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On Quantum Algebras and Coalgebras, Oriented Quantum Algebras and Coalgebras, Invariants of 1–1 Tangles, Knots and Links

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Abstract. We outline a theory of quantum algebras and coalgebras and their resulting invariants of unoriented 1–1 tangles, knots and links, we outline a theory of oriented quantum algebras and coalgebras and their resulting invariants of oriented 1–1 tangles, knots and links, and we show how these algebras and coalgebras are related. Quasitriangular Hopf algebras are examples of quantum algebras and oriented quantum algebras; likewise coquasitriangular Hopf algebras are examples of quantum coalgebras and oriented quantum coalgebras.

Introduction

Since the advent of quantum groups [4] many algebraic structures have been described which are related to invariants of 1–1 tangles, knots, links or 3-manifolds. The purpose of this paper is to outline a theory for several of these structures, which are defined over a field k, and to discuss relationships among them. The structures we are interested in are: quantum algebras, quantum coalgebras, oriented quantum algebras, oriented quantum coalgebras and their "twist" specializations.

Quantum algebras and coalgebras account for regular isotopy invariants of unoriented 1–1 tangles. Twist quantum algebras and coalgebras, which are quantum algebras and coalgebras with certain additional structure, account for regular isotopy invariants of unoriented knots and links. As the terminology suggests,

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oriented quantum algebras and coalgebras account for regular isotopy invariants of oriented 1–1 tangles; twist oriented quantum algebras and coalgebras account for regular isotopy invariants of oriented knots and links.

The notion of quantum algebra was described some time ago [10] whereas the notion of quantum coalgebra (which is more general than dual quantum algebra) was described somewhat later [19]. The notions of oriented quantum algebra and oriented quantum coalgebra have been formulated very recently and initial papers about them [16; 17] have just been circulated. This paper is basically a rough overview of a rather extensive body of joint work by the author and Kauffman [13; 15; 16; 17; 19; 20] on these and related structures. At the time of this writing oriented quantum algebras account for most known regular isotopy invariants of oriented links [20]; thus oriented quantum algebras are important for that reason. Oriented quantum algebras and quantum algebras are related in an interesting way. Every quantum algebra has an oriented quantum algebra structure which is called the associated oriented quantum algebra structure. Not every oriented quantum algebra is an associated oriented quantum algebra structure. However, a quantum algebra can always be constructed from an oriented quantum algebra in a natural way.

A quasitriangular Hopf algebra has a quantum algebra structure and a ribbon Hopf algebra has a twist quantum algebra structure. Hence there are close connections between Hopf algebras, quantum algebras and oriented quantum algebras. Quantum coalgebras and oriented quantum coalgebras are related in the same ways that quantum algebras and oriented quantum algebras are. Coquasitriangular Hopf algebras have a quantum coalgebra structure. There are coquasitriangular Hopf algebras associated with a wide class of quantum coalgebras; therefore there are close connections between Hopf algebras, quantum coalgebras and oriented quantum coalgebras.

In this paper we focus on the algebraic theory of quantum algebras and the other structures listed above and we focus on the algebraic theory of their associated invariants. For a topological perspective on these theories, in particular for a topological motivation of the definitions of quantum algebra and oriented quantum algebra, the reader is encouraged to consult [17; 21]. Here we do not calculate invariants, except to provide a few simple illustrations, nor do we classify them. For more extensive calculations see [12; 13; 14; 15; 18; 19] and [27]. We do, however, describe in detail how the Jones polynomial fits into the context of oriented quantum algebras (and thus quantum coalgebras). A major goal of this paper is to describe in sufficient detail an algebraic context which accounts for many known regular isotopy invariants of knots and links and which may prove to be fertile ground for the discovery of new invariants.

The paper is organized as follows. In Section 1 basic notations are discussed; bilinear forms, the quantum Yang–Baxter and braid equations are reviewed. We assume that the reader has a basic knowledge of coalgebras and Hopf algebras. Good references are [1; 24; 31]. Section 2 deals with Yang–Baxter algebras and the dual concept of Yang–Baxter coalgebras. Yang–Baxter algebras or coalgebras are integral parts of the algebraic structures we study in this paper. Yang–Baxter coalgebras are defined in [2] and are referred to as coquasitriangular coalgebras in [22].

The basic theory of quantum algebras is laid out in Section 3. We discuss the connection between quantum algebras and quasitriangular Hopf algebras and show how ribbon Hopf algebras give rise to oriented quantum algebras. Likewise the basic theory of oriented quantum algebras is outlined in Section 4; in particular we describe the oriented quantum algebra associated to a quantum algebra.

Section 5 gives a rather detailed description of the construction of a quantum algebra from an oriented quantum algebra. Sections 6 and 7 are about the invariants associated to quantum algebras, oriented quantum algebras and their twist specializations. These invariants are described in terms of a very natural and intuitive *bead sliding formalism*. Section 8 makes the important connection between the invariants computed by bead sliding with invariants computed by the well-established categorical method [28; 29]. The reader is encouraged to consult [13; 17; 28; 29] as background material for Sections 6–8.

The material of Section 8 motivates the notion of inner oriented quantum algebra which is discussed in Section 9. The Hennings invariant, which a 3-manifold invariant defined for certain finite-dimensional ribbon Hopf algebras, can be explained in terms of the bead sliding formalism. We comment on aspects of computation of this invariant in Section 10.

Sections 11–13 are the coalgebras versions of Sections 3–5. In Section 14 our paper ends with a discussion of invariants constructed from quantum coalgebras, oriented quantum coalgebras and their twist specializations. There may be a practical advantage to computing invariants using coalgebra structures instead of algebra structures.

Throughout k is a field and all algebras, coalgebras and vector spaces are over k. Frequently we denote algebras, coalgebras and the like by their underlying vector spaces and we denote the set of non-zero elements of k by $k^\star$. Finally, the author would like to thank the referee for his or her very thoughtful suggestions and comments. Some of the comments led to additions to this paper, namely Propositions 1, 4, Corollary 2 and Section 9.

1. Preliminaries

For vector spaces U and V over k we denote the tensor product $U \otimes_k V$ by $U \otimes V$, the identity map of V by 1_V and the linear dual $\mathrm{Hom}_k(V, k)$ of V by V^*. If T is a linear endomorphism of V then an element $v \in V$ is *T-invariant* if $T(v) = v$. The *twist map* $\tau_{U,V} : U \otimes V \longrightarrow V \otimes U$ is defined by $\tau_{U,V}(u \otimes v) = v \otimes u$ for all $u \in U$ and $v \in V$. If V is an algebra over k we let 1_V also denote the unit of k. The meaning 1_V should always be clear from context.

By definition of the tensor product $U \otimes V$ of U and V over k there is a bijective correspondence between the set of bilinear forms $b : U \times V \longrightarrow k$ and the vector space $(U \otimes V)^*$ given by $b \mapsto b_{lin}$, where $b_{lin}(u \otimes v) = b(u, v)$ for all $u \in U$ and $v \in V$. We refer to bilinear forms $b : U \times U \longrightarrow k$ as *bilinear forms on* U.

Let $b : U \times V \longrightarrow k$ be a bilinear form and regard $U^* \otimes V^*$ as a subspace of $(U \otimes V)^*$ in the usual way. Then b is of *finite type* if $b_{lin} \in U^* \otimes V^*$ in which case we write ρ_b for b_{lin}. Thus when b is of finite type $\rho_b(u \otimes v) = b(u, v)$ for all $u \in U$ and $v \in V$. Observe that b is of finite type if and only if one of $b_\ell : U \longrightarrow V^*$ and $b_r : V \longrightarrow U^*$ has finite rank, where $b_\ell(u)(v) = b(u, v) = b_r(v)(u)$ for all $u \in U$ and $v \in V$. Consequently if one of U or V is finite-dimensional b is of finite type. The bilinear form b is *left non-singular* if b_ℓ is one-one, is *right non-singular* if b_r is one-one and is *non-singular* if b is both left and right non-singular.

Suppose that ρ is an endomorphism of $U \otimes U$ and consider the endomorphisms $\rho_{(i,j)}$ of $U \otimes U \otimes U$ for $1 \leq i < j \leq 3$ defined by

$$\rho_{(1,2)} = \rho \otimes 1_U,\ \rho_{(2,3)} = 1_U \otimes \rho \text{ and } \rho_{(1,3)} = (1_U \otimes \tau_{U,U}) \circ (\rho \otimes 1_U) \circ (1_U \otimes \tau_{U,U}).$$

The *quantum Yang–Baxter equation* is

$$\rho_{(1,2)} \circ \rho_{(1,3)} \circ \rho_{(2,3)} = \rho_{(2,3)} \circ \rho_{(1,3)} \circ \rho_{(1,2)} \tag{1–1}$$

and the *braid equation* is

$$\rho_{(1,2)} \circ \rho_{(2,3)} \circ \rho_{(1,2)} = \rho_{(2,3)} \circ \rho_{(1,2)} \circ \rho_{(2,3)}. \tag{1–2}$$

Observe that ρ satisfies (1–1) if and only if $\rho \circ \tau_{U,U}$ satisfies (1–2) or equivalently $\tau_{U,U} \circ \rho$ satisfies (1–2). If ρ is invertible then ρ satisfies (1–1) if and only if ρ^{-1} does and ρ satisfies (1–2) if and only if ρ^{-1} does.

Let (C, Δ, ε) be a coalgebra over k. We use the notation $\Delta(c) = c_{(1)} \otimes c_{(2)}$ for $c \in C$, a variation of the Heyneman–Sweedler notation, to denote the coproduct. Generally we write $\Delta^{(n-1)}(c) = c_{(1)} \otimes \cdots \otimes c_{(n)}$, where $\Delta^{(1)} = \Delta$ and $\Delta^{(n)} = (\Delta \otimes 1_C \otimes \cdots \otimes 1_C) \circ \Delta^{(n-1)}$ for $n \geq 2$. We let C^{cop} denote the coalgebra $(C, \Delta^{\mathrm{cop}}, \varepsilon)$ whose coproduct is given by $\Delta^{\mathrm{cop}}(c) = c_{(2)} \otimes c_{(1)}$ for all $c \in C$. If (M, ρ) is a right C-comodule we write $\rho(m) = m^{<1>} \otimes m^{(2)}$ for $m \in M$.

Let the set of bilinear forms on C have the algebra structure determined by its identification with the dual algebra $(C \otimes C)^*$ given by $b \mapsto b_{lin}$. Observe that $b, b' : C \times C \longrightarrow k$ are inverses if and only if

$$b(c_{(1)}, d_{(1)}) b'(c_{(2)}, d_{(2)}) = \varepsilon(c) \varepsilon(d) = b'(c_{(1)}, d_{(1)}) b(c_{(2)}, d_{(2)})$$

for all $c, d \in C$. In this case we write $b^{-1} = b'$.

For an algebra A over k we let A^{op} denote the algebra whose ambient vector space is A and whose multiplication is given by $a \cdot b = ba$ for all $a, b \in A$. An element tr of A^* is *tracelike* if $\mathsf{tr}(ab) = \mathsf{tr}(ba)$ for all $a, b \in A$. The trace function on the algebra $\mathrm{M}_n(k)$ of $n \times n$ matrices over k is a primary example of a tracelike element.

For a vector space V over k and $\rho \in V \otimes V$ we define

$$V_{(\rho)} = \{(u^* \otimes 1_V)(\rho) + (1_V \otimes v^*)(\rho) \mid u^*, v^* \in V^*\}.$$

For an algebra A over k and invertible $\rho \in A \otimes A$ we let A_ρ be the subalgebra of A generated by $A_{(\rho)} + A_{(\rho^{-1})}$. The following lemma will be useful in our discussion of minimal quantum algebras.

Lemma 1. *Let V be a vector space over k and $\rho \in V \otimes V$.*

(a) *$\rho \in V_{(\rho)} \otimes V_{(\rho)}$ and $V_{(\rho)}$ is the smallest subspace U of V such that $\rho \in U \otimes U$.*
(b) *Suppose that t is a linear endomorphism of V which satisfies $(t \otimes t)(\rho) = \rho$. Then $t(V_{(\rho)}) = V_{(\rho)}$.*

Proof. Part (a) follows by definition of $V_{(\rho)}$. For part (b), we may assume that $\rho \neq 0$ and write $\rho = \sum_{\imath=1}^r u_\imath \otimes v_\imath$, where r is as small as possible. Then $\{u_1, \ldots, u_r\}$, $\{v_1, \ldots, v_r\}$ are linearly independent and the $u_\imath$'s together with the $v_\imath$'s span $V_{(\rho)}$. Since $(t \otimes t)(\rho) = \rho$, or equivalently $\sum_{\imath=1}^r t(u_\imath) \otimes t(v_\imath) = \sum_{\imath=1}^r u_\imath \otimes v_\imath$, it follows that the sets $\{t(u_1), \ldots, t(u_r)\}$ and $\{t(v_1), \ldots, t(v_r)\}$ are also linearly independent, that $\{u_1, \ldots, u_r\}$, $\{t(u_1), \ldots, t(u_r)\}$ have the same span and that $\{v_1, \ldots, v_r\}$, $\{t(v_1), \ldots, t(v_r)\}$ have the same span. Thus $t(V_{(\rho)}) = V_{(\rho)}$. □

2. Yang–Baxter Algebras and Yang–Baxter Coalgebras

Let A be an algebra over the field k and let $\rho = \sum_{\imath=1}^r a_\imath \otimes b_\imath \in A \otimes A$. We set

$$\rho_{1\,2} = \sum_{\imath=1}^r a_\imath \otimes b_\imath \otimes 1, \quad \rho_{1\,3} = \sum_{\imath=1}^r a_\imath \otimes 1 \otimes b_\imath \quad \text{and} \quad \rho_{2\,3} = \sum_{\imath=1}^r 1 \otimes a_\imath \otimes b_\imath.$$

The quantum Yang–Baxter equation for ρ is

$$\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}, \tag{2–1}$$

or equivalently

$$\sum_{\imath,\jmath,\ell=1}^r a_\imath a_\jmath \otimes b_\imath a_\ell \otimes b_\jmath b_\ell = \sum_{\jmath,\imath,\ell=1}^r a_\jmath a_\imath \otimes a_\ell b_\imath \otimes b_\ell b_\jmath. \tag{2–2}$$

Observe that (2–1) is satisfied when A is commutative. The pair (A, ρ) is called a *Yang–Baxter algebra over k* if ρ is invertible and satisfies (2–1).

Suppose that (A, ρ) and (A', ρ') are Yang–Baxter algebras over k. Then $(A \otimes A', \rho'')$ is a Yang–Baxter algebra over k, called the *tensor product of* (A, ρ) *and* (A', ρ'), where $\rho'' = (1_A \otimes \tau_{A,A'} \otimes 1_{A'})(\rho \otimes \rho')$. A *morphism* $f : (A, \rho) \longrightarrow (A', \rho')$ *of Yang–Baxter algebras* is an algebra map $f : A \longrightarrow A'$ which satisfies $\rho' = (f \otimes f)(\rho)$. Note that $(k, 1 \otimes 1)$ is a Yang–Baxter algebra over k. The category of Yang–Baxter algebras over k with their morphisms under composition has a natural monoidal structure.

Since ρ is invertible and satisfies (2–1) then ρ^{-1} does as well as $(\rho^{-1})_{\imath\jmath} = (\rho_{\imath\jmath})^{-1}$ for all $1 \le \imath < \jmath \le 3$. Thus (A, ρ^{-1}) is a Yang–Baxter algebra over k. Note that (A^{op}, ρ) and (A, ρ^{op}) are Yang–Baxter algebras over k also, where $\rho^{\mathrm{op}} = \sum_{\imath=1}^{r} b_\imath \otimes a_\imath$.

An interesting example of a Yang–Baxter algebra for us [16, Example 1] is the following where $A = \mathrm{M}_n(k)$ is the algebra of all $n\times n$ matrices over k. For $1 \le \imath, \jmath \le n$ let $E_{\imath\jmath} \in \mathrm{M}_n(k)$ be the $n\times n$ matrix which has a single non-zero entry which is 1 and is located in the i^{th} row and j^{th} column. Then $\{E_{\imath\jmath}\}_{1\le\imath,\jmath\le n}$ is the standard basis for $\mathrm{M}_n(k)$ and $E_{\imath\jmath}E_{\ell m} = \delta_{\jmath\ell}E_{\imath m}$ for all $1 \le \imath, \jmath, \ell, m \le n$.

Example 1. Let $n \ge 2$, $a, bc \in k^\star$ satisfy $a^2 \ne bc, 1$ and let

$$\mathsf{B} = \{b_{\imath\jmath} \,|\, 1 \le \imath < \jmath \le n\}, \qquad \mathsf{C} = \{c_{\jmath\imath} \,|\, 1 \le \imath < \jmath \le n\}$$

be indexed subsets of $k^\star$ such that $b_{\imath\jmath}c_{\jmath\imath} = bc$ for all $1 \le \imath < \jmath \le n$. Then $(\mathrm{M}_n(k), \rho_{a,\mathsf{B},\mathsf{C}})$ is a Yang–Baxter algebra over k, where

$$\rho_{a,\mathsf{B},\mathsf{C}} = \sum_{1\le\imath<\jmath\le n}\left(\left(a - \frac{bc}{a}\right)E_{\imath\jmath}\otimes E_{\jmath\imath} + b_{\imath\jmath}E_{\imath\imath}\otimes E_{\jmath\jmath} + c_{\jmath\imath}E_{\jmath\jmath}\otimes E_{\imath\imath}\right) + \sum_{\imath=1}^{n} aE_{\imath\imath}\otimes E_{\imath\imath}.$$

That $\rho_{a,\mathsf{B},\mathsf{C}}$ satisfies (2–1) follows by [27, Lemma 4 and (37)]. The notation for the scalar bc is meant to suggest a product. We point out that $\rho_{a,\mathsf{B},\mathsf{C}}$ can be derived from $\beta_{q,P}(A_\ell)$ of [6, Section 5]. See [8] also.

Representations of Yang–Baxter algebras determine solutions to the quantum Yang–Baxter equation. Suppose that A is an algebra over k and $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath \in A\otimes A$. For a left A-module M let ρ_M be the endomorphism of $M\otimes M$ defined by $\rho_M(m\otimes n) = \sum_{\imath=1}^{r} a_\imath{\cdot}m\otimes b_\imath{\cdot}n$ for all $m, n \in M$. Then (A, ρ) is a Yang–Baxter algebra over k if and only if ρ_M is an invertible solution to the quantum Yang–Baxter equation (1–1) for all left A-modules M.

The notions of minimal quantum algebra and minimal oriented quantum algebra are important for theoretical reasons. These notions are based on the notion of minimal Yang–Baxter algebra. A *minimal Yang–Baxter algebra* is a Yang–Baxter algebra (A, ρ) over k such that $A = A_\rho$. Observe that only A_ρ is involved in the definition of ρ_M of the preceding paragraph.

Let (A, ρ) be a Yang–Baxter algebra over k. A *Yang–Baxter subalgebra of* (A, ρ) is a pair (B, ρ) where B is a subalgebra of A such that $\rho, \rho^{-1} \in B\otimes B$; thus $A_\rho \subseteq B$. Consequently (A, ρ) has a unique minimal Yang–Baxter subalgebra which is (A_ρ, ρ). Observe that a Yang–Baxter algebra (B, ρ') is a Yang–Baxter subalgebra of (A, ρ) if and only if B is a subalgebra of A and the inclusion $\imath : B \longrightarrow A$ induces a morphism of Yang–Baxter algebras $\imath : (B, \rho') \longrightarrow (A, \rho)$.

Let I be an ideal of A. Then there is a (unique) Yang–Baxter algebra structure $(A/I, \overline{\rho})$ on the quotient A/I such that the projection $\pi : A \longrightarrow A/I$ determines a morphism $\pi : (A, \rho) \longrightarrow (A/I, \overline{\rho})$ of Yang–Baxter algebras. If K is a field extention of k then $(A\otimes K, \rho\otimes 1_K)$ is a Yang–Baxter algebra over K where we make the identification $\rho\otimes 1_K = \sum_{\imath=1}^{r}(a_\imath\otimes 1_K)\otimes(b_\imath\otimes 1_K)$.

We now turn to Yang–Baxter coalgebras. A *Yang–Baxter coalgebra over k* is a pair (C, b), where C is a coalgebra over k and $b : C{\times}C \longrightarrow k$ is an invertible bilinear form, such that

$$b(c_{(1)}, d_{(1)})b(c_{(2)}, e_{(1)})b(d_{(2)}, e_{(2)}) = b(c_{(2)}, d_{(2)})b(c_{(1)}, e_{(2)})b(d_{(1)}, e_{(1)})$$

for all $c, d, e \in C$. Observe that this equation is satisfied when C is cocommutative.

Let (C, b) and (C', b') be Yang–Baxter coalgebras over k. Then $(C{\otimes}C', b'')$ is a Yang–Baxter coalgebra over k, where $b''(c{\otimes}c', d{\otimes}d') = b(c, d)b'(c', d')$ for all $c, d \in C$ and $c', d' \in C'$. A *morphism* $f : (C, b) \longrightarrow (C', b')$ *of Yang–Baxter coalgebras over* k is a coalgebra map $f : C \longrightarrow C'$ which satisfies $b(c, d) = b'(f(c), f(d))$ for all $c, d \in C$. Observe that (k, b) is a Yang–Baxter coalgebra over k where $b(1, 1) = 1$. The category of all Yang–Baxter coalgebras over k with their morphisms under composition has a natural monoidal structure.

Since (C, b) is a Yang–Baxter coalgebra over k it follows that (C^{cop}, b), (C, b^{-1}) and (C, b^{op}) are as well, where $b^{\mathrm{op}}(c, d) = b(d, c)$ for all $c, d \in C$.

The notions of Yang–Baxter algebra and Yang–Baxter coalgebra are dual as one might suspect. Let (A, ρ) be a Yang–Baxter algebra over k. Then (A^o, b_ρ) is a Yang–Baxter coalgebra over k, where $b_\rho(a^o, b^o) = (a^o{\otimes}b^o)(\rho)$ for all $a^o, b^o \in A^o$, and the bilinear form b_ρ is of finite type. Suppose that C is a coalgebra over k and that $b : C{\times}C \longrightarrow k$ is a bilinear form of finite type, which is the case if C is finite-dimensional. Then (C, b) is a Yang–Baxter coalgebra over k if and only if (C^*, ρ_b) is a Yang–Baxter algebra over k.

Let I be a coideal of C which satisfies $b(I, C) = (0) = b(C, I)$. Then there is a (unique) Yang–Baxter coalgebra structure $(C/I, \overline{b})$ on the quotient C/I such that the projection $\pi : C \longrightarrow C/I$ defines a morphism $\pi : (C, b) \longrightarrow (C/I, \overline{b})$ of Yang–Baxter coalgebras. Now let I the sum of all coideals J of C which satisfy $b(J, C) = (0) = b(C, J)$. Set $C_r = C/I$ and define $b_r : C_r{\times}C_r \longrightarrow k$ by $b_r(c{+}I, d{+}I) = b(c, d)$ for all $c, d \in C$. Then (C_r, b_r) is a Yang–Baxter coalgebra over k. This construction is dual to the construction (A_ρ, ρ) for Yang–Baxter algebras (A, ρ) over k.

Rational representations of Yang–Baxter coalgebras over k determine solutions to the quantum Yang–Baxter equation just as representations of Yang–Baxter algebras do. Suppose that C is a coalgebra over k and that $b : C{\times}C \longrightarrow k$ is a bilinear form. For a right C-comodule M let τ_M be the endomorphism of $M{\otimes}M$ defined by $\tau_M(m{\otimes}n) = m^{<1>}{\otimes}n^{<1>}b(m^{(2)}, n^{(2)})$ for all $m, n \in M$. Then (C, b) is a Yang–Baxter coalgebra over k if and only if τ_M is an invertible solution to the quantum Yang–Baxter equation (1–1) for all right C-comodules M.

Note that τ_M for a Yang–Baxter coalgebra (C, b) can be defined in terms (C_r, b_r). For let (M, ρ) be a right C-comodule and let $\pi : C \longrightarrow C_r$ be the projection. Then (M, ρ_r) is a right C_r-comodule, where $\rho_r = (1_M{\otimes}\pi){\circ}\rho$, and τ_M defined for (M, ρ) is the same as τ_M defined for (M, ρ_r).

Let D be a subcoalgebra of C. Then $(D, b|_{D\times D})$ is a Yang–Baxter coalgebra which we call a *Yang–Baxter subcoalgebra of* (C, b). Observe that Yang–Baxter subcoalgebras of (C, b) are those Yang–Baxter coalgebras (D, b') such that D is a subcoalgebra of C and the inclusion $\imath : D \longrightarrow C$ induces a morphism $\imath : (D, b') \longrightarrow (C, b)$ of Yang–Baxter coalgebras over k. Let K be a field extension of k. Then $(C\otimes K, b_K)$ is a Yang–Baxter coalgebra over K, where $b_K(c\otimes\alpha, d\otimes\beta) = \alpha\beta b(c, d)$ for all $c, d \in C$ and $\alpha, \beta \in K$.

3. Quantum Algebras and Quasitriangular Hopf Algebras

Quantum algebras determine regular isotopy invariants of 1–1 tangles and twist quantum algebras determine regular isotopy invariants of knots and links. The notion of quantum algebra arises in the consideration of the algebra of unoriented knot and link diagrams; see Section 6 for an indication of how the axioms for quantum algebras are related to the diagrams.

In this section we recall the definitions of quantum algebra and twist quantum algebra, discuss basic examples and outline some fundamental results about them. The reader is referred to [19].

A *quantum algebra over* k is a triple (A, ρ, s), where (A, ρ) is a Yang–Baxter algebra over k and $s : A \longrightarrow A^{\mathrm{op}}$ is an algebra isomorphism, such that

(QA.1) $\rho^{-1} = (s\otimes 1_A)(\rho)$ and

(QA.2) $\rho = (s\otimes s)(\rho)$.

Observe that (QA.1) and (QA.2) imply

(QA.3) $\rho^{-1} = (1_A\otimes s^{-1})(\rho)$;

indeed any two of (QA.1)–(QA.3) imply the third.

Quasitriangular Hopf algebras are a basic source of quantum algebras. A *quasitriangular Hopf algebra over* k is a pair (A, ρ), where A is a Hopf algebra with bijective antipode s over k and $\rho = \sum_{\imath=1}^{r} a_\imath\otimes b_\imath \in A\otimes A$, such that:

(QT.1) $\sum_{\imath=1}^{r} \Delta(a_\imath)\otimes b_\imath = \sum_{\imath,\jmath=1}^{r} a_\imath\otimes a_\jmath\otimes b_\imath b_\jmath$,

(QT.2) $\sum_{\imath=1}^{r} \varepsilon(a_\imath) b_\imath = 1$,

(QT.3) $\sum_{\imath=1}^{r} a_\imath\otimes\Delta^{\mathrm{cop}}(b_\imath) = \sum_{\imath,\jmath=1}^{r} a_\imath a_\jmath\otimes b_\imath\otimes b_\jmath$,

(QT.4) $\sum_{\imath=1}^{r} a_\imath\varepsilon(b_\imath) = 1$ and

(QT.5)$(\Delta^{\mathrm{cop}}(a))\rho = \rho(\Delta(a))$ for all $a \in A$.

Our definition of quasitriangular Hopf algebra is another formulation of the definition of quasitriangular Hopf algebra [4, page 811] in the category of finite-dimensional vector spaces over a field. Observe that (QT.1) and (QT.2) imply that ρ is invertible and $\rho^{-1} = (s\otimes 1_A)(\rho)$; apply $(m\otimes 1_A)\circ(s\otimes 1_A\otimes 1_A)$ and $(m\otimes 1_A)\circ(1_A\otimes s\otimes 1_A)$ to both sides of the equation of (QT.1). Using (QT.3) and (QT.4) it follows by a similar argument that $\rho^{-1} = (1_A\otimes s^{-1})(\rho)$. See [23, page 13].

Observe that ρ satisfies (2–1) by (QT.1) and (QT.5), or equivalently by (QT.3) and (QT.5). Thus:

EXAMPLE 2. If (A, ρ) is a quasitriangular Hopf algebra over k then (A, ρ, s) is a quantum algebra over k, where s is the antipode of A.

A fundamental example of a quasitriangular Hopf algebra is the quantum double $(D(A), \boldsymbol{\rho})$ of a finite-dimensional Hopf algebra A with antipode s over k defined in [4]. As a coalgebra $D(A) = A^{*\,cop} \otimes A$. The multiplicative identity for the algebra structure on $D(A)$ is $\varepsilon \otimes 1$ and multiplication is determined by

$$(p \otimes a)(q \otimes b) = p(a_{(1)} \rightharpoonup q \leftharpoonup s^{-1}(a_{(3)})) \otimes a_{(2)} b$$

for all $p, q \in A^*$ and $a, b \in A$; the functional $a \rightharpoonup q \leftharpoonup b \in A^*$ is defined by $(a \rightharpoonup q \leftharpoonup b)(c) = q(bca)$ for all $c \in A$. We follow [26] for the description of the quantum double.

Let $\{a_1, \ldots, a_r\}$ be a linear basis for A and let $\{a^1, \ldots, a^r\}$ be the dual basis for A^*. Then $(D(A), \boldsymbol{\rho})$ is a minimal quasitriangular Hopf algebra, where $\boldsymbol{\rho} = \sum_{\imath=1}^{r} (\varepsilon \otimes a_\imath) \otimes (a^\imath \otimes 1)$. The definition of $\boldsymbol{\rho}$ does not depend on the choice of basis for A.

Coassociativity is not needed for Example 2. A structure which satisfies the axioms for a Hopf algebra over k with the possible exception of the coassociative axiom is called a *not necessarily coassociative Hopf algebra*.

A very important example of a quantum algebra, which accounts for the Jones polynomial when $k = \mathbb{C}$ is the field of complex numbers, is one defined on the algebra $A = \mathrm{M}_2(k)$ of 2×2 matrices over k. The Jones polynomial and its connection with this quantum algebra is discussed in detail in Section 8.

EXAMPLE 3. Let k be a field and $q \in k^\star$. Then $(\mathrm{M}_2(k), \rho, s)$ is a quantum algebra over k, where

$$\rho = q^{-1}(E_{1\,1} \otimes E_{1\,1} + E_{2\,2} \otimes E_{2\,2}) + q(E_{1\,1} \otimes E_{2\,2} + E_{2\,2} \otimes E_{1\,1}) + (q^{-1} - q^3) E_{1\,2} \otimes E_{2\,1}$$

and

$$s(E_{1\,1}) = E_{2\,2}, \;\; s(E_{2\,2}) = E_{1\,1}, \;\; s(E_{1\,2}) = -q^{-2} E_{1\,2}, \;\; s(E_{2\,1}) = -q^2 E_{2\,1}.$$

Let (A, ρ, s) and (A', ρ', s') be quantum algebras over k. The *tensor product of* (A, ρ, s) *and* (A', ρ', s') is the quantum algebra $(A \otimes A', \rho'', s \otimes s')$, where $(A \otimes A', \rho'')$ is the tensor product of the Yang–Baxter algebras (A, ρ) and (A', ρ'). A *morphism* $f : (A, \rho, s) \longrightarrow (A', \rho', s')$ *of quantum algebras* is a morphism $f : (A, \rho) \longrightarrow (A', \rho')$ of Yang–Baxter algebras which satisfies $f \circ s = s' \circ f$. Observe that $(k, 1 \otimes 1, 1_k)$, is a quantum algebra over k. The category of quantum algebras over k and their morphisms under composition has a natural monoidal structure.

Let (A, ρ, s) be a quantum algebra over k. Then (A, ρ) is a Yang–Baxter algebra over k and thus $(A^{\mathrm{op}}, \rho), (A, \rho^{-1})$ and (A, ρ^{op}) are also as we have noted.

It is not hard to see that $(A^{\mathrm{op}}, \rho, s)$, (A, ρ^{-1}, s^{-1}) and $(A, \rho^{\mathrm{op}}, s^{-1})$ are quantum algebras over k.

Certain quotients of A have a quantum algebra structure. Let I be an ideal of A which satisfies $s(I) = I$. Then there exists a (unique) quantum algebra structure $(A/I, \overline{\rho}, \overline{s})$ on the quotient algebra A/I such that the projection $\pi : A \longrightarrow A/I$ determines a morphism $\pi : (A, \rho, s) \longrightarrow (A/I, \overline{\rho}, \overline{s})$ of quantum algebras over k. If $f : (A, \rho, s) \longrightarrow (A'\rho', s')$ is a morphism of quantum algebras, and f is onto, then $(A/I, \overline{\rho}, \overline{s})$ and (A', ρ', s') are isomorphic. We let the reader formulate and prove fundamental homomorphism theorems for quantum algebras.

We say that (A, ρ, s) is a *minimal quantum algebra over* k if $A = A_\rho$. Let B be a subalgebra of A such that $\rho \in B \otimes B$ and $s(B) = B$. Then $(B, \rho, s|_B)$ is a quantum algebra over k which is called a *quantum subalgebra of* (A, ρ, s). Since $\rho = (s \otimes s)(\rho)$ it follows that $s(A_\rho) = A_\rho$ by Lemma 1. Thus $(A_\rho, \rho, s|_{A_\rho})$ is a quantum subalgebra of (A, ρ, s). Observe that A_ρ is generated as an algebra by $A_{(\rho)} + A_{(\rho^{-1})} = A_{(\rho)}$ since $\rho^{-1} = (s \otimes 1_A)(\rho)$.

If $(B, \rho, s|_B)$ is a quantum subalgebra of (A, ρ, s) then $A_\rho \subseteq B$; thus (A, ρ, s) has a unique minimal quantum subalgebra which is $(A_\rho, \rho, s|_{A_\rho})$. Notice that the quantum subalgebras of (A, ρ, s) are the quantum algebras of the form (B, ρ', s'), where B is a subalgebra of A and the inclusion $\imath : B \longrightarrow A$ induces a morphism $\imath : (B, \rho', s') \longrightarrow (A, \rho, s)$.

Quantum algebras account for regular isotopy invariants of (unoriented) 1–1 tangles and twist quantum algebras account for regular isotopy invariants of knots and links. See Section 6.2 for details. A *twist quantum algebra over* k is a quadruple (A, ρ, s, G), where (A, ρ, s) is a quantum algebra over k and $G \in A$ is invertible, such that

$$s(G) = G^{-1} \text{ and } s^2(a) = GaG^{-1}$$

for all $a \in A$. Twist quantum algebras arise from ribbon Hopf algebras and from quantum algebras defined on $A = \mathrm{M}_n(k)$. It is important to note that an essential ingredient for the construction of a knot or link invariant from a twist quantum algebra is an s^*-invariant tracelike functional $\mathsf{tr} \in A^*$.

As the referee has observed, every quantum algebra can be embedded in a twist quantum algebra. Specifically, given a quantum algebra (A, ρ, s) over k there is a twist quantum algebra $(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ over k and an algebra embedding $\imath : A \longrightarrow \boldsymbol{A}$ which induces a morphism of quantum algebras $\imath : (A, \rho, s) \longrightarrow (\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s})$.

We construct $(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ as follows. As a vector space over $\boldsymbol{A} = \oplus_{\ell \in Z} A_\ell$, where $A_\ell = \{(\ell, a) \,|\, a \in A\}$ is endowed with the vector space structure which makes the bijection $A \longrightarrow A_\ell$ defined by $a \mapsto (\ell, a)$ a linear isomorphism. The rule $(\ell, a) \cdot (m, b) = (\ell + m, s^{-2m}(a)b)$ for all $\ell, m \in Z$ and $a, b \in A$ gives $\boldsymbol{A}$ an associative algebra structure. The linear endomorphism $\boldsymbol{S}$ of $\boldsymbol{A}$ determined by $\boldsymbol{S}((\ell, a)) = (-\ell, s^{2\ell+1}(a))$ is an algebra isomorphism $\boldsymbol{S} : \boldsymbol{A} \longrightarrow \boldsymbol{A}^{\mathrm{op}}$. The reader can easily check that $(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ is the desired twist quantum algebra, where

$\imath : A \longrightarrow \boldsymbol{A}$ is defined by $\imath(a) = (0, a)$, $\boldsymbol{\rho} = (\imath \otimes \imath)(\rho)$ and $\mathbf{G} = (1, 1)$. The reader is left to supply the few remaining details of the proof of the following result which describes the pair $(\imath, (\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G}))$.

PROPOSITION 1. *Let (A, ρ, s) be a quantum algebra over the field k. Then the pair $(\imath, (\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G}))$ satisfies the following*:

(a) *$(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ is a twist quantum algebra over k and $\imath : (A, \rho, s) \longrightarrow (\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s})$ is a morphism of quantum algebras.*

(b) *If (A', ρ', s', G') is a twist quantum algebra over k and $f : (A, \rho, s) \longrightarrow (A', \rho', s')$ is a morphism of quantum algebras then there exists a morphism $F : (\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G}) \longrightarrow (A', \rho', s', G')$ of twist quantum algebras uniquely determined by $F \circ \imath = f$.* □

Recall that a ribbon Hopf algebra over k is a triple (A, ρ, v), where (A, ρ) is a quasitriangular Hopf algebra with antipode s over k and $v \in A$, such that

(R.0) v is in the center of A,

(R.1) $v^2 = us(u)$,

(R.2) $s(v) = v$,

(R.3) $\varepsilon(v) = 1$ and

(R.4) $\Delta(v) = (v \otimes v)(\rho^{\mathrm{op}}\rho)^{-1} = (\rho^{\mathrm{op}}\rho)^{-1}(v \otimes v)$.

Ribbon Hopf algebras were introduced and studied by Reshetikhin and Turaev in [29]. The element v is referred to as a special element or ribbon element in the literature. See [11] also.

Let (A, ρ, v) be a ribbon Hopf algebra over k, write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$ and let $u = \sum_{\imath=1}^{r} s(b_\imath)a_\imath$ be the Drinfel'd element of the quasitriangular Hopf algebra (A, ρ). Then u is invertible, $s^2(a) = uau^{-1}$ for all $a \in A$ and $\Delta(u) = (u \otimes u)(\rho^{\mathrm{op}}\rho)^{-1} = (\rho^{\mathrm{op}}\rho)^{-1}(u \otimes u)$ by the results of [3]. Since u is invertible it follows by (R.1) that v is invertible. Thus $G = uv^{-1}$ is invertible, is a grouplike element of A and $s^2(a) = GaG^{-1}$ for all $a \in A$. Since G is a grouplike element of A it follows that $s(G) = G^{-1}$. Collecting results:

EXAMPLE 4. Let (A, ρ, v) be a ribbon Hopf algebra with antipode s over k. Then (A, ρ, s, uv^{-1}) is a twist quantum algebra over k, where u is the Drinfel'd element of the quasitriangular Hopf algebra (A, ρ).

For a detailed explanation of the relationship between ribbon and grouplike elements the reader is referred to [11].

Any algebra automorphism t of $\mathrm{M}_n(k)$ has the form $t(a) = GaG^{-1}$ for all $a \in \mathrm{M}_n(k)$, where $G \in \mathrm{M}_n(k)$ is invertible, by the Norther–Skolem Theorem. See the corollary to [7, Theorem 4.3.1]. Such a G is unique up to scalar multiple. Let a^t be the transpose of $a \in \mathrm{M}_n(k)$.

LEMMA 2. *Let $(\mathrm{M}_n(k), \rho, s)$ be a quantum algebra over k.*

(a) *There exists an invertible* $M \in \mathrm{M}_n(k)$ *such that* $s(a) = Ma^tM^{-1}$ *for all* $a \in \mathrm{M}_n(k)$.
(b) $(\mathrm{M}_n(k), \rho, s, M(M^t)^{-1})$ *is a twist quantum algebra over* k.

PROOF. First observe that $t(a) = s(a^t)$ for all $a \in \mathrm{M}_n(k)$ defines an algebra automorphism t of $\mathrm{M}_n(k)$. Thus $t(a) = MaM^{-1}$ for all $a \in \mathrm{M}_n(k)$ for some invertible $M \in \mathrm{M}_n(k)$ by the preceding remarks. Since $s(a) = t(a^t)$ for all $a \in \mathrm{M}_n(k)$ part (a) follows. Part (b) is the result of a straightforward calculation. □

For the quantum algebra of Example 3 we may take $M = q^{-1}E_{1\,2} - qE_{2\,1}$ and thus $G = -(q^{-2}E_{1\,1} + q^2E_{2\,2})$.

4. Oriented Quantum Algebras

Just as the notion of quantum algebra arises in the consideration of algebra associated with diagrams of unoriented knots and links, the notion of oriented quantum algebra arises in connection with diagrams of oriented knots and links [17]. In Section 7 the reader will begin to see the relationship between the axioms for an oriented quantum algebra and oriented diagrams. For a detailed explanation of the topological motivation for the concept of oriented quantum algebra the reader is referred to [17]. An expanded version of most of what follows is found in [16].

An *oriented quantum algebra* over the field k is a quadruple $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$, where (A, ρ) is a Yang–Baxter algebra over k and $t_{\mathsf{d}}, t_{\mathsf{u}}$ are commuting algebra automorphisms of A, such that

(qa.1) $(1_A \otimes t_{\mathsf{u}})(\rho)$ and $(t_{\mathsf{d}} \otimes 1_A)(\rho^{-1})$ are inverses in $A \otimes A^{\mathrm{op}}$, and

(qa.2) $\rho = (t_{\mathsf{d}} \otimes t_{\mathsf{d}})(\rho) = (t_{\mathsf{u}} \otimes t_{\mathsf{u}})(\rho)$.

Suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ and $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ are oriented quantum algebras over k. Then $(A \otimes A', \rho'', t_{\mathsf{d}} \otimes t'_{\mathsf{d}}, t_{\mathsf{u}} \otimes t'_{\mathsf{u}})$ is an oriented quantum algebra over k, which we refer to as the *tensor product of* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *and* $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$, where $(A \otimes A', \rho'')$ is the tensor product of the Yang–Baxter algebras (A, ρ) and (A', ρ'). A *morphism* $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ *of oriented quantum algebras* is a morphism $f : (A, \rho) \longrightarrow (A', \rho')$ of Yang–Baxter algebras over k which satisfies $t'_{\mathsf{d}} \circ f = f \circ t_{\mathsf{d}}$ and $t'_{\mathsf{u}} \circ f = f \circ t_{\mathsf{u}}$. Note that $(k, 1 \otimes 1, 1_k, 1_k)$ is an oriented quantum algebra over k. The category of oriented quantum algebras over k together with their morphisms under composition has a natural monoidal structure.

An oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ over k is *standard* if $t_{\mathsf{d}} = 1_A$ and is *balanced* if $t_{\mathsf{d}} = t_{\mathsf{u}}$, in which case we write (A, ρ, t) for $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$, where $t = t_{\mathsf{d}} = t_{\mathsf{u}}$.

Standard oriented quantum algebras play an important role in the theory of oriented quantum algebras. There is always a standard oriented quantum algebra associated with an oriented quantum algebra.

PROPOSITION 2. *If* $(A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ *is an oriented quantum algebra over* k *then* $(A,\rho,t_{\mathsf{u}}\circ t_{\mathsf{d}},1_A)$ *and* $(A,\rho,1_A,t_{\mathsf{d}}\circ t_{\mathsf{u}})$ *are also.*

PROOF. Apply the algebra automorphisms $t_{\mathsf{u}}\otimes 1_A$ and $1_A\otimes t_{\mathsf{d}}$ of $A\otimes A^{\mathrm{op}}$ to both sides of the equations of (qa.1). □

The oriented quantum algebra $(A,\rho,1_A,t_{\mathsf{d}}\circ t_{\mathsf{u}})$ is the *standard oriented quantum algebra associated with* $(A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$.

The Yang–Baxter algebra of Example 1 has a balanced oriented quantum algebra structure.

EXAMPLE 5. Let $n\geq 2$, $a,bc\in k^\star$ satisfy $a^2\neq bc,1$ and suppose $\omega_1,\ldots,\omega_n\in k^\star$ satisfy

$$\omega_\imath^2=\left(\frac{a^2}{bc}\right)^{\imath-1}\omega_1^2$$

for all $1\leq\imath\leq n$. Then $(\mathrm{M}_n(k),\rho_{a,\mathsf{B},\mathsf{C}},t)$ is a balanced oriented quantum algebra, where

$$t(E_{\imath\,\jmath})=\left(\frac{\omega_\imath}{\omega_\jmath}\right)E_{\imath\,\jmath}$$

for all $1\leq\imath,\jmath\leq n$.

Example 5 is considered in more generality in [16, Theorem 2].

Suppose that $(A,\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ is an oriented quantum algebra over k and write $\rho=\sum_{\imath=1}^r a_\imath\otimes b_\imath$, $\rho^{-1}=\sum_{\jmath=1}^s \alpha_\jmath\otimes\beta_\jmath\in A\otimes A$. Then axioms (qa.1) and (qa.2) can be formulated

$$\sum_{\imath=1}^r\sum_{\jmath=1}^s a_\imath t_{\mathsf{d}}(\alpha_\jmath)\otimes\beta_\jmath t_{\mathsf{u}}(b_\imath)=1\otimes 1=\sum_{\jmath=1}^s\sum_{\imath=1}^r t_{\mathsf{d}}(\alpha_\jmath)a_\imath\otimes t_{\mathsf{u}}(b_\imath)\beta_\jmath \tag{4–1}$$

and

$$\sum_{\imath=1}^r a_\imath\otimes b_\imath=\sum_{\imath=1}^r t_{\mathsf{d}}(a_\imath)\otimes t_{\mathsf{d}}(b_\imath)=\sum_{\imath=1}^r t_{\mathsf{u}}(a_\imath)\otimes t_{\mathsf{u}}(b_\imath). \tag{4–2}$$

respectively. Alterations to the structure of A determine other oriented quantum algebras.

Observe that $(A^{\mathrm{op}},\rho,t_{\mathsf{d}},t_{\mathsf{u}})$ is an oriented quantum algebra over k in light of (2–2) and (4–1), which we denote by A^{op} as well. We have noted that (A,ρ^{-1}) is a quantum Yang–Baxter algebra over k. Let $t=t_{\mathsf{d}}$ or $t=t_{\mathsf{u}}$. Since $t\otimes t$ is an algebra automorphism of $A\otimes A$ and $\rho=(t\otimes t)(\rho)$ we have $\rho^{-1}=(t\otimes t)(\rho^{-1})$. By applying $t_{\mathsf{d}}^{-1}\otimes t_{\mathsf{u}}^{-1}$ to both sides of the equations of (4–1) we see that $(A,\rho^{-1},t_{\mathsf{d}}^{-1},t_{\mathsf{u}}^{-1})$ is an oriented quantum algebra over k. Notice that $(A,\rho^{\mathrm{op}},t_{\mathsf{u}}^{-1},t_{\mathsf{d}}^{-1})$ is an oriented quantum algebra over k.

Let K be a field extension of k and $A\otimes K$ be the algebra over K obtained by extension of scalars. Then $(A\otimes K,\rho\otimes 1_K,t_{\mathsf{d}}\otimes 1_K,t_{\mathsf{u}}\otimes 1_K)$ is a unoriented quantum algebra structure over K, where $(A\otimes K,\rho\otimes 1_K)$ is the Yang–Baxter algebra described in Section 2.

Certain quotients of A have an oriented quantum algebra structure. Let I be an ideal of A and suppose that $t_{\mathsf{d}}(I) = t_{\mathsf{u}}(I) = I$. Then there is a unique oriented quantum algebra structure $(A/I, \overline{\rho}, \overline{t}_{\mathsf{d}}, \overline{t}_{\mathsf{u}})$ on the quotient algebra A/I such that $\pi : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A/I, \overline{\rho}, \overline{t}_{\mathsf{d}}, \overline{t}_{\mathsf{u}})$ is a morphism, where $\pi : A \longrightarrow A/I$ is the projection. Furthermore, if $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ is a morphism of oriented quantum algebras and $f : A \longrightarrow A'$ is onto, then $(A/\mathrm{ker} f, \overline{\rho}, \overline{t}_{\mathsf{d}}, \overline{t}_{\mathsf{u}})$ and $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ are isomorphic oriented quantum algebras. The reader is left to formulate and prove fundamental homomorphism theorems for oriented quantum algebras.

The oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is a *minimal oriented quantum algebra over* k if $A = A_\rho$. As in the case of quantum algebras, the notion of minimal oriented quantum algebra is theoretically important.

Let B be a subalgebra of A which satisfies $\rho, \rho^{-1} \in B \otimes B$ and $t_{\mathsf{d}}(B) = t_{\mathsf{u}}(B) = B$. Then $(B, \rho, t_{\mathsf{d}}|_B, t_{\mathsf{u}}|_B)$ is an oriented quantum algebra over k which is called an *oriented quantum subalgebra of* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$. Since $\rho = (t_{\mathsf{d}} \otimes t_{\mathsf{d}})(\rho) = (t_{\mathsf{u}} \otimes t_{\mathsf{u}})(\rho)$ it follows that $t_{\mathsf{d}}(A_\rho) = t_{\mathsf{u}}(A_\rho) = A_\rho$ by Lemma 1. Therefore $(A_\rho, \rho, t_{\mathsf{d}}|_{A_\rho}, t_{\mathsf{u}}|_{A_\rho})$ is an oriented quantum subalgebra of $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$.

If $(B, \rho, t_{\mathsf{d}}|_B, t_{\mathsf{u}}|_B)$ is an oriented quantum subalgebra of $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ then $A_\rho \subseteq B$; thus $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ has a unique minimal oriented quantum subalgebra which is $(A_\rho, t_{\mathsf{d}}|_{A_\rho}, t_{\mathsf{u}}|_{A_\rho})$. Oriented quantum subalgebras of $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ are those oriented quantum algebras $(B, \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ over k, where B is a subalgebra of A and the inclusion $\imath : B \longrightarrow A$ determines a morphism $\imath : (B, \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ of oriented quantum algebras.

Minimal Yang–Baxter algebras over k support at most one standard oriented quantum algebra structure. Observe that if (A, ρ) is a Yang–Baxter algebra over k and A is commutative, then $(A, \rho, 1_A, 1_A)$ is an oriented quantum algebra over k.

PROPOSITION 3. *Let* (A, ρ) *be a minimal Yang–Baxter quantum algebra over* k.

(a) *There is at most one algebra automorphism* t *of* A *such that* $(A, \rho, 1_A, t)$ *is a standard oriented quantum algebra over* k.
(b) *Suppose that* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *and* $(A, \rho, t'_{\mathsf{d}}, t'_{\mathsf{u}})$ *are oriented quantum algebras over* k. *Then* $t_{\mathsf{d}} \circ t_{\mathsf{u}} = t'_{\mathsf{d}} \circ t'_{\mathsf{u}}$.
(c) *Suppose that* A *is commutative and* $(A, \rho, 1_A, t)$ *is a standard oriented quantum algebra over* k. *Then* $t = 1_A$.

PROOF. Part (b) follows from part (a) and Proposition 2. We have noted that $(A, \rho, 1_A, 1_A)$ is an oriented quantum algebra over k when A is commutative. Thus part (c) follows from part (a) also.

To show part (a), suppose that $(A, \rho, 1_A, t)$ and $(A, \rho, 1_A, t')$ are standard oriented quantum algebras over k. Then $(1_A \otimes t)(\rho) = (1_A \otimes t')(\rho)$ since both sides of the equation are left inverses of ρ^{-1} in $A \otimes A^{\mathrm{op}}$. This equation together with (qa.2) implies $(t \otimes 1_A)(\rho) = (t' \otimes 1_A)(\rho)$ as well. These two equations imply $t|_{A_{(\rho)}} = t'|_{A_{(\rho)}}$. Now the first two equations also imply $(1_A \otimes t)(\rho^{-1}) =$

$(1_A \otimes t')(\rho^{-1})$ and $(t \otimes 1_A)(\rho^{-1}) = (t' \otimes 1_A)(\rho^{-1})$ since the maps involved are algebra automorphisms of $A \otimes A^{\mathrm{op}}$. Therefore $t|_{A_{(\rho^{-1})}} = t'|_{A_{(\rho^{-1})}}$. Since $A_{(\rho)} + A_{(\rho^{-1})}$ generates A as an algebra it now follows that $t = t'$. $\square$

Every quantum algebra accounts for a standard oriented quantum algebra.

THEOREM 1. *Suppose that (A, ρ, s) is a quantum algebra over the field k. Then $(A, \rho, 1_A, s^{-2})$ is an oriented quantum algebra over k.*

PROOF. Write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$. Now (A, ρ) is a Yang–Baxter algebra over k since (A, ρ, s) is a quantum algebra over k. Since (QA.2) holds for s and ρ it follows that (qa.2) holds for s^{-2} and ρ. The fact that $\rho^{-1} = \sum_{\imath=1}^{r} s(a_\imath) \otimes b_\imath$ translates to

$$\sum_{\imath,\jmath=1}^{r} s(a_\imath) a_\jmath \otimes b_\imath b_\jmath = 1 \otimes 1 = \sum_{\jmath,\imath=1}^{r} a_\jmath s(a_\imath) \otimes b_\jmath b_\imath.$$

Applying $s \otimes 1_A$ to both sides of these equations yields

$$\sum_{\jmath,\imath=1}^{r} s(a_\jmath) s^2(a_\imath) \otimes b_\imath b_\jmath = 1 \otimes 1 = \sum_{\imath,\jmath=1}^{r} s^2(a_\imath) s(a_\jmath) \otimes b_\jmath b_\imath.$$

Since $\rho = (s^2 \otimes s^2)(\rho)$ it follows that

$$\sum_{\jmath,\imath=1}^{r} s(a_\jmath) a_\imath \otimes s^{-2}(b_\imath) b_\jmath = 1 \otimes 1 = \sum_{\imath,\jmath=1}^{r} a_\imath s(a_\jmath) \otimes b_\jmath s^{-2}(b_\imath);$$

that is ρ^{-1} and $(1_A \otimes s^{-2})(\rho)$ are inverses in $A \otimes A^{\mathrm{op}}$. $\square$

As a consequence of part (c) of Proposition 3 and the preceding theorem:

COROLLARY 1. *If (A, ρ, s) is a minimal quantum algebra over k and A is commutative then $s^2 = 1_A$.* $\square$

Let (A, ρ, s) be a quantum algebra over k. Then $(A, \rho, 1_A, s^{-2})$ and $(A, \rho, s^{-2}, 1_A)$ are oriented quantum algebras over k by Theorem 1 and Proposition 2. It may very well be the case that these are the only oriented quantum algebra structures of the form $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$. Sweedler's 4-dimensional Hopf algebra A when the characteristic of k is not 2 illustrates the point. We recall that A is generated as a k-algebra by a, x subject to the relations $a^2 = 1, x^2 = 0, xa = -ax$ and the coalgebra structure of A is determined by $\Delta(a) = a \otimes a$, $\Delta(x) = x \otimes a + 1 \otimes x$.

EXAMPLE 6. Let A be Sweedler's 4-dimensional Hopf algebra with antipode s over k, suppose that the characteristic of k is not 2 and for $\alpha \in k^\star$ let

$$\rho_\alpha = \frac{1}{2}(1 \otimes 1 + 1 \otimes a + a \otimes 1 - a \otimes a) + \frac{\alpha}{2}(x \otimes x + x \otimes ax + ax \otimes ax - ax \otimes x).$$

Then (A, ρ_α, s) is a minimal quantum algebra over k; moreover, $(A, \rho_\alpha, 1_A, s^{-2})$ and $(A, \rho_\alpha, s^{-2}, 1_A)$ are the only oriented quantum algebra structures of the form $(A, \rho_\alpha, t_{\mathsf{d}}, t_{\mathsf{u}})$.

If $(A, \rho, 1_A, t)$ is a standard oriented quantum algebra over k then there may be no quantum algebra of the form (A, ρ, s).

EXAMPLE 7. There is no quantum algebra of the form $(\mathrm{M}_n(k), \rho_{a,\mathsf{B},\mathsf{C}}, s)$, where $n > 2$ and $(\mathrm{M}_n(k), \rho_{a,\mathsf{B},\mathsf{C}}, t)$ is the balanced oriented quantum algebra of Example 5.

However, when $n = 2$ we have the important connection:

EXAMPLE 8. Let $(\mathrm{M}_2(k), \rho_{q^{-1},\{q\},\{q\}}, t)$ be the balanced quantum algebra of Example 5 where $n = 2$ and $\omega_1 = q, \omega_2 = q^{-1}$. Then the associated standard oriented quantum algebra

$$(\mathrm{M}_2(k), \rho_{q^{-1},\{q\},\{q\}}, 1_{\mathrm{M}_2(k)}, t^2) = (\mathrm{M}_2(k), \rho, 1_{\mathrm{M}_2(k)}, s^{-2}),$$

where $(\mathrm{M}_2(k), \rho, s)$ is the quantum algebra over k of Example 3.

Let A be a finite-dimensional Hopf algebra with antipode s over k. The quasitriangular Hopf algebra $(D(A), \boldsymbol{\rho})$ admits an oriented quantum algebra structure $(D(A), \boldsymbol{\rho}, 1_{D(A)}, {s^*}^2 \otimes s^{-2})$ by Theorem 1. If t is a Hopf algebra automorphism of A which satisfies $t^2 = s^{-2}$ then $(D(A), \boldsymbol{\rho}, {t^*}^{-1} \otimes t)$ is a balanced oriented quantum algebra over k.

An oriented quantum algebra defines a regular isotopy invariant of oriented 1–1 tangles as well shall see in Section 7. An oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ with the additional structure of an invertible $G \in A$ which satisfies

$$t_{\mathsf{d}}(G) = t_{\mathsf{u}}(G) = G \quad \text{and} \quad t_{\mathsf{d}} \circ t_{\mathsf{u}}(a) = GaG^{-1}$$

for all $a \in A$ accounts for regular isotopy invariants of knots and links. The quintuple $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ is called a *twist oriented quantum algebra over* k. We note that an important ingredient for the definition of these invariants is a $t_{\mathsf{d}}, t_{\mathsf{u}}$-invariant tracelike element $\mathsf{tr} \in A^*$.

Let $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ and $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}, G')$ be twist oriented quantum algebras over k. Then $(A \otimes A', \rho'', t_{\mathsf{d}} \otimes t'_{\mathsf{d}}, t_{\mathsf{u}} \otimes t'_{\mathsf{u}}, G \otimes G')$ is a twist oriented quantum algebra over k, called the *tensor product of* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ *and* $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}, G')$, where $(A \otimes A', \rho'', t_{\mathsf{d}} \otimes t'_{\mathsf{d}}, t_{\mathsf{u}} \otimes t'_{\mathsf{u}})$ is the tensor product of $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ and $(A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$. A *morphism* $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}, G')$ *of twist quantum oriented algebras* over k is a morphism $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ of oriented quantum algebras which satisfies $f(G) = G'$. The category of twist oriented quantum algebras over k and their morphisms under composition has a natural monoidal structure.

Suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ is a twist oriented quantum algebra over k. Then $(A, \rho, 1_A, t_{\mathsf{d}} \circ t_{\mathsf{u}}, G)$ is as well. There are two important cases in which a standard oriented quantum algebra over k has a twist structure.

Let (A, ρ) be a quasitriangular Hopf algebra with antipode s over k, write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$ and let $u = \sum_{\imath=1}^{r} s(b_\imath) a_\imath$ be the Drinfel'd element of A. Then $(A, \rho, 1_A, s^{-2}, u^{-1})$ is a twist oriented quantum algebra over k.

Now suppose $A = \mathrm{M}_n(k)$ and that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is a standard, or balanced, oriented quantum algebra over k. We have noticed in the remarks preceding Lemma 2 that any algebra automorphism t of A is described by $t(a) = GaG^{-1}$ for all $a \in A$, where $G \in A$ is invertible and is unique up to scalar multiple. Thus $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ is a twist oriented quantum algebra over k for some invertible $G \in A$. For instance the balanced oriented quantum algebra $(\mathrm{M}_n(k), \rho_{a,\mathsf{B},\mathsf{C}}, t)$ of Example 5 has a twist balanced oriented quantum algebra structure $(\mathrm{M}_n(k), \rho_{a,\mathsf{B},\mathsf{C}}, t, t, G)$, where $G = \sum_{\imath=1}^{n} \omega_\imath^2 E_{\imath\,\imath}$.

Just as a quantum algebra can be embedded into a twist quantum algebra, an oriented standard quantum algebra can be embedded into a twist oriented standard quantum algebra. The construction of the latter is a slight modification of the construction of $(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ which precedes the statement of Proposition 1. Here $(\ell, a)\cdot(m, b) = (\ell + m, t^{-m}(a)b)$ and $\boldsymbol{s}$ is replaced by $\boldsymbol{t}$ defined by $\boldsymbol{t}((\ell, a)) = (\ell, t(a))$.

Proposition 4. *Let $(A, \rho, 1_A, t)$ be a standard oriented quantum algebra over the field k. The pair $(\imath, (\boldsymbol{A}, \boldsymbol{\rho}, 1_{\boldsymbol{A}}, \boldsymbol{t}, \boldsymbol{G}))$ satisfies the following*:

(a) *$(\boldsymbol{A}, \boldsymbol{\rho}, 1_{\boldsymbol{A}}, \boldsymbol{t}, \boldsymbol{G})$ is a standard twist oriented quantum algebra over k and $\imath : (A, \rho, 1_A, t) \longrightarrow (\boldsymbol{A}, \boldsymbol{\rho}, 1_{\boldsymbol{A}}, \boldsymbol{t})$ is a morphism of oriented quantum algebras.*
(b) *If $(A', \rho', 1_{A'}, t', G')$ is a twist standard oriented quantum algebra over k and $f : (A, \rho, 1_A, t) \longrightarrow (A', \rho', 1_{A'}, t')$ is a morphism of quantum algebras there exists a morphism $F : (\boldsymbol{A}, \boldsymbol{\rho}, 1_{\boldsymbol{A}}, \boldsymbol{t}, \boldsymbol{G}) \longrightarrow (A', \rho', 1_{A'}, t', G')$ of twist standard oriented quantum algebras uniquely determined by $F \circ \imath = f$.* □

We end this section with a necessary and sufficient condition for a Yang–Baxter algebra over k to have a twist oriented quantum algebra structure.

Proposition 5. *Let (A, ρ) be a Yang–Baxter algebra over the field k, write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$ and $\rho^{-1} = \sum_{\jmath=1}^{s} \alpha_\jmath \otimes \beta_\jmath$, suppose $G \in A$ is invertible and let t be the algebra automorphism of A defined by $t(a) = GaG^{-1}$ for all $a \in A$. Then $(A, \rho, 1_A, t, G)$ is a twist oriented quantum algebra over k if and only if*

(a) *$G \otimes G$ and ρ are commuting elements of the algebra $A \otimes A$,*
(b) *$\sum_{\imath=1}^{r} \sum_{\jmath=1}^{s} a_\imath G \alpha_\jmath \otimes \beta_\jmath b_\imath = G \otimes 1$ and*
(c) *$\sum_{\jmath=1}^{s} \sum_{\imath=1}^{r} \alpha_\jmath G^{-1} a_\imath \otimes b_\imath \beta_\jmath = G^{-1} \otimes 1$.* □

When A is finite-dimensional the conditions of parts (b) and (c) of the proposition are equivalent.

5. Quantum Algebras Constructed from Standard Oriented Quantum Algebras

By Theorem 1 a quantum algebra (A, ρ, s) accounts for a standard oriented quantum algebra $(A, \rho, 1_A, s^{-2})$. By virtue of Example 7 if $(A, \rho, 1_A, t)$ is a standard oriented quantum algebra over k there may be no quantum algebra of the form (A, ρ, s).

Let $(A, \rho, 1_A, t)$ be a standard oriented quantum algebra over k. In this section we show that there is a quantum algebra structure $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s})$ on the direct product $\mathcal{A} = A \oplus A^{\mathrm{op}}$ such that the projection $\pi : \mathcal{A} \longrightarrow A$ onto the first factor determines a morphism of standard oriented quantum algebras $\pi : (\mathcal{A}, \boldsymbol{\rho}, 1_{\mathcal{A}}, \boldsymbol{s}^{-2}) \longrightarrow (A, \rho, 1_A, t)$. Our result follows from a construction which starts with an oriented quantum algebra $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{d}})$ and produces a quantum algebra $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s})$ and oriented quantum algebra $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t}_{\mathsf{d}}, \boldsymbol{t}_{\mathsf{u}})$ which are related by $\boldsymbol{t}_{\mathsf{d}} \circ \boldsymbol{t}_{\mathsf{u}} = \boldsymbol{s}^{-2}$. We follow [15, Section 3].

Let A be an algebra over k and let $\mathcal{A} = A \oplus A^{\mathrm{op}}$ be the direct product of the algebras A and A^{op}. Denote the linear involution of $\mathcal{A}$ which exchanges the direct summands of $\mathcal{A}$ by $\overline{(\ )}$. Thus $\overline{a \oplus b} = b \oplus a$ for all $a, b \in A$. Think of A as a subspace of $\mathcal{A}$ by the identification $a = a \oplus 0$ for all $a \in A$. Thus $\overline{a} = 0 \oplus a$ and every element of $\mathcal{A}$ has a unique decomposition of the form $a + \overline{b}$ for some $a, b \in A$. Note that

$$\overline{(\overline{a})} = a, \quad \overline{ab} = \overline{b}\,\overline{a} \ \text{ and } \ a\overline{b} = 0 = \overline{a}b \tag{5–1}$$

for all $a, b \in A$.

Lemma 3. *Let $(A, \rho, 1_A, t)$ be a standard oriented quantum algebra over k, let $\mathcal{A} = A \oplus A^{\mathrm{op}}$ be the direct product of A and A^{op} and write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$, $\rho^{-1} = \sum_{\jmath=1}^{s} \alpha_\jmath \otimes \beta_\jmath$. Then $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s})$ is a quantum algebra over k, where*

$$\boldsymbol{\rho} = \sum_{\imath=1}^{r} (a_\imath \otimes b_\imath + \overline{a_\imath} \otimes \overline{b_\imath}) + \sum_{\jmath=1}^{s} (\overline{\alpha_\jmath} \otimes \beta_\jmath + \alpha_\jmath \otimes \overline{t^{-1}(\beta_\jmath)})$$

and $\boldsymbol{s}(a \oplus b) = b \oplus t^{-1}(a)$ for all $a, b \in A$.

Proof. Since t is an algebra automorphism of A it follows that t^{-1} is also. Thus $\boldsymbol{s} : \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{op}}$ is an algebra isomorphism. By definition $\boldsymbol{s}(a) = \overline{t^{-1}(a)}$ and $\boldsymbol{s}(\overline{a}) = a$ for all $a \in A$. We have noted in the discussion following (4–2) that $\rho^{-1} = (t^{-1} \otimes t^{-1})(\rho^{-1})$. At this point it is easy to see that $\boldsymbol{\rho} = (\boldsymbol{s} \otimes \boldsymbol{s})(\boldsymbol{\rho})$, or (QA.2) is satisfied for $\boldsymbol{\rho}$ and $\boldsymbol{s}$. Using the equation $\rho^{-1} = (t^{-1} \otimes t^{-1})(\rho^{-1})$ we calculate

$$(s \otimes 1_{\mathcal{A}})(\boldsymbol{\rho}) = \sum_{\imath=1}^{r} \left(\overline{t^{-1}(a_\imath)} \otimes b_\imath + a_\imath \otimes \overline{b_\imath} \right) + \sum_{\jmath=1}^{s} \left(\alpha_\jmath \otimes \beta_\jmath + \overline{\alpha_\jmath} \otimes \overline{\beta_\jmath} \right).$$

Using (5–1), the equation $(t^{-1} \otimes 1_A)(\rho) = (1_A \otimes t)(\rho)$, which follows by (qa.2),

$$\boldsymbol{\rho}((s \otimes 1_{\mathcal{A}})(\boldsymbol{\rho})) = 1 \otimes 1 + \overline{1} \otimes \overline{1} + \overline{1} \otimes 1 + 1 \otimes \overline{1} = 1_{\mathcal{A}} \otimes 1_{\mathcal{A}} = ((s \otimes 1_{\mathcal{A}})(\boldsymbol{\rho}))\boldsymbol{\rho}.$$

Therefore $\boldsymbol{\rho}$ is invertible and $\boldsymbol{\rho}^{-1} = (\boldsymbol{s} \otimes 1_A)(\boldsymbol{\rho})$. We have shown that (QA.1) holds for $\boldsymbol{\rho}$ and $\boldsymbol{s}$.

The fact that $\boldsymbol{\rho}$ satisfies (2–1) is a rather lengthy and interesting calculation. Using the formulation (2–2) of (2–1) one sees that (2–1) for $\boldsymbol{\rho}$ is equivalent to a set of eight equations. With the notation convention $(\rho^{-1})_{\imath\jmath} = \rho^{-1}_{\imath\jmath}$ for

$1 \leq \imath < \jmath \leq 3$, this set of eight equations can be rewritten as set of six equations which are:

$$\rho_{1\,2}\rho_{1\,3}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,3}\rho_{1\,2}, \tag{5–2}$$

$$\rho_{1\,2}\rho_{2\,3}^{-1}\rho_{1\,3}^{-1} = \rho_{1\,3}^{-1}\rho_{2\,3}^{-1}\rho_{1\,2}, \tag{5–3}$$

$$\rho_{1\,3}^{-1}\rho_{1\,2}^{-1}\rho_{2\,3} = \rho_{2\,3}\rho_{1\,2}^{-1}\rho_{1\,3}^{-1}, \tag{5–4}$$

$$\sum_{\ell=1}^{r}\sum_{\jmath,m=1}^{s} a_\ell\alpha_\jmath\otimes\beta_\jmath\alpha_m\otimes t^{-1}(\beta_m)b_\ell = \sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m), \tag{5–5}$$

$$\sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m t^{-1}(\beta_\jmath)\otimes b_\ell\beta_m = \sum_{\ell=1}^{r}\sum_{\jmath,m=1}^{s} a_\ell\alpha_\jmath\otimes t^{-1}(\beta_\jmath)\alpha_m\otimes\beta_m b_\ell \tag{5–6}$$

and

$$\sum_{\jmath,\ell=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath\alpha_\ell\otimes a_m t^{-1}(\beta_\jmath)\otimes b_m t^{-1}(\beta_\ell) = \sum_{\ell,\jmath=1}^{s}\sum_{m=1}^{r}\alpha_\ell\alpha_\jmath\otimes t^{-1}(\beta_\jmath)a_m\otimes t^{-1}(\beta_\ell)b_m. \tag{5–7}$$

By assumption (5–2) holds. Since $\rho_{\imath\,\jmath}$ is invertible and $(\rho_{\imath\,\jmath})^{-1} = (\rho^{-1})_{\imath\,\jmath} = \rho_{\imath\,\jmath}^{-1}$, equations (5–3)–(5–4) hold by virtue of (5–2).

Now t^{-1} is an algebra automorphism of A and $\rho^{-1} = (t^{-1}\otimes t^{-1})(\rho^{-1})$. Thus applying $1_A\otimes t^{-1}\otimes 1_A$ to both sides of the equation of (5–5) we see that (5–5) and (5–6) are equivalent; applying $t^{-1}\otimes 1_A\otimes 1$ to both sides of (5–7) we see that (5–7) is equivalent to $\rho_{2\,3}\rho_{1\,2}^{-1}\rho_{1\,3}^{-1} = \rho_{1\,3}^{-1}\rho_{1\,2}^{-1}\rho_{2\,3}$, a consequence of (5–2). Therefore to complete the proof of the lemma we need only show that (5–5) holds.

By assumption $(1_A\otimes t)(\rho)$ and ρ^{-1} are inverses in $A\otimes A^{\mathrm{op}}$. Consequently ρ and $(1_A\otimes t^{-1})(\rho)$ are inverses in $A\otimes A^{\mathrm{op}}$ since $1_A\otimes t^{-1}$ is an algebra endomorphism of $A\otimes A^{\mathrm{op}}$. Recall that ρ^{-1} satisfies (2–1). Thus

$$\begin{aligned}
&\sum_{\jmath,m=1}^{s}\sum_{\ell=1}^{r}\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m)\\
&=\sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}(a_v\alpha_u)\alpha_\jmath a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m)(t^{-1}(\beta_u)b_v)\\
&=\sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v(\alpha_u\alpha_\jmath)a_\ell\otimes\alpha_m\beta_\jmath\otimes b_\ell t^{-1}(\beta_m\beta_u)b_v\\
&=\sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v(\alpha_\jmath\alpha_u)a_\ell\otimes\beta_\jmath\alpha_m\otimes b_\ell t^{-1}(\beta_u\beta_m)b_v\\
&=\sum_{v,\ell=1}^{r}\sum_{u,\jmath,m=1}^{s}a_v\alpha_\jmath(\alpha_u a_\ell)\otimes\beta_\jmath\alpha_m\otimes(b_\ell t^{-1}(\beta_u))t^{-1}(\beta_m)b_v\\
&=\sum_{v=1}^{r}\sum_{\jmath,m=1}^{s}a_v\alpha_\jmath\otimes\beta_\jmath\alpha_m\otimes t^{-1}(\beta_m)b_v.
\end{aligned}$$

which establishes (5–5). □

Theorem 2. *Let* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *be an oriented quantum algebra over the field* k, *let* $\mathcal{A} = A \oplus A^{\mathrm{op}}$ *be the direct product of* A *and* A^{op} *and write* $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath$, $\rho^{-1} = \sum_{\jmath=1}^{s} \alpha_\jmath \otimes \beta_\jmath$. *Then*:

(a) $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s})$ *is a quantum algebra over* k, *where*

$$\boldsymbol{\rho} = \sum_{\imath=1}^{r} (a_\imath \otimes b_\imath + \overline{a_\imath} \otimes \overline{b_\imath}) + \sum_{\jmath=1}^{s} (\overline{\alpha_\jmath} \otimes \beta_\jmath + \alpha_\jmath \otimes \overline{t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1}(\beta_\jmath)})$$

and $\boldsymbol{s}(a \oplus b) = b \oplus t_{\mathsf{d}}^{-1} \circ t_{\mathsf{u}}^{-1}(a)$ *for all* $a, b \in A$.

(b) $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}})$ *is an oriented quantum algebra over* k, $\boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}}$ *commute with* $\boldsymbol{s}$ *and* $\boldsymbol{t_{\mathsf{d}}} \circ \boldsymbol{t_{\mathsf{u}}} = \boldsymbol{s}^{-2}$, *where* $\boldsymbol{t_{\mathsf{d}}}(a \oplus b) = t_{\mathsf{d}}(a) \oplus t_{\mathsf{d}}(b)$ *and* $\boldsymbol{t_{\mathsf{u}}}(a \oplus b) = t_{\mathsf{u}}(a) \oplus t_{\mathsf{u}}(b)$ *for all* $a, b \in A$.

(c) *The projection* $\pi : \mathcal{A} \longrightarrow A$ *onto the first factor determines a morphism* $\pi : (\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *of oriented quantum algebras*.

Proof. Since $(A, \rho, 1_A, t_{\mathsf{d}} \circ t_{\mathsf{u}})$ is a standard quantum algebra over k by Proposition 2, part (a) follows by Lemma 3. Part (b) is a straightforward calculation which is left to the reader and part (c) follows by definitions. □

Let $\mathcal{C}_q$ be the category whose objects are quintuples $(A, \rho, s, t_{\mathsf{d}}, t_{\mathsf{u}})$, where (A, ρ, s) is a quantum algebra over k and $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra over k such that $t_{\mathsf{d}}, t_{\mathsf{u}}$ commute with s and $t_{\mathsf{d}} \circ t_{\mathsf{u}} = s^{-2}$, and whose morphisms $f : (A, \rho, s, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ are algebra maps $f : A \longrightarrow A'$ which determine morphisms $f : (A, \rho, s) \longrightarrow (A', \rho', s')$ and $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$. The construction $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{d}}})$ of Theorem 2 is a cofree object of $\mathcal{C}_q$. Let $\pi : \mathcal{A} \longrightarrow A$ be the projection onto the first factor.

Proposition 6. *Let* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *be an oriented quantum algebra over the field* k. *Then the pair* $((\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}}), \pi)$ *satisfies the following properties*:

(a) $(\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}})$ *is an object of* $\mathcal{C}_q$ *and* $\pi : (\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *is a morphism of oriented quantum algebras*.

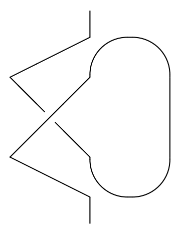
(b) *Suppose that* $(A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ *is an object of* $\mathcal{C}_q$ *and that* $f : (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ *is a morphism of oriented quantum algebras. Then there is a morphism* $F : (A', \rho', s', t'_{\mathsf{d}}, t'_{\mathsf{u}}) \longrightarrow (\mathcal{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{t_{\mathsf{d}}}, \boldsymbol{t_{\mathsf{u}}})$ *uniquely determined by* $\pi \circ F = f$. □

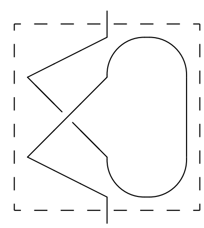
6. Invariants Constructed from Quantum Algebras Via Bead Sliding

We describe a regular isotopy invariant of unoriented 1–1 tangle diagrams determined by a quantum algebra and show how the construction of this tangle invariant is modified to give a regular isotopy invariant of unoriented knot and link diagrams when the quantum algebra is replaced by a twist quantum algebra. Our discussion is readily adapted to handle the oriented case in Section 7; there quantum algebras and twist quantum algebras are replaced by oriented quantum algebras and twist oriented quantum algebras respectively. In both cases the invariants of diagrams determine invariants of 1–1 tangles, knots and links.

What follows is based on [19, Sections 6, 8]. Throughout this section all diagrams are unoriented.

6.1. Invariants of 1–1 **tangles arising from quantum algebras.** We represent 1–1 tangles as diagrams in the plane situated with respect to a fixed vertical. On the left below is a very simple 1–1 tangle diagram

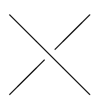
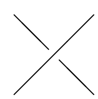




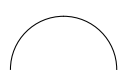
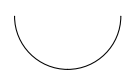
which we refer to as $\mathsf{T}_{\rm curl}$. We require that 1–1 tangle diagrams can be drawn in a box except for two protruding line segments as indicated by the figure on the right above. Let *Tang* be the set of all 1–1 tangle diagrams in the plane situated with respect to the given vertical.

All 1–1 tangle diagrams consist of some or all of the following components:

- crossings;

over crossings  *under crossings* 

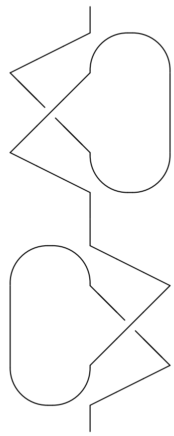
- local extrema;

local maxima  *local minima* 

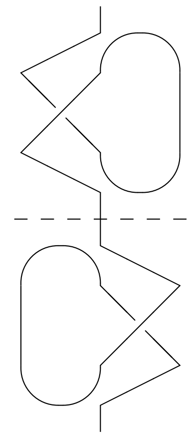
and

- "vertical" lines.

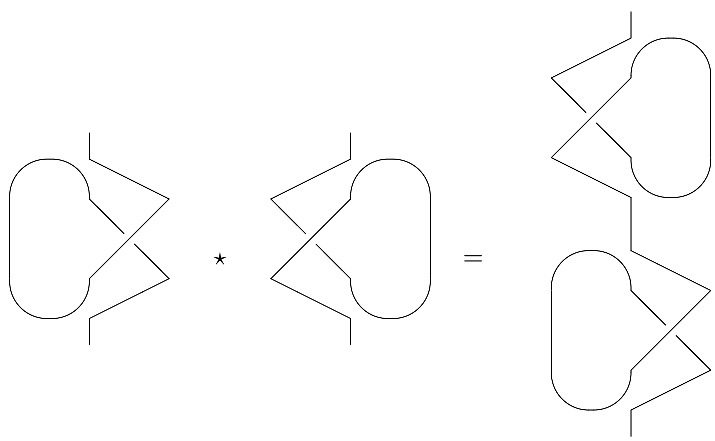
There is a natural product decomposition of 1–1 tangle diagrams in certain situations. When a 1–1 tangle diagram T can be written as the union of two 1–1 tangle diagrams T_1 and T_2 such that the top point of T_1 is the base point of T_2, and the line passing through this common point perpendicular to the vertical otherwise separates T_1 and T_2, then T is called the *product of* T_1 and T_2 and this relationship is expressed by $\mathsf{T} = \mathsf{T}_1 \star \mathsf{T}_2$. For example,



can be split into two parts



and thus



Multiplication is an associative operation. Those $\mathsf{T} \in \mathit{Tang}$ which consist only of a vertical line can be viewed as local neutral elements with respect multiplication in *Tang*.

Let (A, ρ, s) be a quantum algebra over k. We will construct a function $\mathsf{Inv}_A : \mathit{Tang} \longrightarrow A$ which satisfies the following axioms:

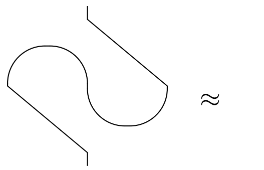
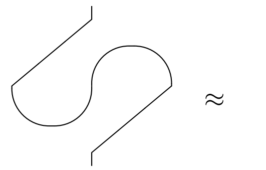
(T.1) If $\mathsf{T}, \mathsf{T}' \in \mathit{Tang}$ are regularly isotopic then $\mathsf{Inv}_A(\mathsf{T}) = \mathsf{Inv}_A(\mathsf{T}')$,

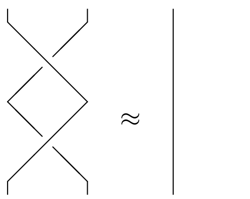
(T.2) If $\mathsf{T} \in \mathit{Tang}$ has no crossings then $\mathsf{Inv}_A(\mathsf{T}) = 1$, and

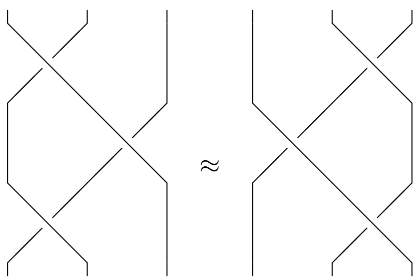
(T.3) $\mathsf{Inv}_A(\mathsf{T}\star\mathsf{T}') = \mathsf{Inv}_A(\mathsf{T})\mathsf{Inv}_A(\mathsf{T}')$ whenever $\mathsf{T}, \mathsf{T}' \in \mathit{Tang}$ and $\mathsf{T}\star\mathsf{T}'$ is defined.

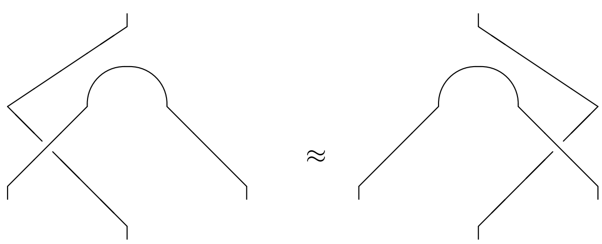
The first axiom implies that Inv_A defines a regular isotopy invariant of 1–1 tangles.

Regular isotopy describes a certain topological equivalence of 1–1 tangle diagrams (and of knot and link diagrams). For the purpose of defining invariants we may view regular isotopy in rather simplistically: $\mathsf{T}, \mathsf{T}' \in \mathit{Tang}$ are *regularly* *isotopic* if T can be transformed to T' by a finite number of local substitutions described in (M.1)–(M.4) below and (M.2rev)–(M.4rev). The symbolism $A \approx B$ in the figures below means that configuration A can be substituted for configuration B and vice versa.

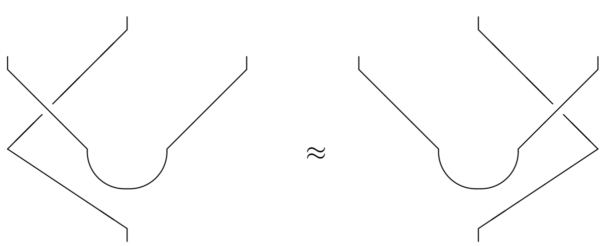
(M.1)  and 

(M.2) 

(M.3) 

(M.4) 

and

 .

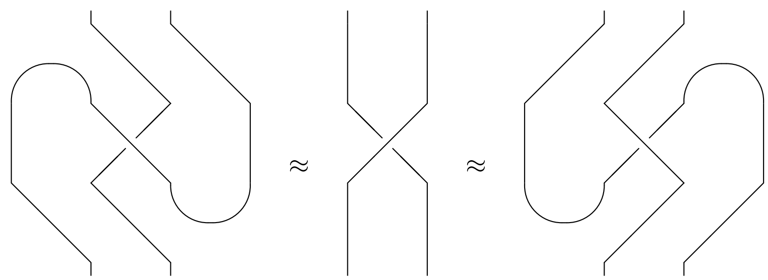
(M.2rev)–(M.4rev) are (M.2)–(M.4) respectively with over crossing lines replaced by under crossing lines and vice versa.

The substitutions of (M.1), (M.2) and (M.3) are known as the Reidemeister moves 0, 2 and 3 respectively. The substitutions described in (M.4) and (M.4rev) are called the *slide moves*.

What is not reflected in our simplistic definition of regular isotopy is the topological fluidity in the transformation of one diagram into another. Subsumed under regular isotopy is contraction, expansion and bending of lines in such a manner that no crossings or local extrema are created. These diagram alterations do not affect the invariants we describe as the reader will see.

The *twist moves*, which are the substitutions of

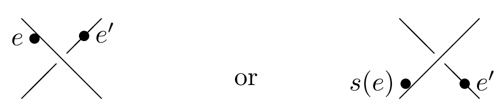
(M.5)



and (M.5rev), are consequences of (M.1), (M.4) and (M.4rev). Observe that crossing type is changed in a twist move.

Let $\mathsf{T} \in Tang$. We describe how to construct $\mathsf{Inv}_A(\mathsf{T})$ in a geometric, formal way which will be seen to be a blueprint for manipulation of certain $2n$-fold tensors.

If T has no crossings set $\mathsf{Inv}_A(\mathsf{T}) = 1$. Suppose that T has $n \geq 1$ crossings. Represent $\rho \in A \otimes A$ by $e \otimes e'$, $f \otimes f', \ldots$. Decorate each crossing of T in the following manner



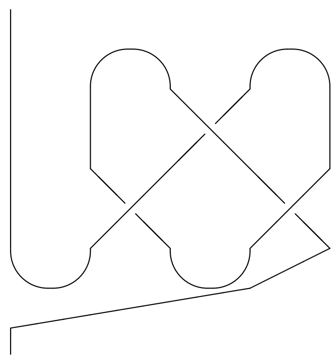
(6–1)

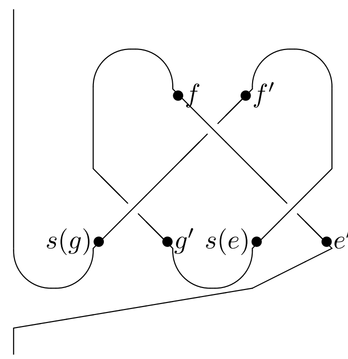
according to whether or not the crossing is an over crossing (left diagram) or an under crossing (right diagram). Thus ρ is associated with an over crossing and ρ^{-1} is associated with an under crossing. Our decorated crossings are to be interpreted as flat diagrams [21] which encode original crossing type.

Think of the diagram T as a rigid wire and think of the decorations as labeled beads which can slide around the wire. Let us call the bottom and top points of the diagram T the starting and ending points.

Traverse the diagram, beginning at the starting point (and thus in the upward direction), pushing the beads so that at the end of the traversal the beads are juxtaposed at the ending point. As a labeled bead passes through a local extrema its label x is altered: if the local extremum is traversed in the counter clockwise direction then x is changed to $s(x)$; if the local extremum is traversed in the clockwise direction then x is changed to $s^{-1}(x)$.

The juxtaposition of the beads with modified labels is interpreted as a formal product $\mathsf{W}_A(\mathsf{T})$ which is read bottom to top. Write $\rho = \sum_{\imath=1}^{r} a_\imath \otimes b_\imath \in A\otimes A$. Substitution of $a_\imath$ and $b_\imath$ for e and e', $a_\jmath$ and $b_\jmath$ for f and f', ... results in an element $\mathsf{w}_A(\mathsf{T}) \in A$. To illustrate the calculation of $\mathsf{W}_A(\mathsf{T})$ and $\mathsf{w}_A(\mathsf{T})$ we use the 1–1 tangle diagram $\mathsf{T}_{\text{trefoil}}$ depicted on the left below.





The crossing decorations are given in the diagram on the right above. Traversal of $\mathsf{T}_{\text{trefoil}}$ results in the formal word

$$\mathsf{W}_A(\mathsf{T}_{\text{trefoil}}) = s^2(e')s^2(f)s(g')s(e)s^{-1}(f')g$$

and thus

$$\mathsf{w}_A(\mathsf{T}_{\text{trefoil}}) = \sum_{\imath,\jmath,k=1}^{r} s^2(b_\imath)s^2(a_\jmath)s(b_k)s(a_\imath)s^{-1}(b_\jmath)a_k.$$

The preceding expression can be reformulated in several ways. Since $\rho = (s\otimes s)(\rho)$ there is no harm in introducing the rule

$$\mathsf{W}_A(\mathsf{T}) = \cdots s^p(x)\cdots s^q(y)\cdots = \cdots s^{p+\ell}(x)\cdots s^{q+\ell}(y)\cdots \tag{6–2}$$

where ℓ is any integer and $x\otimes y$ or $y\otimes x$ represents ρ. Under this rule we have

$$\mathsf{W}_A(\mathsf{T}_{\text{trefoil}}) = s(e')s^3(f)s(g')ef'g$$

and thus

$$\mathsf{w}_A(\mathsf{T}_{\text{trefoil}}) = \sum_{\imath,\jmath,k=1}^{r} s(b_\imath)s^3(a_\jmath)s(b_k)a_\imath b_\jmath a_k$$

as well.

We now give a slightly more detailed description of $\mathsf{W}_A(\mathsf{T})$ which will be useful in our discussion of knots and links. Label the crossing lines of the diagram T by $1, 2, \ldots, 2n$ in the order encountered on the traversal of T. For $1 \le \imath \le 2n$ let $u(\imath)$ be the number of local extrema traversed in the counter clockwise direction minus the number of local extrema traversed in the clockwise direction during the portion of the traversal of the diagram from crossing line $\imath$ to the ending point. Let $x_\imath$ be the decoration on crossing line $\imath$. Then

$$\mathsf{W}_A(\mathsf{T}) = s^{u(1)}(x_1)\cdots s^{u(2n)}(x_{2n}). \tag{6–3}$$

We emphasize that each crossing contributes two factors to the formal product $\mathsf{W}_A(\mathsf{T})$. Let χ be a crossing of T and $\imath < \jmath$ be the labels of the crossing lines of χ. Then χ contributes the $\imath^{th}$ and $\jmath^{th}$ factors to $\mathsf{W}_A(\mathsf{T})$ according to

$$\mathsf{W}_A(\mathsf{T}) = \cdots s^{u(\imath)}(x_\imath)\cdots s^{u(\jmath)}(x_\jmath)\cdots \ .$$

Notice that (6–3) describes a blueprint for computing $\mathsf{w}_A(\mathsf{T})$ in three steps: first, application of a certain permutation of the $2n$ tensorands of $\rho\otimes\cdots\otimes\rho$; second, application of $s^{u(1)}\otimes\cdots\otimes s^{u(2n)}$ to the result of the first step; third, application of the multiplication map $a_1\otimes\cdots\otimes a_{2n}\mapsto a_1\cdots a_{2n}$ to the result of the second step. The reader is left with the exercise of formulating the permutation.

We shall set

$$\mathsf{Inv}_A(\mathsf{T}) = \mathsf{w}_A(\mathsf{T})$$

for all $\mathsf{T}\in$ *Tang*. It is an instructive exercise to show that if $\mathsf{T},\mathsf{T}'\in$ *Tang* are regularly isotopic then $\mathsf{Inv}_A(\mathsf{T}) = \mathsf{Inv}_A(\mathsf{T}')$; that is (T.1) holds. By definition (T.2) holds, and in light of (6–2) it is clear that (T.3) holds. Observe that $\mathsf{Inv}_A(\mathsf{T})$ is s^2-invariant.

We end this section with a result on the relationship between Inv_A and morphisms. To do this we introduce some general terminology for comparing invariants.

Suppose that $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$ are functions with the same domain. Then f *dominates* g if $x, x'\in X$ and $f(x) = f(x')$ implies $g(x) = g(x')$. If f dominates g and g dominates f then f and g are *equivalent*.

PROPOSITION 7. *Let $f : A \longrightarrow A'$ be a morphism of quantum algebras over k. Then $f(\mathsf{Inv}_A(\mathsf{T})) = \mathsf{Inv}_{A'}(\mathsf{T})$ for all $\mathsf{T}\in$ Tang.* □

Thus when $f : A \longrightarrow A'$ is a morphism of quantum algebras over k it follows that Inv_A dominates $\mathsf{Inv}_{A'}$.

6.2. Invariants of knots and links arising from twist quantum algebras. We now turn to knots and links. In this section (A,ρ,s,G) is a twist quantum algebra over k and tr is a tracelike s^*-invariant element of A^*.

Let *Link* be the set of (unoriented) link diagrams situated with respect to our fixed vertical and let *Knot* be the set of knot diagrams in *Link*, that is the set of one component link diagrams in *Link*. If $\mathsf{L}\in$ *Link* is the union of two link diagrams $\mathsf{L}_1,\mathsf{L}_2\in$ *Link* such that the components of L_1 and L_2 do not intersect we write $\mathsf{L} = \mathsf{L}_1\star\mathsf{L}_2$. We shall construct a scalar valued function $\mathsf{Inv}_{A,\mathsf{tr}} : \mathit{Link} \longrightarrow k$ which satisfies the following axioms:

(L.1) If $\mathsf{L}, \mathsf{L}' \in \mathit{Link}$ are regularly isotopic then $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}) = \mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}')$,

(L.2) If $\mathsf{L} \in \mathit{Link}$ is a knot with no crossings then $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}) = \mathsf{tr}(G)$, and

(L.3) $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}{\star}\mathsf{L}') = \mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L})\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}')$ whenever $\mathsf{L}, \mathsf{L}' \in \mathit{Link}$ and $\mathsf{L}{\star}\mathsf{L}'$ is defined.

The first axiom implies that $\mathsf{Inv}_{A,\,\mathsf{tr}}$ defines a regular isotopy invariant of unoriented links.

We first define $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K})$ for knot diagrams $\mathsf{K} \in \mathit{Knot}$. One reason we do this is to highlight the close connection between the function Inv_A and the restriction $\mathsf{Inv}_{A,\,\mathsf{tr}}|_{\mathit{Knot}}$.

Let $\mathsf{K} \in \mathit{Knot}$. We first define an element $w(\mathsf{K}) \in A$. If K has no crossings then $w(\mathsf{K}) = 1$.

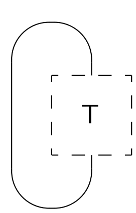
Suppose that K has $n \geq 1$ crossings. Decorate the crossings of K according to (6–1) and choose a point P on a vertical line of K. We refer to P as the starting and ending point. (There is no harm, under regular isotopy considerations, in assuming that K has a vertical line. One may be inserted at an end of a crossing line or at an end of a local extrema.) Traverse the diagram K, beginning at the starting point P in the upward direction and concluding at the ending point which is P again. Label the crossing lines $1, 2, \ldots, 2n$ in the order encountered on the traversal. For $1 \leq \imath \leq 2n$ let $u(\imath)$ be defined as in the case of 1–1 tangle diagrams and let $x_\imath$ be the decoration on crossing line $\imath$. Let $W(\mathsf{K}) = s^{u(1)}(x_1) \cdots s^{u(2n)}(x_{2n})$ and let $w(\mathsf{K}) \in A$ be computed from $W(\mathsf{K})$ in the same manner that $\mathsf{w}_A(\mathsf{T})$ is computed from $\mathsf{W}_A(\mathsf{T})$ in Section 6.1. Then

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K}) = \mathsf{tr}(G^d w(\mathsf{K})), \tag{6–4}$$

where d is the Whitney degree of K with orientation determined by traversal beginning at the starting point in the upward direction. In terms of local extrema, $2d$ is the number of local extrema traversed in the clockwise direction minus the number of local extrema traversed in the counter clockwise direction.

Using a different starting point P' results in the same value for $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K})$. This boils down to two cases: P and P' separated by $m \geq 1$ crossing lines and no local extrema; P and P' separated by one local extrema and no crossing lines. The fact that tr is tracelike is used in the first case and the s^*-invariance of tr is used in the second.

The function $\mathsf{Inv}_{A,\,\mathsf{tr}}|_{\mathit{Knot}}$ can be computed in terms of Inv_A. For $\mathsf{T} \in \mathit{Tang}$ let $\mathsf{K}(\mathsf{T}) \in \mathit{Knot}$ be given by



Every $\mathsf{K} \in \mathit{Knot}$ is regularly isotopic to $\mathsf{K}(\mathsf{T})$ for some $\mathsf{T} \in \mathit{Tang}$.

PROPOSITION 8. *Let (A, ρ, s, G) be a twist quantum algebra over the field k and suppose that tr is a tracelike s^*-invariant element of A^*. Then*

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K}(\mathsf{T})) = \mathsf{tr}(G^d \mathsf{Inv}_A(\mathsf{T}))$$

for all $\mathsf{T} \in \mathit{Tang}$, where d is the Whitney degree of $\mathsf{K}(\mathsf{T})$ with the orientation determined by traversal beginning at the base of T in the upward direction. □

For example, with $\rho = \sum_{\imath=1}^{r}$, observe that

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K}(\mathsf{T}_{\mathrm{trefoil}})) = \sum_{\imath,\jmath,k=1}^{r} \mathsf{tr}(G^{-2} s(b_\imath) s^3(a_\jmath) s(b_k) a_\imath b_\jmath a_k)$$

and

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K}(\mathsf{T}_{\mathrm{curl}})) = \sum_{\imath=1}^{r} \mathsf{tr}(G^0 a_\imath s(b_\imath)) = \sum_{\imath=1}^{r} \mathsf{tr}(a_\imath s(b_\imath)).$$

We now define $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L})$ for $\mathsf{L} \in \mathit{Link}$ with components $\mathsf{L}_1, \dots, \mathsf{L}_r$. Decorate the crossings of L according to (6–1). Fix $1 \leq \ell \leq r$. We construct a formal word $W(\mathsf{L}_\ell)$ in the following manner. If L_ℓ does not contain a crossing line then $W(\mathsf{L}_\ell) = 1$.

Suppose that L_ℓ contains $m_\ell \geq 1$ crossing lines. Choose a point P_ℓ on a vertical line of L_ℓ. We shall refer to P_ℓ as the starting point and the ending point. As in the case of knot diagrams we may assume that L_ℓ has a vertical line. Traverse the link component L_ℓ beginning at the starting point P_ℓ in the upward direction and concluding at the ending point which is also P_ℓ. Label the crossing lines contained in L_ℓ by $(\ell{:}1), \dots, (\ell{:}m_\ell)$ in the order encountered. Let $u(\ell{:}\imath)$ be the counterpart of $u(\imath)$ for *Knot* and let $x_{(\ell:\imath)}$ be the decoration on the crossing line $(\ell{:}\imath)$. Then we set

$$W(\mathsf{L}_\ell) = s^{u(\ell:1)}(x_{(\ell:1)}) \cdots s^{u(\ell:m_\ell)}(x_{(\ell:m_\ell)})$$

and

$$W(\mathsf{L}) = W(\mathsf{L}_1) \otimes \cdots \otimes W(\mathsf{L}_r).$$

Replacing the formal copies of ρ in $W(\mathsf{L})$ with ρ as was done in the case of 1–1 tangle and knot diagrams, we obtain an element $w(\mathsf{L}) = w(\mathsf{L}_1) \otimes \cdots \otimes w(\mathsf{L}_r) \in A \otimes \cdots \otimes A$. The scalar we want is

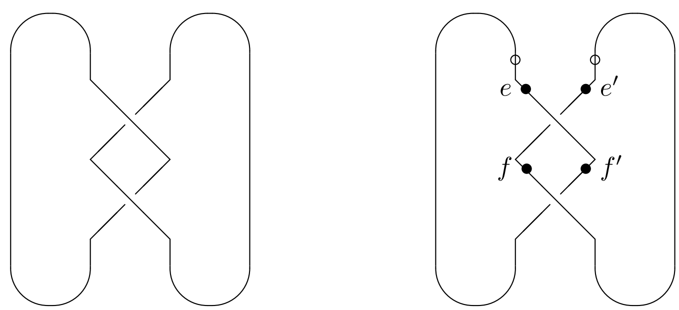
$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}) = \mathsf{tr}(G^{d_1} w(\mathsf{L}_1)) \cdots \mathsf{tr}(G^{d_r} w(\mathsf{L}_r)) \tag{6–5}$$

which is the evaluation of $\mathsf{tr} \otimes \cdots \otimes \mathsf{tr}$ on $G^{d_1} w(\mathsf{L}_1) \otimes \cdots \otimes G^{d_r} w(\mathsf{L}_r)$, where d_ℓ is the Whitney degree of L_ℓ with orientation determined by the traversal which starts at P_ℓ in the upward direction.

The argument that $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{K})$ does not depend on the starting point P for $\mathsf{K} \in \mathit{Knot}$ is easily modified to show that $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L})$ does not depend on the starting points $P_1, \dots, P_r$. The argument that (T.1) holds for Inv_A shows that

(L.1) holds for $\mathsf{Inv}_{A,\,\mathsf{tr}}$ since we may assume that the starting points are not in the local part of L under consideration in (M.1)–(M.4) and (M.2rev)–(M.4rev). That (L.2) and (L.3) hold for $\mathsf{Inv}_{A,\,\mathsf{tr}}$ is a straightforward exercise. Note that (6–5) generalizes (6–4).

For example, consider the Hopf link $\mathsf{L}_{\mathrm{Hopf}}$ depicted below left. The components of L are L_1 and L_2, reading left to right. The symbol $\circ$ designates a starting point.



Observe that $W(\mathsf{L}_{\mathrm{Hopf}}) = f'e \otimes fe'$ and

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}_{\mathrm{Hopf}}) = \sum_{\imath,\jmath=1}^{r} \mathsf{tr}(G^{-1}b_\imath a_\jmath)\mathsf{tr}(Ga_\imath b_\jmath).$$

We end this section by noting how $\mathsf{Inv}_{A,\,\mathsf{tr}}$ and morphisms are related and revisiting the construction of Proposition 1.

Proposition 9. *Let $f : (A, \rho, s, G) \longrightarrow (A', \rho', s', G')$ be a morphism of twist quantum algebras over k and suppose that tr' is a tracelike s'^*-invariant element of A'^*. Then $\mathsf{tr} = \mathsf{tr}' \circ f$ is a tracelike s^*-invariant element of A^* and*

$$\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}) = \mathsf{Inv}_{A',\,\mathsf{tr}'}(\mathsf{L})$$

for all $\mathsf{L} \in Link$. □

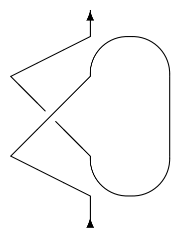
Let (A, ρ, s) be a quantum algebra over k and let $(\boldsymbol{A}, \boldsymbol{\rho}, \boldsymbol{s}, \boldsymbol{G})$ be the twist quantum algebra of Proposition 1 associated with (A, ρ, s). In light of the preceding proposition it would be of interest to know what the $\boldsymbol{s}^*$-invariant tracelike elements Tr of $\boldsymbol{A}^*$ are.

First note that $\mathbf{A}$ is a free right A-module with basis $\{G^n\}_{n \in Z}$. Let $\{\mathsf{tr}_n\}_{n \in Z}$ be a family of functionals of A^* which satisfies $\mathsf{tr}_n \circ s = \mathsf{tr}_{-n}$ and $\mathsf{tr}_n(ba) = \mathsf{tr}_n(as^{2n}(b))$ for all $n \in Z$ and $a, b \in A$. Then the functional $\mathsf{Tr} \in \boldsymbol{A}^*$ determined by $\mathsf{Tr}(G^n a) = \mathsf{tr}_n(a)$ for all $n \in Z$ and $a \in A$ is an $\boldsymbol{s}^*$-invariant and tracelike. All $\boldsymbol{s}^*$-invariant tracelike functionals of $\boldsymbol{A}^*$ are described in this manner.

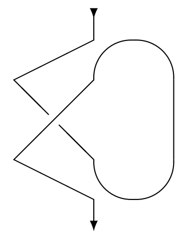
7. Invariants Constructed from Oriented Quantum Algebras Via Bead Sliding

This section draws heavily from the material of the preceding section. The reader is directed to [16; 17] for a fuller presentation of the ideas found in this section.

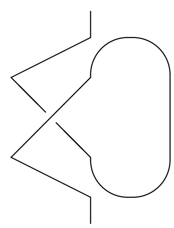
Let **Tang** be the set of all oriented 1–1 tangle diagrams situated with respect to a fixed vertical; that is the set of all diagrams in *Tang* with a designated orientation which we indicate by arrows. For example



and



are elements of **Tang** whose underlying unoriented diagram is



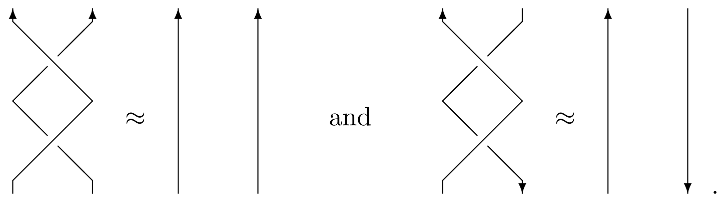
.

We let u : **Tang** $\longrightarrow$ *Tang* be the function which associates to each $\mathbf{T} \in$ **Tang** its underlying unoriented diagram $u(\mathbf{T})$. For $\mathbf{T} \in$ **Tang** we let $\mathbf{T}^{\mathrm{op}}$ be $\mathbf{T}$ with its orientation reversed.

Likewise we let **Link** be the set of all oriented link diagrams situated with respect to the fixed vertical; that is the set of all diagrams of *Link* whose components have a designated orientation. By slight abuse of notation we also let u : **Link** $\longrightarrow$ *Link* be the function which associates to each $\mathbf{L} \in$ **Link** its underlying unoriented diagram $u(\mathbf{L})$. For $\mathbf{L} \in$ **Link** we let $\mathbf{L}^{\mathrm{op}}$ be $\mathbf{L}$ with the orientation on its components reversed.

In this section we redo Section 6 by making minor adjustments which result in a regular isotopy invariant $\mathbf{Inv}_A$: **Tang** $\longrightarrow A$ of oriented 1-1 tangle diagrams, when $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra over k, and in a regular isotopy invariant $\mathbf{Inv}_{A,\,\mathsf{tr}}$: **Link** $\longrightarrow k$ of oriented link diagrams, when $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ is a twist oriented quantum algebra over k and tr is a tracelike t_{d}^*, t_{u}^*-invariant element of A^*.

Regular isotopy in the oriented case is regular isotopy in the unoriented case with all possible orientations taken into account. For example, the one diagram of (M.2) in the unoriented case is replaced by four in the oriented case, two of which are



In Section 7.3 we relate invariants of a quantum algebra and invariants of its associated oriented quantum algebra.

7.1. Invariants of oriented 1–1 tangles arising from oriented quantum algebras. Oriented 1–1 tangle diagrams consist of some or all of the following components:

- oriented crossings;

under crossings

over crossings

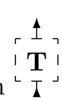
- oriented local extrema;

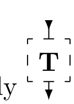
local maxima

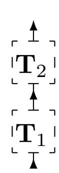
local minima

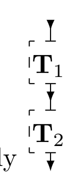
and

- oriented "vertical" lines.

If an oriented 1–1 tangle diagram 

(respectively 

) can be decomposed into two oriented 1–1 tangle diagrams 

(respectively 

) then we write $\mathbf{T} = \mathbf{T}_1 \star \mathbf{T}_2$.

Now suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra over k and let $\mathbf{T} \in$ *Tang*. To define $\mathbf{Inv}_A : \mathbf{Tang} \longrightarrow A$ we will construct formal product $\mathbf{W}_A(\mathbf{T})$ which will determine an element $\mathbf{w}_A(\mathbf{T}) \in A$. In order to do this we need to describe crossing decorations and conventions for sliding labeled beads across local extrema. The sliding conventions are:

and

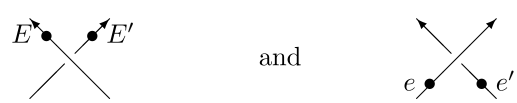
for clockwise motion;

and

for counterclockwise motion. We refer to the oriented local extrema

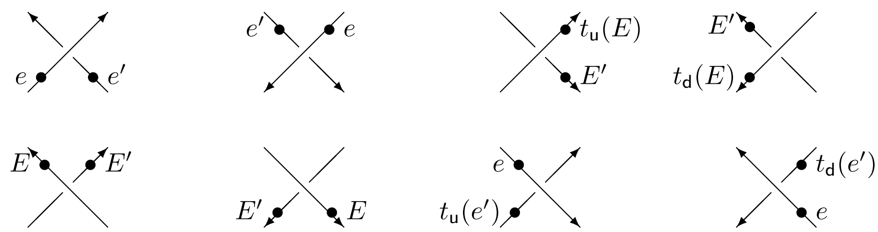
as having type (u_-), (u_+), (d_+) and (d_-) respectively.

There are two crossing decorations



(7–1)

from which all other crossing decorations are derived. Here $E \otimes E'$ and $e \otimes e'$ represent ρ^{-1} and ρ respectively. Compare (7–1) with (6–1). Starting with (7–1), using the above conventions for passing labeled beads across local extrema and requiring invariance under (M.4), crossings are decorated as follows:



Now to define $\mathbf{W}_A(\mathbf{T})$. If $\mathbf{T}$ has no crossings then $\mathbf{W}_A(\mathbf{T}) = 1$. Suppose that $\mathbf{T}$ has $n \geq 1$ crossings. Traverse $\mathbf{T}$ in the direction of orientation and label the crossing lines $1, 2, \ldots, 2n$ in the order in which they are encountered. For $1 \leq \imath \leq 2n$ let $u_{\mathsf{d}}(\imath)$ be the number of local extrema of type (d_+) minus the number of type (d_-) encountered on the portion of the traversal from line $\imath$ to the end of the traversal of $\mathbf{T}$. We define $u_{\mathsf{u}}(\imath)$ in the same way where (u_+) and (u_-) replace (d_+) and (d_-) respectively. Then

$$\mathbf{W}_A(\mathbf{T}) = t_{\mathsf{d}}^{u_{\mathsf{d}}(1)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(1)}(x_1) \cdots t_{\mathsf{d}}^{u_{\mathsf{d}}(2n)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(2n)}(x_{2n}),$$

where $x_\imath$ is the decoration on the crossing line $\imath$. Replacing the formal representations of ρ and ρ^{-1} in $\mathbf{W}_A(\mathbf{T})$ by ρ and ρ^{-1} respectively we obtain an element $\mathbf{w}_A(\mathbf{T}) \in A$.

Set

$$\mathbf{Inv}_A(\mathbf{T}) = \mathbf{w}_A(\mathbf{T})$$

for all $\mathbf{T} \in \mathbf{Tang}$. Observe that $\mathbf{Inv}_A(\mathbf{T})$ is invariant under t_{d} and t_{u}. It can be shown that the oriented counterparts of (T.1)–(T.3) hold for $\mathbf{Inv}_A$; in particular $\mathbf{Inv}_A$ defines a regular isotopy invariant of oriented 1–1 tangles. We have noted that $(A^{\mathrm{op}}, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra over k which we simply refer to as A^{op}. By Theorem 1 that $(A, \rho, 1_A, t_{\mathsf{d}} \circ t_{\mathsf{u}})$ is a standard oriented quantum algebra over k which we denote by A_s. We collect some basic results on the invariant of this section.

Proposition 10. *Let A be an oriented quantum algebra over k. Then*

(a) $\mathbf{Inv}_A(\mathbf{T}^{\mathrm{op}}) = \mathbf{Inv}_{A^{\mathrm{op}}}(\mathbf{T})$ *and*
(b) $\mathbf{Inv}_A(\mathbf{T}) = \mathbf{Inv}_{A_s}(\mathbf{T})$

for all $\mathbf{T} \in \mathbf{Tang}$.

(c) *Suppose that $f : A \longrightarrow A'$ is a morphism of oriented quantum algebras. Then $f(\mathbf{Inv}_A(\mathbf{T})) = \mathbf{Inv}_{A'}(\mathbf{T})$ for all* $\mathbf{T} \in \mathbf{Tang}$.

□

Part (b) of the proposition shows that standard oriented quantum algebras account for the invariants of this section. Part (c) shows that $\mathbf{Inv}_A$ dominates $\mathbf{Inv}_{A'}$ whenever there is a morphism of oriented quantum algebras $f : A \longrightarrow A'$.

Corollary 2. *Suppose that* (A, ρ, s) *is a quantum algebra over the field* k. *Then* $\mathbf{Inv}_{(A,\rho,1_A,s^{-2})}(\mathbf{T}^{\mathrm{op}}) = s(\mathbf{Inv}_{(A,\rho,1_A,s^{-2})}(\mathbf{T}))$ *for all* $\mathbf{T} \in \mathbf{Tang}$.

Proof. Since $s : (A^{\mathrm{op}}, \rho, 1_A, s^{-2}) \longrightarrow (A, \rho, 1_A, s^{-2})$ is a morphism of oriented quantum algebras the corollary follows by parts (a) and (c) of Proposition 10. □

7.2. Invariants of oriented knots and links arising from twist oriented quantum algebras. Throughout this section $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ is a twist oriented quantum algebra over k and tr is a tracelike t^*_{d}, t^*_{u}-invariant element of A^*. The scalar $\mathbf{Inv}_{A,\,\mathsf{tr}}(\mathbf{L})$ for $\mathbf{L} \in \mathbf{Link}$ is defined much in the same manner that $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L})$ is defined for $\mathsf{L} \in \mathit{Link}$ in Section 6.2.

Let $\mathbf{L} \in \mathbf{Link}$ have components $\mathbf{L}_1, \ldots, \mathbf{L}_r$. Decorate the crossings of $\mathbf{L}$ according to the conventions of Section 7.1. For each $1 \leq \ell \leq r$ let d_ℓ be the Whitney degree of the link component $\mathbf{L}_\ell$. We define a formal product $\mathbf{W}(\mathbf{L}_\ell)$ as follows. If $\mathbf{L}_\ell$ contains no crossing lines then $\mathbf{W}(\mathbf{L}_\ell) = 1$. Suppose that $\mathbf{L}_\ell$ contains $m_\ell \geq 1$ crossing lines. Choose a point P_ℓ on a vertical line of $\mathbf{L}_\ell$. We may assume that $\mathbf{L}_\ell$ has a vertical line for the reasons cited in Section 6.2.

Traverse $\mathbf{L}_\ell$ in the direction of orientation beginning and ending at P_ℓ. Label the crossing lines $(\ell{:}1), \ldots, (\ell{:}m_\ell)$ in the order encountered on the traversal. For $1 \leq \imath \leq m$ let $u_{\mathsf{d}}(\ell{:}\imath)$ denote the number of local extrema of type (d_+) minus the number of type (d_-) which are encountered during the portion of the traversal of $\mathbf{L}_\ell$ from the line labeled $\imath$ to its conclusion. Define $u_{\mathsf{u}}(\ell{:}\imath)$ in the same manner, where (u_+) and (u_-) replace (d_+) and (d_-) respectively. Let $x_{(\ell:\imath)}$ be the decoration on the line $(\ell{:}\imath)$. Set

$$\mathbf{W}(\mathbf{L}_\ell) = t_{\mathsf{d}}^{u_{\mathsf{d}}(\ell:1)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(\ell:1)}(x_{(\ell:1)}) \cdots t_{\mathsf{d}}^{u_{\mathsf{d}}(\ell:m)} \circ t_{\mathsf{u}}^{u_{\mathsf{u}}(\ell:m)}(x_{(\ell:m)}),$$

set $\mathbf{W}(\mathbf{L}) = \mathbf{W}(\mathbf{L}_1) \otimes \cdots \otimes \mathbf{W}(\mathbf{L}_r)$ and replace formal copies of ρ and ρ^{-1} in $\mathbf{W}(\mathbf{L})$ to obtain an element $\mathbf{w}(\mathbf{L}) = \mathbf{w}(\mathbf{L}_1) \otimes \cdots \otimes \mathbf{w}(\mathbf{L}_r) \in A \otimes \cdots \otimes A$. We define

$$\mathbf{Inv}_{A,\,\mathsf{tr}}(\mathbf{L}) = \mathsf{tr}(G^{d_1}\mathbf{w}(\mathbf{L}_1)) \cdots \mathsf{tr}(G^{d_r}\mathbf{w}(\mathbf{L}_r)).$$

One can show that the oriented counterparts of (L.1)–(L.3) hold for $\mathbf{Inv}_{A,\,\mathsf{tr}}$; in particular $\mathbf{Inv}_{A,\,\mathsf{tr}}$ defines a regular isotopy invariant of oriented links.

There is an analog of Proposition 8 which we do not state here and there is an analog of part (b) of Proposition 10 and of Proposition 9 which we do record. Let A_s denote the standard twist oriented quantum algebra $(A, \rho, 1_A, t_{\mathsf{d}} \circ t_{\mathsf{u}}, G)$ associated with $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ which we denote by A.

Proposition 11. *Let* $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G)$ *be a twist oriented quantum algebra over the field* k *and let* tr *be a tracelike* t^*_{d}, t^*_{u}*-invariant element of* A^*. *Then*

$$\mathbf{Inv}_{A,\,\mathsf{tr}}(\mathbf{L}) = \mathbf{Inv}_{A_s,\,\mathsf{tr}}(\mathbf{L}) \quad \textit{for all} \quad \mathbf{L} \in \mathbf{Link}. \qquad \Box$$

Thus the invariants of oriented links described in this section are accounted for by standard twist oriented quantum algebras.

PROPOSITION 12. *Suppose that* $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, G) \longrightarrow (A', R', t'_{\mathsf{d}}, t'_{\mathsf{u}}, G')$ *is a morphism of twist oriented quantum algebras over* k *and that* $\mathsf{tr}' \in A^{'*}$ *is a* $t'^{*}_{\mathsf{d}}, t'^{*}_{\mathsf{u}}$*-invariant tracelike element. Then* $\mathsf{tr} \in A^*$ *defined by* $\mathsf{tr} = \mathsf{tr}' \circ f$ *is a* $t^*_{\mathsf{d}}, t^*_{\mathsf{u}}$*-invariant tracelike element and* $\mathbf{Inv}_{A,\,\mathsf{tr}}(\mathbf{L}) = \mathbf{Inv}_{A',\,\mathsf{tr}'}(\mathbf{L})$ *for all* $\mathbf{L} \in \mathbf{Link}$. □

Let $(A, \rho, 1_A, t)$ be a standard oriented quantum algebra over k and let

$$(\boldsymbol{A}, \boldsymbol{\rho}, 1_{\boldsymbol{A}}, \boldsymbol{t}, \boldsymbol{G})$$

be the twist standard quantum algebra of Proposition 4 associated with (A, ρ, s). We describe the tracelike elements Tr of $\boldsymbol{A}^*$.

Observe $\mathbf{A}$ is a free right A-module with basis $\{G^n\}_{n\in Z}$. Let $\{\mathsf{tr}_n\}_{n\in Z}$ be a family of functionals of A^* which satisfies $\mathsf{tr}_n \circ t = \mathsf{tr}_n$ and $\mathsf{tr}_n(ba) = \mathsf{tr}_n(at^n(b))$ for all $n \in Z$ and $a, b \in A$. Then the functional $\mathsf{Tr} \in \boldsymbol{A}^*$ determined by $\mathsf{Tr}(G^n a) = \mathsf{tr}_n(a)$ for all $n \in Z$ and $a \in A$ is an $\boldsymbol{t}^*$-invariant and tracelike; all $\boldsymbol{t}^*$-invariant tracelike functionals of $\boldsymbol{A}^*$ are described in this manner.

7.3. Comparison of invariants arising from quantum algebras and their associated oriented quantum algebras. Let (A, ρ, s) be a quantum algebra over the field k and let $(A, \rho, 1_A, s^{-2})$ be the associated standard oriented quantum algebra. In this brief section we compare the invariants defined for each of them.

THEOREM 3. *Let* (A, ρ, s) *be a quantum algebra over* k*. Then*:

(a) *The equations*

$$\mathbf{Inv}_{(A,\rho,1_A,s^{-2})}(\mathbf{T}) = \mathsf{Inv}_{(A,\rho^{-1},s^{-1})}(u(\mathbf{T}))$$

and

$$\mathsf{Inv}_{(A,\rho,s)}(u(\mathbf{T})) = \mathbf{Inv}_{(A,\rho^{-1},1_A,s^2)}(\mathbf{T})$$

hold for all $\mathbf{T} \in \mathbf{Tang}$ *whose initial vertical line is oriented upward.*

(b) *Suppose further* (A, ρ, s, G^{-1}) *is a twist quantum algebra and* tr *is a tracelike* s^**-invariant element of* A^**. Then*

$$\mathbf{Inv}_{(A,\rho,1_A,s^{-2},G),\,\mathsf{tr}}(\mathbf{L}) = \mathsf{Inv}_{(A,\rho^{-1},s^{-1},G),\,\mathsf{tr}}(u(\mathbf{L}))$$

and

$$\mathsf{Inv}_{(A,\rho,s,G^{-1}),\,\mathsf{tr}}(u(\mathbf{L})) = \mathbf{Inv}_{(A,\rho^{-1},1_A,s^2,G^{-1}),\,\mathsf{tr}}(\mathbf{L})$$

for all $\mathbf{L} \in \mathbf{Link}$.

PROOF. We need only establish the first equations in parts (a) and (b). Generally to calculate a regular isotopy invariant of oriented 1–1 tangle, knot or link diagrams we may assume that all crossing lines are directed upward by virtue of the twist moves, and we may assume that diagrams have vertical lines oriented in the upward direction. We can assume that traversals begin on such lines.

As usual represent ρ by $e \otimes e'$ and ρ^{-1} by $E \otimes E'$. Since $\rho^{-1} = (s \otimes 1_A)(\rho)$ it follows that $e \otimes e' = s^{-1}(E) \otimes E'$. Thus the over crossing and under crossing labels $E\otimes E'$ and $e\otimes e'$ of (7–1) are the over crossing and under crossing labels $E\otimes E'$ and $s^{-1}(E)\otimes E'$ of (6–1). These are the decorations associated with the quantum algebra (A, ρ^{-1}, s^{-1}).

For a crossing decoration representing $x\otimes y$, traversal of the oriented 1–1 tangle or link diagram results in the modification

$$t_{\mathsf{d}}^{u_{\mathsf{d}}}\circ t_{\mathsf{u}}^{u_{\mathsf{u}}}(x)\otimes t_{\mathsf{d}}^{u'_{\mathsf{d}}}\circ t_{\mathsf{u}}^{u'_{\mathsf{u}}}(y) = s^{-2u_{\mathsf{u}}}(x)\otimes s^{-2u'_{\mathsf{u}}}(y)$$

with $(A, \rho, 1_A, s^{-2})$ and traversal in the underlying unoriented diagram results in the modification

$$s^{-(u_{\mathsf{d}}+u_{\mathsf{u}})}(x)\otimes s^{-(u'_{\mathsf{d}}+u'_{\mathsf{u}})}(y) = s^{-2u_{\mathsf{u}}}(x)\otimes s^{-2u'_{\mathsf{u}}}(y)$$

with (A, ρ^{-1}, s^{-1}); the last equation holds by our assumptions on the crossings and traversal. □

8. Bead Sliding Versus the Classical Construction of Quantum Link Invariants

We relate the method of Section 7 for computing invariants of oriented knot and link diagrams to the method for computing invariants by composition of certain tensor products of (linear) morphisms associated with oriented crossings, local extrema and vertical lines. We will see how invariants produced by the composition method are related to those which arise from representations of twist oriented quantum algebras. For a more general discussion of the composition method, which has a categorical setting, the reader is referred to [28; 29]. This section is a reworking of [17, Sections 3, 6] which explicates its algebraic details.

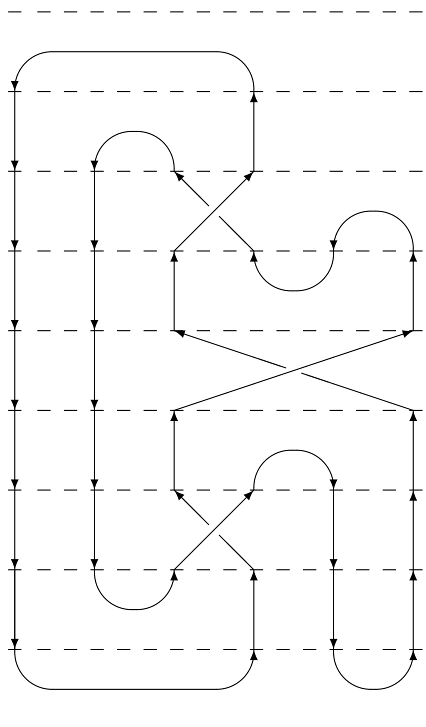
In this section we confine ourselves to the category Vec_k of all vector spaces over k and their linear transformations under composition. We begin with a description of the composition method in this special case.

Let V be a finite-dimensional vector space over the field k, let $\{v_1, \ldots, v_n\}$ be a basis for V and suppose that $\{v^1, \ldots, v^n\}$ is the dual basis for V^*. Observe that

$$\sum_{\imath=1}^{n} v^{\imath}(v)v_{\imath} = v \quad\text{and}\quad \sum_{\imath=1}^{n} v^*(v_{\imath})v^{\imath} = v^* \tag{8–1}$$

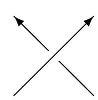
for all $v \in V$ and $v^* \in V^*$. Let $\mathcal{C}_V$ be the full subcategory of Vec_k whose objects are k and tensor powers $U_1\otimes\cdots\otimes U_m$, where $m \geq 1$ and $U_\imath = V$ or $U_\imath = V^*$ for all $1 \leq \imath \leq m$.

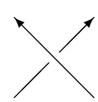
The first step of the composition method is to arrange an oriented link (or knot) diagram $\mathbf{L} \in \mathbf{Link}$ so that all crossing lines are directed upward, which can be done by the twist moves, and so that $\mathbf{L}$ is stratified in such a manner that each stratum consists of a juxtaposition of oriented crossings, local extrema and vertical lines, which we refer to as components of the stratum. One such example is $\mathbf{K}_{\text{trefoil}}$ depicted below.

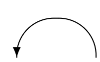


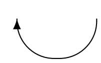
The broken lines indicate the stratification and are not part of the diagram.

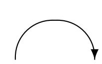
The next step is to associate certain morphisms of $\mathcal{C}_V$ to the components of each stratum. Let $R : V{\otimes}V \longrightarrow V{\otimes}V$ be an invertible solution to the braid equation and let D, U be commuting linear automorphisms of V. We associate

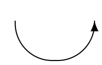
$R : V{\otimes}V \longrightarrow V{\otimes}V$ to 

$R^{-1} : V{\otimes}V \longrightarrow V{\otimes}V$ to 

$\mathcal{D}_+ : V^*{\otimes}V \longrightarrow k$ to 

$\mathcal{D}_- : k \longrightarrow V{\otimes}V^*$ to 

$\mathcal{U}_- : V{\otimes}V^* \longrightarrow k$ to 

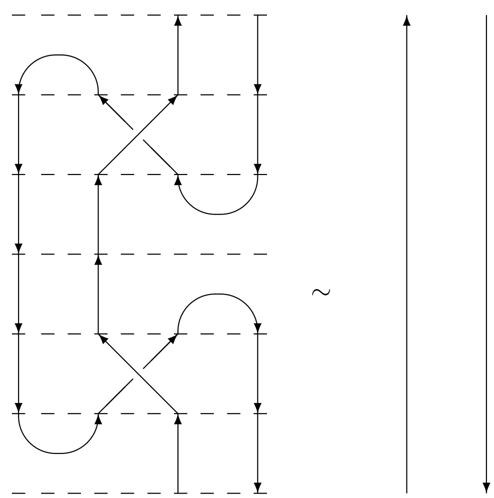
$\mathcal{U}_+ : k \longrightarrow V^*{\otimes}V$ to 

where $\mathcal{D}_+(v^*\otimes v) = v^*(D(v))$ and $\mathcal{U}_-(v\otimes v^*) = v^*(U^{-1}(v))$ for all $v^* \in V^*$, $v \in V$ and

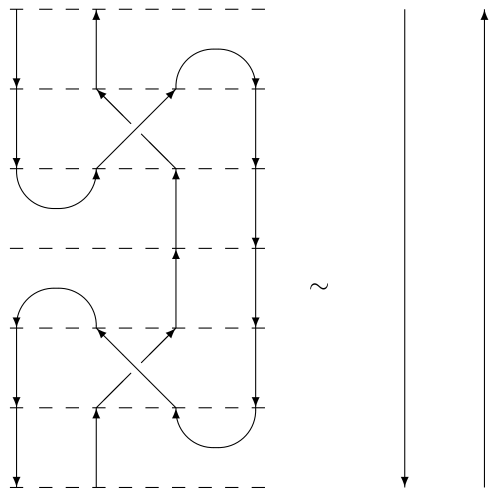
$$\mathcal{D}_-(1) = \sum_{\imath=1}^{n} D^{-1}(v_\imath)\otimes v^\imath, \quad \mathcal{U}_+(1) = \sum_{\imath=1}^{n} v^\imath \otimes U(v_\imath),$$

and finally we associate 1_V and 1_{V^*} to the vertical lines and respectively.

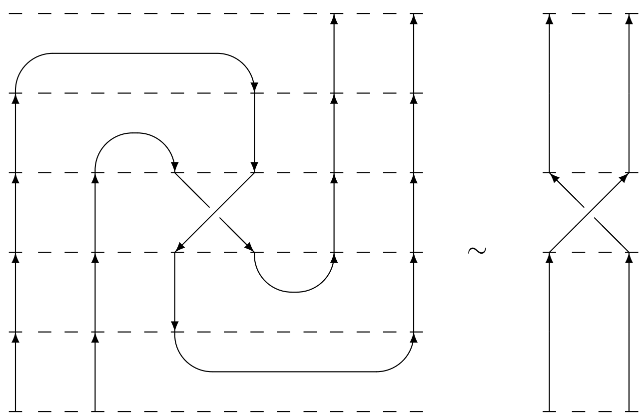
Associate to each stratum the tensor product of the morphisms associated to each of its components, reading left to right. The composition of these tensor products of morphisms, starting with the tensor product associated with the bottom stratum and moving up, determines an endomorphism $\mathrm{INV}_{R,D,U}(\mathbf{L})$ of k, which we identify with its value at 1_k. The scalar valued function $\mathrm{INV}_{R,D,U}$ determines a regular isotopy invariant of oriented link diagrams if and only if



(8–2)

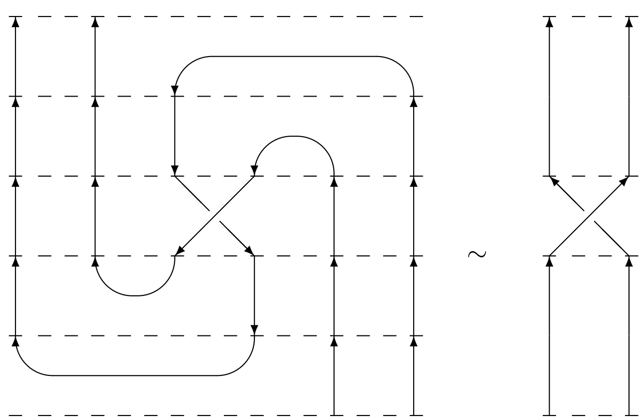


(8–3)

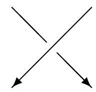


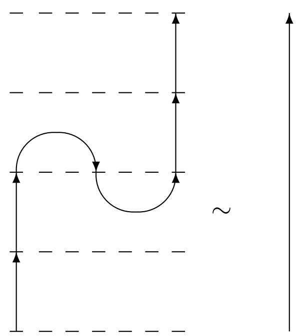
(8–4)

and



(8–5)

where $\sim$ means that the compositions associated with the diagrams are equal and $R^* : V^*\otimes V^* \longrightarrow V^*\otimes V^*$ is associated to  . Compare with [29]. Showing that INV $=$ INV$_{R,D,U}$ is a regular isotopy invariant by evaluating compositions of morphisms directly is an instructive exercise. For example, to show that INV($\mathbf{L}$) is unaffected by the substitution



in $\mathbf{L}$ we need to show that the composition of morphisms associated with the diagram on the left above is the identity map 1_V. Using (8–1) it follows that the composition in question

$$V \xrightarrow{1_V} V = V\otimes k \xrightarrow{1_V\otimes\mathcal{U}_+} V\otimes V^*\otimes V \xrightarrow{\mathcal{U}_-\otimes 1_V} k\otimes V = V \xrightarrow{1_V} V$$

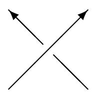
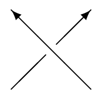
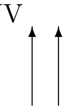
is in fact 1_V. We will show that (8–2)–(8–5) are equivalent to (qa.1)–(qa.2) for a certain oriented quantum algebra structure on $\mathrm{End}(V)^{\mathrm{op}}$.

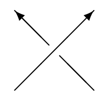
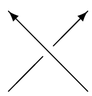
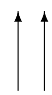
The evaluation of INV($\mathbf{K}_{\mathrm{trefoil}}$) is partially indicated by

$$\begin{aligned}
k = k\otimes k &\xrightarrow{\mathcal{U}_+\otimes\mathcal{U}_+} (V^*\otimes V)\otimes(V^*\otimes V) = V^*\otimes k\otimes V\otimes V^*\otimes V\\
&\xrightarrow{1_{V^*}\otimes\mathcal{U}_+\otimes 1_V\otimes 1_{V^*}\otimes 1_V} V^*\otimes V^*\otimes V\otimes V\otimes V^*\otimes V\\
&\xrightarrow{1_{V^*}\otimes 1_{V^*}\otimes R\otimes 1_{V^*}\otimes 1_V} V^*\otimes V^*\otimes V\otimes V\otimes V^*\otimes V\\
&\qquad\vdots\\
&\xrightarrow{\mathcal{D}_+} k.
\end{aligned}$$

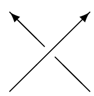
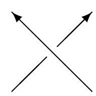
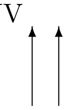
The reader is encouraged to complete the calculation.

When the minimal polynomial of R has degree 1 or 2 there is a way of relating

INV  (**L**), INV  (**L**) and INV  (**L**),

for $\mathbf{L} \in$ **Link**, where the three preceding expressions are INV(**L**), INV($\mathbf{L}'$) and INV($\mathbf{L}''$) respectively, where $\mathbf{L}'$ is **L** with a crossing of the type  replaced by  and $\mathbf{L}''$ is **L** with the same crossing of replaced by .

Suppose that $\alpha R^2 - \gamma R - \beta 1_{V\otimes V} = 0$, where $\alpha, \beta, \gamma \in k$. Then $\alpha R - \beta R^{-1} = \gamma 1_{V\otimes V}$ which implies that

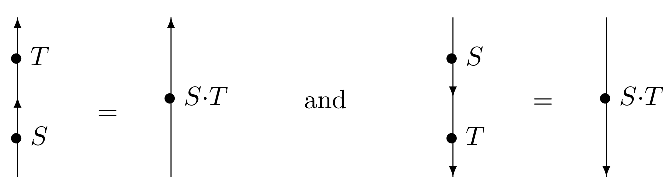
αINV  (**L**) $-$ βINV  (**L**) $=$ γINV  (**L**) . (8–6)

The preceding equation is called a *skein identity* and is the basis of a recursive evaluation of INV when $\alpha, \beta \neq 0$. As we shall see the link invariant associated to Example 1 satisfies a skein identity.

In order to relate the composition method to the bead sliding method of Section 7 we introduce vertical lines decorated by endomorphisms of V. Let $T \in \operatorname{End}(V)$. We associate

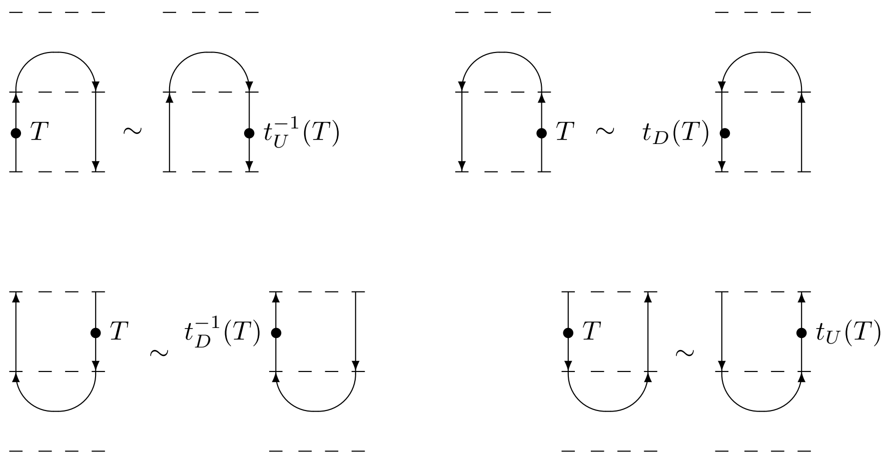
T to  T and T^* to  T

There is a natural way of multiplying decorated vertical lines. For $S, T \in \operatorname{End}(V)$ set $S{\cdot}T = T{\circ}S$. We define



which is consistent with the composition rule for stratified diagrams.

Let t_D, t_U be the algebra automorphisms of End(V) defined by $t_D(T) = D{\circ}T{\circ}D^{-1}$ and $t_U(T) = U{\circ}T{\circ}U^{-1}$ for all $T \in \operatorname{End}(V)$. Then we have the analogs for sliding labeled beads across local extrema:



or in terms of composition formulas:

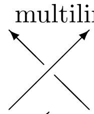
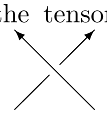
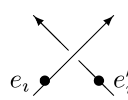
$$\mathcal{U}_-\circ(T\otimes 1_{V^*}) = \mathcal{U}_-\circ(1_V\otimes t_U^{-1}(T)^*),\quad \mathcal{D}_+\circ(1_{V^*}\otimes T) = \mathcal{D}_+\circ(t_D(T)^*\otimes 1_V)$$

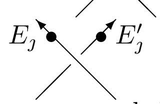
and

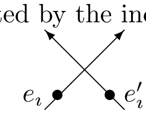
$$(1_V\otimes T^*)\circ\mathcal{D}_- = (t_D^{-1}(T)\otimes 1_{V^*})\circ\mathcal{D}_-,\quad (T^*\otimes 1_V)\circ\mathcal{U}_+ = (1_{V^*}\otimes t_U(T))\circ\mathcal{U}_+ .$$

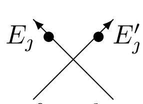
Regard $\mathrm{End}(V\otimes V)$ as $\mathrm{End}(V)\otimes\mathrm{End}(V)$ in the usual way, let $R = \sum_{\imath=1}^r e_\imath\otimes e'_\imath$ and $R^{-1} = \sum_{\jmath=1}^s E_\jmath\otimes E'_\jmath$. Set $\rho = \tau_{V,V}\circ R$. Then ρ and ρ^{-1} satisfy the quantum Yang–Baxter equation (1–1). Observe that

$$\rho(u\otimes v) = \sum_{\imath=1}^r e'_\imath(v)\otimes e_\imath(u) \ \text{ and } \ \rho^{-1}(u\otimes v) = \sum_{\jmath=1}^s E_\jmath(v)\otimes E'_\jmath(u)$$

for all $u, v \in V$. By the multilinearity of the tensor product, to compute $\mathrm{INV}(\mathbf{L})$ we may replace  and  in the diagram $\mathbf{L}$ by 

and 

respectively, where the crossing lines of the last two configurations are regarded as vertical lines decorated by the indicated morphisms. The result of this procedure is indicated by 

and 

. The diagram $\mathbf{L}$ with crossings replaced by these configurations is referred to as a *flat diagram*; see [17, Section 2.1] for example. Usually $e_\imath$ and $e'_\imath$ are denoted by e and e'; likewise $E_\jmath$ and $E'_\jmath$ are denoted by E and E'.

From this point until the end of the section oriented lines, crossings and local extrema will be components of strata. To evaluate $\mathrm{INV}(\mathbf{L})$, first choose an upward directed vertical line (not a crossing line) in each of the components

$\mathbf{L}_1, \ldots, \mathbf{L}_r$ of $\mathbf{L}$ and apply the analogs of the rules for sliding beads across local extrema described above so that the only decorated lines are the chosen verticals and each of these has a single decoration $w(\mathbf{L}_\imath)$. This can be done by bead sliding. Again, we may assume without loss of generality that each component has such a vertical line. Next label the intersections of the diagram $\mathbf{L}$ and the stratification lines by indices (which run over the values $1, \ldots, n$).

For a linear endomorphism T of V we write $T(v_\jmath) = \sum_{\imath=1}^{n} T^\imath_\jmath v_\imath$, where $T^\imath_\jmath \in k$. Thus for endomorphisms S, T of V we have $(T \circ S)^\imath_\jmath = \sum_{\ell=1}^{n} T^\imath_\ell S^\ell_\jmath$. Observe that

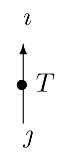
$$\mathcal{D}_+(v^\imath \otimes v_\jmath) = D^\imath_\jmath \quad \text{and} \quad \mathcal{D}_-(1) = \sum_{\imath,\jmath=1}^{n} (D^{-1})^\imath_\jmath v_\imath \otimes v^\jmath,$$

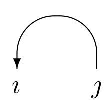
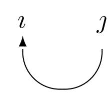
$$\mathcal{U}_+(1) = \sum_{\imath,\jmath=1}^{n} v^\jmath \otimes U^\imath_\jmath v_\imath \quad \text{and} \quad \mathcal{U}_-(v_\jmath \otimes v^\imath) = (U^{-1})^\imath_\jmath.$$

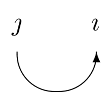
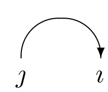
We associate matrix elements to the vertical lines, the crossing lines and to the local extrema of $\mathbf{L}$ whose endpoints are now labeled by indices as follows:

$\delta^\imath_\jmath$ to  or  ,

that is to vertical lines with no line label or crossing lines;

$T^\imath_\jmath$ to 

$D^\imath_\jmath$ to  and $(D^{-1})^\imath_\jmath$ to  ;

$U^\imath_\jmath$ to  and $(U^{-1})^\imath_\jmath$ to  .

The scalar INV($\mathbf{L}$) is obtained by multiplying all of the matrix elements and summing over the indices. It is a product of contributions of the components of $\mathbf{L}$. Each component $\mathbf{L}_\imath$ contributes a factor which is described as follows. Start at the base of the only decorated vertical line in $\mathbf{L}_\imath$ and let $\jmath_1, \ldots, \jmath_m$ be the indices of $\mathbf{L}_\imath$ associated to the intersections of $\mathbf{L}_\imath$ and the stratum lines of $\mathbf{L}$ in the order encountered on a traversal of $\mathbf{L}_\imath$ in the direction of orientation. Observe that $\jmath_1$ and $\jmath_2$, $\jmath_2$ and $\jmath_3$, $\ldots$, $\jmath_{m-1}$ and $\jmath_m$, $\jmath_m$ and $\jmath_1$ label the endpoints of stratum components whose associated matrix elements we will call $(T_1)^{\jmath_2}_{\jmath_1}$, $(T_2)^{\jmath_3}_{\jmath_2}$,

$\dots$, $(T_{m-1})^{\jmath_m}_{\jmath_{m-1}}$, $(T_m)^{\jmath_1}_{\jmath_m}$ respectively. The contribution which the component $\mathbf{L}_\imath$ makes to $\mathrm{INV}(\mathbf{L})$ is

$$\sum_{\jmath_1,\dots,\jmath_m=1}^{n} (T_1)^{\jmath_2}_{\jmath_1}(T_2)^{\jmath_3}_{\jmath_2}\cdots(T_{m-1})^{\jmath_m}_{\jmath_{m-1}}(T_m)^{\jmath_1}_{\jmath_m} = \mathsf{tr}(T_m\circ\cdots\circ T_1) = \mathsf{tr}(G^{-d}w(\mathbf{L}_1)),$$

where $G = D\circ U$ and d is the Whitney degree of $\mathbf{L}_\imath$. If $(\mathrm{End}(V), t_D, t_U)$ is an oriented quantum algebra over k we note $(\mathrm{End}(V), t_D, t_U, D\circ U)$ is a twist oriented quantum algebra over k.

THEOREM 4. *Let V be a finite-dimensional vector space over k. Suppose that D, U are commuting linear automorphisms of V and R is a linear automorphism of $V\otimes V$ which is a solution to the braid equation* (1–2). *Let $\rho = \tau_{V,V}\circ R$ and let t_D, t_U be the algebra automorphisms of* $\mathrm{End}(V)$ *which are defined by $t_D(X) = D\circ X\circ D^{-1}$ and $t_U(X) = U\circ X\circ U^{-1}$ for all $X \in \mathrm{End}(V)$. Then*:

(a) $(\mathrm{End}(V)^{\mathrm{op}}, \rho, t_D, t_U)$ *is an oriented quantum algebra over k if and only if* (8–2)–(8–5) *hold for R, D and U.*
(b) *Suppose that* (8–2)–(8–5) *hold for R, D and U. Then* $\mathrm{INV} = \mathrm{INV}_{R,D,U}$ *is a regular isotopy invariant of oriented link diagrams and*

$$\mathbf{Inv}_{(\mathrm{End}(V)^{\mathrm{op}},\rho,t_D,t_U,(D\circ U)^{-1}),\mathsf{tr}}(\mathbf{L}) = \mathrm{INV}(\mathbf{L})$$

for all $\mathbf{L} \in \mathbf{Link}$.

PROOF. Using bead sliding to evaluate compositions, we note that (8–4) is equivalent to the equation $(t_U\otimes t_U)(\rho) = \rho$ and (8–5) is equivalent to the equation $(t_D^{-1}\otimes t_D^{-1})(\rho) = \rho$. Thus (8–4) and (8–5) are collectively equivalent to (qa.2) for ρ, t_D and t_U.

Assume that (8–4) and (8–5) hold for ρ, t_D and t_U. Let $A = \mathrm{End}(V)^{\mathrm{op}}$. Then (8–2) is equivalent to

$$((t_U\otimes 1)(\rho^{-1}))((1\otimes t_D)(\rho)) = 1\otimes 1$$

in $A\otimes A^{\mathrm{op}}$ and (8–3) is equivalent to

$$((1\otimes t_U)(\rho))((t_D\otimes 1)(\rho^{-1})) = 1\otimes 1$$

in $A\otimes A^{\mathrm{op}}$. Applying the algebra automorphism $t_D\otimes t_U$ of $A\otimes A^{\mathrm{op}}$ to both sides of the first equation we see that (8–2) is equivalent to

$$((t_D\otimes 1)(\rho^{-1}))((1\otimes t_U)(\rho)) = 1\otimes 1$$

in $A\otimes A^{\mathrm{op}}$. We have shown part (a). Note that (8–2) and (8–3) are equivalent since V is finite-dimensional. Part (b) follows from part (a) and the calculations preceding the statement of the theorem. $\square$

Let $(\mathrm{End}(V), \rho, t_D, t_U, G)$ be a twist oriented quantum algebra over k, where $G = D{\circ}U$. We relate the resulting invariant of oriented links with the one described in Theorem 4.

First we identify $\mathrm{End}(V)$ and $\mathrm{M}_n(k)$ via $T \mapsto (T^\imath_\jmath)$ and use this isomorphism to identify $\mathrm{End}(V{\otimes}V) \simeq \mathrm{End}(V){\otimes}\mathrm{End}(V)$ with $\mathrm{M}_n(k){\otimes}\mathrm{M}_n(k)$. Write $\rho = \sum_{\imath,\jmath,k,\ell=1}^n \rho^{\imath\,\jmath}_{k\,\ell} E^\imath_k{\otimes}E^\jmath_\ell$ and let $R_{(\rho)} = \tau_{V,V}{\circ}\rho$ be the solution to the braid equation (1–2) associated to ρ. Then $R_{(\rho)} = \sum_{\jmath,\imath,k,\ell=1}^n \rho^{\jmath\,\imath}_{k\,\ell} E^\imath_k{\otimes}E^\jmath_\ell$ and $R^{-1}_{(\rho)} = \sum_{\jmath,\imath,k,\ell=1}^n (\rho^{-1})^{\imath\,\jmath}_{\ell\,k} E^\imath_k{\otimes}E^\jmath_\ell$.

Let $f : \mathrm{M}_n(k)^{\mathrm{op}} \longrightarrow \mathrm{M}_n(k)$ be the algebra isomorphism defined by $f(x) = x^\tau$, where x^τ is the transpose of $x \in \mathrm{M}_n(k)$. Then $(\mathrm{M}_n(k)^{\mathrm{op}}, \rho_f, t_{D\,f}, t_{U\,f}, f(G))$ is a twist oriented quantum algebra over k, where $\rho_f = (f{\otimes}f)(\rho)$, $t_{D\,f} = f{\circ}t_D{\circ}f^{-1} = t_{(D^{-1})^\tau}$, $t_{U\,f} = f{\circ}t_U{\circ}f^{-1} = t_{(U^{-1})^\tau}$ and $G = D{\circ}U$. Furthermore $f : (\mathrm{M}_n(k)^{\mathrm{op}}, \rho, t_D, t_U, G) \longrightarrow (\mathrm{M}_n(k)^{\mathrm{op}}, \rho_f, t_{D\,f}, t_{U\,f}, f(G))$ is an isomorphism of twist oriented quantum algebras over k. Since $\mathsf{tr} = \mathsf{tr}{\circ}f$ we can use Proposition 12 to conclude that

$$\mathbf{Inv}_{(\mathrm{M}_n(k),\rho,t_D,t_U,G),\mathsf{tr}} = \mathbf{Inv}_{(\mathrm{M}_n(k)^{\mathrm{op}},\rho_f,t_{(D^{-1})^\tau},t_{(U^{-1})^\tau},G^\tau),\mathsf{tr}}. \tag{8–7}$$

We note that $R_{(\rho)}$ and $R_{(\rho_\tau)}$ have the same minimal polynomial. The algebra automorphism of $\mathrm{M}_n(k){\otimes}\mathrm{M}_n(k)$ defined by $F = \tau_{\mathrm{M}_n(k),\mathrm{M}_n(k)}{\circ}(f{\otimes}f)$ satisfies $R_{(\rho_\tau)} = F(R_{(\rho)})$ from which our assertion follows.

Consider the oriented quantum algebra of Example 5 which has a twist structure given by $G = \sum_{\imath=1}^n \omega_\imath^2 E^\imath_\imath$. The resulting invariant of oriented links satisfies a skein identity whenever $\not{b}c$ has a square root in k. In this case

$$\frac{1}{\sqrt{\not{b}c}}R - \sqrt{\not{b}c}R^{-1} = \left(\frac{a}{\sqrt{\not{b}c}} - \frac{\sqrt{\not{b}c}}{a}\right) 1{\otimes}1;$$

thus the invariant $\mathbf{Inv} = \mathbf{Inv}_{(\mathrm{M}_n(k),\rho,t_D,t_U,G),\mathsf{tr}}$ satisfies the skein identity (8–6) with $\alpha = \sqrt{\not{b}c}$, $\beta = 1/\sqrt{\not{b}c}$ and $\gamma = a/\sqrt{\not{b}c} - \sqrt{\not{b}c}/a$. See [16, Section 6] for an analysis of this invariant.

Suppose that further that $k = \mathbb{C}$, $t \in \mathbb{C}$ is transcendental, $q \in \mathbb{C}$ satisfies $t = q^4$, $n = 2$, $a = q^{-1}$, $b_{1\,2} = q$ and $\sqrt{\not{b}c} = q$. Take $\omega_1 = q$ and $\omega_2 = q^{-1}$. In this case the skein identity is the skein identity for the bracket polynomial [10, page 50]. Thus by [10, Theorem 5.2] the Jones polynomial $V_{\mathbf{L}}(t)$ in is given by

$$V_{\mathbf{L}}(t) = V_{\mathbf{L}}(q^4) = \frac{(-q^3)^{\mathrm{writhe}\mathbf{L}}}{\mathsf{tr}(G)}\mathbf{Inv}(\mathbf{L}) = \left(\frac{(-q^3)^{\mathrm{writhe}\mathbf{L}}}{q^2+q^{-2}}\right)\mathbf{Inv}(\mathbf{L}) \tag{8–8}$$

for all $\mathbf{L} \in \mathbf{Link}$.

By virtue of the preceding formula, Example 8 and part (b) of Theorem 3 the Jones polynomial can be computed in terms of the quantum algebra of Example 3. The calculation of the Jones polynomial in terms of Example 3 was done by Kauffman much earlier; see [10, page 580] for example.

9. Inner Oriented Quantum Algebras

The material of the last section suggests designation of a special class of oriented quantum algebras which we will refer to as inner. Formally an *inner oriented quantum algebra* is a tuple (A, ρ, D, U), where $D, U \in A$ are commuting invertible elements such that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra with $t_{\mathsf{d}}(a) = DaD^{-1}$ and $t_{\mathsf{u}}(a) = UaU^{-1}$ for all $a \in A$. Observe that if (A, ρ, D, U) is an inner oriented quantum algebra then $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}, DU)$ is a twist oriented quantum algebra which we denote by (A, ρ, D, U, DU).

An inner oriented quantum algebra (A, ρ, D, U) is *standard* if $D = 1_A$ and is *balanced* if $D = U$. Observe that any standard oriented quantum algebra structure on $A = \mathrm{M}_n(k)$ arises from an inner oriented quantum algebra structure.

If (A, ρ, D, U) is an inner oriented quantum algebra $(A^{\mathrm{op}}, \rho, D^{-1}, U^{-1})$, $(A, \rho^{-1}, D^{-1}, U^{-1})$ and $(A, \rho^{\mathrm{op}}, U^{-1}, D^{-1})$ are as well. See the discussion which follows Example 5.

A *morphism* $f : (A, \rho, D, U) \longrightarrow (A', \rho', D', U')$ *of inner oriented quantum algebras* is an algebra map $f : A \longrightarrow A'$ which satisfies $\rho' = (f \otimes f)(\rho)$, $f(D) = D'$ and $f(U) = U'$; that is $f : (A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}}) \longrightarrow (A', \rho', t'_{\mathsf{d}}, t'_{\mathsf{u}})$ is a morphism of the associated oriented quantum algebras. For inner oriented quantum algebras (A, ρ, D, U) and (A', ρ', D', U') over k we note that $(A \otimes A', \rho'', D \otimes D', U \otimes U')$ is an inner oriented quantum algebra over k, called the *tensor product of* (A, ρ, D, U) *and* (A', ρ', D', U'), where $(A \otimes A', \rho'')$ is the tensor product of the quantum algebras (A, ρ) and (A', ρ'). Observe that $(k, 1 \otimes 1, 1, 1)$ is an inner oriented quantum algebra. Inner oriented quantum algebras together with their morphisms under composition form a monoidal category.

Let $f : A \longrightarrow A'$ be an algebra map. If (A, ρ, D, U) is an inner oriented quantum algebra over k and then $(A', (f \otimes f)(\rho), f(D), f(U))$ is as well and $f : (A, \rho, D, U) \longrightarrow (A', (f \otimes f)(\rho), f(D), f(U))$ is a morphism. In particular if V is a finite-dimensional left A-module and $f : A \longrightarrow \mathrm{End}(V)$ is the associated representation

$$f : (A^{\mathrm{op}}, \rho, D^{-1}, U^{-1}, (DU)^{-1}) \longrightarrow (\mathrm{End}(V)^{\mathrm{op}}, (f \otimes f)(\rho), f(D)^{-1}, f(U)^{-1}, f(DU)^{-1})$$

is a morphism of twist oriented quantum algebras; the latter occurs in part (b) of Theorem 4.

10. The Hennings Invariant

Let (A, ρ, v) be a finite-dimensional unimodular ribbon Hopf algebra with antipode s over the field k and let (A, ρ, s, G) be the twist quantum algebra of Example 4, where $G = uv^{-1}$. Let $\lambda \in A^*$ be a non-zero right integral for A^*. Then $\mathsf{tr} \in A^*$ defined by $\mathsf{tr}(a) = \lambda(Ga)$ for all $a \in A$ is a tracelike s^*-invariant element of A^*.

Suppose further that $\lambda(v), \lambda(v^{-1}) \neq 0$. Then the Hennings invariant for 3-manifolds is defined. It is equal to a normalization of $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L})$ by certain powers of $\lambda(v)$ and $\lambda(v^{-1})$, where $\mathsf{L} \in \mathit{Link}$ is associated with the 3-manifold.

In addition to providing a very rough description of the Hennings invariant, we would like to comment here that Proposition 9 is of very little help in computing this invariant. For suppose that $f : (A, \rho, s, G) \longrightarrow (A', \rho', s', G')$ is a morphism of twist quantum algebras over k, $\mathsf{tr}' \in A'$ is a tracelike s'^{*}-invariant element of A'^{*} and $\mathsf{tr} = \mathsf{tr}' \circ f$. Then $(0) = \lambda(G\ker f)$. The only left or right ideal of A on which λ vanishes is (0). Therefore $\ker f = (0)$, or equivalently f is one-one. Thus the right hand side of $\mathsf{Inv}_{A,\,\mathsf{tr}}(\mathsf{L}) = \mathsf{Inv}_{A',\,\mathsf{tr}'}(\mathsf{L})$ is no easier to compute than the left hand side.

The Hennings invariant [5] was defined originally using oriented links and was reformulated and conceptually simplified using unoriented links [13]. Calculations of the Hennings invariant for two specific Hopf algebras are made in [13], and calculations which are relevant to the evaluation of the Hennings invariant are made in [25].

11. Quantum Coalgebras and Coquasitriangular Hopf Algebras

To define quantum coalgebra we need the notion of coalgebra map with respect to a set of bilinear forms. Let C, D be coalgebras over k and suppose that $\mathcal{B}$ is a set of bilinear forms on D. Then a linear map $T : C \longrightarrow D$ is a *coalgebra map with respect to* $\mathcal{B}$ if $\varepsilon \circ T = \varepsilon$,

$$b(T(c_{(1)}), d)b'(T(c_{(2)}), e) = b(T(c)_{(1)}, d)b'(T(c)_{(2)}, e)$$

and

$$b(d, T(c_{(1)}))b'(e, T(c_{(2)})) = b(d, T(c)_{(1)})b'(e, T(c)_{(2)})$$

for all $c \in C$ and $d, e \in D$. A *quantum coalgebra over* k is a triple (C, b, S), where (C, b) is a Yang–Baxter coalgebra over k and $S : C \longrightarrow C^{\mathrm{cop}}$ is a coalgebra isomorphism with respect to $\{b\}$, such that

(QC.1) $b^{-1}(c, d) = b(S(c), d)$ and

(QC.2) $b(c, d) = b(S(c), S(d))$

for all $c, d \in C$. Observe that (QC.1) and (QC.2) imply

(QC.3) $b^{-1}(c, d) = b(c, S^{-1}(d))$

for all $c, d \in C$; indeed any two of (QC.1)–(QC.3) are equivalent to (QC.1) and (QC.2).

A quantum coalgebra (C, b, S) is *strict* if $S : C \longrightarrow C^{\mathrm{cop}}$ is a coalgebra isomorphism. Not every quantum coalgebra is strict. Let C be any coalgebra over k and suppose that S is a linear automorphism of C which satisfies $\varepsilon = \varepsilon \circ S$. Then (C, b, S) is a quantum coalgebra over k, where $b(c, d) = \varepsilon(c)\varepsilon(d)$ for all $c, d \in C$. There are many quantum coalgebras of this type which are not strict.

Just as quasitriangular Hopf algebras provide examples of quantum algebras, coquasitriangular Hopf algebras provide examples of quantum coalgebras. Recall that a *coquasitriangular Hopf algebra over* k is a pair (A, β), where A is a Hopf algebra over k and $\beta : A{\times}A \longrightarrow k$ is a bilinear form, such that

(CQT.1) $\beta(ab, c) = \beta(a, c_{(1)})\beta(b, c_{(2)})$,

(CQT.2) $\beta(1, a) = \varepsilon(a)$,

(CQT.3) $\beta(a, bc) = \beta(a_{(2)}, b)\beta(a_{(1)}, c)$,

(CQT.4) $\beta(a, 1) = \varepsilon(a)$ and

(CQT.5) $\beta(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = b_{(1)}a_{(1)}\beta(a_{(2)}, b_{(2)})$

for all $a, b, c \in A$. Let (A, β) be a coquasitriangular Hopf algebra over k. The antipode S of A is bijective [30]. By virtue of (CQT.1)–(CQT.4) it follows that β is invertible and $\beta(S(a), b) = \beta^{-1}(a, b) = \beta(a, S^{-1}(b))$ for all $a, b \in A$. Consequently $\beta(a, b) = \beta(S(a), S(b))$ for all $a, b \in A$. (CQT.1) and (CQT.5) imply that (qc.2) for A and β; apply $\beta(\ \ , c)$ to both sides of the equation of (CQT.5). We have shown:

Example 9. If (A, β) is a coquasitriangular Hopf algebra over k then (A, β, S) is a quantum coalgebra over k, where S is the antipode of A.

Associativity is not necessary for the preceding example. A structure which satisfies the axioms for a Hopf algebra over k with the possible exception of the associative axiom is called a *not necessarily associative Hopf algebra over* k.

The notions of strict quantum coalgebra and quantum algebra are dual. Let (A, ρ, s) be a quantum algebra over k. Then (A^o, b_ρ, s^o) is a strict quantum coalgebra over k, and the bilinear form b_ρ is of finite type. Suppose that C is a coalgebra over k, that $b : C{\times}C \longrightarrow k$ is a bilinear form of finite type and that S is a linear automorphism of C. Then (C, b, S) is a strict quantum coalgebra over k if and only if (C^*, ρ_b, S^*) is a quantum algebra over k.

The dual of the quantum algebra described in Example 3 has a very simple description. For $n \geq 1$ let $\mathrm{C}_n(k) = \mathrm{M}_n(k)^*$ and let $\{e_{\imath\jmath}\}_{1\leq\imath,\jmath\leq n}$ be the basis dual to the standard basis $\{E_{\imath\jmath}\}_{1\leq\imath,\jmath\leq n}$ for $\mathrm{M}_n(k)$. Recall that

$$\Delta(e_{\imath\jmath}) = \sum_{\ell=1}^{n} e_{\imath\ell}\otimes e_{\ell\jmath} \ \text{ and } \ \varepsilon(e_{\imath\jmath}) = \delta_{\imath,\jmath}$$

for all $1 \leq \imath, \jmath \leq n$.

Example 10. Let k be a field and $q \in k^\star$. Then $(\mathrm{C}_2(k), b, S)$ is a quantum coalgebra over k, where

$$b(e_{11}, e_{11}) = q^{-1} = b(e_{22}, e_{22}), \ \ b(e_{11}, e_{22}) = q = b(e_{22}, e_{11}),$$

$$b(e_{12}, e_{21}) = q^{-1} - q^3$$

and

$$S(e_{1\,1}) = e_{2\,2},\ \ S(e_{2\,2}) = e_{1\,1},\ \ S(e_{1\,2}) = -q^{-2}e_{1\,2},\ \ S(e_{2\,1}) = -q^{2}e_{2\,1}.$$

Let (C, b, S) and (C', b', S') be quantum coalgebras over k. The *tensor product of* (C, b, S) *and* (C', b', S') is the quantum coalgebra $(C{\otimes}C', b'', S{\otimes}S')$ over k, where $(C{\otimes}C', b'')$ is the tensor product of the Yang–Baxter coalgebras (C, b) and (C', b'). Observe that $(k, b, 1_k)$ is a quantum coalgebra over k, where (k, b) is the Yang–Baxter coalgebra described in Section 2. The category of quantum coalgebras over k with their morphisms under composition has a natural monoidal structure.

Suppose that (C, b, S) is a quantum coalgebra over k. Then (C, b) is a Yang–Baxter coalgebra and thus (C^{cop}, b), (C, b^{-1}) and (C, b^{op}) are as well as noted in Section 2. It is easy to see that (C^{cop}, b, S), (C, b^{-1}, S^{-1}) and $(C, b^{\mathrm{op}}, S^{-1})$ are quantum coalgebras over k.

Suppose that D is a subcoalgebra of C which satisfies $S(D) = D$. Then $(D, b|_{D\times D}, S|_D)$ is a quantum coalgebra over k which is called a *quantum subcoalgebra of* (C, b, S). Observe that the quantum subcoalgebras of (C, b, S) are the quantum coalgebras (D, b', S') over k, where D is a subcoalgebra of C and the inclusion $\imath : D \longrightarrow C$ induces a morphism of quantum coalgebras $\imath : (D, b', S') \longrightarrow (C, b, S)$.

If (C, b) is a Yang–Baxter coalgebra over k then $(D, b|_{D\times D})$ is also where D is any subcoalgebra of C. Thus (C, b) is the sum of its finite-dimensional Yang–Baxter subcoalgebras. There are infinite-dimensional quantum coalgebras (C, b, S) over k such that the only non-zero subcoalgebra D of C which satisfies $S(D) = D$ is C itself; in this case (C, b, S) is not the union of its finite-dimensional quantum subcoalgebras.

Example 11. Let C be the grouplike coalgebra on the set of all integers and let $q \in k^{\star}$. Then (C, b, S) is a quantum coalgebra over k, where

$$b(m, n) = \begin{cases} q & \text{:if } m+n \text{ is even} \\ q^{-1} & \text{:if } m+n \text{ is odd} \end{cases}$$

and $S(m) = m + 1$ for all integers m, n, and if D is a non-zero subcoalgebra of C such that $S(D) = D$ then $D = C$.

Suppose that I is a coideal of C which satisfies $S(I) = I$ and $b(I, C) = (0) = b(C, I)$. Then the Yang–Baxter coalgebra structure $(C/I, \overline{b})$ over k on C/I extends to is a quantum coalgebra structure on $(C/I, \overline{b}, \overline{S})$ over k such that the projection $\pi : C \longrightarrow C/I$ determines a morphism of quantum coalgebras $\pi : (C, b, S) \longrightarrow (C/I, \overline{b}, \overline{S})$. Now suppose that I is the sum of all coideals J of C such that $S(J) = J$ and $b(J, C) = (0) = b(C, J)$. Then the Yang–Baxter coalgebra (C_r, b_r) extends to a quantum coalgebra structure (C_r, b_r, S_r) such that the projection $\pi : C \longrightarrow C_r$ determines a morphism $\pi : (C, b, S) \longrightarrow (C_r, b_r, S_r)$ of quantum coalgebras over k. The quantum coalgebra construction (C_r, b_r, S_r)

is the dual of the construction of the minimal quantum algebra $(A_\rho, \rho, s|_{A_\rho})$ associated to a quantum algebra (A, ρ, s) over k.

Let K be a field extension of k. Then $(C \otimes K, b_K, S \otimes 1_K)$ is a quantum coalgebra over K, where $(C \otimes K, b_K)$ is the Yang–Baxter coalgebra described in Section 2.

We have noted that quantum coalgebras are not necessarily strict. There are natural conditions under which strictness is assured.

LEMMA 4. *A quantum coalgebra (C, b, S) over the field k is strict if b is left or right non-singular.* □

Quantum coalgebra structures pull back under coalgebra maps which are onto.

PROPOSITION 13. *Let (C', b', S') be a quantum coalgebra over the field k and suppose that $f : C \longrightarrow C'$ is an onto coalgebra map. Then there is a quantum coalgebra structure (C, b, S) on C such that $f : (C, b, S) \longrightarrow (C', b', S')$ is a morphism of quantum coalgebras.* □

Every finite-dimensional coalgebra over k is the homomorphic image of $\mathrm{C}_n(k)$ for some $n \geq 1$. As a consequence of the proposition:

COROLLARY 3. *Let (C', b', S') be a quantum coalgebra over k, where C' is finite-dimensional. Then for some $n \geq 1$ there is a quantum coalgebra structure $(\mathrm{C}_n(k), b, S)$ on $\mathrm{C}_n(k)$ and a morphism $\pi : (\mathrm{C}_n(k), b, S) \longrightarrow (C', b', S')$ of quantum coalgebras such that $\pi : \mathrm{C}_n(k) \longrightarrow C'$ is onto.* □

We have noted that coquasitriangular Hopf algebras have a quantum coalgebra structure. In the other direction one can always associate a coquasitriangular Hopf algebra to a quantum coalgebra (C, b, S) over k through the *free coquasitriangular Hopf algebra $(\imath, H(C, b, S), \boldsymbol{\beta})$ on (C, b, S)* whose defining mapping property is described in the theorem below. See [22, Exercise 7.4.4]. A map $f : (A, \beta) \longrightarrow (A', \beta')$ of coquasitriangular Hopf algebras is a Hopf algebra map $f : A \longrightarrow A'$ which satisfies $\beta(a, b) = \beta'(f(a), f(b))$ for all $a, b \in A$.

THEOREM 5. *Let (C, b, S) be a quantum coalgebra over the field k. Then the triple $(\imath, H(C, b, S), \boldsymbol{\beta})$ satisfies the following properties:*

(a) *The pair $(H(C, b, S), \boldsymbol{\beta})$ is a coquasitriangular Hopf algebra over k and $\imath : (C, b, S) \longrightarrow (H(C, b, S), \boldsymbol{\beta}, \boldsymbol{S})$ is a morphism of quantum coalgebras, where $\boldsymbol{S}$ is the antipode of $H(C, b, S)$.*
(b) *If (A', β') is a coquasitriangular Hopf algebra over k and $f : (C, b, S) \longrightarrow (A', \beta', S')$ is a morphism of quantum coalgebras, where S' is the antipode of A', then there exists a map $F : (H(C, b, S), \boldsymbol{\beta}) \longrightarrow (A', \beta')$ of coquasitriangular Hopf algebras over k uniquely determined by $F \circ \imath = f$.* □

Also see [2] in connection with the preceding theorem. The quantum coalgebra (C_r, b_r, S_r) associated with a quantum coalgebra (C, b, S) plays an important role in the theory of invariants associated with (C, b, S). We end this section with a result on (C_r, b_r, S_r) in the strict case. Note that $(C_r)_r = C_r$. Compare part (a) of the following with Corollary 1.

PROPOSITION 14. *Let (C, b, S) be a quantum coalgebra over the field k and suppose that (C_r, b_r, S_r) is strict.*

(a) *If C_r is cocommutative then $S_r^2 = 1_{C_r}$.*
(b) *$S^2(g) = g$ for all grouplike elements g of C_r.*

PROOF. We may assume that $C = C_r$. To show part (a), suppose that C is cocommutative and let b' be the bilinear form on C defined by $b'(c, d) = b(S^2(c), d)$ for all $c, d \in C$. Then b' and b^{-1} are inverses since S is a coalgebra automorphism of C. Thus $b' = b$, or $b(S^2(c), d) = b(c, d)$ for all $c, d \in C$. This equation and (QC.2) imply that $b(c, S^2(d)) = b(c, d)$ for all $c, d \in C$ also. Thus the coideal $I = \mathrm{Im}\,(S^2 - 1_C)$ of C satisfies $b(I, C) = (0) = b(C, I)$. Since S is onto it follows that $S(I) = I$. Thus since $C = C_r$ we conclude $I = (0)$; that is $S^2 = 1_C$. We have established part (a). The preceding argument can be modified to give a proof of part (b). □

With the exception of Example 11 and Theorem 5 the material of this section is a very slight expansion of material found in [19].

12. Oriented Quantum Coalgebras

In this very brief section we define oriented quantum coalgebra and related concepts and discuss a few results about their structure. Most of the material of Section 11 on quantum coalgebras have analogs for oriented quantum coalgebras. The notions of oriented quantum algebra and twist oriented quantum algebra are introduced in [17].

An *oriented quantum coalgebra over k* is a quadruple $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$, where (C, b) is a Yang–Baxter coalgebra over k and T_{d}, T_{u} are commuting coalgebra automorphisms with respect to $\{b, b^{-1}\}$, such that

(qc.1) $b(c_{(1)}, T_{\mathsf{u}}(d_{(2)}))b^{-1}(T_{\mathsf{d}}(c_{(2)}), d_{(1)}) = \varepsilon(c)\varepsilon(d)$,

$b^{-1}(T_{\mathsf{d}}(c_{(1)}), d_{(2)})b(c_{(2)}, T_{\mathsf{u}}(d_{(1)})) = \varepsilon(c)\varepsilon(d)$ and

(qc.2) $b(c, d) = b(T_{\mathsf{d}}(c), T_{\mathsf{d}}(d)) = b(T_{\mathsf{u}}(c), T_{\mathsf{u}}(d))$

for all $c, d \in C$. An oriented quantum coalgebra $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ over k is *strict* if T_{d}, T_{u} are coalgebra automorphisms of C, is *balanced* if $T_{\mathsf{d}} = T_{\mathsf{u}}$ and is *standard* if $T_{\mathsf{d}} = 1_C$. We make the important observation that the T-form structures of [12] are the standard oriented quantum coalgebras over k; more precisely (C, b, T) is a T-form structure over k if and only if $(C, b, 1_C, T^{-1})$ is an oriented quantum coalgebra over k.

Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ and $(C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ be oriented quantum coalgebras over k. Then $(C{\otimes}C', b'', T_{\mathsf{d}}{\otimes}T'_{\mathsf{d}}, T_{\mathsf{u}}{\otimes}T'_{\mathsf{u}})$ is an oriented quantum coalgebra over k, called the *tensor product of* $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ and $(C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$, where $(C \otimes C', b'')$ is the tensor product of the Yang–Baxter coalgebras (C, b) and (C', b'). A *morphism* $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *of oriented quantum coalgebras* is a morphism $f : (C, b) \longrightarrow (C', b')$ of Yang–Baxter coalgebras which satisfies $f \circ T_{\mathsf{d}} = T'_{\mathsf{d}} \circ f$ and $f \circ T_{\mathsf{u}} = T'_{\mathsf{u}} \circ f$. Observe that $(k, b, 1_k, 1_k)$ is an oriented quantum coalgebra over k where (k, b) is the Yang–Baxter coalgebra of Section 2. The category of oriented quantum coalgebras over k and their morphisms under composition has a natural monoidal structure.

The notions of oriented quantum algebra and strict oriented quantum coalgebra are dual as the reader may very well suspect at this point. Suppose that $(A, \rho, t_{\mathsf{d}}, t_{\mathsf{u}})$ is an oriented quantum algebra over k. Then $(A^o, b_\rho, t^o_{\mathsf{d}}, t^o_{\mathsf{u}})$ is a strict oriented quantum coalgebra over k and the bilinear form b_ρ is of finite type. Suppose that C is a coalgebra over k, that b is a bilinear form on C of finite type and that T_{d}, T_{u} are commuting linear automorphisms of C. Then $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is a strict oriented quantum coalgebra over k if and only if $(C^*, \rho_b, T^*_{\mathsf{d}}, T^*_{\mathsf{u}})$ is an oriented quantum algebra over k.

There is an analog of Proposition 2 for oriented quantum coalgebras.

Proposition 15. *If $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra over the field k then $(C, b, T_{\mathsf{d}} \circ T_{\mathsf{u}}, 1_C)$ and $(C, b, 1_C, T_{\mathsf{d}} \circ T_{\mathsf{u}})$ are oriented quantum coalgebras over k.* □

The oriented quantum coalgebra $(C, b, 1_C, T_{\mathsf{d}} \circ T_{\mathsf{u}})$ of the proposition is called the *standard oriented quantum coalgebra associated with* $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$. It may very well be the case that the only oriented quantum coalgebra structures $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ which a Yang–Baxter coalgebra (C, b) supports satisfy $T_{\mathsf{d}} = 1_C$ or $T_{\mathsf{u}} = 1_C$; take the dual of the oriented quantum algebra of Example 6.

As in the case of quantum algebras:

Theorem 6. *Let (C, b, S) be a quantum coalgebra over the field k. Then $(C, b, 1_C, S^{-2})$ is an oriented quantum coalgebra over the field k.* □

It may very well be the case that for a standard oriented quantum coalgebra $(C, b, 1_C, T)$ there is no quantum coalgebra structure of the form (C, b, S); consider the dual of the oriented quantum algebra of Example 7.

To construct knot and link invariants from oriented quantum coalgebras we need a bit more structure. A *twist oriented quantum coalgebra over k* is a quintuple $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}}, G)$, where $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is a strict oriented quantum coalgebra over k and $G \in C^*$ is invertible, such that

$$T^*_{\mathsf{d}}(G) = T^*_{\mathsf{u}}(G) = G \ \text{ and } \ T_{\mathsf{d}} \circ T_{\mathsf{u}}(c) = G^{-1} \rightharpoonup c \leftharpoonup G$$

for all $c \in C$, where $c^* \rightharpoonup c = c_{(1)} c^*(c_{(2)})$ and $c \leftharpoonup c^* = c^*(c_{(1)}) c_{(2)}$ for all $c^* \in C^*$ and $c \in C$. We let the reader work out the duality between twist oriented quantum algebras and twist oriented quantum coalgebras.

The notion of strict (and twist) oriented quantum coalgebra was introduced in [16] and the general notion of quantum coalgebra was introduced in [17]. The results of this section are found in [15, Sections 2, 4].

13. Quantum Coalgebras Constructed from Standard Oriented Quantum Coalgebras

Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k. We consider the dual of the construction discussed in Section 5. The essential part of the construction we describe in this section was made in [12] before the concepts of oriented quantum algebra and coalgebra were formulated. What follows is taken from [15, Section 4].

Let $(C, b, 1_C, T_{\mathsf{d}} \circ T_{\mathsf{u}})$ be the standard oriented quantum coalgebra associated with $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$. Let $\mathcal{C} = C \oplus C^{\mathrm{cop}}$ be the direct sum of the coalgebras C and C^{cop} and think of C as a subcoalgebra of $\mathcal{C}$ by the identification $c = c \oplus 0$ for all $c \in C$. For $c, d \in C$ define $\overline{c \oplus d} = d \oplus c$, let $\boldsymbol{\beta} : \mathcal{C} \otimes \mathcal{C} \longrightarrow k$ be the bilinear form determined by

$$\boldsymbol{\beta}(c, d) = \boldsymbol{\beta}(\overline{c}, \overline{d}), \;\; \boldsymbol{\beta}(\overline{c}, d) = b^{-1}(c, d) \;\text{ and }\; \boldsymbol{\beta}(c, \overline{d}) = b^{-1}(c, (T_{\mathsf{d}} \circ T_{\mathsf{u}})^{-1}(d))$$

and let $\mathbf{S}$ be the linear automorphism of $\mathcal{C}$ defined by

$$\mathbf{S}(c \oplus d) = (T_{\mathsf{d}} \circ T_{\mathsf{u}})^{-1}(d) \oplus c.$$

Since $(C, b, (T_{\mathsf{d}} \circ T_{\mathsf{u}})^{-1})$ is a $(T_{\mathsf{d}} \circ T_{\mathsf{u}})^{-1}$-form structure, it follows by [12, Theorem 1] that $(\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S})$ is a quantum coalgebra over k. It is easy to see that $(\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$ is an oriented quantum coalgebra over k, where

$$\boldsymbol{T}_{\mathsf{d}}(c \oplus d) = T_{\mathsf{d}}(c) \oplus T_{\mathsf{d}}(d) \;\text{ and }\; \boldsymbol{T}_{\mathsf{u}}(c \oplus d) = T_{\mathsf{u}}(c) \oplus T_{\mathsf{u}}(d)$$

for all $c, d \in C$. Observe that the inclusion $\imath : C \longrightarrow \mathcal{C}$ determines a morphism of oriented quantum coalgebras $\imath : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$. Also observe that $\boldsymbol{T}_{\mathsf{d}}$, $\boldsymbol{T}_{\mathsf{u}}$ commute with $\boldsymbol{S}$.

Let $\mathcal{C}_{cq}$ be the category whose objects are quintuples $(C, b, S, T_{\mathsf{d}}, T_{\mathsf{u}})$, where (C, b, S) is a quantum coalgebra over k, $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ is an oriented quantum coalgebra over k and T_{d}, T_{u} commute with S, and whose morphisms

$$f : (C, b, S, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$$

are morphisms of quantum coalgebras $f : (C, b, S) \longrightarrow (C', b', S')$ and morphisms of oriented quantum coalgebras $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$. Our construction gives rise to a free object of $\mathcal{C}_{cq}$.

Proposition 16. *Let $(C, b, T_{\mathsf{d}}, T_{\mathsf{u}})$ be an oriented quantum coalgebra over the field k. Then the pair $(\imath, (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}}))$ satisfies the following properties*:

(a) $(\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$ *is an object of* $\mathcal{C}_{cq}$ *and* $\imath : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}})$ *is a morphism of oriented quantum coalgebras over* k.

(b) *Suppose that* $(C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *is an object of* $\mathcal{C}_{cq}$ *and* $f : (C, b, T_{\mathsf{d}}, T_{\mathsf{u}}) \longrightarrow (C', b', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *is a morphism of oriented quantum coalgebras over* k. *There is a morphism* $F : (\mathcal{C}, \boldsymbol{\beta}, \boldsymbol{S}, \boldsymbol{T}_{\mathsf{d}}, \boldsymbol{T}_{\mathsf{u}}) \longrightarrow (C', b', S', T'_{\mathsf{d}}, T'_{\mathsf{u}})$ *uniquely determined by* $F \circ \imath = f$. □

14. Invariants Constructed from Quantum Coalgebras and Oriented Quantum Coalgebras

The invariants of Section 6 associated with finite-dimensional quantum algebras and twist quantum algebras can be reformulated in terms of dual structures in a way which is meaningful for all quantum coalgebras and twist quantum coalgebras. In this manner regular isotopy invariants of unoriented 1–1 tangles can be constructed from quantum coalgebras and regular isotopy invariants of unoriented knots and links can be constructed from twist quantum coalgebras. In the same way reformulation of the invariants described in Section 7 leads to the construction of regular isotopy invariants of oriented 1–1 tangles from oriented quantum coalgebras and to the construction of regular isotopy invariants of oriented knots and links from oriented twist quantum coalgebras.

14.1. Invariants constructed from quantum coalgebras and twist quantum coalgebras. We assume the notation and conventions of Section 6 in the following discussion. Detailed discussions of the functions Inv_C and $\mathsf{Inv}_{C,\mathsf{tr}}$ are found in [19, Sections 6, 8] respectively. For specific calculations of these functions we refer the reader to [16; 15; 19].

Let (A, ρ, s) be a finite-dimensional quantum algebra over the field k. Then $(C, b, S) = (A^*, b_\rho, s^*)$ is a finite-dimensional strict quantum coalgebra over k. We can regard $\mathsf{Inv}_A(\mathsf{T}) \in A = C^*$ as a functional on C and we can describe $\mathsf{Inv}_A(\mathsf{T})$ in terms of $b = b_\rho$ and $S = s^*$. The resulting description is meaningful for any quantum coalgebra over k.

Let (C, b, S) be a quantum coalgebra over k. We define a function Inv_C : *Tang* $\longrightarrow C^*$ which determines a regular isotopy invariant of unoriented 1–1 tangles. If $\mathsf{T} \in$ *Tang* has no crossings then $\mathsf{Inv}_C(\mathsf{T}) = \varepsilon$.

Suppose that $\mathsf{T} \in$ *Tang* has $n \geq 1$ crossings. Traverse T in the manner described in Section 6.1 and label the crossing lines $1, 2, \ldots, 2n$ in the order encountered. For $c \in C$ the scalar $\mathsf{Inv}_C(\mathsf{T})(c)$ is the sum of products, where each crossing χ contributes a factor according to

$$\mathsf{Inv}_C(\mathsf{T})(c) = \begin{cases} \cdots b(S^{u(\imath)}(c_{(\imath)}), S^{u(\imath')}(c_{(\imath')})) \cdots & :\chi \text{ an over crossing} \\ \cdots b^{-1}(S^{u(\imath)}(c_{(\imath)}), S^{u(\imath')}(c_{(\imath')})) \cdots & :\chi \text{ an under crossing} \end{cases} \tag{14–1}$$

where $\imath$ (respectively $\imath'$) labels the over (respectively under) crossing line of χ.

For example

$$\mathsf{Inv}_C(\mathsf{T}_{\rm trefoil})(c) = b^{-1}(c_{(4)}, S^2(c_{(1)})b(S^2(c_{(2)}, c_{(5)})b^{-1}(c_{(6)}, S^2(c_{(3)})$$

and

$$\mathsf{Inv}_C(\mathsf{T}_{\rm curl})(c) = b(c_{(1)}, S(c_{(2)})).$$

When $(C, b, S) = (A^*, b_\rho, s^*)$ it follows that $\mathsf{Inv}_A(\mathsf{T}) = \mathsf{Inv}_C(\mathsf{T})$ for all $\mathsf{T} \in$ *Tang*. There is a difference, however, in the way in which $\mathsf{Inv}_A(\mathsf{T})$ and $\mathsf{Inv}_C(\mathsf{T})$ are computed which may have practical implications. Computation of $\mathsf{Inv}_A(\mathsf{T})$ according to the instructions of Section 6.1 usually involves non-commutative algebra calculations in tensor powers of A. Computation of $\mathsf{Inv}_C(\mathsf{T})$ based on (14–1) involves arithmetic calculations in the commutative algebra k. There is an analog of Proposition 7 for quantum coalgebras. Thus by virtue of Corollary 3 we can further assume that $C = \mathrm{C}_n(k)$ for some $n \geq 1$.

Final comments about the 1–1 tangle invariant described in (14–1). Even the simplest quantum coalgebras, specifically ones which are pointed and have small dimension, can introduce interesting combinatorics into the study of 1–1 tangle invariants. See [12] for details. Generally quantum coalgebras, and twist quantum coalgebras, seem to provide a very useful perspective for the study of invariants of 1–1 tangles, knots and links.

We now turn to knots and links. Let (C, b, S, G) be a twist quantum coalgebra over k and suppose that c is a cocommutative S-invariant element of C. We construct a scalar valued function $\mathsf{Inv}_{C,\mathsf{c}} : \mathit{Link} \longrightarrow k$ which determines a regular isotopy invariant of unoriented links.

Let $\mathsf{L} \in \mathit{Link}$ be a link diagram with components $\mathsf{L}_1, \ldots, \mathsf{L}_r$. Let $d_1, \ldots, d_r$ be the associated Whitney degrees and set $\mathsf{c}(\ell) = \mathsf{c}{\leftharpoonup}G^{d_\ell}$ for all $1 \leq \ell \leq r$. Let $\omega \in k$ be the product of the $G^{d_\ell}(\mathsf{c}(\ell))$'s such that L_ℓ contains no crossing lines; if there are no such components set $\omega = 1$. The scalar $\mathsf{Inv}_{C,\mathsf{c}}(\mathsf{L})$ is ω times a sum of products, where each crossing contributes a factor according to

$$\mathsf{Inv}_C(\mathsf{T})(c) = \omega \begin{cases} \cdots b(S^{u(\ell:\imath)}(\mathsf{c}(\ell)_{(\imath)}), S^{u(\ell':\imath')}(\mathsf{c}(\ell')_{(\imath')}))\cdots & :\chi \text{ an over crossing} \\ \cdots b^{-1}(S^{u(\ell:\imath)}(\mathsf{c}(\ell)_{(\imath)}), S^{u(\ell':\imath')}(\mathsf{c}(\ell')_{(\imath')}))\cdots & :\chi \text{ an under crossing} \end{cases} \tag{14–2}$$

where $(\ell{:}\imath)$ (respectively $(\ell'{:}\imath')$) labels the over (respectively under) crossing line of χ. We note that $\mathsf{Inv}_{A,\mathsf{tr}}(\mathsf{L}) = \mathsf{Inv}_{C,\mathsf{tr}}(\mathsf{L})$ when (A, ρ, s, G) is a finite-dimensional twist quantum algebra over k and $(C, b, S, G) = (A^*, b_\rho, s^*, G)$.

14.2. Invariants constructed from oriented quantum coalgebras and twist quantum coalgebras. There are analogs $\mathbf{Inv}_C : \mathbf{Tang} \longrightarrow C^*$ and $\mathbf{Inv}_{C,\mathsf{c}} : \mathbf{Link} \longrightarrow k$ of the invariants Inv_C and $\mathsf{Inv}_{C,\mathsf{c}}$ of Section 14.1 for oriented quantum coalgebras and twist oriented quantum coalgebras respectively. There is a version of Theorem 3 for the coalgebra construction of Section 5; see [15, Section 8.3].

The descriptions of $\mathbf{Inv}_C$ and $\mathbf{Inv}_{C,\mathsf{c}}$ are fairly complicated since there are eight possibilities for oriented crossings situated with respect to a vertical. If all crossings of $\mathbf{T} \in \mathbf{Tang}$ and $\mathbf{L} \in \mathbf{Link}$ are oriented in the upward direction, which we may assume by virtue of the twist moves, then the descriptions of $\mathbf{Inv}_C$ and $\mathbf{Inv}_{C,\mathsf{c}}$ take on the character of (14–1) and (14–2) respectively. See [15, Section 8.2] for details.

References

[1] E. Abe, "Hopf Algebras," **74**, *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1980.

[2] Yukio Doi, Braided bialgebras and quadratic algebras, *Comm. in Algebra* **21**, (1993), 1731–1749.

[3] V. G. Drinfel'd, On almost cocommutative Hopf algebras, *Leningrad Math J.* (translation) **1** (1990), 321–342.

[4] V. G. Drinfel'd, Quantum Groups, *Proceedings of the International Congress of Mathematicians*, Berkeley, California, USA (1987), 798–820.

[5] Mark Hennings, Invariants of links and 3-manifolds obtained from Hopf algebras, *J. London Math. Soc.* **54** (1996), 594–624.

[6] Takahiro Hayashi, Quantum groups and quantum determinants, *J. Algebra* **152** (1992), 146–165.

[7] I. N. Herstein, "Noncommutative rings." The Carus Mathematical Monographs, No. 15 Published by The Mathematical Association of America; distributed by John Wiley & Sons, Inc., New York 1968.

[8] M. Jimbo, Quantum R matrix related to the generalized Toda system: An algebraic approach, *in* "Lecture Notes in Physics", **246** (1986), 335–361.

[9] Louis Kauffman, Gauss Codes, quantum groups and ribbon Hopf algebras, *Reviews in Math. Physics* **5** (1993), 735–773.

[10] Louis Kauffman, "Knots and Physics," World Scientific, Singapore/New Jersey/London/Hong Kong, 1991, 1994.

[11] Louis H. Kauffman and David E. Radford, A necessary and sufficient condition for a finite-dimensional Drinfel'd double to be a ribbon Hopf algebra, *J. of Algebra* **159** (1993), 98–114.

[12] Louis H. Kauffman and David E. Radford, A separation result for quantum coalgebras with an application to pointed quantum coalgebras of small dimension, *J. of Algebra* **225** (2000), 162–200.

[13] Louis H. Kauffman and David E. Radford, Invariants of 3-manifolds derived from finite-dimensional Hopf algebras, *J. Knot Theory Ramifications* **4** (1995), 131–162.

[14] Louis H. Kauffman and David E. Radford, On invariants of knots and links which arise from quantum coalgebras, *in preparation.*

[15] Louis H. Kauffman and David E. Radford, Oriented quantum algebras and coalgebras, invariants of knots and links, *in preparation.*

[16] Louis H. Kauffman and David E. Radford, Oriented quantum algebras and invariants of knots and links, *J. of Algebra* (**to appear**), 45 LaTeXpages.

[17] Louis H. Kauffman and David E. Radford, Oriented quantum algebras, categories, knots and links, *J. Knot Theory Ramifications* (**to appear**), 47 LaTeXpages.

[18] Louis H. Kauffman and David E. Radford, Quantum algebra structures on $n \times n$ matrices, *J. of Algebra* **213** (1999), 405–436.

[19] Louis H. Kauffman and David E. Radford, Quantum algebras, quantum coalgebras, invariants of 1–1 tangles and knots, *Comm. Algebra* **28** (2000), 5101–5156.
Quantum algebras, quantum coalgebras, invariants of 1–1 tangles and knots, *Comm. in Alg., to appear* (2000), 62 LaTeXpages.

[20] Louis H. Kauffman and David E. Radford, State sums and oriented quantum algebras, *in preparation.*

[21] Louis H. Kauffman, David E. Radford and Steve Sawin, Centrality and the KRH invariant. *J. Knot Theory Ramifications* **7** (1998), 571–624.

[22] Larry A. Lambe and David E. Radford, "Introduction to the Quantum Yang–Baxter Equation and Quantum Groups: An algebraic Approach," Kluwer Academic Publishers, Boston/Dordrecht/London, 1997.

[23] Shahn Majid, Quasitriangular Hopf algebras and Yang–Baxter equations, *Int. J. Mod. Physics A* **5** (1990), 1–91.

[24] S. Montgomery, "Hopf Algebras and their actions on rings," **82**, *Regional Conference Series in Mathematics*, AMS, Providence, RI, 1993.

[25] D. E. Radford, On Kauffman's knot invariants arising from finite-dimensional Hopf algebras. In *Advances in Hopf algebras (Chicago, Il 1992)*, **148** of *Lecture Notes in Pure and Applied Math. Series*, 205–266. Dekker, New York.

[26] D. E. Radford, Minimal quasitriangular Hopf algebras. *J. Algebra* **157** (1993), 285–315.

[27] D. E. Radford, On a parameterized family of twist quantum coalgebras and the bracket polynomial, *J. of Algebra* **225** (2000), 93–123.

[28] N. Yu. Reshetikhin and V. G. Turaev, Invariants of 3-manifolds via link polynomials and quantum groups, *Invent. Math.* **103** (1991), 547–597.

[29] N. Yu. Reshetikhin and V. G. Turaev, Ribbon graphs and their invariants derived from quantum groups, *Comm. Math. Physics* **127** (1990), 1–26.

[30] P. Schauenburg, On coquasitriangular Hopf algebras and the quantum Yang–Baxter equation, *Algebra Berichte* **67** (1992), 1–76.

[31] M. E. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.

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