Modern Signal Processing
MSRI Publications
Volume **46**, 2003

Sampling of Functions and Sections for Compact Groups

DAVID KEITH MASLEN

ABSTRACT. In this paper we investigate quadrature rules for functions on compact Lie groups and sections of homogeneous vector bundles associated with these groups. First a general notion of band-limitedness is introduced which generalizes the usual notion on the torus or translation groups. We develop a sampling theorem that allows exact computation of the Fourier expansion of a band-limited function or section from sample values and quantifies the error in the expansion when the function or section is not band-limited. We then construct specific finitely supported distributions on the classical groups which have nice error properties and can also be used to develop efficient algorithms for the computation of Fourier transforms on these groups.

CONTENTS

Keywords: Sampling, nonabelian Fourier analysis, compact Lie group.

1. Introduction

The Fourier transform of a function on a compact Lie group computes the coefficients (Fourier coefficients) that enable its expression as a linear combination of the matrix elements from a complete set of irreducible representations of the group. In the case of abelian groups, especially the circle and its lower dimensional products (tori) this is precisely the expansion of a function on these domains in terms of complex exponentials. This representation is at the heart of classical signal and image processing (see [25; 26], for example).

The successes of abelian Fourier analysis are many, ranging from national defense to personal entertainment, from medicine to finance. The record of achievements is so impressive that it has perhaps sometimes led scientists astray, seducing them to look for ways to use these tools in situations where they are less than appropriate: for example, pretending that a sphere is a torus so as to avoid the use of spherical harmonics in favor of Fourier series — a favored mathematical hammer casting the multitudinous problems of science as a box of nails.

There is now however in the applied and engineering communities, a growing awareness, appreciation, and acceptance of the use of the techniques of non-abelian Fourier analysis. A favorite example is the use of spherical harmonics for problems with spherical symmetry. While this is of course classical mathematical technology (see [2; 23], for example), it is only fairly recently that serious attention has been paid to the algorithmic and computational questions that arise in looking for efficient and effective means for their computation [4; 8; 22]. Recent applications include the new analysis of the cosmic microwave background (CMB) data — in this setting, the highest order Fourier coefficients of the function that measures the CMB in all directions from a central point are expected to reveal clues to understanding events in the first moments following the Big Bang [24; 32]. Other examples include the use of spherical harmonic transforms in estimation and control problems on group manifolds [18; 19], and for the solution of nonlinear partial differential equations on the sphere, such as the PDEs of climate modeling [1]. The closely related problem of computing Fourier transforms on the Lie group SO(3) is receiving increased attention for its applicability in volumetric shape matching [13; 14; 17].

In order to bring these new transforms to bear on applications, we must bring the well-studied analytic theory of the representations of compact groups (see [33], for instance) into the realm of the computer. Generally speaking, implementation requires that two problems need to be addressed. On the one hand we need to find a reduction of the a priori continuous data to a finite set of samples of the function, and possibly of its derivatives as well, and we must solve the concomitant problem of function reconstruction, which may only be approximate, from this finite set of samples. This is the *sampling problem.* On the other hand, efficient and reliable algorithms are required in order to turn the discrete data into the Fourier coefficients. These sorts of algorithms go by the name of *Fast Fourier Transforms* or FFTs.

In the abelian case the theory and practice are by now well-known. Shannon sampling is the terminology often used to encompass the solution of the sampling problem for functions on the line, or — and more relevant to this paper — the problem of sampling for a function on the circle, while the associated FFT provides tremendous efficiencies in computation.

In this paper we focus on the sampling problem for compact Lie groups, through an investigation of quadrature rules on these groups. Following the well-known abelian case we distinguish between two situations: the *band-limited* case in which the function in question is known to have only a finite number of nonzero Fourier coefficients, and the *non-band-limited* case. In the former situation it is possible to exactly reconstruct the function from a finite collection of samples, while in the latter, the best we can hope for is an approximation to the Fourier expansion, as well as some measure of how close is this approximation.

We first describe a general setting, *a filtered algebra*, where an extension of the classical notion of band-limited, as in [28], makes sense, and adapt it to the special case of functions on a compact Lie group, G. We define a space of functions $\mathcal{A}_s$ on G, the band-limited functions with band-limit s, in such a way that $\mathcal{A}_s.\mathcal{A}_t$ is contained in $\mathcal{A}_{s+t}$. Then we develop a sampling theorem of the following form:

Assume φ is a distribution on G and f is a continuous function on G that is sufficiently differentiable for the product $f.\varphi$ to exist. There is a canonical projection, P_s, from the space of distributions onto $\mathcal{A}_s$. We describe norms, $\| \; \|$, $\| \; \|_*$, $\| \; \|_{**}$ such that

$$\|P_s(f.(\varphi-\mu))\| \leq M(s,t)\|\varphi-\mu\|_*\|(1-P_s)f\|_{**},$$

provided that $P_{s+t}(\varphi-\mu)=0$, where μ is Haar measure of unit mass on the group and $M(s,t)$ is a function which we explicitly bound in the case of the classical groups.

When f is band-limited this gives a condition on the distribution used to sample f that allows exact computation of the Fourier transform of f from the sampled function. When f is not band-limited it quantifies the error introduced when using the Fourier expansion of $f.\varphi$ to approximate that of f. In particular we show that for sufficiently differentiable functions the projection of the approximate expansion onto a space of band-limited functions closely approximates the projection of the original function onto this space without requiring significantly more sample values than the dimension of the band-limited space. The amount of oversampling is related to the growth function of the algebra generated by the matrix coefficients, and hence to its Gel'fand–Kirillov dimension. This is the content of Section 2.

In Section 3 we extend these results to the expansion of sections of homogeneous vector bundles in terms of basis sections coming from the decomposition of the corresponding induced representation, e.g. the expansion of a tensor field on the sphere in tensor spherical harmonics [16]. Finally in Section 4 we construct finitely supported distributions on the classical groups which are convolutions of distributions supported on one parameter subgroups and which have all the properties required by the sampling theorem, i.e. $P_{s+t}(\varphi - \mu) = 0$ and $\|\varphi - \mu\|_*$ is bounded. These distributions can be used to develop fast algorithms for the computation of Fourier transforms on these groups. A general algebraic approach for such algorithms, which uses efficient algorithms for computing with orthogonal polynomial systems [5], is presented in [21].

Remark. This paper only considers the compact case, but the non-compact is at least as interesting. In this setting G. Chirikjian has pioneered the use of representation theoretic techniques for a broad range of interesting applications including robotics, image processing, and computational chemistry [3].

2. Sampling of Functions

Before going into the general situation it is instructive to consider the familiar case of functions on the 2-sphere S^2, identified with the subalgebra of functions on the compact Lie group SO(3) that right-invariant with respect to translation by SO(2), the subgroup of rotations that leave fixed the North Pole. See Section 2.2.1 for notation.

Example: The Fourier transform on S^2. Let Y_{lm}, with $|m| \leq l$, denote the spherical harmonic on S^2 of order l and degree m (see [23] for explicit definitions). Any continuous function, f, on S^2 has an expansion in spherical harmonics $\sum_{lm} a_{lm} Y_{lm}$ which converges under suitable conditions on f, e.g., when f is C^2. The coefficients a_{lm} are called the *Fourier coefficients* of the function f.

Assume s is a nonnegative integer; then f is said to be *band-limited with band-limit* s if all the coefficients a_{lm} in the expansion of f are zero for $l > s$, i.e. if $f = \sum_{|m| \leq l \leq s} a_{lm} Y_{lm}$. If we now pick $N = (s+1)^2$ points $x_1, \ldots, x_N$ on S^2 in general position, then the function values of f at these points completely determine f provided f is band-limited with band-limit s, so the linear map from function values $(f(x_i))_{1 \leq i \leq N}$ to coefficients $(a_{lm})_{|m| \leq l \leq s}$ is a vector space isomorphism. The numbers a_{lm} can be found from the function f using the formula $a_{lm} = \int_{S^2} f.\overline{Y_{lm}} d\mu$, where μ is the invariant measure on the sphere of unit mass. We can also find these numbers by inverting the equations $f(x_i) = \sum_{|m| \leq l \leq s} a_{lm} Y_{lm}(x_i)$. Another method would be calculate the integrals using sums of the form

$$\sum_{i-1}^{N} f(x_i) \overline{Y_{lm}(x_i)} w_i,$$

where the w_i are numbers, called sample weights, depending only on the points x_i. This is only possible, however, if the w_i and the x_i satisfy

$$\sum_{i=1}^{N} \overline{Y_{lm}}(x_i) w_i = \delta_{(0,0),(l,m)} \qquad \text{for} \quad |m| \le l \le s,$$

which is not usually possible for general sets of $N = (s+1)^2$ points, but is possible for general sets of $N = (2s+1)^2$ points; the condition then determines the sample weights, w_i. This is precisely the condition that we can integrate exactly any band-limited function of band-limit $2s$ using the points and weights, and it follows from the fact that the product of two band-limited functions of band-limit s has band-limit $2s$.

What about functions that may not be band-limited? To treat this more general case we first rewrite this discussion. Let $\mathcal{A}_s$ denote the space of band-limited functions with band-limit s, let $\varphi_s = \sum w_i \delta_{x_i}$ be a finitely supported measure on S^2, and let $b_{lm} = \int_{S^2} f.\overline{Y_{lm}} d\varphi_s$ be the Fourier coefficients of the finite measure $f.\varphi_s$. If f is in $\mathcal{A}_s$ and $\langle \varphi_s - \mu, \mathcal{A}_s^2 \rangle = 0$, then $a_{lm} = b_{lm}$ for $|m| \le l \le s$; to obtain the condition above note that $\mathcal{A}_s^2 = \mathcal{A}_{2s}$. If f is not in $\mathcal{A}_s$, then we can not assume that we will have $a_{lm} = b_{lm}$ for $l \le s$, but we can bound the error. It follows from the example immediately after Theorem 3.7 that, provided $\langle \varphi_s - \mu, \, \mathcal{A}_{2s} \rangle = 0$, we have

$$\sum_{l=0}^{s} (2l+1) \left(\sum_{m=-l}^{l} (b_{lm} - a_{lm})^2 \right)^{1/2} \le 2(s+1)^4 \left(\sum_{i=1}^{N} w_i \right) \sum_{l>s} (2l+1) \left(\sum_{m=-l}^{l} a_{lm}^2 \right)^{1/2}.$$

Let P_s denote the projection from the space of distributions $C^0(S^2)'$ onto $\mathcal{A}_s$ given by truncation of the expansion in spherical harmonics, then we can rewrite the above inequality to obtain

$$\begin{aligned} \|P_s(f.(\varphi_s - \mu))\|_{C_0} \le \|P_s(f.(\varphi_s - \mu))\|_{A_0} &\le 2(s+1)^4 \|\varphi_s\|_{C_0'} \|(1-P_s)f\|_{A_0} \\ &\le K \|\varphi_s\|_{C_0'} \|(1-P_s)f\|_{W_6}, \end{aligned}$$

where $\| \; \|_{A_0}$ is the norm of absolute summability inherited from that on SO(3), $\| \; \|_{W_6}$ is the Sobolev norm on C^6, and K is a positive constant; the last inequality follows from an application of Bernstein's theorem on SO(3) (see [6; 27]). Hence, of f is in C^6, and φ_s is a sequence of measures on S^2 which converges weak-$*$ to μ and for which $\langle \varphi_s, \mathcal{A}_{2s} \rangle = 0$, then $\|P_s(f.(\varphi_s - \mu))\|_{C_0}$ tends to zero as s tends tends to infinity.

This approach to the construction of quadrature rules for functions on S^2, can be generalized, and is the goal of the remainder of this section, which is divided into two parts. First we generalize the band-limited sampling of the introduction to filtered algebras and outline an approach for dealing with functions which are not band-limited. Next we treat the case of continuous functions on a compact Lie group, G. Any such function, f, has a Fourier expansion in terms of the matrix coefficients of irreducible unitary representations of G. The

Fourier transform of f is the collection of all coefficients in this expansion, and may be represented as an element of the space $\prod_\gamma \operatorname{End} V_\gamma$, where γ ranges over the irreducible unitary representations of G, and V_γ is the space on which this representation acts. Sampling a C^m function, f, corresponds to multiplying it by a distribution, φ, of order at most m. By putting norms on the space $\prod_\gamma \operatorname{End} V_\gamma$ we can, under suitable assumptions on φ, bound the difference between a finite number of the Fourier coefficients of f and $f.\varphi$.

In what follows we assume a familiarity with the basic ideas and tools of the representation theory of compact groups. There are many excellent resources for this material. Standard texts include [33; 29].

2.1. An Abstract Framework. Several of the results of this paper fit into a simple framework. Assume $\mathcal{A}$ is a complex algebra and $\{\mathcal{A}_s\}$ is a set of subspaces of $\mathcal{A}$ such that $\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$, where s and t range over some semigroup, which we shall take to be the non-negative integers or reals. Let $\mathcal{A}'$ denote the dual of $\mathcal{A}$, and define a $\mathcal{A}$-module structure of $\mathcal{A}'$ by

$$(a.\varphi)(g) = \varphi(g.a)$$

for any a, g in $\mathcal{A}$, and φ in $\mathcal{A}'$. Let P_s denote the projection from $\mathcal{A}'$ onto $\mathcal{A}'_s$ given by restriction of linear functionals. Then we have the following trivial result.

Lemma 2.1. *Assume φ, μ are linear functionals in $\mathcal{A}'$ such that $P_{s+t}(\varphi - \mu) = 0$. Then*

$$P_s(f.\varphi) = P_s(f.\mu)$$

for any f in $\mathcal{A}_t$.

This lemma simply states that, if the linear functionals, φ and μ, agree on the subspace $\mathcal{A}_{s+t}$, then they also agree on the subspace $\mathcal{A}_s.\mathcal{A}_t$.

Example. Assume $\mathcal{A}$ is a finitely generated $\mathbb{C}$-algebra with identity, and let S be a finite generating set containing the identity. Define $S_0 = \mathbb{C}.1$, and let S_k denote the span of all products of k elements of S. Then $S_k.S_l = S_{k+l}$ for any nonnegative integers k and l.

The lemma above does not necessarily hold for elements, f, which do not belong to $\mathcal{A}_t$. To deal with this case, let us introduce norms on the algebra, $\mathcal{A}$. Assume that $\| \ \|_{\mathcal{A}'_s}$ is a norm on $\mathcal{A}'_s$ and that $\| \ \|_A$, $\| \ \|_B$ are norms on $\mathcal{A}$. Let $\mathcal{A}^{A'}$ be the continuous dual of $\mathcal{A}$ with respect to $\| \ \|_A$, let $\| \ \|'_A$ denote the dual norm, and let $\mathcal{A}^B$ be the completion of $\mathcal{A}$ with respect to $\| \ \|_B$. Now define

$$M(s,t) = \sup\{\|P_s(h.\varphi)\|_{\mathcal{A}'_s} : \|h\|_B = 1,\ \|\varphi\|'_A = 1,\ h \in \mathcal{A},\ \varphi \in \mathcal{A}', P_{s+t}\varphi = 0\}.$$

When there is a possibility of confusion, we shall denote this $M_B^{\mathcal{A}'_s, A}(s,t)$. If $M(s,t) < \infty$ then $P_s(h.\varphi)$ is well defined whenever φ is in the A-continuous dual of $\mathcal{A}$, $P_{s+t}\varphi = 0$, and h is in the B-completion of $\mathcal{A}$. In addition, it only depends on the coset of h modulo $\mathcal{A}_t$.

LEMMA 2.2. *Assume φ, μ are linear functionals in $\mathcal{A}^{A'}$ such that $P_{s+t}(\varphi - \mu) = 0$, and let $h \in \mathcal{A}^B$. Then*

$$\|P_s(f.\varphi) - P_s(f.\mu)\|_{\mathcal{A}'_s} \leq M(s,t)\|\varphi - \mu\|'_A \|f\|_{B/\mathcal{A}_t}$$

where $\| \ \|_{B/\mathcal{A}_t}$ denotes the quotient seminorm on $\mathcal{A}^B/\mathcal{A}_t$.

The next section of this paper is concerned with bounding $M(s,t)$ in the case where $\mathcal{A}$ is the algebra spanned by the matrix coefficients of finite dimensional representations of a compact Lie group. We shall also bound the quantity

$$\overline{M}(s,t) = \sup\{\|e.h\|_{A/\mathcal{A}_{s+t}} : \|e\|_{\mathcal{A}_s} = 1,\ \|h\|_B = 1,\ e \in \mathcal{A}_s,\ h \in \mathcal{A}\}$$

for some particular choices of norms $\| \ \|_{\mathcal{A}_s}$ on $\mathcal{A}_s$. If $\mathcal{A}_s$ is finite dimensional and $\| \ \|_{\mathcal{A}'_s}$ is dual to $\| \ \|_{\mathcal{A}_s}$, then we have $M(s,t) \leq \overline{M}(s,t)$. Weakening $\| \ \|_A$ or $\| \ \|_{\mathcal{A}'_s}$, or strengthening $\| \ \|_B$ or $\| \ \|_{\mathcal{A}_s}$ will decrease $M(s,t)$ and $\overline{M}(s,t)$.

When the algebra $\mathcal{A}$ has a symmetric bilinear form $\langle \ , \ \rangle$ such that $\langle a_1, a_2.a_3\rangle = \langle a_1.a_2, a_3\rangle$, then we have an $\mathcal{A}$-module morphism from $\mathcal{A}$ into $\mathcal{A}'$. Thus we can translate Lemma 2.1 into a statement about subspaces of $\mathcal{A}$.

LEMMA 2.3. (i) $\mathcal{A}_{s+t}^{\perp}.\mathcal{A}_s \subseteq \mathcal{A}_t^{\perp}$.
(ii) *Let $\mathcal{A}_s^- = \cup_{t\leq s}\mathcal{A}_t$, then $\mathcal{A}_{s+t}^{-\perp}.\mathcal{A}_s \subseteq \mathcal{A}_t^{-\perp}$.*

PROOF. Part (ii) holds because $\mathcal{A}_s.\mathcal{A}_t^- \subseteq \mathcal{A}_{s+t}^-$. □

2.2. Sampling of Functions on a Compact Lie Group

2.2.1. Notation and conventions. In what follows, we'll assume G is a connected compact Lie group, with Lie algebra $\mathfrak{g}$. Let T be a maximal torus of G and $\mathfrak{t}$ be it's Lie algebra, then $\mathfrak{h} = \mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Choose a fundamental Weyl chamber and for any dominant integral weight, λ, let Δ_λ be the irreducible Lie algebra representation of highest weight λ. If $\hat{G}$ denotes the unitary dual of G, then the map sending an irreducible unitary representation, ρ, to it's highest weight allows us to identify $\hat{G}$ with a subset of the set the set of all dominant integral weights. For any λ in $\hat{G}$ denote the group representation of highest weight λ by Δ_λ as well, and set $d_\lambda = \dim \Delta_\lambda = \prod_{\alpha\in\Delta^+}(\langle \lambda+\delta, \alpha\rangle / \langle \delta, \alpha\rangle)$ where $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ and $\langle \ , \ \rangle$ is the Killing form, and Δ^+ is the set of positive roots. Let $r = \dim([G,G]\cap T)$ be the semisimple rank of G, l be the dimension of the center of G, and k be the number of positive roots of G. Then $2k+r+l = \dim G$, and d_λ is a polynomial of degree k on $\mathfrak{h}^*$. For any representation, ρ, of G, let $\rho^\vee$ be the representation dual to ρ. This gives an involution, $(\ \ )^\vee$ on $\hat{G}$.

Choose a norm on $\mathfrak{g}$. For any nonnegative integer, m, define a norm on $C^m(G)$, by $\|f\|_{C_m} = \sup\{\|L(X_1 \dots X_p)f\|_\infty : 0 \leq p \leq m,\ X_1, \dots, X_p \in \mathfrak{g},\ \|X_1\| = \dots = \|X_p\| = 1\}$, where L is the left regular representation. Denote the dual norm on $C^m(G)'$, by $\| \ \|_{C_m{}'}$. These norms are all invariant under the right regular representation. If we were to replace the left regular representation by the right regular representation in the above definitions, we would get an equivalent set of norms invariant under the left regular representation. For $0 \leq m \leq \infty$, denote bilinear pairing between $C^m(G)'$ and $C^m(G)$ by $\langle\ ,\ \rangle$. For φ in $C^m(G)'$, and g, h in $C^m(G)$, we have $\langle \varphi, g.h\rangle = \langle \varphi.g, h\rangle$. Define an involution on $C^\infty(G)$ by $\check{f}(x) = f(x^{-1})$, and anti-involutions by $\bar{f}(x) = \overline{f(x)}$, $f^*(x) = \overline{f(x^{-1})}$. These extend to involutions and anti-involutions on $C^\infty(G)'$ by setting $\langle \check{T}, f\rangle = \langle T, \check{f}\rangle$, $\langle \bar{T}, f\rangle = \overline{\langle T, \bar{f}\rangle}$, and $T^* = \check{\bar{T}}$, for any $T \in C^\infty(G)'$ and $f \in C^\infty(G)$. If μ_G denotes Haar measure on G of unit mass, then the map $f \mapsto f.\mu_G$ gives us an inclusion $L^1(G) \subseteq C^0(G)'$, and since G is compact, we also have inclusions $L^p(G) \supseteq L^q(G)$ for $1 \leq p \leq q \leq \infty$. Denote the L^p norm on $L^p(G)$ by $\|\ \|_p$.

Let $\mathcal{A}$ denote the span of all matrix coefficients of finite dimensional unitary representations of G. Then $\mathcal{A}$ is a subalgebra of $C^\infty(G)$ under pointwise multiplication of functions. $\mathcal{A}$ is invariant under the involutions, $^-, ^\vee, ^*$, and the pairing $\langle\ ,\ \rangle$ restricts to a nondegenerate bilinear form on $\mathcal{A}$. The hermitian form $\langle f, \overline{g}\rangle$ is positive definite so the bilinear form is nondegenerate on any subspace of $\mathcal{A}$ closed under $^-$. In particular, if $\overline{\mathcal{A}_s} = \mathcal{A}_s$ then we can use the bilinear form to identify $\mathcal{A}_s'$ with $\mathcal{A}_s$. We shall use $\perp$ to refer to orthogonal complements taken with respect to the bilinear form. For a subspace closed under $^-$ this is the same as the complement taken with respect to the hermitian form. For any $\lambda \in \hat{G}$, let $\mathcal{A}_\lambda$ be the span of the matrix coefficients of Δ_λ. The Schur relations show easily that $\mathcal{A}_\lambda^\perp = \sum_{\mu \in \hat{G}\setminus\{\lambda^\perp\}} \mathcal{A}_\mu$.

2.2.2. The Fourier transform. Let $\mathfrak{F}(\hat{G}) = \prod_{\lambda \in \hat{G}} \operatorname{End} V_\lambda$, where V_λ is the Hilbert space on which Δ_λ acts. Choose a norm on $\mathfrak{h}^*$. For $1 \leq q < \infty$ and $0 \leq m < \infty$, define on $\mathfrak{F}(\hat{G})$ the following norms, which may possibly be infinite:

$$\begin{aligned}
\|A\|_{\mathfrak{F}_q} &= \left(\sum_{\lambda \in \hat{G}} d_\lambda \|A_\lambda\|_{q,\lambda}^q\right)^{1/q}, \\
\|A\|_{\mathfrak{F}_\infty} &= \sup\{\|A_\lambda\|_\infty : \lambda \in \hat{G}\}, \\
\|A\|_{A_m} &= \|A_0\|_{1,0} + \sum_{\lambda \in \hat{G}\setminus\{0\}} d_\lambda \|\lambda\|^m \|A_\lambda\|_{1,\lambda}, \\
\|A\|_{A'_m} &= \sup\{\|\lambda\|^{-m}\|A_\lambda\|_{\infty,\lambda} : \lambda \in \hat{G},\ \lambda \neq 0\} \cup \{\|A_0\|_{\infty,0}\},
\end{aligned}$$

where $\|\ \|_{\infty,\lambda}$ is the operator norm on $\operatorname{End} V_\lambda$ relative to the Hilbert space norm on V_λ, and for $1 \leq q < \infty$, $\|\ \|_{q,\lambda}$ is the norm on $\operatorname{End} V_\lambda$ given by $\|A_\lambda\|_{q,\lambda} = (\operatorname{Tr}(A_\lambda(A_\lambda)^*)^{q/2})^{1/q}$. Let $\mathfrak{F}_q(\hat{G})$, $A_m(\hat{G})$ and $A'_m(\hat{G})$ be the corresponding subspaces of $\mathfrak{F}(\hat{G})$ on which these norms are finite. For general properties of norms of these types see [11].

Recall that if H is a complex Hilbert space, and A is a linear operator on H, then A^* is a linear operator on H, and A^t is a linear operator on its dual space, H', as is $\bar{A} = A^{*t}$. Hence we can define an involution on $\mathfrak{F}(\hat{G})$, by

$(A^t)_\lambda = (A_{\lambda^\vee})^t$, for $A \in \mathfrak{F}(\hat{G})$, $\lambda \in \hat{G}$, and anti-involutions, $(A^*)_\lambda = (A_\lambda)^*$, $(\bar{A})_\lambda = \overline{A_{\lambda^\vee}}$.

We shall now assume that the norm on $\mathfrak{h}^*$ satisfies $\|\lambda^\vee\| = \|\lambda\|$ for any $\lambda \in \hat{G}$. Then the maps $(\ )^t$, $(\ )^*$, and $\overline{(\ )}$ preserve all the above norms on $\mathfrak{F}(\hat{G})$. Define a bilinear pairing between A'_m and A_m, by $\langle A', A\rangle = \sum_{\lambda\in\hat{G}} d_\lambda \operatorname{Tr}(({A'}^t)_\lambda A_\lambda)$. The map $T : A'_m(\hat{G}) \to A_m(\hat{G})'$ given by $T_{A'}(A) = \langle A', A\rangle$ is an isometric isomorphism, and so we shall use this map to identify $A_m(\hat{G})'$ and $A'_m(\hat{G})$ from now on.

Define the Fourier transform to be the map $\mathfrak{F} : C^\infty(G)' \to \mathfrak{F}(\hat{G})$, given by $\langle \varphi, \mathfrak{F}(s)_\lambda v\rangle = \langle s, x \mapsto \langle \varphi, \Delta_\lambda(x)v\rangle\rangle$ for any $\varphi \in V_\lambda^*$ and $v \in V_\lambda$. When f is a function in $L^1(G)$ this becomes $(\mathfrak{F}f)_\lambda = \int_G f(x)\Delta_\lambda(x)d\mu_G(x)$. To make the statement of the next lemma simpler, it is convenient to assume choose the norms on $\mathfrak{h}^*$ and $\mathfrak{g}$ so that $\|\Delta_\lambda(X)\|_{\infty,\lambda} \le \|\lambda\|.\|X\|$; to see that this is possible, just consider the case where the norm on $\mathfrak{g}$ is Ad-invariant. This condition can always be achieved by scaling either the norm on $\mathfrak{h}^*$ or the norm on $\mathfrak{g}$. More specifically, this condition avoids additional constants in the statements of Lemma 2.4((d),(f)).

Lemma 2.4 (Properties of $\mathfrak{F}$). *Assume m is a nonnegative integer, $1 \le q \le 2$ and $1/q + 1/q' = 1$.*

(i) $\mathfrak{F} : C^\infty(G)' \to \mathfrak{F}(\hat{G})$ *is one to one.*
(ii) $\|\mathfrak{F}f\|_{q'} \le \|f\|_q$. *These are the Hausdorff–Young inequalities.*
(iii) $\mathfrak{F}(L^{q'}(G)) \supseteq \mathfrak{F}_q(\hat{G})$, *and for any A in $\mathfrak{F}_q(\hat{G})$ we have* $\|\mathfrak{F}^{-1}(A)\|_{q'} \le \|A\|_q$
(iv) $\mathfrak{F}(C^m(G)) \supseteq A_m(\hat{G})$, *and for any A in $A_m(\hat{G})$ we have* $\|\mathfrak{F}^{-1}A\|_{C_m} \le \|A\|_{A_m}$.
(v) *Assume $T \in C^m(G)'$, $A \in A_m(\hat{G})$, and $f = \mathfrak{F}^{-1}A$. Then* $\langle T, f\rangle = \langle \mathfrak{F}T, \mathfrak{F}f\rangle$.
(vi) *For any s in $C^m(G)'$ we have* $\|\mathfrak{F}s\|_{A'_m} \le \|s\|_{C_m{}'}$
(vii) *For any s in $C^\infty(G)'$ we have* $\mathfrak{F}(\bar{s}) = \overline{\mathfrak{F}s}$, $\mathfrak{F}\check{s} = (\mathfrak{F}s)^t$, *and* $\mathfrak{F}s^* = (\mathfrak{F}s)^*$. *In particular, $\mathfrak{F}$ is real relative to the real structures on $C^\infty(G)'$ and $\mathfrak{F}(\hat{G})$ induced by the anti-involutions, $\overline{(\ )}$, on these spaces.*
(viii) $(\mathfrak{F}(s_1 * s_2))_\lambda = (\mathfrak{F}s_1)_\lambda(\mathfrak{F}s_2)_\lambda$, *for any distributions, s_1, s_2, in $C^\infty(G)'$, and any λ in $\hat{G}$, where $s_1 * s_2$ denotes the convolution of the distributions s_1 and s_2.*
(ix) $\|\mathfrak{F}(s_1 * s_2)\|_{A'_{m_1+m_2}} \le \|\mathfrak{F}s_1\|_{A'_{m_1}}\|\mathfrak{F}s_2\|_{A'_{m_2}}$.

Proof. See [20; 11]. □

The image $\mathfrak{F}\mathcal{A}$ consists of precisely those elements, A, of $\mathfrak{F}(\hat{G})$ such that $A_\lambda = 0$ except for finitely many λ. All the norms defined above are finite on $\mathfrak{F}\mathcal{A}$, and $\mathfrak{F}\mathcal{A}$ is dense in each of these spaces under the corresponding norm. As $\mathfrak{F}$ is one to one, we can transfer the algebra structure on $\mathcal{A}$ to $\mathfrak{F}\mathcal{A}$, and hence obtain a $\mathcal{A}$-module structure on the spaces A_m and A'_m. The map T, is an isomorphism of of $\mathcal{A}$-modules, and we can use same formula to get a dual pairing between $\mathfrak{F}(\hat{G})$ and $\mathfrak{F}\mathcal{A}$, and hence a $\mathcal{A}$-module isomorphism between $(\mathfrak{F}\mathcal{A})'$ and $\mathfrak{F}(\hat{G})$.

2.2.3. Simple bounds for $\overline{M}$(s,t). Let us assume that an increasing set of finite dimensional subspaces $\{\mathcal{A}_s\}$, is given, that $\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$, and that $\bigcup_{s\geq 0} \mathcal{A}_s = \mathcal{A}$. Examples for such subspaces can be obtained from finite dimensional generating sets of $\mathcal{A}$, or as described in Section 2.2.4, from a norm on $\mathfrak{h}^*$. We shall bound $M(s,t)$ for several different choices of norms, $\| \, \|_A$, $\| \, \|_B$, $\| \, \|_{\mathcal{A}_s}$, on $\mathcal{A}$ and $\mathcal{A}_s$.

Using the Leibniz rule one sees that for $f, g \in C^m(G)$, we have $\|fg\|_{C_m} \leq 2^m \|f\|_{C_m} \|g\|_{C_m}$. Therefore

Result. Assume the A, B norms are both $\| \, \|_{C_m}$ and that $\| \, \|_{\mathcal{A}_s}$ is the restriction of $\| \, \|_{C_m}$ to $\mathcal{A}_s$. Then $\overline{M}(s,t) \leq 2^m$

When $m = 0$, this tells us that if φ is a regular bounded complex Borel measure on G satisfying $P_{s+t}(\varphi - \mu_G) = 0$, h is a continuous function on G, and $Y = (g \mapsto \langle \Delta_\lambda(g)u, v\rangle$ is a matrix coefficient in $\mathcal{A}_s$, then $\left|\int_G h.Y \, d\varphi - \int_G h.Y \, d\mu_G\right| \leq \|u\|\|v\|\|\varphi\|_{C_0'} \|h\|_{C_0/\mathcal{A}_t}$. Clearly $\|h\|_{C_0/\mathcal{A}_t}$ tends to zero as t tends to infinity.

In a similar fashion, we can bound $\overline{M}(s,t)$ for weaker choices of the norm $\| \, \|_{\mathcal{A}_s}$ on $\mathcal{A}_s$.

Result. Assume the A, B norms are both $\| \, \|_{C_m}$ and that $\| \, \|_{\mathcal{A}_s}$ is the restriction of $\| \, \|_{C_0'}$ to $\mathcal{A}_s$, then for some $K > 0$, independent of m,

$$\overline{M}(s,t) \leq K^m \left(1 + \sum_{\mathcal{A}_s \cap \mathcal{A}_\lambda \neq \phi} d_\lambda^2 \|\lambda\|^m \right).$$

Consider this for $s = t$. Assume that φ is a distribution of order m on G satisfying $P_{2s}(\varphi - \mu_G) = 0$, and h is a C^m function of G. Then

$$\|P_s(h.\varphi - h.\mu)\|_{C_0} \leq 2^m \left(1 + \sum_{\mathcal{A}_s \cap \mathcal{A}_\lambda \neq \phi} d_\lambda^2 \|\lambda\|^m \right) \|\varphi\|_{C_m'} \|h\|_{C_s/\mathcal{A}_s},$$

but the sum in this bound is bounded from below by a constant times $s^{2k+m+r+l}$, and we are forced to consider higher differentiability conditions on h in order to get convergence of $\|P_s(h.\varphi - h.\mu)\|_{C_0}$ to zero. Doing so leads us naturally to the consider the norms A_m, on $\mathcal{A}$, and more careful arguments with these new norms will give us more refined bounds on $M(s,t)$ in the situation above.

2.2.4. Norms on $\hat{G}$. Let $\| \, \|$ be a norm on $\mathfrak{h}^*$. For any $s \geq 0$ let $\mathcal{A}_s$ be the span of all the matrix coefficients of representations Δ_λ for $\|\lambda\| \leq s$, i.e. $\mathcal{A}_s = \sum_{\|\lambda\| \leq s} A_\lambda$. There are several properties we may require of this norm on $\mathfrak{h}^*$. We say that a norm $\| \, \|$ on $\mathfrak{h}^*$ *has property I* if whenever λ, μ, ν are in $\hat{G}$, and Δ_ν is a summand of $\Delta_\lambda \otimes \Delta_\mu$, then $\|\nu\| \leq \|\lambda\| + \|\nu\|$. We say that $\| \, \|$ *has property II* if $\|\nu'\| \leq \|\lambda\|$ whenever ν' is a weight of Δ_λ.

Lemma 2.5. *$\| \, \|$ has property I if and only if for any $s, t > 0$, $\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$*

Lemma 2.6. (i) *If $\| \, \|$ satisfies property I, and Δ_ν is a summand of $\Delta_\lambda \otimes \Delta_\mu$, then $\big|\|\lambda\| - \|\mu\|\big| \leq \|\nu\|$.*

(ii) $\| \ \|$ *has property I if and only if* $\big|\|\lambda\|-\|\nu\|\big| \leq \|\mu\|$ *whenever* Δ_ν *is a summand of* $\Delta_\lambda \otimes \Delta_\mu$.

PROOF. Part (ii) is a direct consequence of (i). To prove (i), assume I, and suppose Δ_ν is a summand of $\Delta_\lambda \otimes \Delta_\mu$. Then $\mathcal{A}_\nu \subseteq \mathcal{A}_\lambda.\mathcal{A}_\mu$. For any $s \geq 0$, let $\mathcal{A}_s^- = \sum_{\|\rho\|<s} \mathcal{A}_\rho$. Then $\mathcal{A}_\lambda \subseteq \mathcal{A}^{-\perp}_{\|\lambda\|}$, and $\mathcal{A}_\mu \leq \mathcal{A}_{\|\mu\|}$. Assume $\|\lambda\| \geq \|\mu\|$. Lemma 2.3 shows that $\mathcal{A}^{-\perp}_{\|\lambda\|}.\mathcal{A}_\mu \subseteq \mathcal{A}^{-\perp}_{\|\lambda\|-\|\mu\|}$. Hence $\mathcal{A}_\nu \subseteq \mathcal{A}^{-\perp}_{\|\lambda\|-\|\mu\|}$, and so $\|\lambda\| - \|\mu\| \leq \|\nu\|$. □

To show that II implies I, we need the following lemma.

LEMMA 2.7. *Assume* λ, μ, ν *are dominant integral weights. If* Δ_ν *is a summand of* $\Delta_\lambda \otimes \Delta_\mu$, *then* $\nu = \mu + \nu'$ *where* ν' *is a weight of* Δ_λ

PROOF. Follows from Steinberg's formula for the decomposition of tensor products. See [12] □

COROLLARY 2.8. *II implies I*

All the norms on $\mathfrak{h}^*$ which we will use, will satisfy property I. Let us now show that norms satisfying properties I or II really do exist.

Assume $\langle \ , \ \rangle$ is a positive definite Ad-invariant inner product on $\mathfrak{g}^{\mathbb{C}}$. Then define $\|\mu\|_{\mathrm{Ad}} = \sqrt{\langle \mu, \mu \rangle}$. This gives a norm on $\mathfrak{h}^*$ which is invariant under the Weyl group.

For calculations involving the classical groups another set of norms is more convenient. Assume G is a simple classical group and let $\lambda_1, \dots, \lambda_r$ be the fundamental dominant weights with the standard labeling (i.e. that which appears in [12, p. 58]). Define the linear functional, H, on $\mathfrak{h}^*$ by requiring that for $\mu = \sum a_i \lambda_i$, we have

(i) $H(\mu) = \sum_{i=1}^r a_i$ when G is $\mathrm{SU}(r+1)$ or $\mathrm{Sp}(r)$.
(ii) $H(\mu) = \sum_{i=1}^{r-1} a_i + \frac{1}{2} a_r$ when G is $\mathrm{SO}(2r+1)$.
(iii) $H(\mu) = \sum_{i=1}^{r-2} a_i + \frac{1}{2}(a_{r-1} + a_r)$ when G is $\mathrm{SO}(2r)$.

Define a norm $\| \ \|_H$ on $\mathfrak{h}^*$ by requiring that $\|\mu\|_H = H(\mu)$ for any dominant weight and $\| \ \|_H$ is invariant under the Weyl group. Note that in each of the above cases $\| \ \|_H$ is also invariant under $\vee$.

To verify that we indeed have defined norms it is easiest to use a different description. Let $\{e_i\}$ denote the usual basis of $\mathbb{C}^r$. When G is $\mathrm{SU}(r+1)$ we have an isomorphism between $\mathfrak{h}^*$ and $\mathbb{C}^{r+1}/\langle e_1 + \dots e_{r+1} = 0 \rangle$. such that $\lambda_i = \sum_{j=1}^i e_i$. When G is any other simple classical group we have an isomorphism between $\mathfrak{h}^*$ and $\mathbb{C}^r$ with $\lambda_i = \sum_{j=1}^{r-2}$ for $1 \leq i \leq r-2$, and $\lambda_{r-1} = e_1 + \dots + e_{r-1}$, $\lambda_r = e_1 + \cdots e_r$ for $\mathrm{Sp}(r)$, $\lambda_{r-1} = e_1 + \dots + e_{r-1}$, $\lambda_r = \frac{1}{2}(e_1 + \dots e_r)$ for $\mathrm{SO}(2r+1)$, and $\lambda_{r-1} = \frac{1}{2}(e_1 + \dots + e_{r-1} - e_r)$, $\lambda_r = \frac{1}{2}(e_1 + \dots + e_r)$ for $\mathrm{SO}(2r)$. When G is $\mathrm{Sp}(r)$, $\mathrm{SO}(2r+1)$ or $\mathrm{SO}(2r)$, the norm $\| \ \|_H$ corresponds to the sup norm on $\mathbb{C}^r$. When G is $\mathrm{SU}(r+1)$ it corresponds to twice the quotient of the sup norm on $\mathbb{C}^{r+1}$.

LEMMA 2.9. (i) *If $\mathfrak{g}$ is abelian, then any norm on $\mathfrak{h}^*$ has property II.*

(ii) *Assume $\| \|_1$, $\| \|_2$ are norms on $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which both satisfy the same property I or II. Assume $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $\|\lambda_1 + \lambda_2\| = \|\lambda_1\|_1 + \|\lambda_2\|_2$ for any $\lambda_1 \in \mathfrak{h}_1$ and $\lambda_2 \in \mathfrak{h}_2$. Then $\| \|$ satisfies the corresponding property I or II on $\mathfrak{h}^* = \mathfrak{h}_1^* \oplus \mathfrak{h}_2^*$.*

(iii) *$\| \|_{\mathrm{Ad}}$ has property II for any $\mathfrak{g}$.*

(iv) *$\| \|_H$ has property II for any of the simple classical groups.*

PROOF. Parts (i) and (ii) are trivial. For (iii), note that $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$ is an orthogonal direct sum, so we need only prove the result in the case where G is semisimple and $\langle \, , \, \rangle$ on $i\mathfrak{t}$ is simply the Killing form. So let's assume that this is the case, $\lambda \in \hat{G}$, and μ is a weight of λ. Since all elements of the Weyl group are isometries, we may also assume that μ is dominant. Then $\langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle = \langle \lambda + \mu, \lambda - \mu \rangle$, which is greater than 0 because $\lambda + \mu$ is a dominant weight and $\lambda - \mu$ is in the positive root lattice.

Part (iv) is equivalent to the condition that $H(\alpha) \geq 0$ for any simple root α. This is easily checked by inspection of the Cartan matrices of the simple classical lie algebras. □

There is a nice interpretation of $\mathcal{A}_s$ in the case where G is $\mathrm{SU}(r+1)$, $\mathrm{Sp}(r)$ or $\mathrm{SO}(2r+1)$, and $\| \| = \| \|_H$. In this case, $\mathcal{A}_1$ is the span of the matrix coefficients of the representations with highest weight a fundamental analytically integral dominant weight (i.e. an element of a basis for the analytically integral dominant weight over the nonnegative integers) or 0. Hence $\mathcal{A}_1$ is a finite dimensional generating set for $\mathcal{A}$, and for any positive integer s, $\mathcal{A}_s$ is the span of all products of up to s elements of $\mathcal{A}_1$. In particular, $\mathcal{A}_s.\mathcal{A}_t = \mathcal{A}_{s+t}$.

2.2.5. Further bounds for $M(s,t)$. We shall now bound $M(s,t)$, as defined in Section 2.1, where $\| \|_A = \| \|_{A_m}$, $\| \|_B = \| \|_{A_p}$. It is clear that the pairing between A'_m and A_m allows us to identify $\mathfrak{F}\mathcal{A}'_s$ with $\mathfrak{F}\mathcal{A}_s$, and that A_m and A'_m are dual norms on this finite dimensional subspace. In the definition of $M(s,t)$ we shall use $\| \|_{\mathcal{A}_s} = \| \|_{A'_{m_1}}$, $\| \|_{\mathcal{A}'_s} = \| \|_{A_{m_1}}$. The projection, P_s, from $\mathfrak{F}\mathcal{A}' = \mathfrak{F}(\hat{G})$ onto $\mathfrak{F}\mathcal{A}_s$ is given by $(P_s A)_\lambda = 0$ when $\|\lambda\| > s$, and $(P_s A)_\lambda = A_\lambda$ when $\|\lambda\| \leq s$. The quotient norm on $A_p(\hat{G})/\mathfrak{F}\mathcal{A}_t$ is clearly given by $\|f\|_{A_p/\mathfrak{F}\mathcal{A}_t} = \|f - P_t f\|_{A_p}$. Hence

$$M(s,t) = \sup\{\|P_s(h.\varphi)\|_{A_{m_1}} : h, \varphi \in \mathfrak{F}\mathcal{A}, \|h\|_{A_p} = 1, \|\varphi\|_{A'_m} = 1, P_{s+t}\varphi = 0\},$$
$$\overline{M}(s,t) = \sup\{\|e.h - P_{s+t}(e.h)\|_{A_m} : \|h\|_{A_p} = 1, \|e\|_{A'_{m_1}} = 1, e \in \mathfrak{F}\mathcal{A}_s, h \in \mathfrak{F}\mathcal{A}\}.$$

The bounds for $M(s,t)$ depend on the following lemma.

LEMMA 2.10. *Assume f, g are in $A_0(\hat{G})$. Then $f.g$ is well-defined, and*

$$\|f.g\|_{A_0} \leq \|f\|_{A_0} \|g\|_{A_0}.$$

PROOF. See [11]. □

THEOREM 2.11. *Assume the norm on* $\mathfrak{h}^*$ *satisfies property I. Then there is a* $K > 0$ *such that for any non negative integers,* $p \geq m \geq 0$, *and any* $s, t > 1$, *we have*

$$\overline{M}(s,t) \leq K_G s^{2k+2r+l+m_1}(s+t)^m t^{-p}.$$

PROOF. Assume that $e \in \mathfrak{F}\mathcal{A}_s$, $h \in \mathfrak{F}\mathcal{A}$ are such that $\|h\|_{A_p} = 1$, and $\|e\|_{A'_0} = 1$. For any λ in $\hat{G}$, let P_λ denote the projection from $\mathfrak{F}(\hat{G})$ onto the subspace corresponding to $\operatorname{End} V_\lambda$. Let $e_\nu = P_\nu e$, $h_\lambda = P_\lambda h$, and let $\Pi(\nu)$ denote the set of weights of Δ_ν.

Then

$$\begin{aligned}\|e.h\|_{A_m/\mathfrak{F}\mathcal{A}_{s+t}} &\leq \sum_{\|\mu\|>s+t} d_\mu \|\mu\|^m \|P_\mu(e.h)\|_{1,\mu}, \\ &\leq \sum_{\|\mu\|>s+t} d_\mu \|\mu\|^m \sum_{\substack{\|\nu\|\leq s, \lambda-\mu\in\Pi(\nu) \\ \left|\|\lambda\|-\|\mu\|\right|\leq\|\nu\|}} \|P_\mu(e_\nu.h_\lambda)\|_{1,\mu} \\ &\leq \sum_{\|\nu\|\leq s} d_\nu^2 \max\{1, \|\nu\|^{m_1}\} \sum_{\mu,\lambda} \|\mu\|^m d_\lambda \|h_\lambda\|_{1,\lambda},\end{aligned}$$

where we used the inequality

$$\|P_\mu(e_\nu.h_\lambda)\|_{1,\mu} \leq d_\mu^{-1} d_\lambda d_\nu \|h_\lambda\|_{1,\lambda} \|e_\mu\|_{1,\mu} \leq d_\mu^{-1} d_\lambda d_\nu^2 \|h_\lambda\|_{1,\lambda} \|e_\nu\|_{\infty,\nu},$$

which follows directly from Lemma 2.10. Now sum on μ lemma to see that for some $K > 0$, the above quantities are bounded by

$$\begin{aligned}&\sum_{\|\nu\|\leq s} d_\nu^2 \max\{1, \|\nu\|^{m_1}\} \left|\Pi(\nu)\right| \sum_{\|\lambda\|>t} d_\lambda (\|\lambda\| + s)^m \|h_\lambda\|_{1,\lambda} \\ &\quad\leq \sum_{\|\nu\|\leq s} d_\nu^2 \max\{1, \|\nu\|^{m_1}\} \left|\Pi(\nu)\right| (\|\nu\| + t)^m t^{-p} \sum_{\|\lambda\|>t} d_\lambda \|\lambda\|^p \|h_\lambda\|_{1,\lambda} \\ &\quad\leq (s+t)^m s^{m_1} \left(\sum_{\|\nu\|\leq s} d_\nu^2 \left|\Pi(\nu)\right| \right) \sum_{\|\lambda\|>t} d_\lambda \|\lambda\|^p \|h_\lambda\|_{1,\lambda} \\ &\quad\leq K s^{2k+2r+l+m_1} (s+t)^m t^{-p} \sum_{\|\lambda\|>t} d_\lambda \|\lambda\|^p \|h_\lambda\|_{1,\lambda}.\end{aligned}$$

The last inequality holds because there is a constant $C > 0$ such that $\left|\Pi(\nu)\right| \leq C\|\nu\|^r$. This holds for the norm $\| \ \|_{\mathrm{Ad}}$ and hence for any other norm on $\mathfrak{h}^*$. □

When G is abelian we can get a more explicit bound for even more general norms on $\mathfrak{F}\mathcal{A}$. We shall bound $M(s,t)$ for slightly more general choices of $\| \ \|_A$, $\| \ \|_B$ and $\| \ \|_{\mathcal{A}_s}$ than we used above. We have $d_\lambda = 1$, so each $\operatorname{End} V_\lambda$ is naturally and uniquely isomorphic to $\mathbb{C}$. Define norms, on $\mathfrak{F}\mathcal{A}$, for $1 \leq q < \infty$ and

$-\infty \le m < \infty$, by

$$\|A\|_{\mathfrak{F}_q A_m} = \left(|A_0|^q + \sum_{\lambda \in \hat{G} \setminus \{0\}} \left(\|\lambda\|^m \, |A_\lambda| \, \right)^q \right)^{1/q}$$

$$\|A\|_{\mathfrak{F}_\infty A_m} = \sup\{ \|\lambda\|^m \, |A_\lambda| : \lambda \in \hat{G}, \ \lambda \neq 0 \} \cup \{ |A_0| \}.$$

If $1/q + 1/q' = 1$, then $\| \ \|_{\mathfrak{F}_{q'} A_{-m}}$ is the dual norm to $\| \ \|_{\mathfrak{F}_q A_m}$, and when both norms are restricted to $\mathcal{A}_s$, this holds for $q = \infty$ as well. When $m = 0$ we have $\| \ \|_{\mathfrak{F}_q A_0} = \| \ \|_{\mathfrak{F}_q}$, and when q is 1 or ∞, $m \ge 0$, , we have $\| \ \|_{\mathfrak{F}_1 A_m} = \| \ \|_{A_m}$ and $\| \ \|_{\mathfrak{F}_\infty A_{-m}} = \| \ \|_{A'_0}$. Now let $\| \ \|_{A'_s}$ be the restriction of $\| \ \|_{\mathfrak{F}_{q_1} A_{m_1}}$ to $\mathfrak{F}\mathcal{A}_s$, let $\| \ \|_A = \| \ \|_{\mathfrak{F}_{q_2} A_{m_2}}$ and $\| \ \|_B = \| \ \|_{\mathfrak{F}_{q_3} A_{m_3}}$.

THEOREM 2.12. *Assume G is abelian, $1 \le q_1, q_2, q_3 \le \infty$, and s and t are positive integers. Then*

$$M(s,t) \le \left(1 + \sum_{\|\nu\| \le s} (\|\nu\|^{m_1})^{q_1} \right)^{1/q_1} (s+t)^{m_2} t^{-m_3}$$

provided $q_3 \le q_2$ and $m_3 \ge m_2$.

PROOF. Similar to 2.11, except in this case, start with h, φ in $\mathfrak{F}\mathcal{A}$ and expand out the product $h.\varphi$ directly. □

2.2.6. Examples: Sampling for S^1, SO(3), and the simple classical Lie groups

The Simplest Example: Sampling on S^1. Assume m is a nonnegative integer, f is a C^m complex function on S^1, φ is a distribution of order at most m on S^1, and f, φ and $f.\varphi$ have the Fourier expansions $\sum_k c_k x^k$, $\sum_k m_k x^k$ and $\sum_k b_k x^k$ respectively. Then $\|\mathfrak{F}f\|_q = (\sum_k |c_k|^q)^{1/q}$, $\|\mathfrak{F}f\|_{A_m} = \sum_k k^m |c_k|$ and $\|\mathfrak{F}\varphi\|_{A'_m} = \sup\{k^{-m} |m_k| : k \in \mathbb{Z}\}$. Hence

$$\left(\sum_{|k| \le s} |c_k - b_k|^q \right)^{1/q} \le (2s+1)^{1/q} (1 + \frac{s}{t})^m N \sum_{|k|>t} k^m |c_k|$$

$$\le (2s+1)^{1/q} (1 + \frac{s}{t})^m N \frac{\pi}{\sqrt{3}} \left(\sum_{|k|>t} \left| k^{m+1} c_k \right|^2 \right)^{1/2},$$

provided $m_k = 0$ for $0 < |k| \le s + t$ and $m_0 = 1$, and where $N = \sup\{k^{-m} |m_k| : |k| > s + t\}$. The factor $\pi/\sqrt{3}$ could be replaced by a factor of the form $Cb^{-\varepsilon}$ for any ε strictly less than $\frac{1}{2}$. When f is C^{m+1} we can further bound this sum by a Sobolev norm, as

$$\left(\sum_{|k|>b} \left| k^{m+1} c_k \right|^2 \right)^{1/2} = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{d^{m+1}}{d\theta^{m+1}} (f - P_t f)(e^{i\theta}) \right|^2 d\theta \right)^{1/2}.$$

Setting $m = 0$ and $q = \infty$ in the above gives us the results of the introduction.

Example: Sampling on SO(3)**.** For this example we take $G = \mathrm{SO}(3)$. Then the dual $\hat{G}$ can be identified with the set of nonnegative integers. The dimension function is $d_\lambda = 2\lambda + 1$, the rank is $r = 1$, there is only one positive root, and the dimension of the center of SO(3) is zero. Then following the proofs above we find that when the A and B norms are $\| \; \|_{A_m}$, $\| \; \|_{A_p}$, $p \geq m$, and $\| \; \|_{\mathcal{A}_s} = \| \; \|_{A'_0}$, we have

$$\begin{aligned}\overline{M}(s,t) &\leq \left(\sum_{\nu=0}^{s}(2\nu+1)^3\right)\left(1+\frac{s}{t}\right)^m t^{m-p} \\ &\leq (s+1)^2(1+4s+2s^2)\left(1+\frac{s}{t}\right)^m t^{m-p}.\end{aligned}$$

Example: The classical simple Lie groups. Assume G is a classical simple compact Lie group. Let the norm on $\mathfrak{h}^*$ be $\| \; \|_H$, let the A, B, and $\mathcal{A}_s$ norms be $\| \; \|_{A_m}$, $\| \; \|_{A_p}$, and $\| \; \|_{A'_0}$, where $p \geq m$. Let Λ_R be the root lattice, and let B_s denote the closed ball of radius s for $\| \; \|_H$. Then the proofs above, together with property II, show that

$$\overline{M}(s,t) \leq (s+t)^m t^{-p} \sum_{\|\nu\|_H \leq s} d_\nu^2 \left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right|,$$

where the sum is over analytically integral dominant weights. We can bound $\left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right|$ for such ν as follows.

(i) $G = \mathrm{SU}(r+1)$: $\left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right| \leq (s+r+1)^r$.
(ii) $G = \mathrm{Sp}(r)$: $\left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right| \leq 2^{r-1}(s+1)^r$.
(iii) $G = \mathrm{SO}(2r+1)$: $\left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right| = (2s+1)^r$.
(iv) $G = \mathrm{SO}(2r)$: $\left|(\nu + \Lambda_R) \cap B_{\|\nu\|_H}\right| \leq 2(s+1)^2(2s+1)^{r-2}$.

We can use these bounds and the Weyl dimension formula to obtain explicit bounds on $\overline{M}(s,t)$.

(i) $G = \mathrm{SU}(r+1)$:

$$\overline{M}(s,t) \leq \frac{1}{(r+3).r!\prod_{i=1}^{r} i!^2}(s+t)^m t^{-p}\left(s+\frac{r}{3}+\frac{5}{2}\right)^{r^2+3r}.$$

(ii) $G = \mathrm{Sp}(r)$:

$$\overline{M}(s,t) \leq \frac{1}{(r+1)!\prod_{i=1}^{r}(2i-1)!^2}2^{r^2-2}(s+t)^m t^{-p}\left(s+\frac{5r}{12}+\frac{7}{4}\right)^{2r^2+2r}.$$

(iii) $G = \mathrm{SO}(2r+1)$:

$$\overline{M}(s,t) \leq \frac{1}{(r+1)!\prod_{i=1}^{r}(2i-1)!^2}2^{r^2+2r-1}(s+t)^m t^{-p}\left(s+\frac{5r}{12}+\frac{25}{24}\right)^{2r^2+2r}.$$

(iv) $G = \mathrm{SO}(2r+1)$:

$$\overline{M}(s,t) \leq \frac{1}{r.r!\prod_{i=1}^{r-1}(2i)!^2}2^{r^2+2r-2}(s+t)^m t^{-p}\left(s+\frac{5r}{12}+1\right)^{2r^2+2r}, \qquad \text{for } r \geq 3.$$

2.2.7. Differentiability and Sampling. We shall now see how the differentiability of the function being sampled plays a rôle. Define $A_m(G)$ to be the set of all continuous functions, f, on G, such that $\mathfrak{F}f$ is in $A_m(\hat{G})$. Define $\| \ \|_{A_m}$ on $A_m(G)$ by $\|f\|_{A_m} = \|\mathfrak{F}f\|_{A_m}$. Then we have the following result.

LEMMA 2.13. *Assume p is a nonnegative real number and m is a positive integer, and let $X_1, \dots, X_n$ be a basis for the complexified Lie algebra, $\mathfrak{g}^{\mathbb{C}}$ of the connected simple Lie group G. Then*

$$A_{p+m}(G) = \left\{ f \in C^{p+m}(G) : L(X_{i_1} \dots X_{i_m}) f \in A_p(G) \text{ for all } 1 \le i_1, \dots, i_l \le n \right\}$$

and the following norms on A_{p+m} are equivalent

(i) $\|f\|_{A_{p+m}}$.
(ii) $\max\{\|L(X_{i_1} \dots X_{i_j}) f\|_{A_p} : 0 \le j \le m, \text{ and } 1 \le i_1 \dots, i_j \le n\}$
(iii) $\max\left\{\|L(Y_1 \dots Y_j) f\|_{A_p} : 0 \le j \le m,\ Y_1, \dots, Y_j \in \mathfrak{g}^{\mathbb{C}},\ \|Y_1\| = \dots = \|Y_j\| = 1\right\}$.

In addition, this holds when G is an arbitrary compact connected Lie group and m is even.

PROOF. See [20]. □

LEMMA 2.14. *Assume G is a compact group of dimension n and that $m > n/2$. Then $C^m(G) \subseteq A_0(G)$, and this inclusion is continuous relative to the Sobolev norm on $C^m(G)$ given by*

$$\|f\|_{W_m} = \sup\left\{\|L(Y_1 \dots Y_j) f\|_2 : 0 \le j \le m,\ Y_1, \dots, Y_j \in \mathfrak{g}^{\mathbb{C}},\ \|Y_i\| = 1\right\}$$

and the norm $\| \ \|_{A_0}$ on $A_0(G)$.

PROOF. The space $C^m(G)$ is continuously included in the Besov space $\Lambda^{n/2}_{1,2}(G)$, which in turn is continuously included in $A_0(G)$. For definitions and proof, see [27] and [6]. □

Now we can use the bounds we have been obtaining to find convergence conditions on a sequence of measures φ_s and differentiability conditions on a function f, that ensure that $\|\mathfrak{F}P_s(f - f.\varphi)\|_{C_{m_1}}$ tends to zero.

COROLLARY 2.15. *Assume that G is a n-dimensional compact connected Lie group, m, m_1, p are nonnegative integers, and φ_s is a sequence of distributions in $C^m(G)'$ converging weak-$*$ to Haar measure and satisfying $P_{2s}(\varphi - 1) = 0$. Assume f is a function on G.*

(i) *If f is in $C_{\lceil 3n/2 \rceil + r + m + m_1 + p + 1}$, then $s^p \|\mathfrak{F}P_s(f - f.\varphi_s)\|_{A_{m_1}}$ tends to zero as s tends to infinity.*
(ii) *If f is in $C_{\lceil 3n/2 \rceil + r + m + m_1 + p}$ and either G is simple or $n + m + m_1 + r + p$ is even, then $s^p \|\mathfrak{F}P_s(f - f.\varphi_s)\|_{A_{m_1}}$ tends to zero as s tends to infinity.*

PROOF. For clarity, let's just prove the case where $m_1 = p = 0$, and G is simple. Assume that f is in $C_{\lceil 3n/2\rceil+r}$ and φ_s is a sequence of measures in $C^{m\prime}$ converging weak-$*$ to Haar measure and satisfying $P_{2s}(\varphi_s - 1) = 0$.

Then $\|\varphi_s\|_{A'_m}$ is bounded by a constant times $\|\varphi_s\|_{C'_m}$ is bounded, and f is in $A_{n+r+m}(G)$. Hence $\|\varphi_s\|_{A'_m}\|f\|_{A_{n+r+m}/\mathcal{A}_s}$ converges to zero. However, our bounds for $M(s,s)$ show that

$$\|\mathfrak{F}P_s(f - f.\varphi_s)\|_{A_0} \le K2^m s^{n+r+m} s^{-(n+r+m)} \|\varphi_s\|_{A'_m}\|f\|_{A_{n+r+m}/\mathcal{A}_s}. \qquad \square$$

3. Sampling of Sections

It is an easy matter to generalize the above results and obtain a sampling theorem for sections of homogeneous vector bundles. As the theory here follows directly from the sampling theory for groups, I have not been as complete. Assume K is a compact subgroup of the compact Lie group G, τ is a finite dimensional unitary representation of K on E_0, and $E = G \times_\tau E_0$. Then then we can multiply a C^m section of E by a distribution on G/K to obtain a "distributional section" of E, which we will think of as a sampled version of the original section. If we project a sampling distribution on G to a distribution on G/K, then we obtain an appropriate sampling distribution on G/K. For harmonic analysis on homogeneous vector bundles over G/K, where G is compact, see [31].

3.1. Abstract Sampling for Modules. We shall now generalize the situation of Section 2.1. Let $\mathcal{A}$ be a complex algebra. For simplicity we shall assume that $\mathcal{A}$ is commutative. Assume that $\mathcal{M}$, $\mathcal{N}$ are $\mathcal{A}$-modules and that we have a $\mathcal{A}$-bilinear pairing, $\langle\ ,\ \rangle$ between them. Then for any h in $\mathcal{M}$, and φ in $\mathcal{A}'$, we can define $\varphi.h$ in $\mathcal{N}' = \mathrm{Hom}_{\mathbb{C}}(\mathcal{N}'; \mathbb{C})$, by

$$(\varphi.h)(e) = \varphi(\langle e, h\rangle).$$

Let $\mathcal{A}_s$, $\{\mathcal{M}_s\}$, $\{\mathcal{N}_s\}$ be sets of subspaces of $\mathcal{A}$, $\mathcal{M}$, and $\mathcal{N}$, such that $\langle \mathcal{N}_s, \mathcal{M}_t\rangle \le \mathcal{A}_{s+t}$. We set P_s to be the projection from $\mathcal{A}'$ onto $\mathcal{A}'_s$ or from $\mathcal{N}'$ onto $\mathcal{N}'_s$ given by restriction of linear functionals.

LEMMA 3.1. *Assume φ, μ are linear functionals in $\mathcal{A}'$ such that $P_{s+t}(\varphi - \mu) = 0$. Then*

$$P_s(\varphi.h) = P_s(\mu.h)$$

for any h in $\mathcal{M}_t$

EXAMPLE. Assume $\mathcal{M}$ is a finitely generated $\mathcal{A}$-module, X is a finite dimensional generating set for $\mathcal{M}$, and $\mathcal{A}_s.\mathcal{A}_t \subseteq \mathcal{A}_{s+t}$. Let $\mathcal{N} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{M}; \mathcal{A})$, and define

$$\mathcal{M}_s = \mathcal{A}_s.X,$$
$$\mathcal{N}_s = \{f \in \mathcal{N} : f(X) \subseteq \mathcal{A}_s\}.$$

Then $\langle \mathcal{N}_s, \mathcal{M}_t\rangle \subseteq \mathcal{A}_{s+t}$.

We now return to the general situation. Let $\| \|_A$, $\| \|_B$, $\| \|_{\mathcal{N}_s}$, and $\| \|_{\mathcal{N}'_s}$ be norms on $\mathcal{N}$, $\mathcal{M}$, $\mathcal{N}_s$ and $\mathcal{N}'_s$ respectively, and denote their dual norms with a prime. Then we can define

$$N(s,t) = \sup\{\|P_s(h.\varphi)\|_{\mathcal{N}'_s} : \|h\|_B = 1, \|\varphi\|'_A = 1, h \in \mathcal{M}, \varphi \in \mathcal{A}', P_{s+t}\varphi = 0\}$$

When there is a possibility of confusion, we shall write $N_B^{\mathcal{N}'_s,A}$.

Let $\mathcal{M}^{B'}$ denote that continuous dual of $\mathcal{M}$ with respect to $\| \|_B$, and $\mathcal{N}^B$ be the completion of $\| \|_B$ with respect to $\| \|_B$.

Lemma 3.2. *Assume φ, μ are linear functionals in $\mathcal{A}^{A'}$ such that $P_{s+t}(\varphi - \mu) = 0$ and $h \in \mathcal{M}^B$. Then*

$$\|P_s(f.\varphi) - P_s(f.\mu)\|_{\mathcal{N}'_s} \leq N(s,t)\|\varphi - \mu\|'_A\|f\|_{B/\mathcal{M}_t},$$

where $\| \|_{B/\mathcal{M}_t}$ denotes the quotient seminorm on $\mathcal{M}^B/\mathcal{M}_t$.

3.2. Harmonic Analysis of Vector-Valued Functions. Assume E_0 is a finite dimensional complex vector space with norm $\| \|_{E_0}$. Let $C^m(G;E_0)$ be the space of C^m functions on G with values in E_0, and when m is a nonnegative integer, define $\|f\|_{C^m;E_0} = \sup\{\|L(X_1 \dots X_p)f(x)\|_{E_0} : x \in G,\ 0 \leq p \leq m,\ X_1 \dots X_p \in \mathfrak{g},\ \|X_1\| = \dots = \|X_p\| = 1\}$. All norms, $\| \|_{E_0}$, on E_0 will give an equivalent norms $\| \|_{C^m;E_0}$ on $C^m(G;E_0)$. Let $\| \|_{(C^m;E_0)'}$ be the dual norm to $\| \|_{C^m;E_0}$, and $\| \|_{(C^m;E_0^*)'}$ be the norm on $C^m(G;E_0^*)'$, when E_0^* is given the norm dual to that on E_0. The space $C^\infty(G;E_0^*)'$ is the space of all distributions on G with values in E_0, and $C^m(G;E_0^*)'$ is the space of all such distributions of order at most m. We can embed $C^0(G;E_0)$ continuously into $C^0(G;E_0^*)'$ by means of the map $f \mapsto \mu_G.f$, where for any h in $C^0(G;E_0^*)$, we have $\langle \mu_G.f, h\rangle = \langle \mu_G, (x \mapsto \langle h(x), f(x)\rangle)\rangle = \int_G \langle h(x), f(x)\rangle d\mu_G(x)$, and μ_G is Haar measure on G.

Let $\mathfrak{F}(\hat{G};E_0) = \prod_{\gamma\in\hat{G}} (\mathrm{End}(V_\gamma) \otimes E_0)$, and define the Fourier transform, $\mathfrak{F}$, from $C^\infty(G;E_0^*)'$ into $\mathfrak{F}(\hat{G};E_0)$, by

$$\langle X \otimes e^*, (\mathfrak{F}s)_\gamma\rangle = \langle s, (x \mapsto \langle X, \Delta_\gamma(x)\rangle\, e^*)\rangle$$

for any γ in $\hat{G}$, X in $\mathrm{End}(V_\gamma)^*$, e^* in E_0^*, and s in $C^\infty(G;E_0^*)'$. For a continuous function, f, on G with values in E_0, this becomes $(\mathfrak{F}f)_\gamma = \int_G \Delta_\gamma(x) \otimes f(x) d\mu_G(x)$.

We shall define norms on $\mathfrak{F}(\hat{G};E_0)$ which generalize the norms $\| \|_{A_m}$ we had when E_0 was $\mathbb{C}$. Given two finite dimensional complex vector spaces, V and W, and norms $\| \|_V$ on V and $\| \|_W$ on W, define the tensor product of these norms, $\| \|_{V\otimes W}$, to be the operator norm on $V \otimes W = \mathrm{Hom}_{\mathbb{C}}(V^*;W)$ relative to the dual norm $\| \|_{V^*}$ on V^*, and the norm $\| \|_W$ on W. For any γ in $\hat{G}$ let $\| \|_{1,\gamma;E_0}$ denote the norm on $\mathrm{End}(V_\gamma) \otimes E_0$, which is the tensor product of the norms $\| \|_{1,\gamma}$ and $\| \|_{E_0}$. Define a norm $\| \|_{A_m;E_0}$, which is possibly infinite on $\mathfrak{F}(\hat{G};E_0)$, by $\|A\|_{A_m;E_0} = \|A_0\|_{1,0;E_0} + \sum_{\lambda\in\hat{G},\lambda\neq 0} d_\lambda \|\lambda\|^m \|A_\lambda\|_{1,\lambda;E_0}$. Let $A_m(\hat{G};E_0)$ be the subspace of $\mathfrak{F}(\hat{G}; E_0)$ on which this norm is finite. This space is the space of absolutely summable Fourier transforms of distributions on G with values in E_0 whose first m derivatives also have absolutely summable transforms. The map, $\mathfrak{F}$ is one to one, and it's inverse gives a continuous from $A_m(\hat{G}; E_0)$ into $C^m(G; E_0)$.

Now, let $\mathcal{M} = \mathcal{A} \otimes E_0$, $\mathcal{N} = \mathcal{A} \otimes E_0^*$. These naturally embed in $C^\infty(G; E_0)$ and $C^\infty(G; E_0^*)$, and the spaces $\mathfrak{F}\mathcal{M}$, $\mathfrak{F}\mathcal{N}$ are the subspaces of $\mathfrak{F}(\hat{G}; E_0)$ and $\mathfrak{F}(\hat{G}; E_0^*)$ of elements with only finitely many components. Hence we can use $\mathfrak{F}$ to shift any norm on $\mathfrak{F}\mathcal{M}$ over to $\mathcal{M}$. Let $\mathcal{M}_s = \mathcal{A}_s \otimes E_0$, and $\mathcal{N}_s = \mathcal{A}_s \otimes E_0^*$. There is a natural $\mathcal{A}$-bilinear pairing between $\mathcal{M}$ and $\mathcal{N}$. Composing this form with Haar measure gives a $\mathbb{C}$-bilinear pairing between $\mathcal{M}_s$ and $\mathcal{N}_s$, which we shall use to identify $\mathcal{N}_s'$ with $\mathcal{M}_s$.

For calculation of $N(s,t)$, it is more convenient to use the norm $\| \; \|_{A_m \otimes E_0}$ defined on $A_m(\hat{G}) \otimes E_0$, by

$$\|A\|_{A_m \otimes E_0} = \sup\{\| \langle e_0^*, A \rangle_{A_m(\hat{G})} \|_{A_m} : \|e_0^*\|_{E_0^*} = 0\},$$

where $\langle \; , \; \rangle_{A_m(\hat{G})}$ is the natural $A_m(\hat{G})$-bilinear pairing between E_0^* and $\| \; \|_{A_m \otimes E_0}$. It is easy to show that $A_m(\hat{G}) \otimes E_0$ naturally embeds in $A_m(\hat{G}; E_0)$. In fact, these two spaces are equal, as the following lemma will show. First, some terminology. We say that E_0 has dual bases of unit vectors if there is a basis $\{v_i\}$ of unit vectors in E_0, with a dual basis $\{v_i^*\}$ of E_0^* consisting of unit vectors. This happens, for example, when $\| \; \|_{E_0}$ is a Hilbert space norm, or a p-norm in some basis.

LEMMA 3.3. (i) $\| \; \|_{A_m \otimes E_0} \leq \| \; \|_{A_m; E_0}$.
(ii) *If E_0 has dual bases of unit vectors, then* $\| \; \|_{A_m; E_0} \leq (\dim E_0) \| \; \|_{A_m \otimes E_0}$.
(iii) $\| \; \|_{A_m; E_0}$ *and* $\| \; \|_{A_m \otimes E_0}$ *are equivalent norms.*

Define $M(s,t)$ using the A_{m_1}, A_m, A_p norms, as we did in Section 2.2.5. We shall now relate this function to the function $N(s,t)$ for various choices of the norms on $\mathcal{N}_s' = \mathcal{M}_s$, $\mathcal{A}$, and $\mathcal{M}$.

THEOREM 3.4. (i) *If $N(s,t)$ is defined using the $A_{m_1} \otimes E_0$, A_m, $A_p \otimes E_0$ norms on $\mathcal{N}_s'$, $\mathcal{A}$ and $\mathcal{M}$, then*

$$N_{A_p \otimes E_0}^{A_{m_1} \otimes E_0, A_m}(s,t) \leq M_{A_p}^{A_{m_1}, A_m}(s,t).$$

(ii) *If $N(s,t)$ is defined using the $(A_{m_1}; E_0)$, A_m, $(A_p; E_0)$ norms on $\mathcal{N}_s'$, $\mathcal{A}$ and $\mathcal{M}$, then for some $C > 0$,*

$$N_{(A_p; E_0)}^{(A_{m_1}; E_0), A_m}(s,t) \leq C.(\dim E_0) M_{A_p}^{A_{m_1}, A_m}(s,t).$$

When E_0 has dual bases of unit vectors, we may take $C = 1$ in the above inequality.

PROOF. Assume that φ is in $\mathcal{A}$, h is in $\mathcal{M}$, and e_0^* is in E_0^*.

$$\begin{aligned}\| \langle e_0^*, P_s(\varphi.h)\rangle_{\mathcal{A}} \|_{A_{m_1}} &= \|P_s(\varphi.\langle e_0^*, h\rangle_{\mathcal{A}})\|_{A_{m_1}} \\ &\le M(s,t)\|\varphi\|_{A'_m}\| \langle e_0^*, h\rangle_{\mathcal{A}} \|_{A_p} \\ &\le M(s,t)\|\varphi\|_{A'_m}\|e_0^*\|_{E_0^*}\|h\|_{A_p\otimes E_0}.\end{aligned}$$

This proves (i). The second part is an easy corollary of the first. □

The proof of the first part of this theorem did not involve many special properties of the norms A_m; the basic properties used are that $\mathfrak{F}\mathcal{M}$ is dense in the $A_p(\hat{G})\otimes E_0$ and $\mathfrak{F}\mathcal{A}$ is dense in $A_m(\hat{G})'$.

Another approach to bounding $N(s,t)$ uses an analog of Lemma 2.10 to calculate the bound directly. In some circumstances (e.g. when G is abelian), this gives better results than the combination of the previous theorem and the bounds for $M(s,t)$. In particular, we do not use the assumption that E_0 has dual bases of unit vectors.

LEMMA 3.5. *Assume f is a continuous complex function on G, g is in $C^0(G;E_0)$, and $\mathfrak{F}f \in A_0(\hat{G})$, and $\mathfrak{F}g \in A_0(\hat{G};E_0)$. Then*

$$\|\mathfrak{F}(f.g)\|_{A_0;E_0} \le (\dim E_0)\|\mathfrak{F}f\|_{A_0}\|\mathfrak{F}g\|_{A_0;E_0}.$$

PROOF. This has essentially the same proof as for the case when E_0 is simply the complex numbers, as given in [11]. □

Lemma 3.5 implies that if f_λ is in the λ-isotypic subspace of $C^\infty(G)$, g_μ is in the μ-isotypic subspace of $C^\infty(G;E_0)$, under the left regular actions, and ν is in $\hat{G}$, then

$$\|\mathfrak{F}(f_\lambda.g_\mu)\|_{1,\nu;E_0} \le (\dim E_0)\, d_\nu^{-1} d_\lambda d_\mu \|\mathfrak{F}f_\lambda\|_{1,\lambda}\|\mathfrak{F}g_\mu\|_{1,\mu;E_0}.$$

When $E_0 = \mathbb{C}$, this inequality our main ingredient in the bound on $M(s,t)$. The generalization gives us bounds on $N(s,t)$. The second half of the following theorem concerns the case when G is abelian. When G is abelian, define norms on $\mathfrak{F}\mathcal{M}$ for $1 \le q < \infty$ and $-\infty \le m < \infty$ by

$$\|A\|_{\mathfrak{F}_q A_m} = \left(|A_0|^q + \sum_{\lambda\in\hat{G}\setminus\{0\}} \left(\|\lambda\|^m \|A_\lambda\|_{E_0}\right)^q\right)^{1/q},$$
$$\|A\|_{\mathfrak{F}_\infty A_m} = \sup\bigl\{\|\lambda\|^m\|A_\lambda\|_{E_0} : \lambda\in\hat{G},\ \lambda\ne 0\bigr\}\cup\{|A_0|\}.$$

THEOREM 3.6. (i) *Assume G is nonabelian, the norm on $\mathfrak{h}^*$ has property I, and $N(s,t)$ is defined using the $(A_{m_1};E_0)$, A_m, $(A_p;E_0)$ norms on $\mathcal{N}'_s$, $\mathcal{A}$ and $\mathcal{M}$. Then for some K_G depending only on G and the norm on $\mathfrak{h}^*$,*

$$N^{(A_{m_1};E_0),A_m}_{(A_p;E_0)}(s,t) \le (\dim E_0) s^{r+l+m_1+1}(s+t)^{2k+r+m-1}t^{-p}.$$

(ii) *Assume G is abelian, $1 \leq q_1, q_2, q_3 \leq \infty$, and s and t are positive integers. Then we have*

$$N^{(\mathfrak{F}_{q_1}A_{m_1}),(\mathfrak{F}_{q_2}A_{m_2})}_{\mathfrak{F}_{q_3}A_{m_3}}(s,t) \leq \left(1 + \sum_{\|\nu\|\leq s} (\|\nu\|^{m_1})^{q_1}\right)^{1/q_1} (s+t)^{m_2} t^{-m_3},$$

provided $q_3 \leq q_2$ and $m_3 \geq m_2$.

Proof. The key observation in the proof of (i) is that

$$\|P_s(h.\varphi)\|_{A_{m_1};E_0} \leq \sum_{\|\nu\|\leq s} d_\nu (1 + \|\nu\|^{m_1}) \sum_{\substack{\|\mu\|>s+t,\ \|\lambda\|>t, \\ |\|\mu\|-\|\lambda\||\leq\|\nu\|,\ \pi\mu=\pi\nu-\pi\lambda}} d_\nu^{-1} d_\lambda d_\mu^2 \|\mu\|^m \|(\mathfrak{F}h)_\lambda\|_{1,\lambda;E_0} \|\varphi\|_{A'_m},$$

where π is the natural projection from $\mathfrak{h}^*$ onto the dual of the center of $\mathfrak{g}$. Now sum over μ and then ν.

The proof of (ii) is essentially the same as for Theorem 2.12. □

3.3. Homogeneous Vector Bundles. Assume $E = G \times_\tau E_0$ is a homogeneous vector bundle, where τ is a unitary representation of K. E has a G-invariant unitary structure determined by the inner product on E_0. Let $\Gamma^m(E)$ denote the space of C^m sections of E with the norm $\|s\|_{\Gamma^m} = \sup\{\|L(X_1 \ldots X_p)s(x)\|_x : x \in G/K,\ 0 \leq p \leq m,\ X_1 \ldots X_p \in \mathfrak{g}\}$, where $\|\ \|_x$ denotes the norm on the fiber, E_x, determined by the unitary structure of E. If $\delta(G/K)$ is the density bundle and $\mu_{G/K}$ is the invariant density of unit mass on G/K, we obtain a map $\Gamma^0(E) \to \Gamma^0(E \otimes \delta(G/K)) \hookrightarrow \Gamma^0(E^*)'; f \mapsto f.\mu_{G/K}$, allowing us to identify $\Gamma(E)$ with a subspace of $\Gamma^0(E^*)'$. Thus we think of $\Gamma^\infty(E^*)'$ as the space of all distributions, or generalized sections, of E.

There is a representation ψ_τ of K by isometries on each of the spaces $C^m(G;E_0)$ and $C^m(G;E_0^*)'$, defined by $\psi_\tau(k)f(x) = \tau(k)f(x.k)$, on elements of $C(G;E_0)$, and which commutes with the left regular action of G on these spaces. The corresponding spaces of invariant functions or distributions are denoted, $C^m(G;\tau)$ and $C^{m\prime}(G;\tau)$. We then have an isometry[1] $j_\tau : C^{m\prime}(G;\tau) \to \Gamma^m(E^*)'$ which restricts to an isometry between $C^m(G;\tau)$ and $\Gamma^m(E)$. Thus questions about spaces of sections of E can be simply reduced to ones concerning ψ_τ-invariant vector valued functions on G. In particular, the multiplication map $C^m(G/K)' \times \Gamma^m(E) \to \Gamma(E^*)'$ corresponds to the map $C^m(G)'^K \times C^m(G;\tau) \to C^{m\prime}(G;\tau)$ which is the restriction of the scalar multiplication map for distributions on G with functions in $C^m(G;E_0)$.

[1]The space $C^{m\prime}(G;\tau)$ of invariant vectors in $C^m(G;E_0^*)'$ is isometric, via the restriction map, to the space $C^m(G;\tau^\vee)'$. This is because the canonical projection from $C^m(G;E_0^*)'$ onto $C^{m\prime}(G;\tau)$ is the transpose of the projection from $C^m(G;E_0^*)$ onto $C^m(G;\tau^\vee)$, and this last projection is also a contraction.

As in Section 3.2 we set $\mathcal{M} = \mathcal{A} \otimes E_0$ and $\mathcal{N} = \mathcal{A} \otimes E_0^*$. Let $\hat{\mathcal{M}}$, $\hat{\mathcal{N}}$ and $\hat{\mathcal{A}}$ be the subspaces of ψ_τ-, $\psi_{\tau^\vee}$-, and K-invariant vectors in $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{A}$. Let $\hat{\mathcal{M}}_s$, $\hat{\mathcal{N}}_s$, $\hat{\mathcal{A}}_s$, be the intersections of the spaces above with $\mathcal{M}_s$, $\mathcal{N}_s$ and $\mathcal{A}_s$ respectively. Finally, we can use j_τ and $j_{\tau^\vee}$ to obtain corresponding subspaces, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{N}}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{M}}_s$, $\tilde{\mathcal{N}}_s$, $\tilde{\mathcal{A}}_s$ in $\Gamma^\infty(E)$, $\Gamma^\infty(E^*)$ and $C^\infty(G/K)$.

Choosing norms on $\mathcal{N}_s' = \mathcal{M}_s$, $\mathcal{A}$, and $\mathcal{M}$, allows us to define a function $N(s,t)$ as in Section 3.1. If we assume that $\mathcal{N}_s$ is invariant under the projection from $\mathcal{N}$ onto $\tilde{\mathcal{N}}$, then the dual of this projection is an injection from $\tilde{\mathcal{N}}_s'$ into $\mathcal{N}_s'$, and we may restrict the norm on $\mathcal{N}_s'$ to $\tilde{\mathcal{N}}_s'$; in fact, the $\mathbb{C}$-bilinear pairing between $\tilde{\mathcal{N}}_s$ and $\tilde{\mathcal{M}}_s$ is nondegenerate in this case. If we also restrict the norms on $\mathcal{A}$ and $\mathcal{M}$ to $\tilde{\mathcal{A}}$, and $\tilde{\mathcal{M}}$, then we can define another function $\tilde{N}(s,t)$ using these restricted norms.

Theorem 3.7. *Assume that all the subspaces $\mathcal{A}_s$ and the norm on $\mathcal{A}$ are all invariant under the right regular action of K. Then $\tilde{N}(s,t) \le N(s,t)$*

Proof. First note that under these hypotheses, the subspaces $\mathcal{M}_s$, $\mathcal{N}_s$ are invariant under the representations ψ_τ, and $\psi_{\tau^\vee}$, and so the projections onto these spaces commute with the projections from $\mathcal{M}$, and $\mathcal{N}$ onto $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{N}}$. Hence the definition of $\tilde{N}$ makes sense. The projection from $\mathcal{A}$ onto $\tilde{\mathcal{A}}$, P^K, is a contraction with respect to $\| \, \|_A$, and its dual, P^{K*}, is an isometric embedding of the continuous dual of $\tilde{\mathcal{A}}$ with respect to the restricted norm into the continuous dual of $\mathcal{A}$ with its norm. P^K, which is given by integration over K, commutes with the projections, from $\mathcal{A}$ onto $\mathcal{A}_s$, and hence for any φ in the continuous dual of $\tilde{\mathcal{A}}$ such that $P_s\varphi = 0$, we also have $P_s(P^{K*}\varphi) = 0$. This allows us to imbed the calculation of $\tilde{N}(s,t)$ into a calculation involving only the spaces $\mathcal{N}$, $\mathcal{M}$, $\mathcal{A}$ and the subspaces $\mathcal{N}_s$, $\mathcal{M}_s$, and $\mathcal{A}_s$, where it is obvious that $\tilde{\mathcal{N}} \le N$. $\square$

We shall now define the Fourier transform map for spaces of sections of E. The representation, ψ_τ, of K on the γ-isotypic subspace of $C^\infty(G; E_0)$ corresponds, under the Fourier transform $\mathfrak{F}$, to the representation $\mathrm{Id} \otimes \Delta_\gamma^\vee \otimes \tau$, on $\mathrm{End}(V_\gamma) \otimes E_0 = V_\gamma \otimes V_\gamma^* \otimes E_0$. The subspace of invariant vectors of this space is naturally isomorphic to $V_\gamma \otimes \mathrm{Hom}_K(V_\gamma; E_0)$. So the natural space in which to define the Fourier transform of a section of E is $\mathfrak{F}(\hat{E}) = \prod_{\gamma \in \hat{G}} V_\gamma \otimes \mathrm{Hom}_K(V_\gamma; E_0)$. Define norms $\| \, \|_{A_m}$ on $\mathfrak{F}(\hat{E})$ by restricting the norms $\| \, \|_{A_m;E_0}$ on $\mathfrak{F}(\hat{G}; E_0)$, and let $A_m(\hat{E})$ denote the subspace of $\mathfrak{F}(\hat{E})$ on which the corresponding norm is finite. Let P^τ denote both the projection from $C^\infty(G; E_0^*)'$ onto the ψ_τ-invariant subspace, $C^{\infty\prime}(G;\tau)$ and also the projection from $\mathfrak{F}(\hat{G}; E_0)$ onto $\mathfrak{F}(\hat{E})$. Define the Fourier Transform map $\mathfrak{F} : \Gamma^\infty(E^*) \to \mathfrak{F}(E)$ so that $P^\tau \mathfrak{F} = \mathfrak{F} P^\tau$, then $\mathfrak{F}$ maps $\Gamma^m(E)$ into $A_m(\hat{E})$. When τ is the trivial representation, the dual space to $A_m(E)$ corresponds to the space of invariant distributions on G for which A_m', the dual norm previously, is finite. We then have that $\|\mathfrak{F}\varphi\|_{A_m'} \le \|\varphi\|_{(C^m)'}$ for any complex distribution, φ, on G/K. Also note that if φ is a distribution on G satisfying $P_s\varphi = 0$, then $P^K\varphi$ satisfies the same equation in $C^\infty(G/K)'$.

Example: Functions on S^2. Consider the case where $G = \mathrm{SO}(3)$, $K = \mathrm{SO}(2)$, and τ is the trivial representation of SO(2). Then $E = S^2 \times \mathbb{C}$ is the trivial bundle over S^2, and sections of E may be identified with complex functions on S^2. Identify the dual of SO(3) with the set of nonnegative integers. For any $l \geq 0$ we have $\dim \mathrm{Hom}_{\mathrm{SO}(2)}(V_l; \mathbb{C}) = 1$. Choose a $\Delta_l^\vee(\mathrm{SO}(2))$-invariant unit vector, u_l^* in V_l^* for each l. Then the map $v \mapsto v \otimes u_l^*$ gives an isomorphism between V_l and $V_l \otimes \mathrm{Hom}_{\mathrm{SO}(2)}(V_l; \mathbb{C})$. The space $V_l \otimes \mathrm{Hom}_{\mathrm{SO}(2)}(V_l; \mathbb{C})$ is naturally isomorphic to the subspace of $\mathrm{End}\, V_l = V_l \otimes V_l^*$ invariant under $\mathrm{Id} \otimes \Delta_l^\vee$. The composition of these two isomorphisms is map, $v \mapsto A_v$, from V_l into $\mathrm{End}(V_l)$ which is defined by $A_v w = u_l^*(w)v$ for any $w \in V_l$. Assume v is any vector in V_l. We shall now find $\|A_v\|_{q,l}$. Let Pr_v be the self-adjoint projection onto the linear span of v, then $A_v A_v^* = \|v\|^2 \mathrm{Pr}_v$, where $\|v\|$ is the Hilbert space norm, so

$$\|A_v\|_{q,l} = (\mathrm{Tr}\,(A_v A_v^*)^{q/2})^{1/q} = (\mathrm{Tr}(\|v\|^q \,\mathrm{Pr}_v))^{1/q} = \|v\|$$

Using the isomorphisms above, we can identify $\mathfrak{F}(\hat{E})$ with $\prod_{l\geq 0} V_l$, and if $y \in \mathfrak{F}(\hat{E})$, then $\|y\|_{A_m} = \sum_{l\geq 0}(2l+1)\max\{1, l^m\}\|y_l\|$. One can now use the bounds as follows. Assume f is a C^m function on S^2 with $\mathfrak{F}f = y$, and φ is a distribution of order at most m on S^2 satisfying $P_{s+t}(\varphi - 1) = 0$. Let $\mathfrak{F}(\varphi.f) = z$, then for any positive integers s, t, and any $p \geq m$,

$$\sum_{l=0}^{s}(2l+1)\|y_l - z_l\| \leq (s+1)^2(1+4s+2s^2)\Big(1+\frac{s}{t}\Big)^m t^{m-p}\|\mathfrak{F}(\varphi-1)\|_{A'_m}\sum_{l>t}(2l+1)l^p\|y_l\|$$

and $\|\mathfrak{F}(\varphi - 1)\|_{A'_m} = \sup\{l^{-m}\|(\mathfrak{F}s)_l\| : l > s+t\}$.

Example: Line bundles over S^2. For this example take $G = \mathrm{SO}(3)$, $K = \mathrm{SO}(2)$, and let $\tau = \rho_n$ be the representation of SO(2) with weight n, where n is a nonzero integer. Then E is a line bundle over S^2. The space $\mathrm{Hom}_{\mathrm{SO}(2)}(V_l; \rho_n)$ has dimension 1 for $l \geq |n|$ and is zero-dimensional when $0 \leq l < |n|$. When $l \geq |n|$ we may choose a unit vector, w_l^*, in the ρ_n-isotypic space of V_l and obtain an isomorphism, $v \mapsto v \otimes w_l$, between V_l and $\mathrm{Hom}_{\mathrm{SO}(2)}(V_l; \rho_n)$. As before, this allows us to identify $\mathfrak{F}(E)$ with $\prod_{l\geq|n|} V_l$, and for any $y \in \mathfrak{F}(\hat{E})$ we have $\|y\|_{A_m} = \sum_{l\geq|n|}(2l+1)l^m\|y_l\|$. To state the sampling theorem for this situation, assume f is a C^m section of E with $\mathfrak{F}f = y$, and φ is a distribution of order at most m on S^2 satisfying $P_{2b}(s-1) = 0$. Let $\mathfrak{F}(\varphi.f) = z$, and assume s, t are positive integers, and $p \geq m$, then

$$\sum_{l=|n|}^{s}(2l+1)\|y_l - z_l\| \leq (s+1)^2(1+4s+2s^2)\Big(1+\frac{s}{t}\Big)^m t^{m-p}\|\mathfrak{F}(\varphi-1)\|_{A'_m}\sum_{l\geq t+1,|n|}(2l+1)l^p\|y_l\|.$$

4. Construction of Sampling Distributions

4.1. The General Construction. Now we will outline a method for constructing distributions whose Fourier transform vanishes at a given finite set of irreducible representations. These distributions will be finitely supported, have any specified order, and will be of the form $\chi = \psi_1 * \cdots * \psi_n$, where $n = \dim G$ and each of the ψ_i are supported on a finite subset of a 1-parameter subgroup of G. In addition $\psi_1, \ldots, \psi_n$ may be chosen so that χ has bounded A_m norm as the set of irreducible representations at which its Fourier transform must vanish increases. These properties have been chosen as they are required for the development of efficient algorithms for the computation of the Fourier transform of functions sampled on the support of these distributions, as in [21]. The thesis [20] contains a description of these algorithms for functions sampled on the support of the projection of these distributions to the homogeneous spaces $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ and $\mathrm{SU}(n)/\mathrm{SU}(n-1)$; they are generalizations of the algorithm for computing expansions in spherical harmonics developed by Driscoll and Healy in [4]. Here is the general construction.

Assume G is a connected compact Lie group, and K is a connected compact subgroup of G. The Fourier transforms of a distribution, $\psi \in C^\infty(K)'$, and its image $i\psi$ in $C^\infty(G)'$ are simply related; if ρ is a representation of G, then $\rho(i\psi) = (\rho\big|K)(\psi)$. So the relation between the two Fourier transforms is determined by the way that representations of G split on restriction to K.

For any set, Ω_0 of irreducible representations of G, define a two-sided ideal in $C^\infty(G)'$ by

$$\mathfrak{T}_{\Omega_0} = \{f \in C^\infty(G)' : \forall \psi \in \Omega_0 \ \psi(f) = 0\}.$$

We wish to show how for any finite set of representations, Ω_0, we can construct a finitely supported distribution, χ, on G, such that $\chi - 1 \in \mathfrak{T}_{\Omega_0}$. It obviously suffices to consider the case when G is simple and simply connected, the abelian case being trivial. Let us also restrict ourselves to the case when G has a rank one homogeneous space, G/K; this only leaves a few exceptional groups out of our reach.

By induction we can assume that the problem has been solved for K; this is because K is a quotient of a product of abelian groups and semisimple groups which themselves have rank 1 homogeneous spaces. Now let Ω_1 be the set of all irreducible representations of K that are contained in the restriction of some representation in Ω_0 to K. This set is finite, and $\mathfrak{T}_{\Omega_0} \subseteq i(\mathfrak{T}_{\Omega_1})$.

By induction, we can find a finitely supported distribution, $\hat{\chi}$, on K such that $\hat{\chi} - 1_K \in \mathfrak{T}_{\Omega_1}$. Let $\chi_K = i(\hat{\chi})$, then $\chi_K = c_K \pmod{\mathfrak{T}_{\Omega_0}}$, where c_K is the characteristic distribution of the submanifold, K, of G. By polar decomposition, $G = KAK$, where A is a 1 parameter subgroup of G. The idea is to choose a finitely supported distribution, ψ, with support in A, and then let $\chi = \chi_K * \psi * \chi_K$. Then, $\chi = c_K * \psi * c_K = {}^K P^K \psi \pmod{\mathfrak{T}_{\Omega_0}}$, where ${}^K P^K$ is the projection onto bi-invariant distributions. ${}^K P^K \psi$ has an expansion in terms of spherical functions. The polar decomposition allows us to establish an isomorphism of $[-1,1]$ with $K\backslash G/K$ via the obvious composition of maps $[-1,1] \to A \to G \to K\backslash G/K$. So we can lift ${}^K P^K \psi$ up to a finitely supported distribution on $[-1,1]$, where its spherical function expansion corresponds to an expansion in Jacobi polynomials of some sort. By the Chebyshev property of orthogonal polynomials [20, Lemma 3.2], we can choose ψ so that the expansion of ${}^K P^K \psi - 1$ in spherical functions only contains spherical functions corresponding to representations that are not in Ω_0. That is, choose ψ so that ${}^K P^K c = 1 \pmod{\mathfrak{T}_{\Omega_0}}$. Then $\chi - 1 \in \mathfrak{T}_{\Omega_0}$.

An apparent problem with this method, is that the number of distributions in the convolution product for χ is too large. We desire exactly $\dim G$ of these factors, but the method above yields 1 factor for S^1, 3 for $\mathrm{SU}(2)$, 4 for $S(U_2 \times U_1)$, 9 for $\mathrm{SU}(3)$, and $2^k + 2^{k-1} - 3$ for $\mathrm{SU}(k)$, and $\dim \mathrm{SU}(k) = k^2 - 1$. In the examples that follow, we use relations between the ψ_i modulo $\mathfrak{T}_{\Omega_0}$ to reduce the number of factors to $\dim G$, when G is one of the classical groups.

4.1.1. Quadrature Rules. Assume that $\langle \varphi_m \rangle$ is a sequence of orthonormal polynomials relative to the positive measure $w(x)\,dx$ on $[a,c]$. Then a finitely supported distribution satisfying $\langle \psi, \varphi_m \rangle = \delta_{0m}$ for $0 \le m \le n$ is equivalent to a quadrature formula that exactly integrates polynomials of degree at most n with respect to $w(x)dx$. In the case where ψ is a measure supported at the roots of φ_n, this determines the usual Gaussian integration formula, which has the advantages that ψ is positive and $\langle \psi, \varphi_m \rangle\ \delta_{0m}$ for $0 \le m \le 2n+1$. Similarly, by choosing the support of ψ to be the roots of the n-th l-orthogonal polynomial we may find a distribution of order $2l$, supported on these points, such that $\langle \psi, \varphi_m \rangle = \delta_{0m}$ for $0 \le m < (2l+2)n$. For more on this, see [7].

When ψ is a positive measure, satisfying the above conditions, the total variation norm of ψ must be 1. If this measure is pushed onto a Lie group, then the resulting positive measure also has total variation norm 1, and a convolution of such measures has total variation norm 1. The construction above (and in the following examples) can therefore be required to produce measures of total variation 1 on the classical groups. When ψ is supported at the points $\cos(\pi l/n)$, $0 \le l < n$, the total variation norm of ψ tends to 1 as n tends to infinity, provided that w is a nonnegative L^1 function on $[-1,1]$, and $0 < \int_0^\pi w(\cos\theta)d\theta < \infty$ (See [20]).

Together with Lemma 2.4 this shows that the distribution χ of the subsection above can be constructed so it is bounded in the A_m norm as the set Ω_0 varies over finite subsets of $\hat{G}$. To get an explicit formula for χ we need to know how to convolve point distributions on G; this is explained in [20].

4.2. Example: Sampling on $\mathrm{SO}(n)$. The arguments of Section 4.1, when applied to the chain of groups

$$\mathrm{SO}(n) \supseteq \mathrm{SO}(n-1) \supseteq \cdots \supseteq \mathrm{SO}(2),$$

lead to a sampling distribution on $\mathrm{SO}(n)$ that is closely related to the parametrization of that group, by means of Euler angles. Let

$$r_m(\theta) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \cos\theta & \sin\theta & & \\ & & -\sin\theta & \cos\theta & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

where the "rotation block" appears in columns and rows $m-1$ and m. Note that $r_m \smile r_n$ for $|n-m|>1$ and $\mathrm{SO}(n) = \mathrm{SO}(n-1).r_n([0,\pi]).\mathrm{SO}(n-1)$. The highest weight of a representation of $\mathrm{SO}(2r+1)$ is determined by its coordinates $m_{1,2r+1}, \ldots, m_{r,2r+1}$ relative to the basis $\{e_i\}$ described in Section 2.2.4. These numbers range over all sets of integers satisfying

$$m_{1,2r+1} \geq \cdots \geq m_{r,2r+1} \geq 0.$$

The highest weight of a representation of $\mathrm{SO}(2r)$, may also be expressed in the coordinates of Section 2.2.4, and these coordinates are integers, $m_{1,2r}, \ldots, m_{r,2r}$, satisfying

$$m_{1,2r} \geq \cdots \geq |m_{r,2r}|.$$

The "betweenness" relations for the restriction of representations of $\mathrm{SO}(2r+1)$ to $\mathrm{SO}(2r)$ and $\mathrm{SO}(2r)$ to $\mathrm{SO}(2r-1)$ are then

$$m_{1,2r+1} \geq m_{1,2r} \geq m_{2,2r+1} \geq \ldots \geq m_{r,2r+1} \geq |m_{r,2r}|$$

and

$$m_{1,2r} \geq m_{1,2r-1} \geq m_{2,2r} \geq \ldots \geq m_{r-1,2r-1} \geq |m_{r,2r}|,$$

where the $m_{i,j}$ are either all integral or all half integral. For convenience, we'll assume that n is either $2k+1$ or $2k$, that the numbers $m_{1,k}, \ldots m_{k,n}$ satisfy the appropriate restrictions, and that $n>2$ in what follows.

Choose a positive integer, s. We shall construct a distribution, c_n on $\mathrm{SO}(n)$, such that $c_n - 1$ vanishes on representations Δ_λ with $\|\lambda\|_H \leq s$. In terms of the coordinates $m_{i,j}$, this is the same as requiring that $m_{1,n} \leq s$.

The map $[0,\pi] \longleftrightarrow \mathrm{SO}(n-1)\backslash\mathrm{SO}(n)/\mathrm{SO}(n-1) : \theta \mapsto \mathrm{SO}(n-1) r_n(\theta)\,\mathrm{SO}(n-1)$ is a homeomorphism, and its restriction to $(0,\pi)$ is a diffeomorphism. We may therefore identify this double coset space with $[0,\pi]$. The class one representations for $\mathrm{SO}(n)/\mathrm{SO}(n-1)$ have highest weights, $(m,0,\ldots,0)$, where m is a nonnegative integer, and the corresponding spherical functions are Gegenbauer polynomials in $\cos(\theta)$, where $\theta \in [0,\pi]$, namely

$$\varphi_m^n = \frac{\Gamma(n-2)m!}{\Gamma(n+m-2)}.C_m^{(n-2)/2}(\cos\theta).$$

See [30] for a proof of this. For fixed n, the sequence of functions $C_m^{(n-2)/2}$ is a sequence of real orthogonal polynomials, so the sequence of functions φ_m^n is an extended Chebyshev system.

Choose real finitely supported distributions, $\tilde{\psi}_{i,k}$, on $[0,\pi]$, for $2 < i \leq k \leq n$ which each satisfy

$$\left\langle \tilde{\psi}_{i,k}, \varphi_m^i \right\rangle = \delta_{0m} \text{ for } 0 \leq m \leq s.$$

A lot of choices are involved here. In particular, the support, F, of $\tilde{\psi}_{i,k}$ may be any nonempty finite subset of $[0,\pi]$, and the order, p, of $\tilde{\psi}_{i,k}$ is likewise arbitrary provided that $(p+1)\,|F| \geq s+1$.

For the case $n=2$, choose $\tilde{\psi}_{2,k}$ to be a real distribution supported on a finite subset of $[0,2\pi)$ such that

$$\left\langle \tilde{\psi}_{2,k}, e^{im.(\ )} \right\rangle = \delta_{0m} \text{ for } |m| \leq s.$$

Define $\psi_{i,k} = (r_i)_*(\tilde{\psi}_{i,k})$ for $2 \leq i \leq k \leq n$, i.e. $\langle \psi_{i,k}, f\rangle = \langle \tilde{\psi}_{i,k}, f \circ r_k\rangle$, for any C^∞ function, f, on G. Finally we can define our sampling distributions:

$$\begin{aligned} c_2 &= \psi_{2,2}, \\ c_n &= \psi_{2,n} * \cdots * \psi_{n,n} * c_{n-1}. \end{aligned}$$

The convolution product for c_n has $\dim \mathrm{SO}(n) = \frac{n(n-1)}{2}$ factors. It is clear that we can choose the $s_{i,k}$ so that the order of c_n is 0 and c_n has support of size at most $(2s+1)^{n-1}s^{(n-1)(n-2)/2}$. If we allow c_n to have a higher order, then we can decrease the size of its support.

Theorem 4.1. *If $\|\lambda\|_H \leq s$, then $\Delta_\lambda(c_n - 1) = 0$.*

Proof. Let

$$\begin{aligned} \Omega_s^n &= \{\lambda \in \widehat{\mathrm{SO}(n)} : \|\lambda\|_H \leq s\} \\ &= \{\Delta_{(m_{1,n},\ldots,m_{k,n})} : |m_{1,n}| \leq s\}. \end{aligned}$$

Using the embeddings $C^\infty(\mathrm{SO}(2))' \hookrightarrow \cdots \hookrightarrow C^\infty(\mathrm{SO}(n))'$ and the betweenness relations for the restriction of representations of $\mathrm{SO}(n)$ to $\mathrm{SO}(n-1)$, it is obvious that $\mathfrak{T}_{\Omega_s^2} \subseteq \cdots \subseteq \mathfrak{T}_{\Omega_s^n}$ We shall show, using induction, that $c_n = c_{\mathrm{SO}(n)} \pmod{\mathfrak{T}_{\Omega_s^n}}$, for all n. Now, from the general arguments given previously, we know that if we define $\hat{c}_k$ by

$$\begin{aligned} \hat{c}_2 &= \psi_{2,2}, \\ \hat{c}_k &= \hat{c}_{k-1} * \psi_{k,k} * \hat{c}_{k-1}, \end{aligned}$$

then $\hat{c}_s = c_{\mathrm{SO}(k)} \pmod{\mathfrak{T}_{\Omega_s^k}}$, for all k. We need to show that $\hat{c_n} = c_n$. To prove this, it suffices to show that if $\psi_2, \ldots \psi_n$ are distributions with the support of ψ_k contained in $r_k(\mathbb{R})$, and satisfying $c_{\mathrm{SO}(k-1)} * \psi_k * c_{\mathrm{SO}(k-1)} = c_{\mathrm{SO}(k)} \pmod{\mathfrak{T}_{\Omega_s^k}}$, then $\hat{c_n} = \psi_2 * \cdots \psi_n * c_{n-1} \pmod{\mathfrak{T}_{\Omega_s^n}}$. By induction, we assume that this is true for numbers less than n. Then for any $\psi_2, \ldots, \psi_n$ as above, we have

$$\begin{aligned}\hat{c}_n &= c_{n-1} * \psi_n * c_{\mathrm{SO}(n-1)} \pmod{\mathfrak{T}_{\Omega_s^n}} \\ &= (\psi_2 * \cdots * \psi_{n-1} * c_{n-2}) * \psi_n * c_{\mathrm{SO}(n-1)} \pmod{\mathfrak{T}_{\Omega_s^n}} \\ &= \psi_2 * \cdots * \psi_{n-1} * \psi_n * c_{\mathrm{SO}(n-2)} * c_{\mathrm{SO}(n-1)} \pmod{\mathfrak{T}_{\Omega_s^n}} \\ &= \psi_2 * \cdots * \psi_n * c_{n-1} \pmod{\mathfrak{T}_{\Omega_s^n}},\end{aligned}$$

where we have used the facts that $c_{\mathrm{SO}(n-2)} * c_{\mathrm{SO}(n-1)} = c_{\mathrm{SO}(n-1)}$, and $c_{n-2} \smile \psi_n$. □

The distribution $P^{\mathrm{SO}(n-1)}(\psi_{2,n} * \cdots * \psi_{n,n})$ on $S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1)$ is zero on the associated spherical functions coming from representations of $\mathrm{SO}(n)$ satisfying $|m_{1,n}| \leq s$. In [20], it is shown that a fast transform is possible for functions sampled on the support of this distribution.

A similar argument leads to the parametrization of $\mathrm{SO}(n)$ by Euler angles.

4.3. Example: Sampling on $\mathrm{SU}(n)$. In this case, the appropriate chain of subgroups to use is,

$$\mathrm{SU}(n) \subseteq S(U_{n-1} \times U_1) \subseteq \mathrm{SU}(n-1) \subseteq \cdots \subseteq S(U_1 \times U_1).$$

Let $r_k(\theta)$ be the same matrix as was used in the case of $\mathrm{SO}(n)$, but also define $q_k(\theta) = \mathrm{Diag}(e^{-i\theta}, \ldots, e^{-i\theta}, e^{ik\theta}, 1, \ldots, 1)$. where there are exactly k entries of the form $e^{-i\theta}$. Note that $q_k(\theta) \smile \mathrm{SU}(k)$, that the q_k generate the usual choice of maximal torus in $\mathrm{SU}(n)$, and that

$$\begin{aligned}S(U_{n-1} \times U_1) &= q_{n-1}([0, 2\pi]).\,\mathrm{SU}(n-1), \\ \mathrm{SU}(n) &= S(U_{n-1} \times U_1).r_n([0, \pi/2]).S(U_{n-1} \times U_1).\end{aligned}$$

In fact, the map

$$[0, \pi/2] \to S(U_{n-1} \times U_1)\backslash \mathrm{SU}(n)/S(U_{n-1} \times U_1) : \\ \theta \mapsto S(U_{n-1} \times U_1) r_n(\theta) S(U_{n-1} \times U_1)$$

is a homeomorphism, and its restriction to $(0, \pi/2)$ is a diffeomorphism.

Let $\lambda_{1,n}, \ldots \lambda_{n-1,n}$ be the coordinates of the highest weight of a representation of $\mathrm{SU}(n)$ relative to the basis, $\{e_i\}$ of the dual of the usual Cartan subalgebra, as given in Section 2.2.4. Then

$$\lambda_{1,n} \geq \cdots \geq \lambda_{n-1,n} \geq 0.$$

Representations of the group $S(U_{n-1} \times U_1)$, are determined by a collection of numbers $(\lambda_{1,n-1}, \ldots \lambda_{n-2,n-1}; \lambda_{n-1,n-1})$, where $(\lambda_{1,n-1}, \ldots \lambda_{n-2,n-1})$ is the highest weight of the restriction to $\mathrm{SU}(n-1)$, and $\lambda_{n-1,n-1}$ is the weight of the restriction to the subgroup $q_{n-1}(\mathbb{R})$. The relations giving the representations of $S(U_{n-1} \times U_1)$ arising are

$$\begin{aligned}\lambda_{1,n-1} &= \mu_1 - \mu_{n-1},\\ &\dots\\ \lambda_{n-2,n-1} &= \mu_{n-2} - \mu_{n-1},\\ \lambda_{n-1,n-1} &= (n-1)\sum_{j=1}^{n-1} \lambda_{j,n} - n\sum_{j=1}^{n-1} \mu_j,\end{aligned}$$

where the μ_j are integers satisfying

$$\lambda_{1,n} \geq \mu_1 \geq \lambda_{2,n} \geq \dots \lambda_{n-1,n} \geq \mu_{n-1} \geq 0.$$

In the case $n = 2$ the appropriate relation is $\lambda_{1,2} \geq |\lambda_{1,1}|$, where $\lambda_{1,2} - \lambda_{1,1}$ must be even. To restrict to $\mathrm{SU}(n-1)$ from $S(U_{n-1} \times U_1)$ simply throw away $\lambda_{n-1,n-1}$. If we now define for $m \geq 2$

$$\begin{aligned}\Omega_s^m &= \{\Delta_\lambda : \|\lambda\|_H \leq s\} = \{\Delta_{(\lambda_{1,m},\dots,\lambda_{m-1,m})} : \lambda_{1,m} \leq s\}\\ \breve{\Omega}_s^{m-1} &= \{\Delta_{(\lambda;\lambda_{m-1,m-1})} : \|\lambda\|_H \leq s, |\lambda_{m-1,m-1}| \leq (m-1)s\}\\ &= \{\Delta_{(\lambda_{1,m-1},\dots,\lambda_{m-2,m-1};\lambda_{m-1,m-1})} : \lambda_{1,m-1} \leq s, |\lambda_{m-1,m-1}| \leq (m-1)s\}\\ \breve{\Omega}_s^1 &= \{\Delta_{(\lambda_{1,1})} : |\lambda_{1,1}| \leq s\},\end{aligned}$$

then using the embeddings

$$C^\infty(S(U_1 \times U_1))' \hookrightarrow C^\infty(\mathrm{SU}(2))' \hookrightarrow \dots \hookrightarrow C^\infty(S(U_{n-1} \times U_1))' \hookrightarrow C^\infty(\mathrm{SU}(n))'$$

and the restriction relations given above, we see that

$$\mathfrak{T}_{\breve{\Omega}_s^1} \subseteq \mathfrak{T}_{\Omega_s^2} \subseteq \cdots \mathfrak{T}_{\Omega_s^{n-1}} \subseteq \mathfrak{T}_{\breve{\Omega}_s^{n-1}} \subseteq \mathfrak{T}_{\Omega_s^n}.$$

The class 1 representations of $\mathrm{SU}(n)$ relative to $S(U_{n-1} \times U_1)$ have highest weights of the form $(2m, m, \dots)$, where $m \geq 0$, and using the map $[0, \pi/2] \longleftrightarrow S(U_{n-1} \times U_1)\backslash \mathrm{SU}(n)/S(U_{n-1} \times U_1)$ specified above, have corresponding spherical functions which are Jacobi polynomials in $\cos 2\theta$,

$$\varphi_m^n = \frac{(n-2)!m!}{(n+m-2)!}.P_m^{n-2,0}(\cos 2\theta).$$

For a proof of this, see [20].

For $2 \leq i \leq k \leq n$ choose be a real finitely supported distribution, $\tilde{\psi}_{i,k}$, on $[0, \pi/2]$, that satisfies $\langle \tilde{\psi}_{i,k}, \varphi_m^i \rangle = \delta_{0,m}$ for $0 \leq m \leq \lfloor \frac{s}{2} \rfloor$. For $1 \leq j \leq k < n$, choose a real finitely supported distribution, $\tilde{\zeta}_{j,k}$, on $[0, 2\pi)$ that satisfies $\langle \tilde{\zeta}_{j,k}, e^i m(\ ) \rangle = \delta_{0,m}$ for $|m| \leq jb$. Define $\tilde{\zeta}'_{n-1,n}$ in the same way, with $j = n-1$. Then set $\psi_{i,k} = (r_i)_*(\tilde{\psi}_{i,k})$, $\zeta_{j,k} = (q_j)_*(\tilde{\zeta}_{j,k})$, and define $\zeta'_{n-1,n}$ similarly. Finally define

$$\begin{aligned}c_2 &= \zeta_{1,2} * \psi_{2,2} * \zeta'_{1,2},\\ c_n &= (\zeta_{1,n} * \psi_{2,n}) * \cdots * (\zeta_{n-1,n} * \psi_{n,n}) * \zeta'_{n-1,n} * c_{n-1}.\end{aligned}$$

THEOREM 4.2. $c_n = c_{\mathrm{SU}(n)} \pmod{\mathfrak{T}_{\Omega_s^n}}$ *and* $c_n * \zeta'_{n-1,n} = c_{S(U_n\times U_1)} \pmod{\mathfrak{T}_{\breve{\Omega}_s^n}}$.

PROOF. We use induction on n. It suffices to show that if the ψ_k are distributions supported on $r_k(\mathbb{R})$, and ζ_k, ζ'_k satisfy $c_{S(U_{k-1}\times U_1)} * \psi_k * c_{S(U_{k-1}\times U_1)} = c_{\mathrm{SU}(k)}$, and $\zeta'_k = \zeta_k = c_{q_k(\mathbb{R})}$ modulo $\mathfrak{T}_{\Omega_s^n}$, then

$$c_{\mathrm{SU}(n)} = (\zeta_1 * \psi_2) * \cdots * (\zeta_{n-2} * \psi_{n-1}) * \zeta'_{n-2} * c_{\mathrm{SU}(n-1)} \pmod{\mathfrak{T}_{\Omega_s^n}}.$$

By induction, we can assume this holds for numbers less than n. Let Q_n be the subgroup of $\mathrm{SU}(n)$ given by $Q_n = \{\mathrm{Diag}(e^{i(n-1)\theta}, e^{-i\theta}, \dots e^{-i\theta}) : \theta \in \mathbb{R}\}$, and note that $\zeta_{n-2} * c_{\mathrm{SU}(n-2)} * \zeta_{n-1} = \zeta_{n-1} * c_{\mathrm{SU}(n-2)} * c_{Q_n} \pmod{\mathfrak{T}_{\Omega_s^n}}$. Therefore, working modulo $\mathfrak{T}_{\Omega_s^n}$, we have

$$\begin{aligned} c_{\mathrm{SU}(n)} &= c_{\mathrm{SU}(n-1)} * \zeta_{n-1} * \psi_{n,n} * \zeta'_{n-1} * c_{\mathrm{SU}(n-1)} \\ &= \zeta_1 * \psi_2 * \cdots * \psi_{n-1} * (\zeta'_{n-2} * c_{\mathrm{SU}(n-2)} * \zeta_{n-1}) * \psi_n * \zeta'_{n-1} * c_{\mathrm{SU}(n-1)} \\ &= \zeta_1 * \cdots * \psi_{n-1} * (\zeta_{n-1} * c_{\mathrm{SU}(n-2)} * c_{Q_n}) * \psi_n * \zeta'_{n-1} * c_{\mathrm{SU}(n-1)} \\ &= \zeta_1 * \cdots * \psi_{n-1} * \zeta_{n-1} * \psi_n * c_{\mathrm{SU}(n-2)} * c_{Q_n} * \zeta'_{n-1} * c_{\mathrm{SU}(n-1)} \\ &= \zeta_1 * \psi_2 * \cdots * \psi_{n-1} * \zeta_{n-1} * \psi_{n,n} * \zeta'_{n-1,n} * c_{\mathrm{SU}(n-1)}, \end{aligned}$$

where we used the fact that $Q_n \subseteq S(U_{n-1} \times U_1)$. □

The distribution, $P^{\mathrm{SU}(n-1)}(\zeta_{1,n} * \psi_{2,n} * \cdots * \zeta_{n-1,n} * \psi_{n,n} * \zeta'_{n-1,n})$, on $S^{2n-1} = \mathrm{SU}(n-1)/\mathrm{SU}(n-1)$, is zero on associated spherical functions coming from representations whose highest weight, $(\lambda_{1,n}, \dots, \lambda_{n-1,n})$, satisfies $\lambda_{1,n} \le b$. In [20] is is shown how to perform fast transforms for functions sampled on the support of this distribution. By commutativity, $(\zeta_{1,n} * \psi_{2,n}) * \cdots * (\zeta_{n-1,n} * \psi_{n,n}) = (\zeta_{1,n} * \cdots * \zeta_{n-1,n}) * \psi_{2,n} * \cdots \psi_{n,n})$, so by replacing $\zeta_{1,n} * \cdots * \zeta_{n-1,n}$ by an appropriate distribution on the maximal torus of $\mathrm{SU}(n)$, we can obtain yet more distributions on $\mathrm{SU}(n)$, which satisfy the above theorem.

The same commutativity relations can applied to the subgroups q_i and r_j of $\mathrm{SU}(n)$. This yields a parametrization of $\mathrm{SU}(n)$, which is analogous to the Euler angles for $\mathrm{SO}(n)$.

4.4. Example: Sampling on $\mathrm{Sp}(n)$**.** $\mathrm{Sp}(n) = \{A \in M_n(\mathbb{H}) : A^*A = \mathrm{Id}\}$, where $\mathbb{H}$ denotes the division ring of quaternions. By elementary geometry, one can see that $\mathrm{Sp}(n)/(\mathrm{Sp}(n-1)\times \mathrm{Sp}(1))$ is isomorphic to the right quaternionic projective space, $\mathbf{P}^{n-1}\mathbb{H}$ and that the map

$$\begin{aligned} &[0,\pi/2] \to (\mathrm{Sp}(n-1)\times \mathrm{Sp}(1))\backslash \mathrm{Sp}(n)/(\mathrm{Sp}(n-1)\times \mathrm{Sp}(1)) \\ &\qquad : \theta \mapsto (\mathrm{Sp}(n-1)\times \mathrm{Sp}(1)).r_n(\theta).(\mathrm{Sp}(n-1)\times \mathrm{Sp}(1)) \end{aligned}$$

is a homeomorphism, and its restriction to $(0,\pi/2)$ is a diffeomorphism. Note that $\mathrm{Sp}(1) \leftrightarrow \mathrm{SU}(2)$.

Let

$$R_n = \left\{ \begin{pmatrix} 1 & & \\ & \ddots & \\ & & a \end{pmatrix} : a \in \mathrm{Sp}(1) \right\},$$

so that $\mathrm{Sp}(n-1)\times\mathrm{Sp}(1)=\mathrm{Sp}(n-1).R_n$.

Working in the basis $\{e_i\}$ of Section 2.2.4, the highest weights of representations of $\mathrm{Sp}(n)$ are determined by integers $m_{1,n},\dots,m_{n,n}$, where

$$m_{1,n}\geq\cdots\geq m_{n,n}\geq 0.$$

The highest weights, $\nu=(m_{1,n-1},\dots,m_{n-1,n-1})$, of those representations occurring in the restriction of the representation, $\Delta_{(m_{1,n},\dots,m_{n,n})}$, of $\mathrm{Sp}(n)$ to $\mathrm{Sp}(n-1)$ satisfy

$$p_1\geq m_{1,n-1}\geq p_2\geq\cdots\geq m_{n-1,n-1}\geq p_n,$$

where

$$m_{1,n}\geq p_1\geq\cdots\geq m_{n,n}\geq p_n\geq 0,$$

but the corresponding multiplicities may be greater than one. The restriction of $\Delta_{(m_{1,n},\dots,m_{n,n})}$ to $\mathrm{Sp}(n-1)\times\mathrm{Sp}(1)$ is precisely

$$\sum_{\nu}\left(\Delta_\nu\otimes\left(\bigotimes_{i=1}^{n}\Delta_{(\min\{m_{i-1,n-1},m_{i,n}\}-\max\{m_{i,n-1},m_{i+1,n}\})}\right)\right),$$

where $m_{n+1,n}=m_{n,n-1}=0$, $m_{0,n-1}=+\infty$, and ν ranges over the highest weights of irreducible representations of $\mathrm{Sp}(n)$ appearing in the restriction of $\Delta_{(m_{1,n},\dots,m_{n,n})}$ to $\mathrm{Sp}(n-1)$; see [33]. Hence, highest weights, m, of the representations occurring in the restriction from $\mathrm{Sp}(n)$ to R_n satisfy $m_{1n}\geq m$. It should be clear then, that if we define, for any positive integer s,

$$\Omega_s^n=\{\Delta_\lambda:\|\lambda\|_H\leq s\}=\{\Delta_{(m_{1,n},\dots,m_{n,n})}:m_{1,n}\leq s\},$$

then $\mathfrak{T}_{\Omega_s^1}\subseteq\cdots\subseteq\mathfrak{T}_{\Omega_s^n}$. Also, let $\Omega_s^{\mathrm{SU}(2)}$ be the set of all irreducible representations, Δ_m, of $\mathrm{SU}(2)$ such that $0\leq m\leq b$, and denote the corresponding set of representations of R_n by $\Omega_s^{R_n}$. Using the embedding $C^\infty(R_n)'\hookrightarrow C^\infty(\mathrm{Sp}(n))'$, we see that $\mathfrak{T}_{\Omega_s^{R_n}}\subseteq\mathfrak{T}_{\Omega_s^n}$.

For any $1\leq k\leq n$, we can construct, using previous techniques, a finitely supported measure, $\upsilon_{k,n}$, on $R_n\leftrightarrow\mathrm{SU}(2)$, such that $\upsilon_{k,n}=c_{R_k}\pmod{\mathfrak{T}_{\Omega_s^{R_k}}}$. Now assume that $n\geq 2$. The class one representations of $\mathrm{Sp}(n)$ relative to $\mathrm{Sp}(n-1)\times\mathrm{Sp}(1)$ have highest weights of the form $(m,m,0,\dots)$, where m is a nonnegative integer, and the corresponding spherical functions can be written using the map $[0,\pi/2]\to(\mathrm{Sp}(n-1)\times\mathrm{Sp}(1))\backslash\mathrm{Sp}(n)/(\mathrm{Sp}(n-1)\times\mathrm{Sp}(1))$, in the form

$$\varphi_m^n=\frac{(2n-3)!m!}{(m+2n-3)!}.P_m^{2n-3,1}(\cos 2\theta).$$

For a proof of this, see [15]. Let $\tilde{\psi}_{k,n}$ be a real finitely supported distribution on $[0,\pi/2]$ that satisfies $\langle\tilde{\psi}_{k,n},\varphi_m^k\rangle=\delta_{0,m}$ for $0\leq m\leq s$, and set $\psi_{k,n}=(r_k)_*(\tilde{\psi}_{k,n})$. Then define c_n inductively by

$$c_1=\upsilon_{1,1},$$
$$c_n=\upsilon_{1,n}*(\psi_{2,n}*\upsilon_{2,n})*\cdots*(\psi_{n,n}*\upsilon_{n,n})*c_{n-1}.$$

This finitely supported measure is the convolution product of $\dim \mathrm{Sp}(n) = 2n^2 + n$ factors each supported on a 1-parameter subgroup of $\mathrm{Sp}(n)$, and it is easy to prove the following theorem.

THEOREM 4.3. $c_n = c_{\mathrm{Sp}(n)} \pmod{\mathfrak{T}_{\Omega_s^n}}$.

PROOF. Similar to the $\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ cases. □

Acknowledgments

I thank Dan Rockmore for reorganizing this paper, and for rewriting the introduction to bring it up to date. I would like to thank Persi Diaconis, and Dennis Healy for many discussions and a lot of encouragement and advice along the way. I would also like to thank the Harvard University Department of Mathematics, and the Max-Planck-Institut für Mathematik, which supported me during the writing of this paper.

References

[1] J. P. Boyd, *Chebyshev and Fourier Spectral Methods*, Lecture Notes in Engineering **49**, Springer, New York, 1989.

[2] W. Byerly, *An elementary treatise on Fourier's series and spherical, cylindrical, and ellipsoidal harmonics: with applications to problems in mathematical physics*, Ginn, Boston, 1893.

[3] G. Chirikjian and A. Kyatkin, *Engineering applications of noncommutative harmonic analysis: with emphasis on rotation and motion groups*, CRC Press, Boca Raton (FL), 2000.

[4] J. R. Driscoll and D. M. Healy Jr., "Computing Fourier transforms and convolutions on the 2-sphere", *Adv. Applied Math.* **15** (1994), 202–250.

[5] J. R. Driscoll, D. M. Healy Jr., and D. Rockmore, "Fast spherical transforms on distance transitive graphs", *SIAM J. Comput.*, **26**:4, 1997, 1066–1099.

[6] G. I. Gaudrey and R. Pini, "Bernstein's theorem for compact, connected Lie groups", *Math. Proc. Cambridge Philosoph. Soc.* **99** (1986), 297–305.

[7] A. Ghizetti and A. Ossicini, *Quadrature Formulae*, Birkhäuser, Basel, 1970.

[8] D. Healy, D. Rockmore, P. Kostelec, and S. Moore, "FFTs for the 2-sphere: Improvements and variations", *J. Fourier Anal. Appl.* **9**:4 (2003), 341–384.

[9] D. Healy, P. Kostelec, and D. Rockmore, "Safe and effective higher order Legendre transforms", to appear in *App. Comp. Math.*

[10] S. Helgason, *Groups and geometric analysis*, Academic Press, New York, 1984.

[11] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Grundlehren der mathematischen Wissenschaften **152**, Springer, Berlin, 1963.

[12] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer, New York, 1980.

[13] M. Kazhdan, T. Funkhouser and S. Rusinkiewicz, "Rotation invariant spherical harmonic representation of 3D shape descriptors", to appear in *Symposium on Geometry Processing* (2003).

[14] M. Kazhdan and T. Funkhouser, "Harmonic 3D Shape Matching", Technical Sketch, *SIGGRAPH* (2002).

[15] A. U. Klimyk and N.J. Vilenkin, "Relations between spherical functions of compact groups", *J. Math. Phys.* **30**:6 (June 1989), 1219–1225.

[16] P. Kostelec, D. K. Maslen, D. M. Healy, and D. Rockmore, "Computational harmonic analysis for tensor fields on the two-sphere", *J. Comput. Phys.* **162**:2 (2000), 514–535.

[17] P. Kostelec and D. Rockmore, "FFTs for SO(3)", in preparation.

[18] J. T. Lo and L. R. Eshleman, "Exponential Fourier densities on S^2 and optimal estimation and detection for directional processes", *IEEE Trans. Inform. Theory*, **23**:3 (May 1977), 321–336.

[19] J. T. Lo and L. R. Eshleman, "Exponential Fourier densities on SO(3) and optimal estimation and detection for rotational processes", *SIAM J. Appl. Math.* **36**:1 (Feb. 1979), 73–82.

[20] D. K. Maslen, *Fast transforms and sampling for compact groups*, Ph.D. Thesis, Harvard University, MA (1993).

[21] D. K. Maslen, "Efficient computation of Fourier transforms on compact groups", *J. Fourier Anal. Appl.* **4**:1 (1998), 19–52.

[22] M. P. Mohlenkamp, "A fast transform for spherical harmonics", *J. Fourier Anal. Appl.* **5**:3 (1999), 159–184..

[23] P. M. Morse and H. Feshbach, *Methods of theoretical physics*, McGraw-Hill, New York (1953).

[24] S. P. Oh, D.N. Spergel, G. Hinshaw, "An efficient technique to determine the power spectrum from the cosmic microwave background sky maps", *Astrophysical Journal* **510** (1999), 551–563.

[25] A. Oppenheim and R. Schafer, *Digital signal processing*, Prentice-Hall, Englewood Cliffs (NJ), 1975.

[26] A. Papoulis, *The Fourier integral and its applications*, McGraw-Hill, NY, 1962.

[27] R. Pini, "Bernstein's Theorem on SU(2)", *Bollettino Un. Mat. Ital.* (6) **4**-A (1985), 381–389.

[28] D. Slepian, "Some comments on Fourier analysis, uncertainty and modeling", *SIAM Review* **25**:3 (July 1983), 379–393.

[29] M. Sugiura, *Unitary representations and harmonic analysis: an introduction*, North-Holland, Amsterdam, 1990.

[30] N. J. Vilenkin, *Special functions and the theory of group representations*, Translations of Mathematical Monographs **22**, Amer. Math. Soc., Providence, 1968.

[31] N. R. Wallach, *Harmonic analysis on homogeneous spaces*, Dekker, New York, 1973.

[32] M. Zaldarriaga and U. Seljak, "An all-sky analysis of polarization in the microwave background", *Phys. Rev. D* **55** (1997), 1830–1840.

[33] D. P. Želobenko, *Compact Lie groups and their representations*, Translations of Mathematical Monographs **40**, Amer. Math. Soc., Providence, 1973.

David Keith Maslen
Susquehanna International Group
401 City Ave., Suite 220
Bala Cynwyd, PA 19004
U.S.A.
david@maslen.net