overline{g}$, i.e., $\overline{g}(h) = g^{-1}hg, h \in H$. Similarly, if λ is a group-like element of H^*, then λ induces a Hopf automorphism of H, denoted by $\overline{\lambda}$ where $\overline{\lambda}(h) = \sum \lambda(h_{(1)})h_{(2)}\lambda^{-1}(h_{(3)}),\ h \in H$. Since $G(D(H)) = G(H^*) \times G(H)$ ([40, Prop.9]) and $\overline{g}$ commutes with $\overline{\lambda}$ in $\mathrm{Aut}(H)$, we have a homomorphism θ:

$$G(D(H)) \longrightarrow \mathrm{Aut}(H), \quad (\lambda, g) \mapsto \overline{g}\overline{\lambda}.$$

Let $K(H)$ denote the subgroup of $G(D(H))$ consisting of elements

$$\{(\lambda, g) \mid \overline{g^{-1}}(h) = \overline{\lambda}(h), \forall h \in H\}.$$

Lemma 5.4. *Let H be a faithfully projective Hopf algebra. Then $K(H) \cong G(D(H)^*)$.*

Proof. By [40, Prop.10], an element $g \otimes \lambda$ is in $G(D(H)^*)$ if and only if $g \in G(H), \lambda \in G(H^*)$ and g, λ satisfy the identity:

$$g(\lambda \rightharpoonup h) = (h \leftharpoonup \lambda)g,\ \forall h \in H,$$

where, $\lambda \rightharpoonup h = \sum h_{(1)}\lambda(h_{(2)})$ and $h \leftharpoonup \lambda = \sum h_{(2)}\lambda(h_{(1)})$. Let $g \in G(H), \lambda \in G(H^*)$, for any $h \in H$, we have

$$\begin{aligned}\sum gh_{(1)}\lambda(h_{(2)}) = \sum \lambda(h_{(1)})h_{(2)}g &\Longleftrightarrow \sum h_{(1)}\lambda(h_{(2)}) = \sum \lambda(h_{(1)})g^{-1}h_{(2)}g \\ &\Longleftrightarrow \sum \lambda^{-1}(h_{(1)})h_{(2)}\lambda(h_{(3)}) = \sum g^{-1}hg.\end{aligned}$$

This means $g \otimes \lambda$ is in $G(D(H)^*)$ if and only if $(\lambda, g) \in K(H)$. Therefore $K(H) = G(D(H)^*)$. □

Applying Lemmas 5.1–5.4, one may able to show that the group homomorphism θ can be embedded into the following long exact sequence:

Theorem 5.5 [52, Thm. 5]. *Let H be a faithfully projective Hopf algebra over k. The following sequence is exact*:

$$1 \longrightarrow G(D(H)^*) \longrightarrow G(D(H)) \xrightarrow{\theta} \mathrm{Aut}(H) \xrightarrow{\omega} \mathrm{BQ}(k, H), \tag{5–1}$$

where $\theta(\lambda, g) = \overline{\lambda}\overline{g}$ and $\omega(\alpha) = A_{\alpha^{-1}} = \mathrm{End}(H_{\alpha^{-1}})$.

As a consequence of the theorem, we rediscover the Deegan–Caenepeel's embedding theorem for a commutative and cocommutative Hopf algebra [8; 17].

Corollary 5.6. *Let H be a faithfully projective Hopf algebra such that $G(H)$ and $G(H^*)$ are contained in the centers of H and H^* respectively. Then the map ω in the sequence* (5–1) *is a monomorphism. In particular, if H is a commutative and cocommutative faithfully projective Hopf algebra over k, then* $\mathrm{Aut}(H)$ *can be embedded into* $\mathrm{BQ}(k, H)$.

Note that in this case, $G(D(H)^*) = G(D(H))$. *It follows that the homomorphism* θ *is trivial*, *and hence the homomorphism* ω *is a monomorphism*.

In the following, we present two examples of the exact sequence (5–1).

Example 5.7. Let H be the Sweedler Hopf algebra over a field k in Subsection 4.4. H is a self-dual Hopf algebra, i.e., $H \cong H^*$ as Hopf algebras. It is straightforward to show that the Hopf automorphism group $\mathrm{Aut}(H)$ is isomorphic to $k^\bullet = k\backslash 0$ via:

$$f \in \mathrm{Aut}(H), f(g) = g, f(h) = zh, z \in k^\bullet.$$

Considering the group $G(D(H))$ of group-like elements, it is easy to see that

$$G(D(H)) = \{(\varepsilon, 1), (\lambda, 1), (\varepsilon, g), (\lambda, g)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

where $\lambda = p_1 - p_g$, and p_1, p_g is the dual basis of $1, g$. One may calculate that the kernel of the map θ is given by:

$$K(H) = \{(\varepsilon, 1), (\lambda, g)\} \cong \mathbb{Z}_2$$

The image of θ is $\{\overline{1}, \overline{g}\}$ which corresponds to the subgroup $\{1, -1\}$ of k^*. Thus by Theorem 5.5 we have an exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow k^\bullet \longrightarrow \mathrm{BQ}(k, H),$$

It follows that $k^\bullet/\mathbb{Z}_2$ can be embedded into the Brauer group $\mathrm{BQ}(H)$. In particular, if $k = \mathcal{R}$, the real field, then $Br(\mathcal{R}) = \mathbb{Z}_2 \subset \mathrm{BQ}(\mathcal{R}, H)$, and $\mathcal{R}^\bullet/\mathbb{Z}_2$ is a non-torsion subgroup of $\mathrm{BQ}(\mathcal{R}, H)$.

In the previous example, the subgroup $k^\bullet/\mathbb{Z}_2$ of the Brauer group $\mathrm{BQ}(k, H)$ is still an abelian group. The next example shows that the general linear group $GL_n(k)$ modulo a finite group of roots of unity for any positive number n may be embedded into the Brauer group $\mathrm{BQ}(k, H)$ of some finite dimensional Hopf algebra H.

Example 5.8. Let $m > 2, n$ be any positive numbers. Let H be Radford's Hopf algebra of dimension $m2^{n+1}$ over $\mathbb{C}$ (complex field) generated by $g, x_i, 1 \le i \le n$ such that

$$g^{2m} = 1, {x_i}^2 = 0, gx_i = -x_i g, x_i x_j = -x_j x_i.$$

The coalgebra structure Δ and the counit ε are given by

$$\Delta g = g \otimes g, \quad \Delta x_i = x_i \otimes g + 1 \otimes x_i, \quad \varepsilon(g) = 1, \quad \varepsilon(x_i) = 0, \quad 1 \le i \le n.$$

By [40, Prop.11], the Hopf automorphism group of H is $GL_n(\mathbb{C})$. Now we compute the groups $G(D(H))$ and $G(D(H)^*)$. It is easy to see that $G(H) = (g)$ (see also [40, p353]) is a cyclic group of order $2m$. Let $\omega_i, 1 \le i \le m$ be the m-th roots of 1, and let ζ_j be the *m-th* roots of -1. Define the algebra maps η_i and λ_i from H to $\mathbb{C}$ as follows:

$$\eta_i(g) = \omega_i g, \quad \eta_i(x_j) = 0, 1 \le i, j \le m,$$

and

$$\lambda_i(g) = \zeta_i g, \quad \lambda_i(x_j) = 0, 1 \le i, j \le m.$$

One may check that $\{\eta_i, \lambda_i\}_{i=1}^n$ is the group $G(H^*)$. It follows that $G(D(H)) = G(H) \times G(H^*) \cong (g) \times U$, where U is the group of $2m$-th roots of 1. To compute $G(D(H)^*)$ it is enough to calculate $K(H)$. Since

$$\overline{g^i} = \begin{cases} \text{id if } i \text{ is even,} \\ \nu \ \text{ if } i \text{ is odd,} \end{cases}$$

where $\nu(g) = g, \quad \nu(x_j) = -x_j, 1 \le j \le n$, and

$$\overline{\eta_i}(g) = g, \quad \overline{\eta_i}(x_j) = \omega_i x_j, \quad 1 \le i, j \le n,$$
$$\overline{\lambda_i}(g) = g, \quad \overline{\lambda_i}(x_j) = \zeta_i x_j, \quad 1 \le i, j \le n.$$

It follows that

$$K(H) = \{(\varepsilon, g^{2i}), (\psi, g^{2i-1}), 1 \le i \le m\},$$

where ψ is is given by:

$$\psi(g) = -g, \quad \psi(x_i) = 0.$$

Consequently $G(D(H)^*) \cong U$, Since the base field is $\mathbb{C}$, $(g) \cong U$, and we have an exact sequence

$$1 \longrightarrow U \longrightarrow U \times U \longrightarrow GL_n(\mathbb{C}) \longrightarrow \mathrm{BQ}(\mathbb{C}, H).$$

The two examples above highlight the interest of the study of the Brauer group of a Hopf algebra. In Example 5.8, even though the classical Brauer group $Br(\mathbb{C})$ is trivial, the Brauer group $\mathrm{BQ}(\mathbb{C}, H)$ is still large enough.

In the rest of this section, we consider a natural action of $\mathrm{Aut}(H)$ on $\mathrm{BQ}(k, H)$. Let A be an H-Azumaya algebra, and α a Hopf algebra automorphism of H. Consider the YD H-module algebra $A(\alpha)$, which equals A as a k-algebra, but with H-structures of $(A(\alpha), \rightharpoonup, \chi')$ given by

$$h \rightharpoonup a = \alpha(h) \cdot a \text{ and } \chi'(a) = \sum a_{(0)} \otimes \alpha^{-1}(a_{(1)}) = (1 \otimes \alpha^{-1})\chi(a)$$

for all $a \in A(\alpha), \ h \in H$.

Lemma 5.9. *Let A, B be H-Azumaya algebras. If α is a Hopf algebra automorphism of H, then $A(\alpha)$ is an H-Azumaya algebra, and $(A\#B)(\alpha) \cong A(\alpha)\#B(\alpha)$.*

The action of $\mathrm{Aut}(H)$ on $\mathrm{BQ}(k, H)$ is an inner action. Indeed, we have:

Theorem 5.10 [12, Thm. 4.11]. $\mathrm{Aut}(H)$ *acts innerly on* $\mathrm{BQ}(k, H)$, *more precisely, for any H-Azumaya algebra B and $\alpha \in \mathrm{Aut}(H)$, we have* $[B(\alpha)] = [A_\alpha][B][A_{\alpha^{-1}}]$.

Theorem 5.10 yields the multiplication rule for two elements $[B]$ and $\omega(\alpha) = [A_\alpha]$ where B is any H-Azumaya algebra and α is in $\mathrm{Aut}(H)$. In particular, if T is a subgroup such that T is invariant (or stable) under the action of $\mathrm{Aut}(H)$, then the subgroup generated by T and $\omega(\mathrm{Aut}(H))$ is $(\iota \otimes \omega)(T \rtimes \mathrm{Aut}(H))$, where $\rtimes$ is the usual semi-direct product of groups.

Example 5.11. Let (H_4, R_0) be the CQT Hopf algebra described in Subsection 4.4. The automorphism group $\mathrm{Aut}(H_4)$ of H_4 is isomorphic to $k^\bullet$. If A is an R_0-Azumaya algebra and $\alpha \in \mathrm{Aut}(H_4)$, then $A(\alpha)$ is still an R_0-Azumaya algebra as the automorphism α does not affect the induced action (1–5). Thus the subgroup $\mathrm{BC}(k, H_4, R_0)$ is stable under the action of $\mathrm{Aut}(H_4)$. By Example 5.7 we get a non-abelian subgroup of $\mathrm{BQ}(k, H_4)$ (see [56]):

$$\begin{aligned}(\iota \otimes \omega)(\mathrm{BC}(k, H_4, R_0) \rtimes k^\bullet) &\cong \mathrm{BC}(k, H_4, R_0) \rtimes (k^\bullet/\mathbb{Z}_2)\\ &\cong \mathrm{BW}(k) \times (k, +) \rtimes (k^\bullet/\mathbb{Z}_2).\end{aligned}$$

6. The Second Brauer Group

In the classical Brauer group theory of a commutative ring k, an Azumaya algebra can be characterized as a central separable algebra over k. However this is not the case when we deal with the Brauer group of structured algebras. For instance, in the Brauer–Wall group of a commutative ring k, a representative (i.e., a $\mathbb{Z}_2$-graded Azumaya algebra) is not necessarily a central separable algebra, instead it is a graded central and graded separable algebra. Motivated by the example of the Brauer–Wall group, one may be inspired to try to define the Brauer group of a Hopf algebra H by using H-separable algebras in a natural way as we did for the Brauer-long group of $\mathbb{Z}_2$-dimodule algebras in section 2. In 1974, B. Pareigis for the first time defined two Brauer groups in a symmetrical category (see [38]). The first Brauer group was defined in terms of Morita equivalence whereas the second Brauer group was defined by so called ‘central separable algebras’ in the category. The two Brauer groups happen to be equal if the unit of the symmetrical category is a projective object. This is the case if the category is the $\mathbb{Z}_2$-graded module category with the graded product (2–2).

However the dimodule category of a finite abelian group (or a finite commutative cocommutative Hopf algebra) is not a symmetric category, instead a braided monoidal category ([32]). Therefore, the definition of the second Brauer group due to Pareigis can not be applied to the dimodule category. Nevertheless, we are able to modify Pareigis's definitions to get the proper definitions of the two Brauer groups for a braided monoidal category (see [52]) so that they allow to recover all known Brauer groups. We are not going into the details of the categorical definitions given in [52]. Instead we will focus our attention on the Yetter Drinfel'd module category of a Hopf algebra.

Like the Brauer–Wall group $\mathrm{BW}(k)$, the Brauer–Long group $\mathrm{BD}(k, H)$ of a finite commutative and cocommutative Hopf algebra H over a commutative ring

k can be defined by central separable algebras in the category when k is nice (e.g., k is a field with characteristic 0. However, a counter example exists when k is not so nice, e.g., $\mathrm{BD}(k, \mathbb{Z}_2)$ when 2 is not a unit in k (see [7]).

When a Hopf algebra is not commutative and cocommutaive, even if k is a field with $\mathrm{ch}(k) = 0$, the second Brauer group defined by H-separable algebras turns out to be smaller than the Brauer group of H-Azumaya algebras. In other words, an H-Azumaya algebra is not necessarily an H-separable algebra. Some examples of this will be presented. Let H be a Hopf algebra over k and let A^e (or ${}^eA$) be the H-enveloping algebra $A\#\overline{A}$ (or $\overline{A}\#A$).

Definition 6.1. Let A be a YD H-module algebra. A is said to be *H-separable* if the following exact sequence splits in ${}_{A^e}\mathcal{Q}^H$:

$$A^e \xrightarrow{\tilde{\pi}_A} A \longrightarrow 0.$$

In this section π_A is the usual multiplication of A. We will often use M_0 to stand for $M^H \bigcap M^{coH}$, the intersection of the invariants and the coinvariants of YD H-module M. A H-separable algebra can be described by separability idempotent elements.

Proposition 6.2. *Let A be a* YD H-module algebra. The following statements are equivalent:

(1) A is H-separable.
(2) There exists an element $e_l \in A_0^e$ such that $\pi_A(e_l) = 1$ and $(a\#1)e_l = (1\#\overline{a})e_l$ for all $a \in A$.
(3) There exists an element $e \in (A\#A)_0$ such that $\pi_A(e_l) = 1$ and $(a\#1)e = e(1\#a)$ for all $a \in A$.
(4) There exists an element $e_r \in {}^eA_0$ such that $\pi_A(e_r) = 1$ and $e_r(\overline{a}\#1) = e_r(1\#a)$ for all $a \in A$.
(5) $\pi_A : {}^eA \longrightarrow A \longrightarrow 0$ splits in $\mathcal{Q}^H_{{}^eA}$.

The proof of these statements is straightforward. We emphasize that H-separable algebras are k-separable by the statement (3). However a separable YD H-module algebra is not necessarily H-separable. For instance, the H_4-Azumaya algebra $\left\langle\frac{1,-1}{k}\right\rangle$ is a separable algebra and $k\mathbb{Z}_2$-separable, but not a H_4-separable algebra. We also have that H_4 itself is an H_4-Azumaya algebra, but it is certainly not a separable algebra over any field.

If $e_l = \sum x_i\#\overline{y}_i$ is a separability idempotent in $A\#\overline{A}$, we may choose $e = \sum x_i\#y_i$ and $e_r = \sum \overline{x}_i\#y_i$. Thus we may write e_A for e_l and e'_A for e_r without ambiguity. Since e_A is an idempotent element in each of the above cases, it follows that if A is H-separable then $M^A = e_A \rightharpoonup M$ and ${}^AN = N \leftharpoonup e'_A$ for $M \in {}_{A^e}\mathcal{Q}^H$ and $N \in \mathcal{Q}^H_{{}^eA}$ respectively. In particular, $A^A = e_A \rightharpoonup A$ and ${}^AA = A \leftharpoonup e'_A$. For a YD H-module algebra A we shall call A^A and ${}^AA$ the *left* and the *right H-center* of A respectively. In case $A^A = k$ or ${}^AA = k$ we shall say that A is *left* or *right central* respectively, and A is *H-central* if A is both left and right central.

Proposition 6.3. (1) *Let $f : A \longrightarrow B$ be an epimorphism of* YD H-module algebras. If A is H-separable then B is H-separable.

(2) Let E be a commutative k-algebra. If A is H-separable then $E \otimes_k A$ is an $E \otimes_k H$-separable E-algebra.

(3) If A is H-separable, then $\overline{A}$ is H-separable. If in addition, A is left (or right) central then $\overline{A}$ is right (or left) central respectively.

(4) If A, B are H-separable, so is $A\#B$. If in addition, A and B are left (or right) central, then $A\#B$ is left (or right) central.

Like the classical case, the ground ring k is an H-direct summand of of a left (or right) central H-separable algebra.

Lemma 6.4. *Let A be a left or right H-central H-separable algebra. Then the inclusion map embeds k as a direct summand of A in $\mathcal{Q}^H$.*

Proof. Let e be an H-separability idempotent of A. Then the map $T_e : A \longrightarrow k$ given by $T_e(a) = e \rightharpoonup a$ for $a \in A$ is a YD H-module map. We have $e \rightharpoonup 1 = \pi_A(e) = 1$. □

The map T_e described above is a section for the inclusion map $\iota : k \hookrightarrow A$ in $\mathcal{Q}^H$, that is, $T_e \circ \iota = \mathrm{id}$. We will call a YD H-module map $T : A \longrightarrow k$ an *H-trace map* of a YD H-module algebra A if $T(1) = 1$. Notice that usually a trace map is an onto map but does not necessarily carry the unit to the unit. We will show later that an H-Azumaya algebra A is an H-central H-separable algebra if and only if A has an H-trace map. It follows that H-trace maps in a one-to-one way correspond to H-separability idempotents when A is an H-Azumaya algebra.

A YD H-module algebra A is said to be *H-simple* if A has no proper YD H-module ideal (simply H-ideal). This is equivalent to A being simple in ${}_{A^e}\mathcal{Q}^H$ or $\mathcal{Q}^H_{{}^eA}$.

Proposition 6.5. *Let A be a left (or right) H-central H-separable algebra. Then A is H-simple if and only if k is a field.*

Proof. Suppose that A is H-simple and I is a non-zero ideal of k. IA is an H-ideal of A and $IA = A$. Let t be the H-trace map described in Lemma 6.4. Then $t(IA) = t(A)$ implies $I = k$. It follows that k is a field.

Conversely, suppose that A is an H-separable algebra over a field k. Since H-separability implies k-separability, A is semisimple artinian. Let M be an H-ideal of A, then there exists a central idempotent $c \in A$ such that $M = cA = Ac$. c must be in A_0. Now for any $a \in A$, we have

$$\sum a_{(0)}(a_{(1)} \cdot c) = ac = ca, \quad \sum c_{(0)}(c_{(1)} \cdot a) = ca = ac.$$

Thus c is in both A^A and ${}^AA$. Now if A is left or right H-central H-separable algebra then $c \in k$, and hence $M = cA = A$. □

LEMMA 6.6. *If A is a left or right H-central H-separable algebra, then for any maximal H-ideal I of A there exists a maximal ideal α of k such that $I = \alpha A$ and $I \cap k = \alpha$.*

In view of Proposition 6.5 and Lemma 6.6, one may use an arqument similar to the classical case [14, 2.8] to obtain that a H-central H-separable algebra is an H-Azumaya algebra with an H-trace map. In fact, we have the following:

THEOREM 6.7. *A YD H-module algebra A is an H-central H-separable algebra if and only if A is an H-Azumaya algebra with an H-trace map.*

PROOF. By the foregoing remark, it is sufficient to show that an H-Azumaya algebra with an H-trace map is H-central and H-separable. Assume that A is an H-Azumaya algebra with an H-trace map T. Since A is H-Azumaya we have the isomorphism $A\#\overline{A} \cong \mathrm{End}(A)$. In this way we may view T as an element in $A\#\overline{A}$. In fact T is in $(A\#\overline{A})_0$ since T is H-linear and H-colinear. We now can show that T is an H-separability idempotent of A^e. Now $\pi_A(T) = T(1) = 1$, and for any $a, x \in A$,

$$(a\#1)T(x) = aT(x) = (1\#\overline{a})T(x),$$

because $T(x) \in k$. It follows from the foregoing equalities that we have $(a\#1)T = (1\#\overline{a})T$ for any $a \in A$. Therefore, A is H-separable. □

In general, an H-Azumaya algebra is not necessarily H-separable, in other words, an H-Azumaya algebra need not have an H-trace map. For example, H_4 is not a separable algebra, but it is an H_4-Azumaya algebra (see [56]). For this reason, we call an H-central H-separable algebra a *strongly* H-Azumaya algebra (for short we say that it is strong).

COROLLARY 6.8. *Let A, B be H-Azumaya algebras. If $A\#B$ is strong, so are A and B.*

PROOF. By Theorem 6.7 it is enough to show that both A and B have an H-trace map. This is the case since $A\#B$ has an H-trace map T and the restriction map $T_A(a) = T(a\#1)$ and T_B are clearly H-trace maps of A and B respectively. □

This corollary indicates that even the trivial H-Azumaya algebra $\mathrm{End}(M)$, for M a faithfully projective YD H-module, is not necessarily strong. For example, if A is non-strongly H-Azumaya, e.g., $A = H_4$, then $\overline{A}$ is not strong, and hence $\mathrm{End}(A) \cong A\#\overline{A}$ is not strong by Corollary 6.8. So a strongly H-Azumaya algebra may be Brauer equivalent to a non-strongly H-Azumaya algebra. Now a natural question arises. What condition has to be imposed on H so that any H-Azumaya algebra is strongly H-Azumaya? We have a complete answer for a faithfully projective Hopf algebra, and a partial answer for an infinite Hopf algebra over a field with characteristic 0.

Proposition 6.9. *Let H be a faithfully projective Hopf algebra over k. The following are equivalent*:

(1) *Any H-Azumaya algebra is strongly H-Azumaya.*
(2) *Any elementary H-Azumaya algebra is strongly H-Azumaya.*
(3) *There exist an integral $t \in H$ and an integral $\varphi \in H^*$ such that $\varepsilon(t) = 1$ and $\varphi(1) = 1$.*
(4) *k is a projective object in $\mathcal{Q}^H$.*

Proof. (1) $\Longleftrightarrow$ (2) due to Corollary 6.8. To prove that (2) $\Longrightarrow$ (3), we take the faithfully projective YD H-module M which is the left regular H-module of H itself, with the H-comodule structure given by

$$\rho(h) = \sum h_{(2)} \otimes h_{(3)} S^{-1}(h_{(1)})$$

for any $h \in M$. Now let A be the elementary H-Azumaya algebra $\mathrm{End}(M)$. Since A is strong, A has an H-trace map, say, $T : A \longrightarrow k$. Since A is faithfully projective, A^* is a YD H-module. We may view T as an element in A_0^* as T is H-linear and H-colinear. Identify A^* with $M \tilde{\otimes} {}^*M$ as a YD H-module. One may easily check that the k-module ${A^*}^H$ of H-invariants of A^* consists of elements of form

$$\left\{ \sum t_{(1)} \otimes t_{(2)} \rightharpoonup f \mid t \in \int_l, \quad f \in {}^*M \right\}$$

where $\int_l$ is the rank one k-module of left integrals of H and $(t_{(2)} \rightharpoonup f)(h) = f(S^{-1}(t_{(2)})h)$ for any $h \in M$. It follows that $T = \sum t_{(1)} \otimes t_{(2)} \rightharpoonup f$ for some left integral $t \in H$ and an element f in ${}^*M$. Let $\{m_i \otimes p_i\}$ be a dual basis of M so that $\sum m_i \otimes p_i = 1_A$. Since $T(1_A) = 1_k$, we obtain:

$$T(1_A) = \sum p_i(t_{(1)}) f(S^{-1}(t_{(2)})m_i) = \sum f(S^{-1}(t_{(2)})t_{(1)}) = \varepsilon(t) f(1) = 1.$$

So $\varepsilon(t)$ is a unit of k, and one may choose a left integral t' to replace t so that $\varepsilon(t') = 1$.

Similarly, if we choose a faithfully projective YD H-module M as follows: $M = H$ as a right H-comodule with the comultiplication as the right comodule structure and with the adjoint left H-action given by

$$h \cdot m = \sum h_{(2)} m S^{-1}(h_{(1)}),$$

then one may find a left integral $\varphi \in H^*$ such that $\varphi(1) = 1$.

(3) $\Longleftrightarrow$ (4) is the Maschke theorem. Since H is a faithfully projective Hopf algebra, the quantum double Hopf algebra $D(H)$ is faithfully projective over k, and $D(H)$ is a projective object in $\mathcal{Q}^H$. Assume that there are two left integrals $t \in H$ and $\varphi \in H^*$ such that $\varepsilon(t) = 1$ and $\varphi(1) = 1$. The counit of $D(H)$ is a YD H-module map which is split by the YD H-module map $\iota' : k \longrightarrow D(H)$ sending the unit 1 to the element $\varphi \bowtie t$. So k is a YD H-module direct summand of $D(H)$, and hence it is projective in $\mathcal{Q}^H$. The converse holds as the foregoing argument can be reversed.

Finally, we show that (3) $\Longrightarrow$ (1). Assume $\varphi \bowtie t$ is a left integral of $D(H)$ such that $\varepsilon(t) = 1 = \varphi(1)$. If A is an H-Azumaya algebra, then the multiplication map

$$\pi : A\#\bar{A} \longrightarrow A$$

splits as a left $A\#\bar{A}$-module. Let $\mu : A \longrightarrow A\#\bar{A}$ be the split map, and let $e = \mu(1)$. Then $(\varphi \bowtie t) \cdot e$ is an H-separability element of A. So A is strongly H-Azumaya. □

If H is not a faithfully projective Hopf algebra, we have a sufficient condition which requires that the antipode of H be involutory.

Proposition 6.10. *Let k be a field with $\mathrm{ch}(k) = 0$ and let H be a Hopf algebra over k. If the antipode S of H is involutory, then any H-Azumaya algebra is strongly H-Azumaya.*

Proof. By Corollary 6.8, it is enough to show that any elementary H-Azumaya algebra is strong. Let M be a faithfully projective YD H-module, and let $A = \mathrm{End}(M)$. We show that A has an H-trace map. Identify A with $M \otimes M^*$ as YD H-modules. Let tr be the normal trace map of A which sends the element $m \otimes m^*$ to $m^*(m)$. Since $\mathrm{ch}(k) = 0$, we have that $\mathrm{tr}(1_A) = n$ for some integer is a unit. We show that tr is a YD H-module map, then the statement follows. Indeed, if $h \in H$, $m \in M$ and $m^* \in M^*$, we have

$$\begin{aligned}\mathrm{tr}(h \cdot (m \otimes m^*)) &= \sum \mathrm{tr}(h_{(1)} \cdot m \otimes h_{(2)} \cdot m^*) = \sum (h_{(2)} \cdot m^*)(h_{(1)} \cdot m) \\ &= \sum m^*(S(h_{(2)})h_{(1)} \cdot m) = \varepsilon(h)\,\mathrm{tr}(m).\end{aligned}$$

Similarly, tr is H-colinear as well. □

Note that when $\mathrm{ch}(k) = 0$ the condition $S^2 = \mathrm{id}$ is equivalent to the condition (3) in Proposition 6.9 if H is finite dimensional (see [28]). However, this is not the case when H is not finite. There is an example of a Hopf algebra that is involutory (e.g., $T(V)$, the universal enveloping Hopf algebra), but without integrals. Nevertheless, it remains open whether k is a projective object in $\mathcal{Q}^H$ if and only if $S^2 = \mathrm{id}$ in case $\mathrm{ch}(k) = 0$.

As mentioned in the title of this section, we are able to define the second Brauer group of strongly H-Azumaya algebras as a result of proposition 6.3. That is, the second Brauer group, denoted $\mathrm{BQs}(k, H)$, consists of isomorphism classes of strongly H-Azumaya algebras modulo the same Brauer equivalence, where the elementary H-Azumaya algebras $\mathrm{End}(M)$ are required to be H-separable. It is evident from Theorem 6.7 that the second Brauer group $\mathrm{BQs}(k, H)$ is a subgroup of $\mathrm{BQ}(k, H)$, which contains the usual Brauer group $\mathrm{Br}(k)$ as a normal subgroup. Let us summarize it as follows:

Corrollary 6.11. *The subset $\mathrm{BQs}(k, H)$ represented by the strongly H-Azumaya algebras is a subgroup of $\mathrm{BQ}(k, H)$.*

Proof. The only thing left to check is the coincidence of the two Brauer equivalence relations. Assume that A and B are two strongly H-Azumaya algebras and $[A] = [B]$ in $\mathrm{BQ}(k, H)$. Then $A\#\overline{B} \cong \mathrm{End}(M)$ for some faithfully projective YD H-module M. It follows that $A\#\mathrm{End}(B) \cong B\#\mathrm{End}(M)$. By Proposition 6.3, $\mathrm{End}(B)$ and $\mathrm{End}(M)$ are strongly H-Azumaya algebras, and we obtain that $[A] = [B]$ in $\mathrm{BQs}(k, H)$. $\square$

Now the question arises: when is $\mathrm{BQs}(k, H)$ equal to $\mathrm{BQ}(k, H)$? When a Hopf algebra satisfies the assumption in Proposition 6.9 or Proposition 6.10, $\mathrm{BQ}(k, H) = \mathrm{BQs}(k, H)$. However, since a strongly H-Azumaya algebra may be Brauer equivalent to a non-strongly H-Azumaya algebra, there is a possibility that for some Hopf algebra H, any $\mathrm{BQ}(k, H)$ element can be represented by a strongly H-Azumaya algebra, but at the same time there may exist non-strongly H-Azumaya algebras.

Note that for a quasitriangular or coquasitriangular Hopf algebra H, The H-central and H-separable or strongly H-Azumaya algebras are special cases of those above. For example, If (H, R) is a cosemisimple-like coquasitriangular Hopf algebra, then $\mathrm{BCs}(k, H, R) = \mathrm{BC}(k, H, R)$. If G a finite abelian group, $H = kG$ with a bilinear map $\varphi : G \times G \longrightarrow k$, Then H is a cosemisimple-like coquasitriangular Hopf algebra. Thus all graded (H-) Azumaya algebra are strongly H-Azumaya [14; 15].

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