Modern Signal Processing
MSRI Publications
Volume **46**, 2003

An Invitation to Matrix-Valued Spherical Functions: Linearization of Products in the Case of Complex Projective Space $P_2(\mathbb{C})$

F. ALBERTO GRÜNBAUM, INÉS PACHARONI, AND JUAN TIRAO

Abstract. The classical (scalar-valued) theory of spherical functions, put forward by Cartan and others, unifies under one roof a number of examples that were very well-known before the theory was formulated. These examples include special functions such as like Jacobi polynomials, Bessel functions, Laguerre polynomials, Hermite polynomials, Legendre functions, which had been workhorses in many areas of mathematical physics before the appearance of a unifying theory. These and other functions have found interesting applications in signal processing, including specific areas such as medical imaging.

The theory of matrix-valued spherical functions is a natural extension of the well-known scalar-valued theory. Its historical development, however, is different: in this case the theory has gone ahead of the examples. The purpose of this article is to point to some examples and to interest readers in this new aspect in the world of special functions.

We close with a remark connecting the functions described here with the theory of matrix-valued orthogonal polynomials.

1. Introduction and Statement of Results

The theory of matrix-valued spherical functions (see [GV; T]) gives a natural extension of the well-known theory for the scalar-valued case, see [He]. We start with a few remarks about the scalar-valued case.

The classical (scalar-valued) theory of spherical functions (put forward by Cartan and others after him) allows one to unify under one roof a number of examples that were very well known before the theory was formulated. These examples include many special functions like Jacobi polynomials, Bessel functions, Laguerre polynomials, Hermite polynomials, Legendre functions, etc.

This paper is partially supported by NSF grants FD9971151 and 1-443964-21160 and by CONICET grant PIP 655-98.

All these functions had "proved themselves" as the work-horse in many areas of mathematical physics before the appearance of a unifying theory. Many of these functions have found interesting applications in signal processing in general as well as in very specific areas like medical imaging. It suffices to recall, for instance, that Cormack's approach [C] — for which he got the 1979 Nobel Prize in Medicine, along with G. Hounsfield — was based on classical orthogonal polynomials and that the work of Hammaker and Solmon [HS] as well as that of Logan and Shepp [LS] is based on the use of Chebychev polynomials.

The crucial property here is the fact that these functions satisfy the integral equation that characterizes spherical functions of a homogeneous space. For a review on some of these topics the reader can either look at some of the specialized books on the subject such as [He] or start from a more introductory approach as that given in either [DMcK] and [T1, vol. I].

This integral equation is actually satisfied by all Gegenbauer polynomials and not only those corresponding to symmetric spaces. This point is fully exploited in [DG] where this property is put to use to show that different weight functions can be used in carrying out the usual tomographic operations of projection and backprojection. This works well for parallel beam tomography but has never been made to work for fan beam tomography because of a lack of an underlying group theoretical formulation in this case. For a number of issues in this area, including a number of open problems, see [G2].

For a variety of other applications of spherical functions one can look at [DMcK; T1].

We now come to the main issue in this article.

The situation with the matrix-valued extension of this theory is entirely different. In this case the theory has gone ahead of the examples and, in fact, to the best of our knowledge, the first examples involving nonscalar matrices have been given recently in [GPT1; GPT2; GPT3]. For scalar-valued instances of nontrivial type, see [HeSc].

The issue of how useful these functions may turn out to be as a tool in areas like geometry, mathematical physics, or signal processing in the broad sense is still open. From a historical perspective one could argue, rather tautologically, that the usefulness of the classical spherical functions rests on the many interesting properties they all share. With that goal in mind, it is natural to try to give a glimpse at these new objects and to illustrate some of their properties. The rather mixed character of the audience attending these lectures gives us an extra incentive to make this material accessible to people that might normally not look in the specialized literature.

The purpose of this contribution is thus to present very briefly the essentials of the theory and to describe one example in some detail. This is not the appropriate place for a complete description, and we refer the interested reader to the papers [GPT1; GPT2; GPT3].

We hope to pique the curiosity of some readers by exploring the extent to which the property of "positive linearization of products" holds in the case of the spherical functions associated to $P_2(\mathbb{C})$. This result has been important in the scalar case, including its use in the proof of the Bieberbach conjecture, see [AAR]. The property in question is illustrated well by considering the case of Legendre polynomials: the product of any two such is expressed as a linear combination involving other Legendre polynomials with degrees ranging from the absolute value of the difference to the sum of the degrees of the two factors involved. Moreover, the coefficients in this expansion are positive.

We should stress that the intriguing property described here is one enjoyed by a *matrix-valued function* put together from different spherical functions of a given type. In the classical *scalar-valued* case these two notions agree and the warning is not needed. This combination of spherical functions has already been seen, see [GPT1; GPT2; GPT3] to enjoy a natural form of the *bispectral property*. For an introduction to this expanding subject we could consult, for instance, [DG1; G12]. The roots of this problem are too long to trace in this short paper, but the reader may want to take a look at [S1]. For off-shoots that have yet to be explored further one can also see [G13; G15]. The short version of the story is that some remarkably useful *algebraic* properties that have surfaced first in signal processing and which one would like to extend and better understand have a long series of connections with other parts of mathematics. For a collection of problems arising in this area see [HK].

The issue of linearization of products, without insisting on any positivity results, plays (in the scalar-valued case) an important role in fairly successful applications of mathematics. For example, the issue of expressing the product of spherical harmonics of different degrees as a sum of spherical harmonics plays a substantial role in both theoretical and practical algorithms for the harmonic analysis of functions on the sphere. For some developments in this area see [DH] as well as [KMHR].

In the context of quantum mechanics this discussion is the backbone of the *addition rule for angular momenta* as can be seen in any textbook on the subject.

In the last section we make a brief remark connecting the functions described here with the theory of matrix-valued orthogonal polynomials, as developed for instance in [D] and [DVA].

2. Matrix-Valued Spherical Functions

Let G be a locally compact unimodular group and let K be a compact subgroup of G. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of K; for each $\delta \in \hat{K}$, let ξ_δ denote the character of δ, $d(\delta)$ the degree of δ, i.e. the dimension of any representation in the class δ, and $\chi_\delta = d(\delta)\xi_\delta$.

Given a homogeneous space G/K a zonal spherical function ([He]) φ on G is a continuous complex valued function which satisfies $\varphi(e) = 1$ and

$$\varphi(x)\varphi(y) = \int_K \varphi(xky)\, dk \qquad\qquad x, y \in G. \tag{2–1}$$

The following definition gives a fruitful generalization of this concept.

DEFINITION 2.1 [T; GV]. A spherical function Φ on G of type $\delta \in \hat{K}$ is a continuous function on G with values in $\mathrm{End}(V)$ such that

(i) $\Phi(e)$ equals I, the identity transformation.
(ii) $\Phi(x)\Phi(y) = \int_K \chi_\delta(k^{-1})\Phi(xky)\, dk$, for all $x, y \in G$.

The connection with differential equations of the group G comes from the property below.

Let $D(G)^K$ denote the algebra of all left invariant differential operators on G which are also invariant under all right translation by elements in K. If (V, π) is a finite dimensional irreducible representation of K in the equivalence class $\delta \in \hat{K}$, a spherical function on G of type δ is characterized by:

(i) $\Phi : G \longrightarrow \mathrm{End}(V)$ is analytic.
(ii) $\Phi(k_1 g k_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$.
(iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

We will be interested in the specific example given by the complex projective plane. This can be realized as the homogeneous space G/K, where $G = \mathrm{SU}(3)$ and $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$. In this case iii) above can be replaced by: $[\Delta_2\Phi](g) = \lambda_2\Phi(g)$, $[\Delta_3\Phi](g) = \lambda_3\Phi(g)$ for all $g \in G$ and for some $\lambda_2, \lambda_3 \in \mathbb{C}$. Here Δ_2 and Δ_3 are two algebraically independent generators of the polynomial algebra $D(G)^G$ of all differential operators on G which are invariant under left and right multiplication by elements in G.

The set $\hat{K}$ can be identified with the set $\mathbb{Z} \times \mathbb{Z}_{\geq 0}$. If $k = \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix}$, with $A \in \mathrm{U}(2)$ and $a = (\det A)^{-1}$, then

$$\pi(k) = \pi_{n,l}(A) = (\det A)^n \; A^l,$$

where A^l denotes the l-symmetric power of A, defines an irreducible representation of K in the class $(n, l) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0}$.

For simplicity we restrict ourselves in this brief presentation to the case $n \geq 0$. The paper [GPT1] deals with the general case. The representation $\pi_{n,l}$ of $\mathrm{U}(2)$ extends to a unique holomorphic multiplicative map of $\mathrm{M}(2, \mathbb{C})$ into $\mathrm{End}(V_\pi)$, which we shall still denote by $\pi_{n,l}$. For any $g \in \mathrm{M}(3, \mathbb{C})$, we shall denote by $A(g)$ the left upper 2×2 block of g, i.e.

$$A(g) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}.$$

For any $\pi = \pi_{(n,l)}$ with $n \geq 0$ let $\Phi_\pi : G \longrightarrow \mathrm{End}(V_\pi)$ be defined by

$$\Phi_\pi(g) = \Phi_{n,l}(g) = \pi_{n,l}(A(g)).$$

It happens that Φ_π is a spherical function of type (n, l), one that will play a very important role in the construction of *all* the remaining spherical functions of the same type.

Consider the open set

$$\mathcal{A} = \{\, g \in G : \det A(g) \neq 0 \,\}.$$

The group $G = \mathrm{SU}(3)$ acts in a natural way in the complex projective plane $P_2(\mathbb{C})$. This action is transitive and K is the isotropy subgroup of the point $(0, 0, 1) \in P_2(\mathbb{C})$. Therefore $P_2(\mathbb{C}) = G/K$. We shall identify the complex plane $\mathbb{C}^2$ with the affine plane $\{\, (x, y, 1) \in P_2(\mathbb{C}) : (x, y) \in \mathbb{C}^2 \,\}$.

The canonical projection $p : G \longrightarrow P_2(\mathbb{C})$ maps the open dense subset $\mathcal{A}$ onto the affine plane $\mathbb{C}^2$. Observe that $\mathcal{A}$ is stable by left and right multiplication by elements in K.

To determine all spherical functions $\Phi : G \longrightarrow \mathrm{End}(V_\pi)$ of type $\pi = \pi_{n,l}$, we use the function Φ_π introduced above in the following way: in the open set $\mathcal{A}$ we define a function H by

$$H(g) = \Phi(g)\, \Phi_\pi(g)^{-1},$$

where Φ is suppose to be a spherical function of type π. Then H satisfies:

(i) $H(e) = I$.
(ii) $H(gk) = H(g)$, for all $g \in \mathcal{A}, k \in K$.
(iii) $H(kg) = \pi(k)H(g)\pi(k^{-1})$, for all $g \in \mathcal{A}, k \in K$.

Property ii) says that H may be considered as a function on $\mathbb{C}^2$.

The fact that Φ is an eigenfunction of Δ_2 and Δ_3 makes H into an eigenfunction of certain differential operators D and E on $\mathbb{C}^2$.

We are interested in considering the differential operators D and E applied to a function $H \in C^\infty(\mathbb{C}^2) \otimes \mathrm{End}(V_\pi)$ such that $H(kp) = \pi(k)H(p)\pi(k)^{-1}$, for all $k \in K$ and p in the affine complex plane $\mathbb{C}^2$. This property of H allows us to find ordinary differential operators $\tilde{D}$ and $\tilde{E}$ defined on the interval $(0, \infty)$ such that

$$(D\,H)(r, 0) = (\tilde{D}\tilde{H})(r), \qquad (E\,H)(r, 0) = (\tilde{E}\tilde{H})(r),$$

where $\tilde{H}(r) = H(r, 0)$.

Introduce the variable $t = (1 + r^2)^{-1}$, which converts the operators $\tilde{D}$ and $\tilde{E}$ into new operators D and E.

The functions $\tilde{H}$ turn out to be diagonalizable. Thus, in an appropriate basis of V_π, we can write $\tilde{H}(r) = H(t) = (h_0(t), \ldots, h_l(t))$.

We find it very convenient to introduce two integer parameters w, k subject to the following three inequalities: $0 \leq w$, $0 \leq k \leq l$, which give a very convenient parametrization of the irreducible spherical functions of type (n, l). In fact, for

each pair (l,n), there are a total of $l+1$ *families* of matrix-valued functions of t and w. In this instance these matrices are diagonal and one can put these diagonals together into a *full matrix-valued function* as we will do in the next two sections. It appears that this function, which coincides with the usual spherical function in the scalar case, enjoys some interesting properties.

The reader can consult [GPT1] to find a fairly detailed description of the entries that make up the matrices mentioned up to now. A flavor of the results is given by the following statement.

For a given $l \geq 0$, the spherical functions corresponding to the pair (l,n) have components that are expressed in terms of generalized hypergeometric functions of the form ${}_{p+2}F_{p+1}$, namely

$$ {}_{p+2}F_{p+1}\left(\begin{matrix} a, b, s_1+1, \ldots, s_p+1 \\ c, s_1, s_2, \ldots, s_p \end{matrix}; t\right) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{j!(c)_j}(1+d_1 j+\cdots+d_p j^p)t^j. $$

3. The Bispectral Property

For given nonnegative integers n, l and w consider the matrix whose rows are given by the vectors $H(t)$ corresponding to the values $k = 0, 1, 2, \ldots, l$ discussed above. Denote the corresponding matrix by

$$\Phi(w,t).$$

As a function of t, $\Phi(w,t)$ satisfies two differential equations

$$ D\Phi(w,t)^t = \Phi(w,t)^t \Lambda, \quad E\Phi(w,t)^t = \Phi(w,t)^t M \ . $$

Here Λ and M are diagonal matrices with

$$ \begin{aligned} \Lambda(i,i) &= -w(w+n+i+l+1)-(i-1)(n+i), \\ M(i,i) &= \Lambda(i,i)(n-l+3i-3)-3(i-1)(l-i+2)(n+i), \end{aligned} $$

for $1 \leq i \leq l+1$; D and E are the differential operators introduced earlier. Moreover we have

THEOREM 3.1. *There exist matrices A_w, B_w, C_w, independent of t, such that*

$$ A_w\Phi(w-1,t)+B_w\Phi(w,t)+C_w\Phi(w+1,t) = t\Phi(w,t) \ . $$

The matrices A_w and C_w consist of two diagonals each and B_w is tridiagonal. Assume, for convenience, that these vectors are normalized in such a way that for $t=1$ the matrix $\Phi(w,1)$ consists of all ones.

For details on these matrices as well as for a full proof of this statement, which was conjectured in [GPT1], the reader can consult [GPT2] and [PT].

4. Linearization of Products

The property in question states that the product of members of certain families of (scalar-valued) orthogonal polynomials is given by an expansion of the form

$$P_i P_j = \sum_{k=|j-i|}^{j+i} a_k P_k$$

and that the coefficients in the expansion are all nonnegative.

For a nice and detailed account of the situation in the scalar case, see for instance [A], [S]. Very important contributions on these and related matters are [G] and [K].

It is important to note that the property in question is not true for all families of orthogonal polynomials, in fact it is not even true for all Jacobi polynomials $P_w^{(\alpha,\beta)}$, *normalized by the condition* $P_w^{(\alpha,\beta)}(1)$ *positive*. For our purpose it is important to recall that nonnegativity is satisfied if $\alpha \geq \beta$ and $\alpha+\beta \geq 1$.

The case $l = 0$, $n > 1$.

From [GPT1] we know that when $l = 0$ and $n \geq 0$ the appropriate eigenfunctions (without the standard normalization) are given by

$$\Phi(w,t) = {}_2F_1\left(\begin{matrix}-w, & w+n+2 \\ & n+1\end{matrix}; t\right).$$

This means that with the usual convention that the Jacobi polynomials are positive for $t = 1$ we are dealing with the family

$$P_w^{(1,n)}(t).$$

If $n = 0$ or $n = 1$ the family $P_w^{(1,n)}$ meets the sufficient conditions for nonnegativity given above. For $n = 0$ the coefficients a_k are all strictly positive; in the case $n = 1$ the coefficients $a_{|i-j|+2k}$, are strictly positive while the coefficients $a_{|i-j|+k}$, k odd, are zero, as the example below illustrates.

We now turn our attention to the case $n > 1$.

Conjecture 4.1. *For n an integer larger than one, the coefficients in the expansion for the product $P_i P_j$ above alternate in sign.*

This conjecture is backed up by extensive experiments, one of which is shown below. It deals with the case of w (that is, i and j) equal to 3 and 4. Richard Askey supplied a proof of this conjecture. This gives us a new chance to thank him for many years of encouragement and help.

The product of the (scalar-valued, and properly normalized) functions $\Phi(3,t)$ and $\Phi(4,t)$ is given by the expansion

$$\Phi(3,t)\Phi(4,t) = a_1\Phi(1,t)+a_2\Phi(2,t)+a_3\Phi(3,t)+a_4\Phi(4,t) \\ +a_5\Phi(5,t)+a_6\Phi(6,t)+a_7\Phi(7,t),$$

with coefficients given by the expressions

$$a_1 = \frac{(n+2)(n+3)(n+4)}{(n+8)(n+9)(n+10)},$$

$$a_2 = -\frac{6(n-1)(n+3)(n+4)(n+6)^2}{(n+7)(n+8)(n+9)(n+10)(n+11)},$$

$$a_3 = \frac{3(n+4)(n+5)(7n^3+52n^2+67n+162)}{(n+7)(n+9)(n+10)(n+11)(n+12)},$$

$$a_4 = -\frac{4(n-1)(n+6)(11n^3+123n^2+436n+648)}{(n+8)(n+9)(n+11)(n+12)(n+13)},$$

$$a_5 = \frac{3(n+5)(n+6)(n+7)(19n^3+155n^2+162n+504)}{(n+8)(n+9)(n+10)(n+11)(n+13)(n+14)},$$

$$a_6 = -\frac{42(n-1)(n+5)(n+6)^2(n+7)(n+8)}{(n+9)(n+10)(n+11)(n+12)(n+13)(n+15)},$$

$$a_7 = \frac{14(n+5)(n+6)^2(n+7)^2(n+8)}{(n+10)(n+11)(n+12)(n+13)(n+14)(n+15).}$$

This shows that even in the scalar-valued case, as soon as we are dealing with nonclassical spherical functions we encounter an interesting sign alternating property that is quite different from the more familiar case. Here and below we see that things become different once n is an integer larger than one.

Now we explore the picture in the case of general l.

The case $l > 0$, $n > 1$

CONJECTURE 4.2. *If $i \leq j$ then the product of $\Phi(i,t)$ and $\Phi(j,t)$ allows for a (unique) expansion of the form*

$$\Phi(i,t)\Phi(j,t) = \sum_{k=\min\{j-i-l,0\}}^{j+i+l} A_k \Phi(k,t).$$

Here the coefficients A_k are matrices and the matrix-valued function $\Phi(w,t)$ is the one introduced in Section 3. *This conjecture holds for all nonnegative n and is well known for $l = 0$ and $n = 0$.*

In the case of $l = 0$ we obtain the usual range in the expansion coefficients ranging from $j-i$ to $j+i$ as in the case of addition of angular momenta. For larger values of l we see that *extra* terms appear at each end of the expansion.

CONJECTURE 4.3. *If $i < j$ then the coefficients A_k in the expansion*

$$\Phi(i,t)\Phi(j,t) = \sum_{k=\min\{j-i-l,0\}}^{j+i+l} A_k \Phi(k,t).$$

with k in the range $j-i, j+i$ have what we propose to call "the hook alternating property."

We will explain this conjecture by displaying one example. First notice that we exclude those coefficients that are not in the *traditional* or *usual* range discussed above.

At this point it may be appropriate in the name of *truth in advertisement* to admit that we have no concrete evidence of the significance of the property alluded to above and displayed towards the end of the paper. We trust that the reader will find the property cute and intriguing. It would be very disappointing if nobody were to find some use for it.

The results illustrated below have been checked for many values of $l > 0$, but are displayed here for $l = 1$ only.

Recall that from [GPT1] the rows that make up the matrix-valued function $H(t, w)$ are given as follows: the first row is obtained from the column vector

$$H(t) = \begin{pmatrix} \left(1 - \dfrac{\lambda}{n+1}\right) {}_3F_2\left(\begin{matrix} -w,\ w+n+3,\ \lambda-n \\ n+2,\ \lambda-n-1 \end{matrix}; t\right) \\ {}_2F_1\left(\begin{matrix} -w,\ w+n+3 \\ n+1 \end{matrix}; t\right) \end{pmatrix}$$

with

$$\lambda = -w(w+n+3)$$

and the second row comes from the column vector

$$H(t) = \begin{pmatrix} {}_2F_1\left(\begin{matrix} -w,\ w+n+4 \\ n+2 \end{matrix}; t\right) \\ -(n+1)\, {}_3F_2\left(\begin{matrix} -w-1,\ w+n+3,\ \lambda \\ n+1,\ \lambda-1 \end{matrix}; t\right) \end{pmatrix}$$

with

$$\lambda = -w(w+n+4) - n - 2.$$

The product of the matrices $\Phi(2,t)$ and $\Phi(6,t)$ is given by the expansion

$$\Phi(2,t)\Phi(6,t) = A_3\Phi(3,t) + A_4\Phi(4,t) + A_5\Phi(5,t) + A_6\Phi(6,t) \\ + A_7\Phi(7,t) + A_8\Phi(8,t) + A_9\Phi(9,t),$$

where

$$A_3 = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{16(n+4)(n+5)(n+6)^2(n+7)^2}{(n+11)(n+12)(n+13)(n+14)(n+15)(n+16)} \end{pmatrix};$$

$$A_4 = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \text{ with}$$

$$L_{11} = \frac{15(n+5)^2(n+6)(n+8)}{2(n+12)(n+13)(n+14)(n+15)},$$

$$L_{12} = \frac{5(n+5)(n+6)(4\,n^2+55\,n+216)}{6(n+13)(n+14)(n+15)(n+16)},$$

$$L_{21} = \frac{(n+5)(n+6)(n+7)(8\,n^2+153\,n+724)}{2(n+12)(n+13)(n+14)(n+15)(n+16)},$$

$$L_{22} = -\frac{5(n+6)(n+7)(248\,n^4+4665\,n^3+27202\,n^2+45137\,n-23252)}{12(n+11)(n+13)(n+14)(n+15)(n+16)(n+17)};$$

$$A_5 = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \text{ with}$$

$$M_{11} = -\frac{(n+5)(n+6)(185\,n^3+3284\,n^2+15732\,n+10368)}{6(n+7)(n+12)(n+14)(n+15)(n+16)},$$

$$M_{12} = -\frac{(n+5)(85\,n^4+1817\,n^3+11380\,n^2+7072\,n-93460)}{7(n+7)(n+13)(n+15)(n+16)(n+17)},$$

$$M_{21} = -\frac{(n+6)^2(170\,n^4+4735\,n^3+42068\,n^2+99767\,n-168628)}{12(n+7)(n+12)(n+14)(n+15)(n+16)(n+17)},$$

$$M_{22} = \frac{\begin{array}{r}4327\,n^7+163698\,n^6+2480127\,n^5+19091004\,n^4+78090428\,n^3\\ +163454544\,n^2+172290528\,n+132098688\end{array}}{14(n+7)(n+12)(n+13)(n+15)(n+16)(n+17)(n+18)};$$

$$A_6 = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \text{ with}$$

$$N_{11} = \frac{2(193\,n^5+5832\,n^4+65284\,n^3+328884\,n^2+727621\,n+634422)}{7(n+8)(n+13)(n+14)(n+16)(n+17)},$$

$$N_{12} = \frac{171\,n^5+4729\,n^4+45764\,n^3+188570\,n^2+442336\,n+1133640}{8(n+8)(n+14)(n+15)(n+17)(n+18)},$$

$$N_{21} = \frac{171\,n^6+7071\,n^5+116213\,n^4+959879\,n^3+4245034\,n^2+10640548\,n+15755112}{7(n+8)(n+13)(n+14)(n+16)(n+17)(n+18)},$$

$$N_{22} = -\frac{\begin{array}{r}4269\,n^7+169934\,n^6+2677678\,n^5+21066480\,n^4+85737209\,n^3\\ +169428298\,n^2+129986220\,n-46794888\end{array}}{8(n+8)(n+13)(n+14)(n+15)(n+17)(n+18)(n+19)};$$

$$A_7 = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \text{ with}$$

$$P_{11} = -\frac{3(n+5)(129\,n^4+3710\,n^3+36430\,n^2+129960\,n+76536)}{8(n+9)(n+14)(n+15)(n+16)(n+18)},$$

$$P_{12} = -\frac{(n+5)(n+10)(57\,n^3+917\,n^2+2274\,n-11268)}{3(n+9)(n+15)(n+16)(n+17)(n+19)},$$

$$P_{21} = \frac{-3(57\,n^6+2505\,n^5+44489\,n^4+389955\,n^3+1576582\,n^2+1465908\,n-4434696)}{8(n+9)(n+14)(n+15)(n+16)(n+18)(n+19)},$$

$$P_{22} = \frac{\begin{array}{r}2(n+10)(829\,n^6+27979\,n^5+352571\,n^4+2024521\,n^3\\ +5197384\,n^2+5712396\,n+5004720)\end{array}}{3(n+9)(n+14)(n+15)(n+16)(n+17)(n+19)(n+20)};$$

$$A_8 = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \text{ with}$$

$$Q_{11} = \frac{5(n+5)(n+6)(21\,n^2+401\,n+1920)}{6(n+15)(n+16)(n+17)(n+18)},$$

$$Q_{12} = \frac{15(n+5)(n+6)(n+8)(n+11)}{2(n+16)(n+17)(n+18)(n+19)},$$

$$Q_{21} = \frac{5(n+6)(10\,n^4+329\,n^3+4942\,n^2+36611\,n+96300)}{6(n+15)(n+16)(n+17)(n+18)(n+20)},$$

$$Q_{22} = -\frac{3(n+6)(n+11)(430\,n^4+9773\,n^3+67728\,n^2+129129\,n-59220)}{4(n+15)(n+16)(n+17)(n+18)(n+19)(n+21)};$$

$$A_9 = \begin{pmatrix} 0 & 0 \\ T_{21} & T_{22} \end{pmatrix} \text{ with}$$

$$T_{21} = \frac{99(n+4)(n+6)(n+7)(n+10)}{4(n+16)(n+17)(n+18)(n+19)(n+20)},$$

$$T_{22} = \frac{165(n+4)(n+6)(n+7)(n+8)(n+10)(n+12)}{2(n+16)(n+17)(n+18)(n+19)(n+20)(n+21)}.$$

Notice that if we concentrate our attention on the coefficients within the *traditional* range we see that the first matrix A_4 has its first *hook* made up of positive entries, the second hook (which in this example consists of only one entry) has negative signs. The second matrix A_5 has its first hook negative, the second hook positive. The third matrix A_6 repeats the behavior of the first one, the fourth one A_7 imitates the second one, and so on.

Extensive experimentation shows that this *double alternating property* holds for values of l greater than zero. For coefficient matrices in the traditional expansion range, the first matrix has its first hook positive, the second one negative, the third one positive, etc. The second matrix has the same alternating pattern of signs for the hooks but its first hook is negative. The third matrix imitates the first, etc.

The following picture captures the phenomenon described above for n larger than one and when the index k is in the traditional range.

$$\begin{array}{ll} +++\cdots+ & ---\cdots- \\ +--\cdots- & -++\cdots+ \\ +-+\cdots+ & -+-\cdots- \\ +-+ & -+- \\ \vdots\ \vdots\ \vdots & \vdots\ \vdots\ \vdots \\ +-+ & -+- \end{array} \qquad \text{etc.}$$

5. The Relation with Matrix-Valued Orthogonal Polynomials

We close the paper remarking, once again, that our matrix-valued spherical functions are orthogonal with respect to a nice inner product and have polynomial entries. Yet, they do not fit directly into the existing theory of matrix-valued orthogonal polynomials as given for instance in [D] and [DVA].

It is however possible to establish such a connection: define the matrix-valued function $\Psi(j,t)$ by means of the relation

$$\Phi(j,t) = \Psi(j,t)\Phi(0,t).$$

It is now a direct consequence of the definitions that the family $\Psi(j,t)$ satisfies all the standard requirements in [DVA] and not only satisfies a three term recursion relation but also $\Psi(j,t)^t$ satisfies a fixed differential equation with matrix coefficients and only the "eigenvalue matrix" depends on j. In other words the family $\Psi(j,t)$ meets all the conditions given at the beginning of Section 3 and meets also the conditions of the standard theory in [DVA] giving an example of a *classical family of matrix-valued orthogonal polynomials*. In particular, the coefficients in the differential operator D (obtained by conjugation from the one in [GPT1]) are matrix polynomials of degree going with the order of differentiation. For a nice introduction to this circle of ideas, see the pioneering work in [D].

Acknowledgments. We are much indebted to the editors for suggesting a number of places where the exposition could be improved. Grünbaum acknowledges a useful conversation with A. Duran that steered him in the direction to Section 5 above.

References

[A] R. Askey, *Orthogonal polynomials and special functions,* SIAM, (1975).

[AAR] G. Andrews, R. Askey and R. Roy, *Special functions,* Encyclopedia of Mathematics and its applications, Cambridge University Press, 1999.

[C] A. Cormack, "Representation of a function by its line integrals, with some radiological applications I", J. Appl. Physics **34** (1963), 2722–2727.

[D] A. Duran, "Matrix inner product having a matrix symmetric second order differential operators", Rocky Mountain Journal of Mathematics **27**:2 (1997).

[DG] M. E. Davison and F. A. Grünbaum, "Tomographic reconstructions with arbitrary directions", Comm. Pure and Appl. Math. **34** (1981), 77–120.

[DH] J. Driscoll and D. Healy, Jr., "Computing Fourier transforms and convolutions on the 2-sphere", Advances in Applied Mathematics **15** (1994), 202–250.

[DMcK] H. Dym and H. P. McKean, Jr., *Fourier series and integrals,* Academic Press.

[DVA] A. Duran and W. Van Assche, "Orthogonal matrix polynomials and higher order recurrence relations", Linear algebra and its applications **219** (1995), 261–280.

[G] G. Gasper, "Positive sums of the classical orthogonal polynomials", SIAM J. Math. Anal. **8** (1977), 423–447.

[G2] F. A. Grünbaum, "Backprojections in tomography, spherical functions and addition formulas: a few challenges", pp. 143–152 in *Inverse problems, image analysis, and medical imaging*, Contemporary Mathematics **313**, edited by M. Z. Nashed and O. Scherzer, Amer. Math. Soc., Providence, 2002.

[GPT1] F. A. Grünbaum, I. Pacharoni and J. A. Tirao, "Matrix valued spherical functions associated to the complex projective plane", J. Functional Analysis **188** (2002) 350–441.

[GPT2] F. A. Grünbaum, I. Pacharoni and J. Tirao, "A matrix valued solution to Bochner's problem", J. Physics A Math. Gen. **34** (2001), 10647–10656.

[GPT3] F. A. Grünbaum, I. Pacharoni and J. A. Tirao, "Matrix valued spherical functions associated to the three dimensional hyperbolic space", Int. J. Math. **13**:7 (2002), 727–784.

[GV] R. Gangolli and V. S. Varadarajan, *Harmonic analysis of spherical functions on real reductive groups,* Ergebnisse der Mathematik **101**, Springer, Berlin, 1988.

[DG1] J. Duistermaat and F. A. Grünbaum, "*Differential equations in the spectral parameter",* Commun. Math. Phys. **103** (1986), 177–240.

[G12] F. A. Grünbaum, "*Time-band limiting and the bispectral problem",* Comm. Pure Appl. Math. **47** (1994), 307–328.

[G13] F. A. Grünbaum, "*A new property of reproducing kernels of classical orthogonal polynomials",* J. Math. Anal. Applic. **95** (1983), 491–500.

[G15] F. A. Grünbaum, "*Some explorations into the mystery of band and time limiting",* Adv. Appl. Math. **13** (1992), 328–349.

[HS] Ch. Hamaker and D. Solmon, "The angles between the null-spaces of X-rays", J. Math. Anal. Appl. **62** (1978), 1–23.

[HK] J. Harnad and A. Kasman (editors), *The bispectral problem,* CRM proceedings and lectures notes **14**, Amer. Math. Soc., Providence, 1998.

[HeSc] G. Heckman and H. Schlicktkrull, *Harmonic analysis and special functions on symmetric spaces,* Perspective in mathematics **16**, Academic Press, San Diego, 1994.

[He] S. Helgason, *Groups and geometric analysis,* Mathematical Surveys and Monographs **83**, Amer. Math. Soc., Providence, 2000

[K] T. Koornwinder, "Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula", J. London Math. Society **18**:2 (1978), 101–114.

[KMHR] P. Kostelec, D. Maslen, D. Healy, Jr. and D. Rockmore, "Computational harmonic analysis for tensor fields on the two-sphere", J. Comput. Phys. **162** (2000), 514–535.

[LS] B. Logan and L. Shepp, "Optimal reconstruction of a function from its projections", Duke Math. J. (1975), 645–659.

[PT] I. Pacharoni and J. A. Tirao, "Three term recursion relation for functions associated to the complex projective plane", to appear in Mathematical Physics, Analysis and Geometry, 2003.

[S] R. Szwarc, "Orthogonal polynomials and a discrete boundary value problem, II", Siam J. Math. Anal. **23** (1992), 965–969.

[S1] D. Slepian, "*Some comments on Fourier analysis, uncertainty and Modeling",* SIAM Review **25**:3 (July 1983).

[T1] A. Terras, *Harmonic analysis on symmetric spaces and applications,* 2 vol., Springer, NY, 1985 and 1988.

[T] J. Tirao, "Spherical functions", Rev. Unión Matem. Argentina **28** (1977), 75–98.

F. Alberto Grünbaum
Departament of Mathematics
University of California
Berkeley CA 94720
grunbaum@math.berkeley.edu

Inés Pacharoni
CIEM-FaMAF
Universidad Nacional de Córdoba
Córdoba 5000
Argentina
pacharon@mate.uncor.edu

Juan Tirao
CIEM-FaMAF
Universidad Nacional de Córdoba
Córdoba 5000
Argentina
tirao@mate.uncor.edu