Inside Out: Inverse Problems
MSRI Publications
Volume **47**, 2003

Sojourn Times, Singularities of the Scattering Kernel and Inverse Problems

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Abstract. We study inverse problems in the scattering by obstacles in odd-dimensional Euclidean spaces. In general, such problems concern the recovery of the geometric properties of the obstacle from the information related to the scattering amplitude $a(\lambda, \omega, \theta)$, related to the wave equation in the exterior of the obstacle with Dirichlet boundary condition. It turns out that all singularities of the Fourier transform of $a(\lambda, \omega, \theta)$, the so-called scattering kernel, are given by the sojourn (traveling) times of scattering rays in the exterior of the obstacle. Apart from that these sojourn times are a naturally observable data. The purpose of this survey is to describe several results in obstacle scattering obtained in the last twenty years concerning sojourn times of scattering rays, and to motivate further study of related inverse scattering problems.

1. Introduction

The scattering operator $S(\lambda)$ presents a mathematical model for the data observed experimentally in many branches of physics, chemistry and mathematics. The operator $S(\lambda)$ is related to behavior as the time $t \to \pm\infty$ of the solutions of an unperturbed operator L_0 and to its perturbation L. The kernel of $S(\lambda) - I$, the so called *scattering amplitude* $a(\lambda, \omega, \theta)$, contains the information related to the perturbation of L_0 and this kernel is the leading term of the asymptotic of an outgoing solution $v_s(r\theta, \lambda)$ of $Lv_s = 0$ as $|x| = r \to \infty$. Obstacle scattering problems arise in many physical phenomena and concern the perturbation caused by a bounded obstacle K with connected complement Ω. In general the inverse scattering problems deal with recovering geometric properties of K from information related to the scattering amplitude.

Schiffer's result (see [12], [2]) implies that the obstacle K is uniquely determined if we know the scattering amplitude $a(\lambda, \omega, \theta)$ for $\lambda \in (\alpha, \beta) \subset \mathbb{R}^+$ and all $\omega, \theta \in \mathbf{S}^{n-1}$. Some more precise results concerning uniqueness in this inverse scattering problem are known under weaker assumptions (see [2], [7], [11], [26] for more details and references.) On the other hand, in general in experiments one cannot determine the scattering amplitude for all (outgoing) directions $\theta \in \mathbf{S}^{n-1}$ or all (incoming) directions $\omega \in \mathbf{S}^{n-1}$, while the *sojourn times* or *traveling times* of the so-called (ω, θ)-rays in the exterior of the obstacle give a physically observable data. This naturally leads to the consideration of inverse scattering problems involving such rays. In fact, it turns out that all singularities of the Fourier transform $s(t, \omega, \theta)$ of $a(\lambda, \omega, \theta)$, the so-called *scattering kernel*, have the form $-T_\gamma$, where T_γ are sojourn times of (ω, θ)-rays γ. Moreover, for (ω, θ) in a set of full measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ the singularities of $s(t, \omega, \theta)$ are precisely the numbers of the form $-T_\gamma$, that is the so-called *Poisson relation* becomes an equality (see Section 5). This leads to some interesting geometrical observations. The purpose of this survey is to describe several results in obstacle scattering obtained in the last twenty years concerning sojourn times of (ω, θ)-rays, and to motivate further study of related inverse scattering problems.

The scattering amplitude is defined in Section 2. The case of a convex obstacle is then considered in details, and the leading term of the asymptotic of the scattering amplitude as $\lambda \to +\infty$ is derived. Section 3 is devoted to the Fourier transform of the scattering amplitude, the so-called scattering kernel $s(t, \theta, \omega)$, where $t \in \mathbb{R}$ and θ, $\omega \in \mathbf{S}^{n-1}$. It turns out that the singularities of $s(t, \theta, \omega)$ in t are very much related to the geometry of the obstacle K. Namely, these are given by sojourn (traveling) times of scattering rays in the exterior of the obstacle incoming with direction ω and outgoing with direction θ. This is particularly easy to see in the case of a convex obstacle, where a scattering ray can have at most one reflection at the boundary ∂K of the obstacle. In the general case a typical scattering ray is a mutiply reflecting ray with reflections at ∂K. Moreover there are other, more complicated rays, that have to be taken into account when studying the singularities of the scattering kernel; some of these contain gliding segments on ∂K which are simply geodesics with respect to the metric on ∂K induced by the Euclidean structure. All these are generalized bicharacteristics in the sense of Melrose and Sjöstrand [20]. Their definition is sketched in Section 3, and at the end of that section the leading term of the singularity of $s(t, \theta, \omega)$ at $t \sim -T$ is described, where T is the sojourn time of a scattering ray satisfying some nondegeneracy properties.

Section 4 is purely geometrical. Here we give a simple definition of a reflecting (ω, θ)-ray, and show that for almost all $(\omega, \theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$, the reflecting (ω, θ)-rays in the exterior of K have no tangencies to ∂K and any two of them have different sojourn times. These properties, together with nondegeneracy of the differential cross-sections, play an important role in the analysis of the singularities of the scattering kernel. The latter is dealt with in Section 5. The central point here is the so-called Poisson relation for the scattering kernel, and the first half of Section 5 is devoted to the idea of its proof. We then proceed to discuss the question of how often this relation becomes an equality. One of the problems to do this is to show that (under certain nondegeneracy assumptions about the obstacle) for almost all $(\omega, \theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$, the (ω, θ)-rays in the exterior of K are reflecting rays, i.e. they do not contain gliding segments on the boundary. Combining this with previous results gives that the Poisson relation becomes an equality for almost all $(\omega,\theta)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$.

In Section 6 we discuss the existence of simply reflecting nondegenerate scattering rays with sojourn times tending to infinity. This leads to some interesting results concerning the behavior of the modified resolvent of the Laplacian.

Finally, in Section 7 the inverse scattering problem is considered of recovering geometric information about the obstacle from its scattering length spectrum, i.e. from the set of sojourn times of scattering rays in the exterior of the obstacle[1]. Pairs of obstacles K, L are considered such that for (almost) all $(\omega,\theta)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$ the sets of sojourn times of (ω,θ)-rays in the exteriors of K and L are the same. It then turns out that the generalized geodesic flows in the nontrapping parts of the cotangent bundles of the exteriors of K and L are conjugated by a time preserving conjugacy which is almost everywhere smooth and symplectic. Various geometric relationships between K and L are derived.

2. Scattering Amplitude for Strictly Convex Obstacles

Let $K\subset\mathbb{R}^n$, $n\geq 3$, n odd, be a bounded domain with C^∞ boundary ∂K and connected complement $\Omega=\overline{\mathbb{R}^n\setminus K}$. Such K is called an *obstacle* in $\mathbb{R}^n$. Throughout this paper we deal with the Dirichlet problem for the Laplacian but similar considerations can be applied to other boundary value problems. To introduce the scattering amplitude $a(\lambda,\theta,\omega)$, $(\theta,\omega)\in\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$, consider the *outgoing solution* $v_s=v_s(x,\lambda)$ of the problem

$$\begin{cases}(\Delta+\lambda^2)v_s=0 & \text{in } \mathring{\Omega},\\ v_s+e^{-i\lambda\langle x,\omega\rangle}=0 & \text{on } \partial K\end{cases}$$

satisfying the so-called $(i\lambda)$ - outgoing Sommerfeld radiation condition. This condition means that as $|x|=r\to\infty$ we have

$$v_s(r\theta,\lambda)=\frac{e^{-i\lambda r}}{r^{(n-1)/2}}\big(a(\lambda,\theta,\omega)+O(r^{-1})\big),\quad x=r\theta.$$

We can interpret $v_i=e^{-i\lambda\langle x,\omega\rangle}$ as an *incoming plane wave*, while $v_s(x,\lambda)$ is the *outgoing wave* obtained after the impact of v_i on ∂K. To obtain a formula for the leading term $a(\lambda,\theta,\omega)$ we apply the Green formula combined with the outgoing condition and deduce the representation

$$v_s(x,\lambda)=\int_{\partial K}\Big(E_\lambda(x-y)\frac{\partial v_s}{\partial\nu}(y,\lambda)-\frac{\partial E_\lambda}{\partial\nu}(x-y)v_s(y,\lambda)\Big)\,dS_y, \tag{2–1}$$

[1] According to the Poisson relation, this is equivalent to trying to obtain information about the obstacle from the singularities of the scattering kernel.

where $E_\lambda(x)$ is the outgoing Green function

$$E_\lambda(x) = \frac{(i\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \frac{e^{-i\lambda r}}{r^{(n-1)/2}} + O\big(r^{-(n+1)/2}\big)$$

and $\nu(x)$ is the unit normal to $x \in \partial K$ pointing into Ω. Next, we multiply (2–1) by $e^{i\lambda r} r^{(n-1)/2}$, put $x = r\theta$, and taking the limit $r \to \infty$, we get

$$a(\lambda, \theta, \omega) = \frac{(i\lambda)^{(n-3)/2}}{2(2\pi)^{(n-1)/2}} \int_{\partial K} \Big(i\lambda \langle \nu(x), \theta\rangle e^{i\lambda\langle x, \theta - \omega\rangle} + e^{i\lambda\langle x, \theta\rangle} \frac{\partial v_s}{\partial \nu}(x, \lambda)\Big)\, dS_x,$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the scalar product in $\mathbb{R}^n$.

Following the physical literature, $a(\lambda, \theta, \omega)$ is called the *scattering amplitude.* The analysis of the leading term of its asymptotic as $\lambda \to +\infty$ has a long tradition in mathematical physics. The simplest case to deal with is when $\theta \neq \omega$ and K is a strictly convex obstacle. In this case the integral

$$I(\lambda) = \frac{(i\lambda)^{(n-1)/2}}{2(2\pi)^{(n-1)/2}} \int_{\partial K} \langle \nu(x), \theta\rangle e^{i\lambda\langle x, \theta - \omega\rangle} dS_x$$

is rather easy to study. The phase function $\langle x, \theta - \omega\rangle|_{x \in \partial K}$ has two critical points $x_\pm$ with

$$\langle x_+, \theta - \omega\rangle = \max_{y \in \partial K} \langle y, \theta - \omega\rangle, \quad \langle x_-, \theta - \omega\rangle = \min_{y \in \partial K} \langle y, \theta - \omega\rangle,$$
$$\nu(x_\pm) = \pm \frac{\theta - \omega}{|\theta - \omega|}.$$

Here x^+ denotes the point in the *illuminated region* (see Figure 1)

$$\partial K_+(\omega) = \{y \in \partial K : \langle \nu(y), \omega\rangle < 0\}$$

related to ω, while x^- lies in the *shadow region*

$$\partial K_-(\omega) = \{y \in \partial K : \langle \nu(y), \omega\rangle > 0\},$$

and we have used the convention that the obstacle lies in the half-space

$$\{x \in \mathbb{R}^n : \langle x,\, \theta - \omega\rangle < 0\}.$$

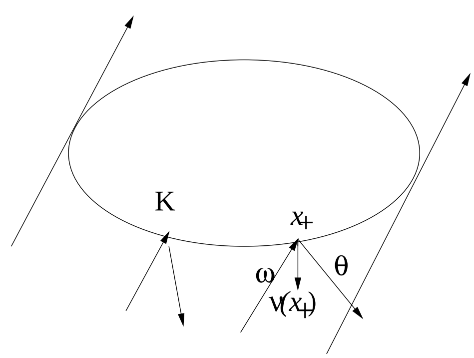

Figure 1.

Applying a stationary phase argument for the integral over $\partial K_+(\omega)$, one gets

$$\frac{(i\lambda)^{(n-1)/2}}{2(2\pi)^{(n-1)/2}}\int_{\partial K_+(\omega)}\langle\nu(x),\theta\rangle e^{i\lambda\langle x,\theta-\omega\rangle}dS_x$$
$$=\tfrac{1}{2}e^{i\lambda\langle x_+,\theta-\omega\rangle}\mathcal{K}(x_+)^{-1/2}\frac{\langle\nu(x_+),\theta\rangle}{|\theta-\omega|^{(n-1)/2}}+O(|\lambda|^{-1}),$$

$\mathcal{K}(y)>0$ being the Gauss curvature at $y\in\partial K$. We get a similar expression for the integral over $\partial K_-(\omega)$.

The analysis of the term involving $\partial v_s/\partial\nu$ is more complicated. In mathematical physics many efforts have been concerned with construction of an approximate outgoing solution $w_0(x,\lambda)$ of the problem

$$\begin{cases}(\Delta+\lambda^2)w_0=f(x,\lambda) & \text{in } \mathring{\Omega},\\ w_0+e^{-i\lambda\langle x,\omega\rangle}=g(x,\lambda) & \text{on } \partial K,\end{cases}$$

with $f(x,\lambda)\in C^\infty(\Omega)$ and $g(x,\lambda)\in C^\infty(\partial K)$. This leads to considerable difficulties when one has to describe the form of the solution w_0 in a domain close to the grazing submanifold

$$G(\omega)=\{y\in\partial K:\langle\nu(y),\omega\rangle=0\}.$$

The progress of the microlocal analysis in the seventies led to the investigation of the above problem without a precise information for w_0 in a neighborhood of $G(\omega)$. This was done by Majda [14] exploiting the works of Hörmander [9], Taylor [30] and Melrose [17] for the propagation of the singularities. Below we present the idea of the approach of Majda and refer to [14] for more details.

Consider the boundary problem

$$\begin{cases}(\partial_t^2-\Delta)u_0=F(t,x) & \text{in } \mathbb{R}\times\mathring{\Omega},\\ u_0+\delta(t-\langle x,\omega\rangle)=G(t,x) & \text{on } \mathbb{R}\times\partial K,\end{cases}$$

where $F(t,x)\in C^\infty(\mathbb{R}\times\Omega)$ vanishes for $t\le -t_0$, $G(t,x)\in C_0^\infty(\mathbb{R}\times\partial K)$ and t_0 is chosen so that

$$\operatorname{supp}_t\delta\big(t-\langle x,\omega\rangle|_{x\in\partial K}\big)\subset\{t:|t|\le t_0\}.$$

Taking a partition of unity $\{\psi_j(t,x)\}_{j=1}^M$ on $[-t_0,t_0]\times\partial K$, we pass to the analysis of the solutions of the localized problems

$$\begin{cases}(\partial_t^2-\Delta)u_j=F_j(t,x) & \text{in } \mathbb{R}\times\mathring{\Omega},\\ u_j+\psi_j\delta(t-\langle x,\omega\rangle)=G_j(t,x) & \text{on } \mathbb{R}\times\partial K\ ,\end{cases}\tag{2–2}$$

with $F_j(t,x)\in C^\infty(\mathbb{R}\times\Omega)$, $G_j(t,x)\in C_0^\infty(\mathbb{R}\times\partial K)$ and $F_j=0$ for $t\le t_0$. Then using the decay of local energy for strictly convex obstacles we get

$$\frac{\partial v_s}{\partial\nu}\bigg|_{\partial K}=\sum_{j=1}^M\int e^{-i\lambda t}\,\frac{\partial u_j(t,x)}{\partial\nu}\bigg|_{\mathbb{R}\times\partial K}dt+O(|\lambda|^{-N})\quad\text{for all } N.$$

The results on the propagation of the wave front set $WF(u_j)$ of the solutions of (2–2) (see [30], [17]) say that

$$WF\Big(\frac{\partial u_j}{\partial \nu}\Big|_{\mathbb{R}\times\partial K}\Big) \subset WF\big(\psi_j\delta(t-\langle x,\omega\rangle)|_{\mathbb{R}\times\partial K}\big). \tag{2–3}$$

In the case when $\operatorname{supp}\psi_j \cap \big(\mathbb{R}\times G(\omega)\big)=\varnothing$ the above relation follows from the pseudo-local property of pseudo-differential operators [10] since we have, modulo smooth terms, the representation

$$\frac{\partial u_j}{\partial \nu}\Big|_{\mathbb{R}\times\partial K} = -B_j\big(\psi_j\delta(t-\langle x,\omega\rangle)|_{\mathbb{R}\times\partial K}\big),$$

B_j being a first order pseudo-differential operator. In the case where $\operatorname{supp}\psi_j$ overlaps with $\mathbb{R}\times G(\omega)$ we apply the results of Taylor [30] and Melrose [17] for diffraction problems. Thus we are going to study the expression

$$\sum_j \iint_{\partial K} e^{-i\lambda(t-\langle x,\theta\rangle)}\frac{\partial u_j}{\partial \nu}\,dt\,dS_x, \tag{2–4}$$

where the integral is interpreted in the sense of distributions. From the definition of the wave front it is easy to see that the condition

$$(t,y',d_t\Phi,d'_y\Phi)\cap WF(u)=\varnothing \quad \text{for } y'\in D\subset\mathbb{R}^{n-1}$$

implies

$$\int_{\mathbb{R}}\int_D e^{-i\lambda\Phi(y',t)}u(y',t)\,dt\,dy' = O(|\lambda|^{-N}) \quad \text{for all } N.$$

In order to exploit this property, assume that in local coordinates $U_j\cap\partial K$ is given by

$$y_n=g(y'),\quad y'=(y_1,\dots,y_{n-1})\in D\subset\mathbb{R}^{n-1}.$$

Then (2–3) yields

$$WF\Big(\frac{\partial u_j}{\partial \nu}\Big|_{\mathbb{R}\times\partial K}\Big) \subset \big\{(t,y,\tau,\xi)\in T^*(\mathbb{R}\times\partial K): t=\langle y,\omega\rangle \\ \text{with } y\in\operatorname{supp}\psi_j\big(y,\langle y,\omega\rangle\big) \text{ and } (\xi,\tau)=\pm\big(-\omega'-\nabla g(y')\omega_n,1\big)\big\}.$$

Clearly, for the phase function $\Phi=t-\langle y,\theta\rangle|_{y\in U_j\cap\partial K}$ we have

$$d_{y',t}\Phi=(-\theta'-\nabla g(y')\theta_n,1),$$

which coincides with the directions of the wave front of $(\partial u_j/\partial\nu)\big|_{\mathbb{R}\times\partial K}$ only in the case

$$-\omega'-\nabla g(y')\omega_n=-\theta'-\nabla g(y')\theta_n.$$

Thus we deduce immediately

$$\frac{\theta-\omega}{|\theta-\omega|}=\pm\nu(y',g(y')).$$

The assumption $\theta \neq \omega$ implies that for $y \in G(\omega)$ the last condition is impossible. Moreover, the same argument shows that $\operatorname{supp} \psi_j(y, \langle y, \omega\rangle)$ must be included in a small neighborhood $U_\pm$ of $x_\pm$ with $\psi_j(y, \langle y, \omega\rangle) = 1$ in a neighborhood of $x_\pm$.

Since x_- lies in the shadow region, we have $\langle \nu(x_-), \omega\rangle > 0$ and the solution of the wave equation which is smooth for $t < 0$ in a small neighborhood of $(\langle x_-, \omega\rangle, x_-)$ has the form $u_- = -\delta(t - \langle x, \omega\rangle)$. Thus we obtain

$$\frac{\partial v_s}{\partial \nu}\bigg|_{U_- \cap \partial K} = i\lambda \langle \nu, \omega\rangle e^{-i\lambda\langle x, \omega\rangle}|_{U_- \cap \partial K},$$

and replacing $(\partial v_s/\partial \nu)|_{U_- \cap \partial K}$ in expression (2–4), we see that the shadow region makes no contribution to $a(\lambda, \theta, \omega)$ because

$$\langle \nu(x_-), \theta + \omega\rangle = 0.$$

Passing to the illuminated region, denote by ψ_+ and B_+ the cutoff function and the pseudo-differential operator related to U_+. Then for the formally adjoint operators B_+^* we obtain

$$-\iint_{U_+} B_+^*\big(e^{-i\lambda(t - \langle y', \theta'\rangle - g(y')\theta_n)}\big)\psi_+ \delta\big(t - \langle y', \omega'\rangle - g(y')\omega_n\big)\big(1 + |\nabla g(y')|^2\big)^{1/2} dt dy'$$
$$= -\lambda \int_{U_+} e^{i\lambda(\langle y', \theta' - \omega'\rangle + g(y')(\theta_n - \omega_n))} b_+(y', \theta) dy' + O(1)$$

with

$$b_+(y', \theta) = -i\beta_+\big(y', -1, \theta' + \nabla g(y')\theta_n\big)\big(1 + |\nabla g(y')|^2\big)^{1/2},$$

$i\beta_+$ being the principal symbol of B_+. Thus our task is reduced to the study of an integral having the same form as $I(\lambda)$.

Without loss of generality we can assume that $\nabla g(x'_+) = 0$. From the construction of the asymptotic solution in a neighborhood of x_+ we obtain

$$\beta_+(x'_+, -1, \theta') = \langle \nu(x_+), \theta\rangle > 0$$

and we conclude that

$$\frac{1}{2}\Big(\frac{i\lambda}{2\pi}\Big)^{(n-1)/2} \int_{U_+} e^{i\lambda(\langle y', \theta' - \omega'\rangle + g(y')(\theta_n - \omega_n))} b_+(y', \theta) dy'$$
$$= \tfrac{1}{2} e^{i\lambda\langle x_+, \theta - \omega\rangle} \mathcal{K}(x_+)^{-1/2} \frac{\langle \nu(x_+), \theta\rangle}{|\theta - \omega|^{(n-1)/2}} + O(|\lambda|^{-1}).$$

Taking the sum of all contributions, one gets

$$a(\lambda, \theta, \omega) = e^{i\lambda\langle x_+, \theta - \omega\rangle} \mathcal{K}(x_+)^{-1/2} \langle \nu(x_+), \theta\rangle |\theta - \omega|^{(1-n)/2} + O(|\lambda|^{-1}).$$

Finally, in the illuminated region we have

$$\frac{\langle \nu(x_+), \theta\rangle}{|\theta - \omega|} = \frac{\langle \theta - \omega, \theta\rangle}{|\theta - \omega|^2} = \frac{1}{2}$$

and

$$a(\lambda,\theta,\omega) = \tfrac{1}{2}e^{i\lambda\langle x_+,\theta-\omega\rangle}\mathcal{K}(x_+)^{-1/2}|\theta-\omega|^{(3-n)/2} + O(|\lambda|^{-1}).$$

Thus from the limit

$$|a(\omega,\theta)| = \lim_{\lambda\to\infty} |a(\lambda,\omega,\theta)|$$

we can determine the Gauss curvature $\mathcal{K}(x_+)$ at x_+. When (ω,θ) runs over a set

$$V \subset \mathbf{S}^{n-1}\times\mathbf{S}^{n-1} \setminus \{(\omega,\omega) : \omega\in\mathbf{S}^{n-1}\}\ ,$$

we can recover the Gauss curvature $\mathcal{K}(y)$ at every point $y\in\partial K$, provided the map

$$V \ni (\omega,\theta) \to \frac{\theta-\omega}{|\theta-\omega|} \in \mathbf{S}^{n-1}$$

is onto. On the other hand, the knowledge of the Gauss curvature at all points of ∂K determines uniquely ∂K (see [14] for more details).

The case $\omega = \theta$ is more complicated since the singularities associated to diffracted rays must be taken into account. See [19] and [31] for results in this direction.

3. Singularities of the Scattering Kernel

Throughout this section we assume that $\theta \neq \omega$. To study the general case of nonconvex obstacles it is more convenient to consider the *scattering kernel* $s(t,\theta,\omega)$ defined as the Fourier transform of the scattering amplitude:

$$s(t,\theta,\omega) = \mathcal{F}_{\lambda\to t}\left(\left(\frac{\lambda}{2\pi i}\right)^{(n-1)/2}\overline{a(\lambda,\theta,\omega)}\right),$$

where $(\mathcal{F}_{\lambda\to t}\varphi)(t) = (2\pi)^{-1}\int e^{it\lambda}\varphi(\lambda)\,d\lambda$ for functions $\varphi\in\mathcal{S}(\mathbb{R})$. Let $V(t,x;\omega)$ be the solution of the problem

$$\begin{cases} (\partial_t^2-\Delta)V = 0 & \text{in } \mathbb{R}\times\mathring{\Omega}, \\ V+\delta(t-\langle x,\omega\rangle) = 0 & \text{on } \mathbb{R}\times\partial K, \\ V|_{t<-t_0} = 0. \end{cases}$$

Then we have

$$s(\sigma,\theta,\omega) = (-1)^{(n+1)/2}2^{-n}\pi^{1-n}\int_{\partial K}\partial_t^{n-2}\partial_\nu V(\langle x,\theta\rangle-\sigma,x;\omega)\,dS_x,$$

where the integral is interpreted in the sense of distributions. Our aim will be to examine the singularities of $s(t,\theta,\omega)$ with respect to t.

First we define the so-called reflecting (ω,θ)-rays. Given two directions θ,ω in $\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$, consider a curve $\gamma\in\Omega$ having the form

$$\gamma = \bigcup_{i=0}^{m} l_i, \qquad m\geq 1,$$

where $l_i = [x_i, x_{i+1}]$ are finite segments for $i = 1, \dots, m-1$, $x_i \in \partial K$, and l_0 (resp. l_m) is the infinite segment starting at x_1 (resp. at x_m) and having direction $-\omega$ (resp. θ). The curve γ is called a *reflecting* (ω, θ)-ray in Ω if for $i = 0, 1, \dots, m-1$ the segments l_i and l_{i+1} satisfy the law of reflection at x_{i+1} with respect to ∂K. The points $x_1, \dots, x_m$ are called *reflection points* of γ and this ray is called *ordinary reflecting* (or *simply reflecting*) if γ has no segments tangent to ∂K.

Next, we define two important notions related to (ω, θ)-rays (also-called *scattering rays*). Fix an arbitrary open ball U_0 with radius $a > 0$ containing K. For $\xi \in \mathbf{S}^{n-1}$ introduce the hyperplane Z_ξ orthogonal to ξ and such that ξ is pointing into the interior of the open half space H_ξ with boundary Z_ξ containing U_0. Let $\pi_\xi : \mathbb{R}^n \to Z_\xi$ be the orthogonal projection. For a reflecting (ω, θ)-ray γ in Ω with successive reflecting points $x_1, \dots, x_m$ the *sojourn time* T_γ of γ is defined by

$$T_\gamma = \|\pi_\omega(x_1) - x_1\| + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| + \|x_m - \pi_{-\theta}(x_m)\| - 2a.$$

Obviously, $T_\gamma + 2a$ coincides with the length of this part of γ which lies in $H_\omega \cap H_{-\theta}$ (see Figure 2). In fact, the sojourn time T_γ does not depend on the choice of the ball U_0 since it follows easily that

$$\|\pi_\omega(x_1) - x_1\| = a + \langle x_1, \omega\rangle, \|x_m - \pi_{-\theta}(x_m)\| = a - \langle x_m, \theta\rangle\ ,$$

therefore

$$T_\gamma = \langle x_1, \omega\rangle + \sum_{i=1}^{m-1} \|x_i - x_{i+1}\| - \langle x_m, \theta\rangle.$$

Given an ordinary reflecting (ω, θ)-ray γ set $u_\gamma = \pi_\omega(x_1)$. There exists a small neighborhood W_γ of u_γ in Z_ω such that for every $u \in W_\gamma$ there are a unique direction $\theta(u) \in \mathbf{S}^{n-1}$ and points $x_1(u), \dots, x_m(u)$ which are the successive reflection points of a reflecting $(u, \theta(u))$-ray in Ω with $\pi_\omega(x_1(u)) = u$. This defines a smooth map

$$J_\gamma : W_\gamma \ni u \to \theta(u) \in \mathbf{S}^{n-1}$$

and $dJ_\gamma(u_\gamma)$ is called a *differential cross section* related to γ. We say that γ is *nondegenerate* if

$$\det dJ_\gamma(u_\gamma) \neq 0.$$

The notion of sojourn time as well as that of differential cross section are well known in the physical literature. The definitions given above are due to Guillemin [5].

For strictly convex obstacles all (nontrivial) reflecting rays have only one reflection point x_1 and the corresponding sojourn time is equal to $\langle x_1, \omega - \theta\rangle$.

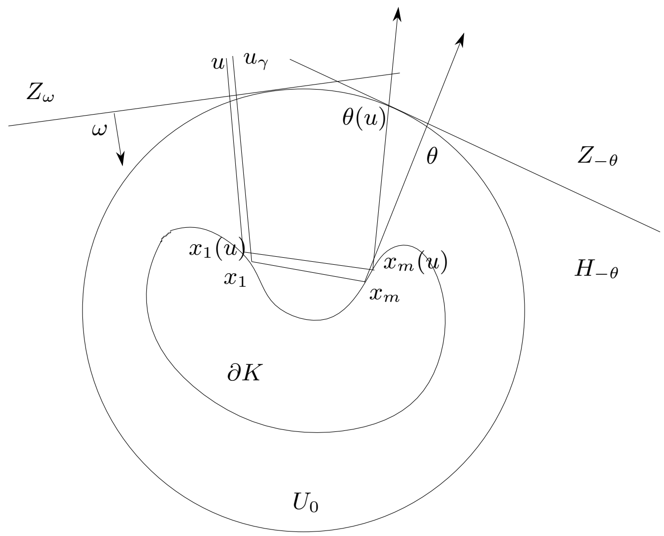

Figure 2.

Moreover, the stationary phase argument of the previous section implies that $\overline{a(\lambda,\omega,\theta)}$ has a complete asymptotic expansion

$$\overline{a(\lambda,\omega,\theta)} = e^{i\langle x_+,\omega-\theta\rangle}\sum_{j=0}^{N} c_j\lambda^{-j} + O(|\lambda|^{-N-1}) \quad \text{for all } N\in\mathbb{N},$$

which gives

$$\operatorname{sing\,supp} s(t,\theta,\omega) = \{-T_+\},$$

$T_+ = \langle x_+,\omega-\theta\rangle$ being the sojourn time of the (ω,θ)-ray γ_+ reflecting at x_+. A simple geometric argument implies that

$$|\det dJ_{\gamma_+}(u_{\gamma_+})| = 4|\theta-\omega|^{(n-3)}\mathcal{K}(x_+),$$

and for t close to $-T_+$ we have

$$s(t,\theta,\omega) = \left(\frac{-1}{2\pi}\right)^{(n-1)/2}\left|dJ_{\gamma_+}(u_{\gamma_+})\right|^{-1/2}\delta^{(n-1)/2}(t+T_+) + \text{l.o.s.}$$

(the abbrevation stands for "lower order singularities").

For strictly convex obstacles T_+ is an isolated singularity of $s(t,\theta,\omega)$ related to an ordinary reflecting ray. This situation can be generalized for generic obstacles if we consider the back scattering direction $\theta=-\omega$. Without loss of the generality we may assume that K lies in the half space $\{x\in\mathbb{R}^n : \langle x,\omega\rangle > 0\}$. Then the function

$$\partial K \ni x \to \langle x,\omega\rangle \in \mathbb{R}^+$$

has a positive minimum $\rho(\omega)$ and there exists at least one reflecting $(\omega,-\omega)$-ray γ with sojourn time $T_\gamma = 2\rho(\omega)$. Of course we could have many $(\omega,-\omega)$-rays with the same minimal sojourn time. A geometric argument based on Sard's theorem shows that there exists a subset $\mathcal{B} \subset \mathbf{S}^{n-1}$ with full measure such that for every $\omega \in \mathcal{B}$ we have only a finite number of reflecting (ω, θ)-rays with sojourn time $2\rho(\omega)$. Moreover, each of these rays $\gamma_1, \dots, \gamma_M$, has only one reflection point $x_k \in \partial K, k = 1, \dots, M$, and ∂K has a nonvanishing Gauss curvature $\mathcal{K}(x_k) \neq 0$ for every $k = 1, \dots, M$. Thus, repeating the argument from Section 2, it follows that for $\omega \in \mathcal{B}$ the sojourn time $T = -2\rho(\omega)$ is an isolated singularity of the scattering kernel $s(t, -\omega, \omega)$, and for such ω we have

$$\max \operatorname{sing} [\operatorname{supp}_t s(t, -\omega, \omega)] = -2\rho(\omega),$$

and for t close to $-2\rho(\omega)$,

$$s(t, -\omega, \omega) = 2^{1-n} \Big(-\frac{1}{\pi}\Big)^{(n-1)/2} \sum_{k=1}^{M} \big|\mathcal{K}(x_k)\big|^{-1/2} \delta^{(n-1)/2}(t + 2\rho(\omega)) + \text{l.o.s.}$$

This result is due to Majda [15]. From the maximal singularity of the back scattering kernel one obtains that the *convex hull* of the obstacle is given by

$$\hat{K} = \bigcap\nolimits_\omega \{x : \langle x, \omega \rangle \geq \rho(\omega)\}.$$

Thus one can recover the geometry of a convex obstacle.

It is much more complicated to get similar results in the case of nonconvex obstacles. Now the information obtained by means of rays having only one reflection is no longer sufficient. One needs to consider multiple reflecting (ω, θ)-rays leading to isolated singularities of $s(t, \theta, \omega)$. Roughly speaking, the singularities of the scattering kernel are amongst the sojourn times of (ω, θ)-rays, however now one has to consider not only simply reflecting (ω, θ)-rays but all generalized geodesics *incoming* with direction ω and *outgoing* with direction θ (see [22, Chapter 9] and [18]); these are simply called (ω, θ)-rays. In general, there exist (ω, θ)-rays with grazing or gliding segments (see Figure 3).

The precise definition of an (ω, θ)-ray is based on the notion of a generalized bicharacteristic of the operator $\Box = \partial_t^2 - \Delta_x$ given as trajectories of the generalized Hamilton flow $\mathcal{F}_t$ in Ω generated by the symbol $\sum_{i=1}^n \xi_i^2 - \tau^2$ of $\Box$ (see [20] for a precise definition). In general, $\mathcal{F}_t$ is not smooth and in some cases there may exist two different integral curves issued from the same point in the phase space (see [30] for an example). To avoid this situation we assume that the following generic condition is satisfied.

$(\mathcal{G})$ If for $(x, \xi) \in T^*(\partial K)$ the normal curvature of ∂K vanishes of infinite order in direction ξ, then ∂K is convex at x in direction ξ.

We will now sketch the definition of a generalized bicharacteristics of $\Box$. Let $p(x, \xi)$ be the restriction of the principal symbol of $\Box$ to the level surface $\tau = 1$ (this is the case of motion with unit speed along geodesics). Notice that in this case the so-called *zero bicharacteristic set* $\Sigma = p^{-1}(0)$ coincides with the

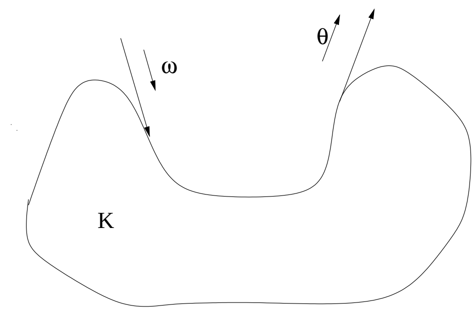

Figure 3.

cosphere bundle $S^*(\Omega)$ of Ω. Given a point $x \in \partial K$, we choose local coordinates

$$x = (x_1, \dots, x_n), \xi = (\xi_1, \dots, \xi_n)$$

in $T^*(\mathbb{R}^n)$ so that locally ∂K is given by $x_1 = 0$ and Ω by $x_1 \geq 0$. The coordinates (x, ξ) can be chosen so that, up to a nonzero smooth factor, $p(x, \xi)$ has the form

$$p(x, \xi) = \xi_1^2 - r(x, \xi')$$

with $x' = (x_2, \dots, x_n), \xi' = (\xi_2, \dots, \xi_n)$ and $r(x, \xi')$ homogeneous of order two in ξ'. Introduce the sets

$$\begin{aligned} \Sigma_0 &= \{(x, \xi) \in T^*(\mathbb{R}^n) \setminus \{0\} : x_1 > 0\}, \\ H &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') > 0\}, \\ G &= \{(x, \xi) \in \Sigma : x_1 = 0, r(0, x', \xi') = 0\}. \end{aligned}$$

The sets H and G are called *hyperbolic* and *glancing* set, respectively. Next consider the symbols

$$r_0(x', \xi') = r(0, x', \xi'), r_1(x', \xi') = \frac{\partial r}{\partial x_1}(0, x', \xi'),$$

and define the *diffractive* and *gliding* sets by

$$\begin{aligned} G_d &= \{(x, \xi) \in G : r_1(x', \xi') > 0\}, \\ G_g &= \{(x, \xi) \in G : r_1(x', \xi') < 0\}, \end{aligned}$$

respectively. The generalized bicharacteristics are related to the Hamilton vector fields

$$\begin{aligned} H_p &= \sum_{j=1}^{n} \Big(\frac{\partial p}{\partial \xi_j} . \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} . \frac{\partial}{\partial \xi_j} \Big), \\ H_{r_0} &= \sum_{j=2}^{n} \Big(\frac{\partial r_0}{\partial \xi_j} . \frac{\partial}{\partial x_j} - \frac{\partial r_0}{\partial x_j} . \frac{\partial}{\partial \xi_j} \Big). \end{aligned}$$

We have $d_\xi p(x, \xi) \neq 0$ on $S^*(\Omega)$ and $d_{\xi'} r_0(x', \xi') \neq 0$ on G. Moreover, the above definitions are independent on the choice of the local coordinates. Using the above local coordinates the generalized bicharecteristics of $\Box$ are defined as follows.

Let $I \subset \mathbb{R}$ be an open interval. A curve $\gamma : I \to S^*(\Omega)$ is called a *generalized bicharacteristic* of $\Box$ if there exists a discrete subset $B \subset I$ such that the following conditions hold:

(i) If $\gamma(t_0) \in \Sigma_0 \cup G_d$ for some $t_0 \in I \setminus B$, then γ is differentiable at t_0 and

$$\frac{d}{dt}\gamma(t_0) = H_p(\gamma(t_0)).$$

(ii) If $\gamma(t_0) \in G \setminus G_d$ for some $t_0 \in I \setminus B$, then

$$\gamma(t) = (x_1(t), x'(t), \xi_1(t), \xi'(t))$$

is differentiable at t_0 and

$$\frac{dx_1}{dt}(t_0) = \frac{d\xi_1}{dt}(t_0) = 0, \qquad \frac{d}{dt}(x'(t), \xi'(t))_{|t=t_0} = H_{r_0}(\gamma(t_0)).$$

(iii) If $t_0 \in B$, then $\gamma(t) \in \Sigma_0$ for all $t \neq t_0, t \in I$ with $|t - t_0|$ sufficiently small. Moreover, in this case for $\xi_1^{\pm}(x', \xi') = \pm\sqrt{r_0(x', \xi')}$ we have

$$\lim_{t \to t_0, \pm(t - t_0) > 0} \gamma(t) = \big(0, x'(t), \xi_1^{\pm}(x'(t_0)), \xi'(t_0)\big) \in H.$$

The functions $x(t)$, $\xi'(t)$, $|\xi_1(t)|$ are continuous on I, while the function $\xi_1(t)$ has a jump discontinuity at any point $t \in B$. Finally, under the condition $(\mathcal{G})$ a generalized bicharacteristic $\gamma : \mathbb{R} \to S^*(\Omega)$ of $\Box$ is *uniquely extendible* in the sense that for each $t \in \mathbb{R}$ the only generalized bicharacteristic (up to the change of parameter t) passing through $\gamma(t)$ is γ ([20]; see also [10, vol. III]).

More generally, working with the restriction of the principal symbol of $\Box$ to a level surface $\tau = \tau_0 \neq 0$, one defines generalized bicharacetristics on the set $\dot{T}^*(\Omega)$ of all $(x, \xi) \in T^*(\Omega)$ such that $\xi \neq 0$. Given $\sigma = (x, \xi) \in \dot{T}^*(\Omega)$, there exists a unique generalized bicharacteristic $(x(t), \xi(t)) \in \dot{T}^*(\Omega)$ such that $x(0) = x$ and $\xi(0) = \xi$. Set $\mathcal{F}_t(x, \xi) = (x(t), \xi(t))$ for all $t \in \mathbb{R}$. This defines a flow $\mathcal{F}_t : \dot{T}^*(\Omega) \to \dot{T}^*(\Omega)$ ([20]) which is sometimes called the *generalized geodesic flow* on $\dot{T}^*(\Omega)$. Obviously, it leaves the *cosphere bundle* $S^*(\Omega)$ invariant. At points of transversal reflection at $\dot{T}^*_{\partial K}(\Omega)$ the flow $\mathcal{F}_t$ is discontinuous. To make it continuous, consider the *quotient space* $\dot{T}^*_b(\Omega) = \dot{T}^*(\Omega)/\sim$ of $\dot{T}^*(\Omega)$ with respect to the following equivalence relation: $\rho \sim \sigma$ if and only if $\rho = \sigma$ or $\rho, \sigma \in T^*_{\partial K}(\Omega)$ and either $\lim_{t \nearrow 0} \mathcal{F}_t(\rho) = \sigma$ or $\lim_{t \searrow 0} \mathcal{F}_t(\rho) = \sigma$. Let $S^*_b(\Omega)$ be the image of $S^*(\Omega)$ in $\dot{T}^*_b(\Omega)$. Melrose and Sjöstrand ([20]) proved that the natural projection of $\mathcal{F}_t$ on $\dot{T}^*_b(\Omega)$ is continuous.

After these definitions a curve $\gamma = \{x(t) \in \Omega : t \in \mathbb{R}\}$ is called an (ω, θ)*-ray* if there exist real numbers $t_1 < t_2$ such that

$$\tilde{\gamma}(t) = (x(t), \xi(t)) \in S^*(\Omega)$$

is a *generalized bicharacteristic* of $\Box$ and

$$\xi(t) = \begin{cases} \omega & \text{for } t \le t_1, \\ \theta & \text{for } t \ge t_2, \end{cases}$$

provided that the time t increases when we move along $\tilde{\gamma}$. Denote by $\mathcal{L}_{\omega,\theta}(\Omega)$ the set of all (ω,θ)-rays in Ω. The *sojourn time* T_δ of $\delta \in \mathcal{L}_{\omega,\theta}(\Omega)$ is defined as the length of the part of δ lying in $H_\omega \cap H_{-\theta}$.

Turning to the problem of the behavior of $s(t,\theta,\omega)$ near singularities, assume that γ is a fixed *nondegenerate ordinary reflecting* (ω,θ)-ray such that

$$T_\gamma \ne T_\delta \quad \text{for every } \delta \in \mathcal{L}_{\omega,\theta}(\Omega) \setminus \{\gamma\}. \tag{3–1}$$

By using the continuity of the generalized Hamilton flow, it is easy to show that

$$(-T_\gamma - \varepsilon, -T_\gamma + \varepsilon) \cap \operatorname{sing\,supp} s(t,\theta,\omega) = \{-T_\gamma\}$$

for $\varepsilon > 0$ sufficiently small. The singularity of $s(t,\theta,\omega)$ at $t = -T_\gamma$ can be investigated using a global construction of an asymptotic solution as a Fourier integral operator ([6], [21], Chapter 9 in [22]).

Theorem 3.1 [21]. *Under the assumption* (3–1) *we have*

$$-T_\gamma \in \operatorname{sing\,supp} s(t,\theta,\omega)$$

and for t close to $-T_\gamma$ the scattering kernel has the form

$$s(t,\theta,\omega) = \left(\frac{1}{2\pi i}\right)^{(n-1)/2} (-1)^{m_\gamma - 1} \exp\left(i\frac{\pi}{2}\beta_\gamma\right) \times \left|\frac{\det dJ_\gamma(u_\gamma)\langle \nu(q_1),\omega\rangle}{\langle \nu(q_m),\theta\rangle}\right|^{-1/2} \delta^{(n-1)/2}(t+T_\gamma) + \text{l.o.s.}, \tag{3–2}$$

where m_γ is the number of reflections of γ and q_1, q_m are the first and last reflection points, *respectively*, *of γ and $\beta_\gamma \in \mathbb{Z}$.*

For strictly convex obstacles we have $\beta_\gamma = -(n-1)/2$, $q_1 = q_m$ and $\theta - \omega$ is parallel to $\nu(q_1)$.

4. Properties of Reflecting (ω, θ)-Rays

To apply the result of the previous section we need the condition (3–1) and it is desirable to prove that there exists a subset $\mathcal{S} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ with zero Lebesgue measure such that for all directions $(\omega,\theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \setminus \mathcal{S}$ the corresponding (ω,θ)-rays satisfy (3–1). Here one has to deal with all (generalized) (ω,θ)-rays and this makes the problem rather difficult. We start with a result concerning the *ordinary reflecting* (ω,θ)-rays only.

Theorem 4.1 [23]. *For every $\omega \in \mathbf{S}^{n-1}$ there exists a set $S(\omega) \subset \mathbf{S}^{n-1}$ the complement of which is a countable union of compact subsets of $\mathbf{S}^{n-1}$ of measure zero such that if $\theta \in S(\omega)$, then any two different ordinary reflecting (ω,θ)-rays in Ω have distinct sojourn times.*

Sketch of proof. Let U_0 be an open ball with center 0 and radius a containing K and let $Z = Z_\omega$ be the hyperplane introduced in Section 3. Given an integer $k \geq 1$, denote by U_k the set of those $u \in Z$ for which the trajectory $\gamma(u)$ of the generalized Hamiltonian flow starting in u with direction ω is an ordinary reflecting ray with exactly k reflection points. Let $J_k(u) \in \mathbf{S}^{n-1}$ be the direction of $\gamma(u)$ after the last reflection. Obviously, U_k is open in Z and the map

$$J_k : U_k \ni z \to J_k(u) \in \mathbf{S}^{n-1}$$

is smooth.

Now let us fix two arbitrary integers $k \geq 1, s \geq 1$. For $u \in U_k$ denote by $f(u)$ the sojourn time of the scattering ray determined by $\gamma(u)$. In the same way denote by $g(v)$ the sojourn time of the scattering ray with s reflections determined by $v \in V_s$. The functions $f : U_k \to \mathbb{R}, g : V_s \to \mathbb{R}$ are smooth.

For $u \in U_k$ denote by $x_1(u), \dots, x_k(u)$ the successive reflection points of $\gamma(u)$. The corresponding maps $x_i : U_k \to \partial K$ are smooth and for every $y \in \partial K$ we denote by $N(y)$ the unit normal to ∂K pointing into Ω. Thus for $u \in U_k$ we obtain

$$J_k(u) = \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|} - 2\left\langle \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|}, N(x_k(u))\right\rangle N(x_k(u)),$$

and

$$f(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| + t - 2a,$$

with the convention that $x_0(u)$ (resp. $x_{k+1}(u)$) denotes the orthogonal projection of $x_1(u)$ (resp. $x_k(u)$) on Z (resp. $Z_{-J_k(u)}$), and where $t = \|x_k(u) - x_{k+1}(u)\|$. We obtain easily $t = a - \langle J_k(u), x_k \rangle$, so

$$f(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| - \langle x_k(u), J_k(u)\rangle - a.$$

For $v \in V_s$ the successive reflection points of $\gamma(v)$ will be denoted by $y_1(v), \dots,$ $y_s(v)$. Next we set $y_0(v) = v$ and we define $y_{s+1}(v)$ in the same way as $x_{k+1}(u)$. Now denote by $W(k,s)$ the set of those $(u,v) \in U_k \times V_s$ for which

$$J_k(u) = J_s(v), f(u) = g(v)$$

and

$$\operatorname{rank} dJ_k(u) = \operatorname{rank} dJ_s(v) = n - 1.$$

Lemma 4.2. *$W(k,s)$ is a smooth $(n-2)$-dimensional submanifold of $U_k \times U_s$.*

Proof of Lemma 4.2. Consider a point $w_0 = (u_0, v_0) \in W(k,s)$. Since $\operatorname{rank} dJ_k(u_0) = \operatorname{rank} dJ_s(v_0) = n-1$, there exists a neighborhood U of w_0 in $U_k \times V_s$ such that for every $(u,v) \in U$ we have

$$\operatorname{rank} dJ_k(u) = \operatorname{rank} dJ_s(v) = n - 1.$$

Define the map $L : U \to \mathbb{R}^n$ by

$$L(u,v) = \big(\lambda(u,v),\ (\chi^{(j)}(u,v))_{1\le j\le n-1}\big)$$

with

$$\lambda(u,v) = f(u) - g(v), \chi(u,v) = J_k(u) - J_s(v).$$

Clearly, $W(k,s) \cap U \subset L^{-1}(0)$ and to prove that $W(k,s)$ is a smooth $(n-2)$-dimensional submanifold of $U_k \times V_s$ it is sufficient to show that L is a submersion at any point w_0 of $L^{-1}(0)$. For this purpose we assume without loss of generality that $\theta_n \neq 0$. Suppose that

$$\sum_{j=1}^{n-1} A_j \operatorname{grad} \chi^{(j)}(w_0) + C \operatorname{grad} \lambda(w_0) = 0$$

with some constants A_j, C. Calculating the derivatives involved above and using the geometrical meaning of f, g, J_k and J_s, one derives $A_1 = \cdots = A_{n-1} = C = 0$. Thus L is a submersion at w_0. See [23] for more details. □

Consider the map $\varphi : U_k \times V_s \to \mathbf{S}^{n-1}$ given by $\varphi(u,v) = J_k(u)$. This map is smooth and dim $W(k,s) = n-2$ shows that $\varphi(W(k,s))$ is a countable union of compact subsets of $\mathbf{S}^{n-1}$ of measure zero. Clearly

$$F_k = \{u \in U_k : \operatorname{rank} dJ_k(u) \le n-2\}$$

is a countable union of compact subsets. By Sard's theorem, $J_k(F_k)$ has measure zero in $\mathbf{S}^{n-1}$ for all k, so $F = \bigcup_k J_k(F_k)$ also has measure zero in $\mathbf{S}^{n-1}$. Hence the subset

$$S(\omega) = \mathbf{S}^{n-1} \setminus \big(F \cup \textstyle\bigcup_{k,s} J_k(W(k,s))\big)$$

of $\mathbf{S}^{n-1}$ has the desired properties. This concludes the proof of Theorem 4.1. □

Setting $\mathcal{S} = \{(\omega,\theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} : \theta \in \mathcal{S}(\omega)\}$, we see that for $(\omega,\theta) \in \mathcal{S}$ any two different ordinary reflecting rays in Ω have distinct sojourn times and the complement of $\mathcal{S}$ in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ has measure zero.

To deal with reflecting rays with tangent segments, we introduce a more general type of trajectories. A curve γ in $\mathbb{R}^n$ is called an (ω,θ)-*trajectory* for Ω if it has the form $\gamma = \bigcup_{i=0}^{s} l_i$, where $l_i = [x_i, x_{i+1}]$, i ranges from 1 through $s-1$, $x_i \in \partial K$ for $i = 1, \dots, s$, while l_0 (resp. l_s) is the infinite ray starting at x_1 (resp. x_s) with direction $-\omega$ (resp. θ) and, for every $i = 0, 1, \dots, s-1$, l_i and l_{i+1} satisfy the law of reflection at x_i with respect to ∂K. It is clear that every reflecting (ω,θ)-ray is an (ω,θ)-trajectory, but the converse is not true in general since some (ω,θ)-trajectory may intersect transversally ∂K. On the other hand, every (ω,θ)-reflecting ray with tangent segment is an (ω,θ)-trajectory. We have the following.

Theorem 4.3 [23]. *There exists $\mathcal{T} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{T}$ all (ω,θ)-trajectories for Ω are ordinary.*

PROOF. We follow the idea of the proof of Theorem 4.1. For simplicity set $\partial K = X$. Fix two integers k and s so that $s \geq 1, 0 \leq k \leq s$. Let $M(s,k)$ be the set of those

$$\zeta = (\omega; x; y; \theta) \in M_s = \mathbf{S}^{n-1} \times X^{(s)} \times X \times \mathbf{S}^{n-1}$$

with $x = (x_1, \dots, x_s)$ such that there exists an (ω, θ)-trajectory for X with successive transversal reflection points $x_1, \dots, x_s$, the segment $[x_k, x_{k+1}]$ of which is tangent to X at $y \in (x_k, x_{k+1})$. Here

$$X^{(s)} = \{(x_1, \dots, x_s) \in X^s : x_i \neq x_j, i \neq j\}$$

and x_0 (resp. x_{s+1}) is the orthogonal projection of x_1 on Z_ω (resp. of x_s on $Z_{-\theta}$).

The main step in the proof is to show that $M(s,k)$ is a smooth submanifold of M_s of dimension $2n-3$. This follows from a specially adapted parametrization of $M(s,k)$; see [23] for details. Using this one obtains Theorem 4.3 easily. Consider the projection

$$\pi_s : M_s = \mathbf{S}^{n-1} \times X^{(s)} \times X \times \mathbf{S}^{n-1} \to \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$$

given by

$$\pi_s(\omega; x; y; \theta) = (\omega, \theta),$$

and introduce the open subsets of M_s

$$U_r(s,k) = \{(\omega; x; y; \theta) \in M_s : x_k^{(r)} \neq x_{k+1}^{(r)}\}, r = 1, \dots, n.$$

Then $M_r(s,k) = M(s,k) \cap U_r(s,k)$ is a smooth submanifold of M_s of dimension $2n-3 < \dim(\mathbf{S}^{n-1} \times \mathbf{S}^{n-1})$. Since π_s is smooth, the set

$$L_r(s,k) = \pi_s\big(M_r(s,k)\big) \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$$

has measure zero. Consequently, for the covering $M_r(s,k) = \bigcup_{j=1}^\infty K_j$ with K_j compact, one gets that

$$L_r(s,k) = \bigcup_{j=1}^\infty \pi_s(K_j)$$

is a countable union of compact subsets of $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ of measure zero. Setting

$$\mathcal{T} = \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} \setminus \bigcup_{0 \leq k \leq s} \bigcup_{r=1}^\infty L_r(s,k),$$

completes the proof of Theorem 4.3. □

Finally, we find a subset $\mathcal{U} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega, \theta) \in \mathcal{T} \cap \mathcal{U}$ all reflecting (ω, θ)-rays are ordinary and nondegenerate. So there exists a subset $\mathcal{A} = \mathcal{T} \cap \mathcal{U} \cap \mathcal{S}$ of $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ of full measure so that for every $(\omega, \theta) \in \mathcal{A}$ the corresponding (ω, θ)-reflecting rays are ordinary, nondegenerate and with distinct sojourn times.

The study of the generalized (ω, θ)-rays leads to many difficulties. However it is quite natural to expect that for almost all (ω, θ) in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ there are no generalized (ω,θ)-rays different from reflecting ones. This will be discuss in details in the next section.

5. Poisson Relation for the Scattering Kernel

Let K be an obstacle in $\mathbb{R}^n, n \geq 3, n$ odd, with C^∞ boundary ∂K so that

$$K \subset \{x \in \mathbb{R}^n : |x| \leq \rho_0\}$$

and let $\Omega = \overline{\mathbb{R}^n \setminus K}$. In what follows we assume that K satisfies the condition $(\mathcal{G})$ from Section 3. Let $\pi : T^*(\mathbb{R}\times\Omega) \to \Omega$ be the natural projection.

The following result of [21], [1] (see also [22, Chapter 8] and [18]) shows that for $\omega \neq \theta$ all singularities in t of $s(t,\theta,\omega)$ are given by (negative) sojourn times.

Theorem 5.1 [21], [1]. *For $\omega \neq \theta$ we have*

$$\text{sing supp } s(t,\theta,\omega) \subset \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}. \tag{5–1}$$

In analogy with the well-known Poisson relation for the Laplacian on Riemannian manifolds, (5–1) is called the *Poisson relation for the scattering kernel*, while the set of all T_γ, where $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$, $(\omega,\theta) \in \mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$, is called the *scattering length spectrum* of K.

Sketch of proof. The proof uses results on the propagation of singularities along generalized bicharactaristics, and some properties of oscillatory integrals. Consider a fixed t_0 so that

$$-t_0 \notin \{-T_\gamma : \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}.$$

Take $T > 0$ with $|t_0| < T$ and introduce the set

$$\Gamma_T = \{T_\gamma : |T_\gamma| \leq T, \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega)\}.$$

The continuity of the generalized Hamiltonian flow implies that Γ_T is closed, so we can choose $\varepsilon_0 > 0$ so that

$$T_\gamma \notin [t_0-\varepsilon_0,\ t_0+\varepsilon_0] \quad \text{for all } \gamma \in \mathcal{L}_{(\omega,\theta)}(\Omega).$$

Let $\rho(t) \in C_0^\infty(\mathbb{R}), \rho(t) = 1$ for $|t| \leq 1/2, \rho(t) = 0$ for $|t| \geq 1$. Set $\rho_\delta(t) = \rho(t/\delta)$ for $0 < \delta \leq \varepsilon_0/2$. To prove that $t_0 \notin \text{sing supp}\, s(t,\theta,\omega)$, it is sufficient to show that the integral

$$\begin{aligned} J(\lambda) &= \left\langle s(t,\theta,\omega),\ \rho_\delta(t+t_0)e^{-i\lambda t}\right\rangle \\ &= \sum_{k=0}^{n-2} c_k(-i\lambda)^{n-2-k}\int_{\mathbb{R}}\int_{\partial\Omega} e^{i\lambda(t-\langle x,\theta\rangle)}\frac{d^k\rho_\delta}{dt^k}(\langle x,\theta\rangle - t + t_0)\frac{\partial w}{\partial\nu}(t,x;\omega)\,dt\,dS_x, \end{aligned}$$

with c_k a constant, is rapidly decreasing with respect to λ. Here $w(t,x;\omega) = V(t,x;\omega)+\delta(t-\langle x,\omega\rangle)$, where $V(t,x;\omega)$ is defined in Section 3. Let us treat the term with $k=0$, the other ones can be examined by a similar argument.

Without loss of generality we may assume that $\omega = (0, \dots, 0, 1)$. Set

$$Z(\tau) = \{x \in \mathbb{R}^n : x_n = \tau\},$$

where $\tau < -\rho_0$ and let $\mathbb{R}^+_\tau = \{t \in \mathbb{R} : t > \tau\}$. To localize the problem, introduce a partition of unity on $Z(\tau)$ given by functions

$$\varphi_j(x') \in C_0^\infty(\mathbb{R}^{n-1}), x' = (x_1, \dots, x_{n-1}).$$

Consider the problems

$$\begin{cases} \Box\, v_j = 0 \quad \text{in } \mathbb{R}^+_\tau \times \mathbb{R}^n_x, \\ v_j(\tau, x) = \varphi_j(x')\delta(\tau - x'), \\ \dfrac{\partial v_j}{\partial t}(\tau, x) = \varphi_j(x')\delta'(\tau - x_n), \end{cases} \qquad \begin{cases} \Box\, W_j = 0 \quad \text{in } \mathbb{R} \times \mathring{\Omega}, \\ W_j = 0 \qquad \text{on } \mathbb{R} \times \partial\Omega, \\ W_j(\tau, x) = \varphi_j(x')\delta(\tau - x'), \\ \dfrac{\partial W_j}{\partial t}(\tau, x) = \varphi_j(x')\delta'(\tau - x_n). \end{cases}$$

Clearly, there exists a compact set $F_0' \subset \mathbb{R}^{n-1}$ such that if $\operatorname{supp} \varphi_j \cap F_0' = \varnothing$, then the straight lines issued from $(x', \tau), x' \in \operatorname{supp} \varphi_j$, with direction ω do not meet $\partial\Omega$. For such j and $\omega \neq \theta$ we have

$$WF\left(\left(\frac{\partial W_j}{\partial \nu} \right)_{|\mathbb{R} \times \partial\Omega} \right) \cap \left\{ (t, x, 1, -\theta_{|T_x(\partial\Omega)}) : |t| \leq T + \rho_0 + 1, x \in \partial\Omega \right\} = \varnothing. \tag{5–2}$$

This implies easily

$$\int_{\mathbb{R}} \int_{\partial\Omega} e^{i\lambda(t - \langle x, \theta \rangle)} \rho_\delta(\langle x, \theta \rangle - t + t_0) \frac{\partial W_j}{\partial \nu} \, dt \, dS_x = O(|\lambda|^{-m}) \quad \text{for all } m \in \mathbb{N}. \tag{5–3}$$

Now set $F_0 = \{x \in \mathbb{R}^n : x' \in F_0', x_n = \tau\}$ and denote by $l(u_0)$ the straight line passing through $u_0 \in F_0$ with direction ω. There are three cases:

(i) $\varnothing \neq l(u_0) \cap \overline{K} \subset \partial\Omega$;
(ii) $l(u_0)$ meets transversally $\partial\Omega$ at $x_1(u_0)$;
(iii) $l(u_0)$ is tangent to $\partial\Omega$ at $x_1(u_0)$ and ω is an asymptotic direction for $\partial\Omega$ at $x_1(u_0)$.

In the case (i) the generalized bicharecteristic γ_0 with Im $(\pi \circ \gamma_0) = l(u_0)$ is uniquely extendible, and results on propagation of singularities lead to (5–2) which in turn gives (5–3). To deal with the case (ii), set $t_1(u) = |u - x_1(u)|$, $u \in F_0$. The solution v_j with such j is given by an oscillatory integral and $WF(v_j)$ is included in the set of all $(t, x, \pm\sigma, \mp\omega) \in T^*(\mathbb{R}^{n+1}) \setminus \{0\}$ such that $\sigma > 0$ and there exist $\hat{x} \in Z(\tau)$, $\hat{x}' \in \operatorname{supp}\varphi_j, s \geq 0$ with $t = \tau \pm \sigma s$, $x = \hat{x} \pm \sigma s \omega$. We modify v_j on the intersection of a small neighborhood of $x_1(u_0)$ with the interior of K so that the modified function $\tilde{v}_j$ satisfies (for some $\varepsilon > 0$) the properties

$$\tilde{v}_j = \begin{cases} v_j & \text{for } t < t_1 + \varepsilon, \\ 0 & \text{for } t > t_1 + 2\varepsilon. \end{cases}$$

Here $t_1 = \max\{t_1(u) : u \in O(u_0)\}$, where $O(u_0)$ is a sufficiently small neighborhood of u_0 with supp $\varphi_j \subset O(u_0)$ and ε is small enough. Moreover, we preserve the condition

$$\Box\, \tilde{v}_j = 0 \quad \text{in } \mathbb{R}^+_\tau \times \mathring{\Omega}.$$

Set $h_j = (\tilde{v}_j)_{|\mathbb{R}^+_\tau \times \partial\Omega}$ and notice that $h_j = 0$ for t sufficiently close to τ. We extend h_j as 0 for $t < \tau$ and consider the solution w_j of the problem

$$\begin{cases} \Box\, w_j = 0 & \text{in } \mathbb{R} \times \mathring{\Omega}, \\ w_j + h_j = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ w_j = 0 & \text{for } t < \tau. \end{cases}$$

We have $(\partial/\partial t)(w_j + \tilde{v}_j)_{|\mathbb{R}^+_\tau \times \partial\Omega} = 0$ and we are going to study the integrals

$$I_{j,\delta}(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{i\lambda(t - \langle x,\theta\rangle)} \rho_\delta\big(\langle x,\theta\rangle - t + t_0\big) \Big(\frac{\partial}{\partial\nu} - \langle \nu,\theta\rangle \frac{\partial}{\partial t}\Big) \tilde{v}_j \, dt\, dS_x,$$

$$J_{j,\delta}(\lambda) = \int_{\mathbb{R}} \int_{\partial\Omega} e^{i\lambda(t - \langle x,\theta\rangle)} \rho_\delta\big(\langle x,\theta\rangle - t + t_0\big) \Big(\frac{\partial}{\partial\nu} - \langle \nu,\theta\rangle \frac{\partial}{\partial t}\Big) w_j \, dt\, dS_x.$$

This study is based on certain information about the generalized wave front set

$$WF_b(v) \subset T^*(\mathbb{R} \times \mathring{\Omega}) \cup T^*(\mathbb{R} \times \partial\Omega) = \tilde{T}^*(\mathbb{R} \times \Omega),$$

where the map $\sim$ is the one introduced in Section 3 (see [20] for the properties of $WF_b(u)$). For $x \in \partial\Omega$ we have

$$\sim\, : T^*(\mathbb{R} \times \Omega) \ni (t, x, \tau, \xi) \to (t, x, \tau, \xi_{|T_x(\partial\Omega)}) \in T^*(\mathbb{R} \times \partial\Omega).$$

The crucial step in the analysis of $I_{j,\delta}(\lambda)$ and $J_{j,\delta}(\lambda)$ is the following.

Proposition 5.2. *Set $T_1 = \rho_0 + |t_0| + 1$ and suppose that there exists $\eta > 0$ such that*

$$WF_b(w_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \mu = \sim (t, x, 1, -\theta), T_1 + \eta \le t \le T_1 + 2\eta\} = \varnothing,$$

$$WF_b(\tilde{v}_j) \cap \{\mu \in \tilde{T}^*(\mathbb{R} \times \Omega) : \mu = \sim (t, x, 1, -\theta), T_1 + \eta \le t \le T_1 + 2\eta\} = \varnothing.$$

Then

$$I_{j,\delta}(\lambda) = O(|\lambda|^{-m}),\ J_{j,\delta}(\lambda) = O(|\lambda|^{-m}) \quad \textit{for all } m \in \mathbb{N}.$$

A similar argument can be applied in case (iii), which completes the proof of Theorem 5.1. □

While in general the relation (5–1) is not an equality, it turns out that there exists a set $\mathcal{R}$ of full measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega, \theta) \in \mathcal{R}$ the Poisson relation becomes an equality. This is rather important for some inverse scattering problems.

It is proved in [27] that for each $T > 0$, $S^*(\Omega)$ can be represented as a countable union of Borel subsets S_i such that on each S_i , $\{\mathcal{F}_t\}_{0 \le t \le T}$ coincides with the restriction of an one-parameter family $\mathcal{G}^{(i)}_t$ of Lipschitz maps defined in a neighborhood of S_i in $\dot{T}^*(\Omega)$, taking values in $T^*(\mathbb{R}^n)$ and such that for all but finitely many t, $\mathcal{G}_t^{(i)}$ is smooth and its restriction to smooth local cross-sections is a contact transformation. As a consequence of this regularity property one gets the following.

THEOREM 5.3 [27]. *The generalized geodesic flow $\mathcal{F}_t$ preserves the Hausdorff dimension of Borel subsets of $S^*(\Omega)$.*

This would have been a trivial fact if the maps $\mathcal{F}_t$ were Lipschitz. However, it is well-known and easy to see that this not the case. Locally near a point $\rho \in S^*(\Omega)$, the map $\mathcal{F}_t$ is Lipschitz on a neighborhood of ρ for small $|t|$ when $\rho \notin S^*_{\partial K}(\Omega)$ or ρ is a transversal reflection point. Whenever $\rho \in G$, the map $\mathcal{F}_t$ is not Lipschitz (see [20] or [10, vol. III]). For example, in the simplest case of a diffractive tangent point $\rho \in G_d$, the map $\mathcal{F}_t$ has a singularity of "square root type" at ρ, so it is clearly not Lipschitz.

Let $\Gamma : I \to S^*(\Omega)$ be a generalized geodesic in Ω. We say that Γ is *gliding* on ∂K if the set of those $t \in I$ such that $\Gamma(t) \in G_g$ is dense in I. In this case the trajectory $\{\Gamma(t) : t \in I\}$ is called a *gliding segment* on ∂K.

Given $T > 0$, denote by $\mathcal{T}_T$ *the set of those* $\rho \in S^*(\Omega)$ *such that* $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\} \cap G_g \neq \varnothing$, that is, the trajectory $\{\mathcal{F}_t(\rho) : 0 \leq t \leq T\}$ contains a nontrivial gliding segment on ∂K.

LEMMA 5.4. ([27]) *Let $\mathcal{L}_0$ be an isotropic submanifold of $S^*(\Omega) \setminus S^*_{\partial K}(\Omega)$ of dimension $n-1$ such that $H_p(\rho)$ is not tangent to $\mathcal{L}_0$ at any $\rho \in \mathcal{L}_0$. Then for every $T > 0$ we have $\dim_H \mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0) \leq n-2$. Moreover, if for a given T we have $\mathcal{F}_T(\mathcal{L}_0) \subset S^*(\Omega) \setminus S^*_{\partial K}(\Omega)$, then there exists a countable family $\{\mathcal{I}_m\}$ of smooth $(n-2)$-dimensional isotropic submanifolds of $S^*(\Omega)$ such that $\mathcal{F}_T(\mathcal{T}_T \cap \mathcal{L}_0) \subset \bigcup_m \mathcal{I}_m$.*

Using Theorems 3.1, 4.1, 4.3, 5.1 and Lemma 5.4, one obtains:

THEOREM 5.5 [27]. *There exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for each $(\omega, \theta) \in \mathcal{R}$ the only (ω, θ)-rays in Ω are reflecting (ω, θ)-rays and*

$$\operatorname{sing\,supp} s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}(\Omega)\}.$$

SKETCH OF PROOF. It follows from the results of Melrose and Sjöstrand [20] (see also Theorem 24.3.9 in [10], vol. III) that every (ω, θ)-ray γ in Ω that does not contain gliding segments is a reflecting (ω, θ)-ray, that is, it consists of finitely many straight line segments in Ω (see Section 3).

We will show that there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for each $(\omega, \theta) \in \mathcal{R}$ the only (ω, θ)-rays in Ω are reflecting (ω, θ)-rays.

As before, denote by $U_0 = \{x \in \mathbb{R}^n : |x| < \rho_0\}$ an open ball in $\mathbb{R}^n$ containing the obstacle K and let C be the boundary sphere of $\mathcal{U}_0$. Fix $\omega \in \mathbf{S}^{n-1}$, $x_0 \in C$ and consider the generalized geodesic $(x(t), \xi(t)) = \mathcal{F}_t(x_0, \omega)$. Let $T > 0$ be such that $x(T) \in C$. Set

$$S_0 = \{(x,\xi) \in S^*(\Omega) : x \in C,\ \xi \text{ is transversal to } C\}.$$

Since $\Sigma = p^{-1}(0) = S^*(\Omega)$, using the notation $S^*_C(\Omega) = \{(x,\xi) \in S^*(\Omega) : x \in C\}$, we have

$$S_0' = S_0 \cap \Sigma = \{(x,\xi) \in S^*_C(\Omega) : \xi \text{ is transversal to } C\}.$$

Then S_0' is a symplectic submanifold of S. Let $\mathcal{P} : S_0 \to S_0$ be the local map defined in a neighborhood of (x_0, ω) using the shift along the flow $\mathcal{F}_t$; then $\mathcal{P}(S_0') \subset S_0'$. Consider the Lagrangian submanifold

$$\mathcal{L}_0 = \{(x,\xi) \in S_0' : \xi = \omega\}$$

of S_0'. Setting $\mathcal{T} = \mathcal{T}_T$ and applying Lemma 5.4 to $\mathcal{L}_0$ gives that $\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T})$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. Since locally near (x_0,ω) the map $\mathcal{F}_T : S_0 \to \mathcal{F}_T(S_0)$ is smooth, $\mathcal{F}_T(S_0)$ is a $(2n-1)$-dimensional submanifold of S transversal to the flow $\mathcal{F}_t$ at $\mathcal{F}_T(x_0,\omega)$. Consequently, locally near $\mathcal{F}_T(x_0,\omega) \in \mathcal{F}_T(S_0) \cap S_0$ the shift $\mathcal{Q}$ along $\mathcal{F}_t$ from $\mathcal{F}_T(S_0)$ to S_0 (forwards or backwards) is a smooth map. Moreover $\mathcal{Q}$ maps $\mathcal{F}_T(S_0')$ into S_0' (since $p^{-1}(0)$ is invariant under the flow $\mathcal{F}_t$), the restriction $\mathcal{Q} : \mathcal{F}_T(S_0') \to S_0'$ is a local symplectic map, and $\mathcal{P} = \mathcal{Q} \circ \mathcal{F}_T$. Hence the set $\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}) = \mathcal{Q}(\mathcal{F}_T(\mathcal{L}_0 \cap \mathcal{T}))$ is contained in a countable union of isotropic $(n-2)$-dimensional submanifolds of S. The projection $j : S_0' \to \mathbf{S}^{n-1}$, $j(x,\xi) = \xi$, is smooth, so Sard's theorem gives now that the set $j(\mathcal{P}(\mathcal{L}_0 \cap \mathcal{T}))$ has Lebesgue measure zero in $\mathbf{S}^{n-1}$. Hence there exists a neighborhood U of x_0 in C and a subset $\mathcal{R}_\omega(U) = \mathbf{S}^{n-1} \setminus j(\mathcal{P}(\mathcal{L} \cap \mathcal{T}))$ of full Lebesgue measure in $\mathbf{S}^{n-1}$ such that for $x \in U$ every generalized (ω,θ)-ray in Ω passing through x with $\theta \in \mathcal{R}_\omega(U)$ is a reflecting (ω,θ)-ray. Covering C by a finite family of neighborhoods U_i, we find a subset $\mathcal{R}_\omega = \bigcap_i \mathcal{R}_\omega(U_i)$ of full Lebesgue measure in $\mathbf{S}^{n-1}$ such that every (ω,θ)-ray in Ω with $\theta \in \mathcal{R}_\omega$ is a reflecting (ω,θ)-ray. It now follows from Fubini's theorem that

$$\mathcal{R}' = \{(\omega,\theta) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1} : \theta \in \mathcal{R}_\omega\}$$

is a subset of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. Moreover it is clear that for $(\omega,\theta) \in \mathcal{R}'$, all (ω,θ)-rays in Ω are reflecting ones.

According to Theorems 4.1 and 4.3 above, there exists a subset $\mathcal{R}'' = \mathcal{T} \cap \mathcal{S}$ of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for $(\omega,\theta) \in \mathcal{R}''$ every reflecting (ω,θ)-ray in Ω has no tangencies to ∂K and $T_\gamma \neq T_\delta$ whenever γ and δ are different reflecting (ω,θ)-rays in Ω. Then $\mathcal{R} = \mathcal{R}' \cap \mathcal{R}''$ has full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$. Given $(\omega,\theta) \in \mathcal{R}$, it follows from Theorem 3.1 that $-T_\gamma \in \text{sing supp}\ s(t,\theta,\omega)$ for all $\gamma \in \mathcal{L}_{\omega,\theta}(\Omega)$. Combining this with Theorem 5.1 completes the proof of the theorem. □

Using Theorem 5.5 we will now derive a simple but rather important property of obstacles ([12]; see also [27, Proposition 2.3]): most rays incoming from infinity are not trapped by the obstacle K. Here it is essential that we consider points in the set

$$S_C^*(\Omega) = \{(x,\xi) \in S^*(\Omega) : x \in C\},$$

where C as before is the boundary sphere of an open ball U_0 containing K. In general it is not true that the trapped points $(x,\xi) \in S^*(\Omega_K)$ with x near K form a set of Lebesgue measure zero in $S^*(\Omega_K)$. Example 7.1 below, due to M. Livshitz, shows that in some cases the set of trapped points may even contain a nontrivial open subset of $S^*(\Omega_K)$.

PROPOSITION 5.6. *The set of those $(x,\xi) \in S_C^*(\Omega)$ such that the trajectory $\{\mathcal{F}_t(x,\xi) : t \geq 0\}$ is bounded has Lebesgue measure zero in $S_C^*(\Omega)$.*

PROOF. For $(x,\omega) \in S_C^*(\Omega)$, let $\delta(x,\omega)$ be the generalized geodesic in Ω_K issued from x in direction ω. Assume that there exists a subset W of positive Lebesgue measure in $S_C^*(\Omega)$ such that $\delta(x,\omega) \subset \mathcal{U}_0$ for all $(x,\omega) \in W$. According to Theorem 4.3 and to an argument from the proof of Theorem 5.5 above (or using Lemma 5.4 directly), we may assume that for all $(x,\omega) \in W$ the generalized geodesic $\delta(x,\omega)$ does not contain gliding segments on ∂K and has only transversal reflections at ∂K. Given $(x,\omega) \in W$, denote by x' the first common point of $\delta(x,\omega)$ with ∂K and by ω' the reflected direction of $\delta(x,\omega)$ at x', i.e. $\omega' = \omega - 2\langle\omega,\nu(x')\rangle\nu(x')$, where $\nu(x')$ is the outer unit normal to K at x'. Then the set $W' = \{(x',\omega') \in S_{\partial K}^*(\Omega) : (x,\omega) \in W\}$ is a subset of positive Lebesgue measure in $S_{\partial K}^*(\Omega)$.

Denote by $M \subset S_{\partial K}^*(\Omega)$ the set of those $(y,\eta) \in S_{\partial K}^*(\Omega)$ for which the standard billiard ball map B is well-defined. The map B (as a local map) preserves the so-called Liouville's measure μ on M which is absolutely continuous with respect to the Lebesgue measure on $S_{\partial K}^*(\Omega)$.

Next, we use the argument from the proof of the Poincaré Recurrence Theorem in ergodic theory. It follows from the definition of W' that $B^k(W') \subset M$ and $\mu(B^k(W')) = \mu(W') > 0$ for all $k = 0,1,2,\ldots$. On the other hand, in the situation under consideration we clearly have $\mu\bigl(\bigcup_{k=0}^{\infty} B^k(W')\bigr) < \infty$. Therefore there exist nonnegative integers $k < m$ with $B^k(W') \cap B^m(W') \neq \varnothing$. Since B is invertible, this means that there exists $(x',\omega') \in W' \cap B^{m-k}(W')$. Then $(x',\omega') = B(y,\eta)$ for some $(y,\eta) \in B^{m-k-1}(W') \subset M$. Now the choice of W and the definition of W' show that W' has no common points with $B(M)$. This is a contradiction which proves the proposition. □

6. Existence of Scattering Rays with Sojourn Times Tending to Infinity

In this section we study the existence of (ω,θ)-rays for trapping obstacles. The image $S_b^*(\Omega) = \sim(S^*(\Omega))$ of the characteristic set $S^*(\Omega)$ is called the *compressed characteristic set* and the image $\tilde{\gamma} = \sim(\gamma)$ of a generalized bicharacteristic defined in Section 3 is called a *compressed generalized bicharacteristic*.

Let again U_0 be an open ball containing K and C be its boundary sphere. Given a point $z = (x, \xi) \in S_b^*(\Omega)$, consider the compressed generalized bicharacteristic

$$\gamma_z(t) = \big(x(t), \xi(t)\big) \in S_b^*(\Omega)$$

parametrized by the time t and passing through z for $t = 0$. Denote by $T(z) \in \mathbb{R}^+ \cup \infty$ the maximal $T > 0$ such that $x(t) \in U_0$ for $0 \le t \le T(z)$. We introduce the *trapping set*

$$\Sigma_\infty = \big\{(x, \xi) \in S_b^*(\Omega) : x \in C, T(z) = \infty\big\}.$$

It follows from the continuity of the generalized Hamiltonian flow that Σ_∞ is closed in Σ. The obstacle K is called *trapping* if $\Sigma_\infty \neq \varnothing$. We have the following.

THEOREM 6.1 [23]. *Let the obstacle K be trapping and satisfy the condition $(\mathcal{G})$. Then there exists a sequence of ordinary reflecting nondegenerate scattering rays γ_m with sojourn times $T_{\gamma_m} \to \infty$.*

PROOF. It is easy to see that $\Sigma_\infty \neq S_b^*(\Omega)$, hence the boundary $\partial\Sigma_\infty$ of Σ_∞ in $S_b^*(\Omega)$ is not empty. Take a point $\hat{z} \in \partial\Sigma_\infty$. Since $S_b^*(\Omega) \setminus \Sigma_\infty \neq \varnothing$, there exists a sequence $z_m = (x_m, \xi_m) \in S_b^*(\Omega), x_m \in C$, such that $z_m \notin \Sigma_\infty$ for all m and $z_m \to \hat{z}$. Consider the compressed generalized bicharacteristics $\gamma_{z_m}(t) = (z_m, \xi_m(t))$ passing through z_m for $t = 0$ and such that $T(z_m) < \infty$. The sequence $\{T(z_m)\}$ is unbounded, since otherwise we will have $T(\hat{z}) < \infty$ in contradiction with $\hat{z} \in \Sigma_\infty$. Thus we may assume that $\lim_{m\to\infty} T(z_m) = +\infty$. Set $y_m = x_m(T(z_m)) \in C$, $\omega_m = \xi_m(T(z_m)) \in \mathbf{S}^{n-1}$. Taking a subsequence, we may assume that $y_m \to u \in C$ and $\omega_m \to \omega \in \mathbf{S}^{n-1}$. For the generalized bicharacteristics $\gamma_\mu(t) = (y(t), \xi(t))$ issued from $\mu = (u, \omega)$ we have $T(\mu) = \infty$ and $y(t) \in U_0$ for $t \ge 0$.

Let Z_ω be the hyperplane passing through u and orthogonal to ω and let Z_∞ be the set of those points $y \in Z_\omega$ for which the generalized bicharacteristic γ_{μ_y} passing through $\mu_y = (y, \omega)$ has the property $T(\mu_y) = \infty$. The set Z_∞ is closed in Z_ω, $Z_\omega \neq \varnothing$ and $Z_\infty \neq Z_\omega$. Thus there exists a sequence of points $u_m \to y_0$ for some $y_0 \in Z_\omega$ with $u_m \in Z_\omega \setminus Z_\infty$ such that $T(\mu_{u_m}) < \infty$ for all m and $T(\mu_{u_m}) \to \infty$. Applying Proposition 5.6, we can approximate γ_{u_m} by ordinary reflecting rays γ_{δ_m} with sojourn times going to infinity and by a second approximation we may choose the ordinary reflecting rays γ_{δ_m} to be nondegenerate.

Now consider a fixed ordinary reflecting (ω'_m, θ'_m)-ray with sojourn time T_m which is nondegenerate. In general it is possible to have other (generalized) (ω'_m, θ'_m)-rays with the same sojourn time and T_m could be a nonisolated point in sing supp $s(t, \omega'_m, \theta'_m)$. Let $\mathcal{A} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ be the set introduced at the end of Section 4 and let $\mathcal{R} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ be the set of Theorem 5.5. Let

$$\Xi = \mathcal{R} \cap \mathcal{A} \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}.$$

Then for $(\omega, \theta) \in \Xi$ each (ω, θ)-ray is ordinary reflecting and nondegenerate. By applying the inverse mapping theorem, it is easy to see that we may approximate (ω'_m, θ'_m) by a pair $(\omega''_m, \theta''_m) \in \Xi$ sufficiently close to (ω'_m, θ'_m) so that there exist ordinary reflecting nondegenerate (ω''_m, θ''_m)-rays with sojourn times $T''_m \to \infty$ (see [23] for more details). □

The sojourn times T''_m are isolated points in sing supp $s(t, \omega''_m, \theta''_m)$ and the argument of Section 3 based on (3–2) implies that following.

THEOREM 6.2. *Under the assumptions of Theorem* 6.1 *there exists a sequence* $(\omega_m, \theta_m) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ *and ordinary reflecting nondegenerate* (ω_m, θ_m)*-rays with sojourn times* $T_m \to \infty$ *so that*

$$-T_m \in \operatorname{sing\,supp} s(t, \omega_m, \theta_m) \quad \textit{for all } m \in \mathbb{N}. \tag{6–1}$$

Relation (6–1) was called property (S) in [24], and there we conjectured that every trapping obstacle has the property (S). The above result shows that for generic obstacles this conjecture is true. Moreover, the above argument implies that for each $m \in \mathbb{N}$ there exists a set $\Pi_m \subset \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ with positive measure $\varepsilon_m > 0$ so that the (ω, θ)-rays with $(\omega, \theta) \in \Pi_m$ produce singularities $-\tau_m \le -m$ of the scattering kernel $s(t, \omega, \theta)$. Thus for obstacles satisfying (S) some sojourn times can be observed after a sufficiently long time.

The property (S) leads to some interesting results concerning the behavior of the modified resolvent of the Laplacian [23]. For $\operatorname{Im} \lambda > 0$ consider the outgoing resolvent $R(\lambda) = (-\Delta - \lambda^2)^{-1}$ of the Laplacian in Ω with Dirichlet boundary conditions on $\partial\Omega$. The outgoing condition means that for $f \in C_0^\infty(\Omega)$ there exists $g(x) \in C_0^\infty(\mathbb{R}^n)$ so that we have

$$R(\lambda) f(x) = R_0(\lambda) g(x) \quad \text{as } |x| \to \infty,$$

where

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1} : L^2_{\text{comp}}(\mathbb{R}^n) \to H^2_{\text{loc}}(\mathbb{R}^n)$$

is the outgoing resolvent of the free Laplacian in $\mathbb{R}^n$ related to the outgoing Green function introduced in Section 2. The operator

$$R(\lambda) : L^2_{\text{comp}}(\Omega) \ni f \to R(\lambda) f \in H^2_{\text{loc}}(\Omega)$$

has a meromorphic continuation in $\mathbb{C}$ with poles λ_j such that $\operatorname{Im} \lambda_j < 0$, called *resonances* ([12], [25]). Let $\chi_1(x), \chi_2(x) \in C_0^\infty(\mathbb{R}^n)$ be cutoff functions such that $\chi_1(x) = \chi_2(x) = 1$ on a neighborhood of K and $\chi_1(x) = 1$ on supp $\chi_2(x)$. It is easy to see that the *modified resolvent*

$$\tilde{R}(\lambda) = \chi_1 R(\lambda) \chi_2$$

has a meromorphic continuation in $\mathbb{C}$. The poles of $\tilde{R}(\lambda)$ are independent of the choice of χ_i and they coincide with their multiplicities with those of the resonances (see [12], [25]). On the other hand, the scattering amplitude $a(\lambda, \omega, \theta)$ also admits a meromorphic continuation in $\mathbb{C}$ and the poles of this continuation and their multiplicities are the same as those of the resonances (see [12]). From the general results on propagation of singularities ([20]) it follows that if K is nontrapping, there exist $\varepsilon > 0$ and $d > 0$ so that $\tilde{R}(\lambda)$ has no poles in the domain

$$U_{\varepsilon,d} = \{\lambda \in \mathbb{C} : d - \varepsilon \log(1+|\lambda|) \leq \operatorname{Im}\lambda \leq 0\}.$$

For trapping obstacles we expect to have poles in all domains $U_{\varepsilon,d}$. For the moment this is an open problem and we have a weaker result.

Theorem 6.3 [23]. *Assume that there exists a sequence of ordinary reflecting* (ω_m, θ_m)*-rays in* Ω *with sojourn times* $T_m \to \infty$. *Let* $\Phi \in C_0^\infty(\mathbb{R})$ *be such that supp* $\Phi \subset (-1,1)$ *and* $\Phi(t) = 1$ *for* $|t| \leq \frac{1}{2}$. *Assume that there exists a sequence* $\gamma_m \to 0$ *of nonzero real numbers and an integer* k *independent on* m *such that*

$$\left|\mathcal{F}_{t\to\lambda}\left(\Phi\Big(\frac{t+T_m}{\gamma_m}\Big)s(t,\omega_m,\theta_m)\right)\right| \geq (c_m - o_m(1))|\lambda|^k \qquad \text{as } |\lambda| \to \infty,$$

where $c_m > 0$. *Then there are two possibilities*:

(i) *For each* $\varepsilon > 0$ *and each* $d > 0$, *the modified resolvent* $\tilde{R}(\lambda)$ *has poles in the domain* $U_{\varepsilon,d}$.

(ii) *For some* $\varepsilon > 0$ *and* $d > 0$ *the modified resolvent* $\tilde{R}(\lambda)$ *is holomorphic in* $U_{\varepsilon,d}$ *but for all* $\alpha \geq 0, p \in \mathbb{N}, k \in \mathbb{N}$ *we have*

$$\sup_{\substack{\lambda \in U_{\varepsilon,d} \\ \|\varphi\|_{H^k(\Omega)}=1}} (1+|\lambda|)^{-p} e^{-\alpha|\operatorname{Im}\lambda|} \|\tilde{R}(\lambda)\varphi\|_{H^1(\Omega)} = +\infty.$$

It is natural to make the conjecture that under the assumptions of Theorem 6.3, condition (i) always takes place.

7. Rigidity of the Scattering Length Spectrum

Fix again a large open ball U_0 in $\mathbb{R}^n$, $n \geq 3$, n odd[2], and let $C = \partial U_0$. Throughout this section we consider obstacles K in $\mathbb{R}^n$ contained in U_0 with smooth boundaries ∂K that satisfy the condition $(\mathcal{G})$ from Section 3 and such that $\gamma_K(\sigma)$ is a nondegenerate simply reflecting ray for almost all $\sigma \in S_C^*(\Omega)$ such that $\gamma_K(\sigma) \cap \partial K \neq \varnothing$. Denote by $\mathcal{K}_0$ the class of obstacles with these properties. One can derive from [22] (see Chapter 3 there) that $\mathcal{K}_0$ is of second Baire category (with respect to the C^∞ Whitney topology; see [8]) in the class of all obstacles with smooth boundaries.

Since in this section we deal with more than one obstacle, it is convenient to replace the notation Ω, $\mathcal{F}_t$, $s(t,\omega,\theta)$, $\dot{T}_b^*(\Omega)$ and $S_b^*(\Omega)$ used so far (see Section 3 for the latter two) by Ω_K, ${\mathcal{F}^{(K)}}_t$, $s_K(t,\omega,\theta)$, $\dot{T}_b^*(\Omega_K)$ and $S_b^*(\Omega_K)$, respectively.

A point $\sigma = (x,\omega) \in \dot{T}^*(\Omega_K)$ is called a *trapped point* if at least one of the curves $\{\mathrm{pr}_1({\mathcal{F}^{(K)}}_t(\sigma)) : t \leq 0\}$ and $\{\mathrm{pr}_1({\mathcal{F}^{(K)}}_t(\sigma)) : t \geq 0\}$ in Ω_K is bounded.

[2]In fact, most of the considerations in this section are purely geometrical and apply also in the case when n is even, $n \geq 2$.

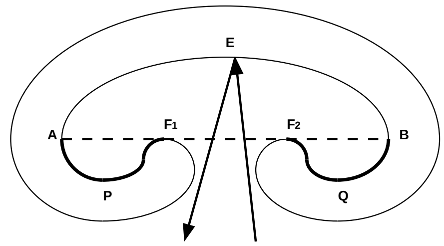

Figure 4. Livshits' example. Adapted from [18, Chapter 5].

Here we use the notation $\mathrm{pr}_1(y,\eta) = y$ and $\mathrm{pr}_2(y,\eta) = \eta$. Denote by $\mathrm{Trap}\,\Omega_K$ the *set of all trapped points* in $\dot{T}^*(\Omega_K)$. Notice that the set Σ_∞ used in Section 6 coincides with $\mathrm{Trap}\,\Omega_K \cap S^*_C(\Omega_K)$. It is easy to see that $\Sigma_\infty \neq \varnothing$ if and only if $\mathrm{Trap}\,\Omega_K \neq \varnothing$. So, if $\mathrm{Trap}\,\Omega_K = \varnothing$, then K is a nontrapping obstacle. It is known for example that all star-shaped obstacles are nontrapping.

The *scattering length spectrum* (SLS) of K is by definition the family of sets of real numbers $\mathcal{S}L_K = \{\mathcal{S}L_K(\omega,\theta)\}_{(\omega,\theta)}$ where (ω,θ) runs over $\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$ and $\mathcal{S}L_K(\omega,\theta)$ is the set of sojourn times T_γ of all (ω,θ)-rays γ in Ω_K. Thus, $\mathcal{S}L_K$ is a map which assigns to each pair of directions (ω,θ) a set $\mathcal{S}L_K(\omega,\theta)$ of real numbers.

In this section we discuss the problem of recovering information about the geometry of the obstacle K from its SLS. Two obstacles K and L in $\mathbb{R}^n$ are said to have *almost the same SLS* if there exists a subset $\mathcal{R}$ of full Lebesgue measure in $\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}$ such that $\mathcal{S}L_K(\omega,\theta) = \mathcal{S}L_L(\omega,\theta)$ for all $(\omega,\theta)\in\mathcal{R}$. We will say that a property P of obstacles in $\mathbb{R}^n$ can be recovered by the SLS of the obstacle if whenever K and L have almost the same SLS and K has property P, then L has property P as well.

It follows from results of A. Majda [15] (see also Majda and Ralston [16]) and P. Lax and R. Phillips [13] that the convex hull $\hat{K}$ of K can be recovered from $\mathcal{S}L_K$. Consequently, in the class of convex obstacles and also in the class of connected obstacles with real analytic boundaries, K is completely determined by its SLS.

Example 7.1. The following example of M. Livshits (Chapter 5 in [18]) shows that in general $\mathcal{S}L_K$ does not determine K uniquely. Here the part E is half an ellipse with foci F_1 and F_2. The ellipse has the property that any ray intersecting the segment connecting the foci, after reflection at the boundary, intersects the same segment again. It is now clear that no scattering ray in the exterior of the obstacle K has a common point with the parts P and Q, so these two "pockets" cannot be recovered from the SLS of the obstacle. It should be mentioned that this example is in $\mathbb{R}^2$ and no examples like this in higher dimensions are known to the authors.

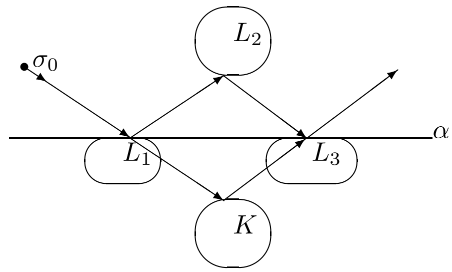

Figure 5.

The problem considered at the beginning of this section is of a global nature. The following simple example shows that in the corresponding local problem there is no uniqueness (unless possibly some nondegeneracy conditions are imposed).

Example 7.2. Consider two obstacles K and $L = L_1 \cup L_2 \cup L_3$ in $\mathbb{R}^n$, $n \geq 2$, as shown in Figure 5. Here K and L_2 are (strictly) convex domains, while L_1 and L_3 are convex domains. Moreover K and L_2 are symmetric with respect to the hyperplane α containing the flat "top parts" of ∂L_1 and ∂L_3. The rays on the figure are generated by some σ_0 (far from K and L). For any σ close to σ_0 we have $\mathcal{F}^{(K)}{}_t(\sigma) = \mathcal{F}^{(L)}{}_t(\sigma)$ for $t \gg 0$ and both trajectories have common points with the corresponding obstacles (and are nondegenerate). On the other hand, $K \cap L = \varnothing$. It should be mentioned however that the obstacles K and L in this example do not satisfy the condition $\mathcal{G}$. Whether such examples exist with K and L satisfying $\mathcal{G}$ is an open problem.

It turns out that if two obstacles K and L have almost the same SLS, then their generalized geodesic flows are conjugate with a time preserving conjugacy on the nontrapping parts of their phase spaces.

Theorem 7.3 [29]. *If the obstacles $K, L \in \mathcal{K}_0$ have almost the same SLS, then there exists a homeomorphism*

$$\Phi : \dot{T}_b^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K \to \dot{T}_b^*(\Omega_L) \setminus \operatorname{Trap} \Omega_L$$

with the following properties:

(i) Φ *defines a symplectic map on an open dense subset of* $\dot{T}_b^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K$;
(ii) Φ *maps* $S_b^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K$ *onto* $S_b^*(\Omega_L) \setminus \operatorname{Trap} \Omega_L$;
(iii) $\mathcal{F}^{(L)}{}_t \circ \Phi = \Phi \circ \mathcal{F}^{(K)}{}_t$ *for all* $t \in \mathbb{R}$;
(iv) $\Phi(x, \xi) = (x, \xi)$ *for any* $(x, \xi) \in \dot{T}_b^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K = \dot{T}_b^*(\Omega_L) \setminus \operatorname{Trap} \Omega_L$ *such that* $x \notin U_0$.

Conversely, it is not difficult to show that if $K, L \in \mathcal{K}_0$ are two obstacles for which there exists a homeomorphism $\Phi : S_b^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K \to S_b^*(\Omega_L) \setminus \operatorname{Trap} \Omega_L$ such that $\mathcal{F}^{(L)}{}_t \circ \Phi = \Phi \circ \mathcal{F}^{(K)}{}_t$ for all $t \in \mathbb{R}$ and $\Phi = \mathrm{id}$ on $S^*(\mathbb{R}^n \setminus U_0) \setminus \operatorname{Trap} \Omega_K$, then K and L have the same SLS ([29]).

There is a clear analogy between the property described above and the *lens equivalence* of geodesic flows on Riemannian manifolds without boundary (see [3] and the references there).

Sketch of proof of Theorem 7.3. Assume that the obstacles K and L have almost the same SLS. The existence of the conjugacy Φ follows easily from the following main lemma.

Lemma 7.4. *Assume that* $\sigma \in S^*(\mathbb{R}^n \setminus U_0)$ *and* $t \in \mathbb{R}$ *with* $\mathcal{F}^{(K)}{}_t(\sigma) \in S^*(\mathbb{R}^n \setminus U_0)$. *Then* $\mathcal{F}^{(K)}{}_t(\sigma) = \mathcal{F}^{(L)}{}_t(\sigma)$.

Given $\sigma \in \dot{T}^*(\Omega) \setminus \mathrm{Trap}\, \Omega_K$, take $t \in \mathbb{R}$ so large that $\mathcal{F}^{(K)}{}_t(\sigma) \in S^*(\mathbb{R}^n \setminus U_0)$. Then define $\Phi(\sigma) = \mathcal{F}^{(L)}{}_{-t} \circ \mathcal{F}^{(K)}{}_t(\sigma)$. It follows from the above lemma that the definition of Φ is correct and moreover $\mathcal{F}^{(L)}{}_t \circ \Phi = \Phi \circ \mathcal{F}^{(K)}{}_t$ for all $t \in \mathbb{R}$ and $\Phi(\sigma) = \sigma$ for $\sigma \in \dot{T}^*(\mathbb{R}^n \setminus U_0) \setminus \mathrm{Trap}\, \Omega_K$. Clearly Φ is a homeomorphism and it follows from the properties of the generalized geodesic flows ([20]) that it is a symplectic map on an open dense subset of $\dot{T}^*_b(\Omega_K) \setminus \mathrm{Trap}\, \Omega_K$. This shows how Theorem 7.3 is derived from Lemma 7.4.

Proof of Lemma 7.4. Fix for a moment an arbitrary $(\omega_0, \theta_0) \in \mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$, and let δ be a nondegenerate simply reflecting (ω_0, θ_0)-ray in Ω_K with reflection points $x_1, \dots, x_k$ $(k \geq 1)$ and δ' is a nondegenerate simply reflecting (ω_0, θ_0)-ray in Ω_L with reflection points $y_1, \dots, y_m$ $(m \geq 1)$. Using the nondegeneracy of δ and the Inverse Mapping Theorem one derives the existence of a neighborhood U of (ω_0, θ_0) in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that for each $(\omega, \theta) \in U$ there are a unique reflecting (ω, θ)-ray $\delta(\omega, \theta)$ in Ω_K with reflection points $x_1(\omega, \theta), \dots, x_k(\omega, \theta)$ close to $x_1, \dots, x_k$, respectively, and a unique reflecting (ω, θ)-ray $\delta'(\omega, \theta)$ in Ω_L with reflection points $y_1(\omega, \theta), \dots, y_m(\omega, \theta)$ close to $y_1, \dots, y_m$, respectively.

Lemma 7.5. *Under the preceding assumptions, suppose in addition that* $T_{\delta(\omega,\theta)} = T_{\delta'(\omega,\theta)}$ *for all* $(\omega, \theta) \in U$. *Then for each* $(\omega, \theta) \in U$ *there exist real numbers* $\lambda(\omega, \theta)$ *and* $\mu(\omega, \theta)$ *such that*

$$y_1(\omega, \theta) = x_1(\omega, \theta) + \lambda(\omega, \theta)\omega, \qquad y_m(\omega, \theta) = x_k(\omega, \theta) + \mu(\omega, \theta)\theta. \tag{7–1}$$

Proof. Let $(\omega, \theta) = (\omega(u), \theta(v))$, $(u, v) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$, be a smooth parametrization of U and set $x_j(u, v) = x_j(\omega(u), \theta(v))$ and $y_j(u, v) = y_j(\omega(u), \theta(v))$. For the functions

$$f(u, v) = \langle \omega(u), x_1(u, v) \rangle + \sum_{i=1}^{k-1} \| x_i(u, v) - x_{i+1}(u, v) \| - \langle x_k(u, v), \theta(v) \rangle,$$

$$g(u, v) = \langle \omega(u), y_1(u, v) \rangle + \sum_{i=1}^{m-1} \| y_i(u, v) - y_{i+1}(u, v) \| - \langle y_m(u, v), \theta(v) \rangle,$$

we have $f(u,v) = g(u,v)$ for all (u,v), therefore the derivatives of these two functions coincide. A simple calculation gives

$$\frac{\partial f}{\partial u_j}(u) = \Big\langle \frac{\partial \omega}{\partial u_j}, x_1 \Big\rangle + \Big\langle \omega, \frac{\partial x_1}{\partial u_j} \Big\rangle + \sum_{i=1}^{k-1} \Big\langle \frac{x_{i+1}-x_i}{\|x_{i+1}-x_i\|}, \frac{\partial x_{i+1}}{\partial u_j} - \frac{\partial x_i}{\partial u_j} \Big\rangle - \Big\langle \frac{\partial x_k}{\partial u_j}, \theta \Big\rangle.$$

Using the notation $e_i = \dfrac{x_{i+1}-x_i}{\|x_{i+1}-x_i\|}$ and the reflection law at the points $x_1, \dots, x_{k-1}$, we find

$$\begin{aligned}\frac{\partial f}{\partial u_j}(u) &= \Big\langle \frac{\partial \omega}{\partial u_j},\, x_1 \Big\rangle + \Big\langle \omega - e_1,\, \frac{\partial x_1}{\partial u_j} \Big\rangle + \Big\langle e_1 - e_2,\, \frac{\partial x_2}{\partial u_j} \Big\rangle + \cdots \\ &\qquad + \Big\langle e_{k-2} - e_{k-1},\, \frac{\partial x_{k-1}}{\partial u_j} \Big\rangle + \Big\langle e_{k-1} - \theta,\, \frac{\partial x_k}{\partial u_j} \Big\rangle \\ &= \Big\langle \frac{\partial \omega}{\partial u_j},\, x_1 \Big\rangle.\end{aligned}$$

In the same way one gets $\dfrac{\partial g}{\partial u_j} = \Big\langle \dfrac{\partial \omega}{\partial u_j}, y_1 \Big\rangle$. Hence

$$\Big\langle \frac{\partial \omega}{\partial u_j}, x_1 \Big\rangle = \Big\langle \frac{\partial \omega}{\partial u_j}, y_1 \Big\rangle$$

for all $j = 1, \dots, n-1$, so $y_1 - x_1 = \lambda\omega$ for some $\lambda \in \mathbb{R}$.

Similarly, $y_m = x_k + \mu\theta$ for some $\mu \in \mathbb{R}$. □

We continue the proof of Lemma 7.4. As usual, we denote by $\mathring{M}$ the *interior* (largest open subset) of a subset M of $\mathbb{R}^n$. Let $\mathcal{R}$ be a subset of full Lebesgue measure in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ such that

$$\mathcal{S}L_K(\omega,\theta) = \mathcal{S}L_L(\omega,\theta), \quad (\omega,\theta) \in \mathcal{R}. \tag{7–2}$$

Shrinking $\mathcal{R}$ if necessary, we will assume that $(\omega,\omega) \notin \mathcal{R}$ for any $\omega \in \mathbf{S}^{n-1}$. Then for $(\omega,\theta) \in \mathcal{R}$, any (ω,θ)-ray in Ω_K (and in fact in the exterior of any obstacle) must have at least one reflection point. Furthermore, using Theorems 4.3, 5.1 and 5.5 above, we may assume that the set $\mathcal{R}$ is chosen in such a way that: (i) for $(\omega,\theta) \in \mathcal{R}$ all (ω,θ)-rays in Ω_K (resp. Ω_L) are nondegenerate simply reflecting (ω,θ)-rays; (ii) if $(\omega,\theta) \in \mathcal{R}$ and γ and δ are (ω,θ)-rays in Ω_K (resp. Ω_L), then $T_\gamma \neq T_\delta$.

It follows from [13] and [15] (see also [16]) that $\hat{K} = \hat{L}$.

Let $\sigma_0 = (u_0, \omega_0) \in S^*(\mathring{\Omega}_{\hat{K}})$ and $t_0 \in \mathbb{R}$ be such that ${\mathcal{F}^{(K)}}_{t_0}(\sigma_0) \notin S^*(\mathring{\Omega}_{\hat{K}})$. We will show that ${\mathcal{F}^{(K)}}_{t_0}(\sigma_0) = {\mathcal{F}^{(L)}}_{t_0}(\sigma_0)$. Using various results from [20], [23] and [29], one derives that it is enough to consider the case when σ_0 is nontrapped and $(\omega_0, \theta_0) \in \mathcal{R}$. Then $\delta = \gamma_K(\sigma_0)$ is a nondegenerate simply reflecting (ω_0,θ_0)-ray in Ω_K.

The essential case to consider is when $\gamma_K(\sigma_0) \cap \partial K \neq \varnothing$. Then there exists $s_0 \in \mathbb{R}$ with ${\mathcal{F}^{(K)}}_{s_0}(\sigma_0) = (x_0, \xi_0)$, $x_0 \in \partial K$, and without loss of generality we will assume $s_0 > 0$ and moreover that s_0 is the minimal positive number with $\mathrm{pr}_1({\mathcal{F}^{(K)}}_{s_0}(\sigma_0)) \in \partial K$. Let $x_1 = x_0, x_2, \dots, x_k$ be the successive reflection points of δ. According to (7–2), there exists a reflecting (ω_0, θ_0)-ray δ' in Ω_L with $T_{\delta'} = T_\delta$. Let $y_1, \dots, y_m$ be the successive reflection points of δ'. The choice of $\mathcal{R}$ and $(\omega_0, \theta_0) \in \mathcal{R}$ imply that δ' is nondegenerate. From the latter one derives that there exist a neighborhood U of (ω_0, θ_0) in $\mathbf{S}^{n-1} \times \mathbf{S}^{n-1}$ and a neighborhood U_i of x_i in ∂K for each $i = 1, \dots, k$ such that for every $(\omega, \theta) \in U$ there is a unique reflecting (ω, θ)-ray $\delta(\omega, \theta)$ in Ω_K with reflection points $x_1(\omega, \theta) \in U_1, \dots, x_k(\omega, \theta) \in U_k$ smoothly depending on (ω, θ). Similarly, there exists a neighborhood U'_j of y_j in ∂L for each $j = 1, \dots, m$ such that for every $(\omega, \theta) \in U$ there is a unique reflecting (ω, θ)-ray $\delta'(\omega, \theta)$ in Ω_L with reflection points $y_1(\omega, \theta) \in U'_1, \dots, y_m(\omega, \theta) \in U'_m$ smoothly depending on (ω, θ). Moreover $\delta(\omega_0, \theta_0) = \delta$ and $\delta'(\omega_0, \theta_0) = \delta'$.

According to (7–2), for each $(\omega, \theta) \in \mathcal{R} \cap U$ there exists a unique reflecting (ω, θ)-ray $\delta''(\omega, \theta)$ in Ω_L with

$$T_{\delta''(\omega,\theta)} = T_{\delta(\omega,\theta)}. \tag{7–3}$$

Assuming U is small enough, it then follows that $\delta''(\omega, \theta) = \delta'(\omega, \theta)$ for each $(\omega, \theta) \in \mathcal{R} \cap U$. Otherwise there exists a sequence $\{(\omega_p, \theta_p)\}_{p=1}^{\infty} \subset \mathcal{R} \cap U$ converging to (ω_0, θ_0) such that $\delta''(\omega_p, \theta_p) \neq \delta'(\omega_p, \theta_p)$ for all p. Let $Z = Z_{\omega_0}$. Denote by u_p the (incoming) intersection point of $\delta''(\omega_p, \theta_p)$ with Z; then $\delta''(\omega_p, \theta_p) = \gamma_L(u_p, \omega_p)$. Considering an appropriate subsequence, we may assume that $u_p \to u \in Z$ as $p \to \infty$. Then $\delta'' = \gamma_L(u, \omega_0)$ is an (ω_0, θ_0)-ray in Ω_L and clearly $T_{\delta''} = \lim_p T_{\delta''(\omega_p,\theta_p)} = T_{\delta''(\omega_0,\theta_0)}$. Now (7–3) implies $T_{\delta''} = T_{\delta(\omega_0,\theta_0)} = T_\delta$ and therefore $T_{\delta''} = T_{\delta'(\omega_0,\theta_0)} = T_{\delta'}$. This and $(\omega_0, \theta_0) \in \mathcal{R}$ give $\delta'' = \delta'$. Hence u belongs to $\delta' = \delta'(\omega_0, \theta_0)$ and therefore for large p, the ray $\delta''(\omega_p, \theta_p)$ has m reflection points belonging to the neighborhoods U'_j, respectively. From the choice of U and the uniqueness of the (ω, θ)-rays $\delta'(\omega, \theta)$ for $(\omega, \theta) \in U$, it now follows that $\delta''(\omega_p, \theta_p) = \delta'(\omega_p, \theta_p)$. This is a contradiction with the choice of the sequence $\{(\omega_p, \theta_p)\}_p$ which proves that $\delta''(\omega, \theta) = \delta'(\omega, \theta)$ for all $(\omega, \theta) \in \mathcal{R} \cap U$. Hence

$$T_{\delta'(\omega,\theta)} = T_{\delta(\omega,\theta)} \tag{7–4}$$

for $(\omega, \theta) \in \mathcal{R} \cap U$. This gives that (7–4) holds for all $(\omega, \theta) \in U$, and then Lemma 7.5 implies that equations (7–1) hold for some real numbers $\lambda(\omega, \theta)$ and $\mu(\omega, \theta)$ for all $(\omega, \theta) \in U$. In particular, $\delta' = \gamma_L(\sigma_0)$.

Let ${\mathcal{F}^{(K)}}_{t_0}(\sigma_0) = (z, \zeta)$. Then either $\zeta = \omega_0$ and $z = x_1 + s\omega_0$ for some $s < 0$, or $\zeta = \theta_0$ and $z = x_k + s\theta_0$ for some $s > 0$. The same holds for ${\mathcal{F}^{(L)}}_{t_0}(\sigma_0) = (z', \zeta')$. In both cases (7–1) and (7–4) imply $(z, \zeta) = (z', \zeta')$, i.e. ${\mathcal{F}^{(K)}}_{t_0}(\sigma_0) = {\mathcal{F}^{(L)}}_{t_0}(\sigma_0)$. □

Using the existence of the conjugacy Φ and the fact that it is measure preserving with respect to the canonical measures on $S_b^*(\Omega_K)$ and $S_b^*(\Omega_L)$, one derives the following.

Corollary 7.6. *Let the obstacles K and L have almost the same SLS. If the sets of trapped points of both K and L have Lebesgue measure zero, then* $\operatorname{Vol} K = \operatorname{Vol} L$.

Livshits' example shows that the above conclusion is not true without any assumption about the sets of trapped points. Notice that far from the obstacle the trapping set is relatively small. For example, if C is a large sphere in $\mathbb{R}^n$ (i.e. it contains K in its interior), a slight modification of the proof of Proposition 5.6 shows that $\dim\bigl(S^*_C(\Omega_K) \cap \operatorname{Trap}\Omega_K\bigr) \le 2n-3$. On the other hand, in some cases (as in Livshits' example) we have $\dim\bigl(\operatorname{Trap}\Omega_K \cap S^*_b(\Omega_K)\bigr) = 2n-1 = \dim\bigl(S^*_b(\Omega_K)\bigr)$.

Another simple consequence of Theorem 7.3 concerns backscattering rays. Denote by $\operatorname{Trap}^{(n)} \partial K$ the set of those $x \in \partial K$ such that $(x, \nu_K(x)) \in \operatorname{Trap}\Omega_K$, where $\nu_K(x)$ is the *outward unit normal* to ∂K at x.

Suppose that K and L are obstacles with almost the same SLS. Let Φ be the conjugacy from Theorem 7.3. Given $x \in \partial K \setminus \operatorname{Trap}^{(n)} \partial K$, take an arbitrary $t > 0$ such that $(z,\zeta) = \mathcal{F}^{(K)}{}_t(x, \nu_K(x)) \in S^*(\mathbb{R}^n \setminus U_0)$. Then $\mathcal{F}^{(K)}{}_t(z,-\zeta) = (x, \nu_K(x))$ and $\mathcal{F}^{(K)}{}_{2t}(z,-\zeta) = (z,\zeta)$. Therefore

$$(z,\zeta) = \Phi(z,\zeta) = \Phi \circ \mathcal{F}^{(K)}{}_{2t}(z,-\zeta) = \mathcal{F}^{(L)}{}_{2t} \circ \Phi(z,-\zeta) = \mathcal{F}^{(L)}{}_{2t}(z,-\zeta),$$

so for $(y,\eta) = \mathcal{F}^{(L)}{}_t(z,-\zeta)$ we must have $y \in \partial L$ and $\eta \perp \partial L$ at y. Thus, $\Phi(x,\nu_K(x)) = (y, \nu_L(y))$ for some $y \in \partial L \setminus \operatorname{Trap}^{(n)} \partial L$. Setting $\varphi(x) = y$, one gets a homeomorphism

$$\varphi : \partial K \setminus \operatorname{Trap}^{(n)} \partial K \to \partial L \setminus \operatorname{Trap}^{(n)} \partial L$$

such that $\varphi(x) = y$ whenever $\Phi(x,\nu_K(x)) = (y,\nu_L(y))$. In particular, assuming that $\dim \operatorname{Trap}^{(n)} \partial K < n-2$ and $\dim \operatorname{Trap}^{(n)} \partial L < n-2$, it follows that K and L must have the same number of connected components.

Here we denote by $\dim X$ the *topological dimension* of X (see [4], for example). Since $\dim X \le \dim_H X$, where $\dim_H X$ is the Hausdorff dimension of the metric space X (see [4], for example), all assumptions of the form $\dim X \le a$ can be replaced by $\dim_H X \le a$.

It seems natural to conjecture that in the case of nontrapping obstacles the SLS uniquely determines the obstacle. While this is still an open problem, using Theorem 7.3 and backscattering rays as above, one can prove this conjecture at least for star-shaped obstacles (as mentioned above, these are necessarily nontrapping).

Proposition 7.7 [29]. *Let K and L have almost the same SLS. If K is star-shaped, $L = K$.*

Even though the trapping set is relatively small far from the obstacle, in general it may be big enough to *topologically divide* $S^*_C(\Omega_K)$, i.e. it may happen that $S^*_C(\Omega_K) \setminus \operatorname{Trap}\Omega_K$ has more than one connected component.

We will denote by $\partial K^{(\mathrm{ob})}$ the union of all connected components of ∂K that have a common point with at least one scattering ray in Ω_K, and call it the *observable part* of the boundary ∂K. The obstacle K will be called *observable*, if $\partial K = \partial K^{(\mathrm{ob})}$.

THEOREM 7.8 [28]. *Let K, L be obstacles in $\mathbb{R}^n$ with real analytic boundaries that have almost the same SLS. If K is such that $\operatorname{Trap}\Omega_K$ does not topologically divide $S^*_C(\Omega_K)$, then $\partial K^{(\mathrm{ob})} = \partial L^{(\mathrm{ob})}$. If in addition both K and L are observable, then $K = L$.*

The idea of the proof of Theorem 7.8 is rather simple. Let Y be the union of all connected components of $\partial K^{(\mathrm{ob})}$ that do not coincide with connected components of L. Assuming $Y \neq \varnothing$, one finds $\sigma \in S^*(\mathbb{R}^n \setminus U)$ such that $\gamma_K(\sigma) = \left\{\mathrm{pr}_1(\mathcal{F}^{(K)}{}_t(\sigma)) : t \in \mathbb{R}\right\}$ has a common point with Y. Consider a smooth curve $\sigma(s)$ in $S^*(\mathbb{R}^n \setminus U)$ that connects σ to a point $\sigma(0) = \sigma_0$ generating a *free ray*, i.e. a ray without common points with K. After some regularization of the curve $\sigma(s)$ (imposing some transversality conditions on it), we choose the smallest s with $\gamma_K(\sigma(s)) \cap Y \neq \varnothing$. For $\rho = \sigma(s)$, the scattering ray $\gamma_K(\rho)$ has only one common point y with Y which is a tangent point, and all transversal reflection of its occur at connected components of ∂K that coincide with connected components of L. Then we show that $y' \in \partial L$ for a dense set of points y' in a neighborhood of y in Y. Thus, $\partial K = \partial L$ near y which is a contradiction with the definition of Y. See [28] for details.

It is not clear how restrictive the condition that $\operatorname{Trap}\Omega_K$ does not topologically divide $S^*_C(\Omega_K)$ is. It turns out ([28]) that this condition is satisfied when K is a finite disjoint union of strictly convex domains with C^∞ boundaries. This and Theorem 7.8 imply the following.

COROLLARY 7.9. ([28]) *If K is a finite disjoint union of strictly convex domains, K and L have almost the same SLS and both ∂K and ∂L are real analytic, then $K = L$.*

It is an open problem whether the statement of Corollary 7.9 remains true for obstacles with C^∞ boundaries ∂K and ∂L.

Next, we describe a few results from [29] involving scattering rays having tangencies to the boundary.

Denote by $\mathcal{K}^{(fin)}$ the class of obstacles $K \in \mathcal{K}_0$ such that the normal curvature of K does not vanish of infinite order. From now on until the end of this section we assume that $K, L \in \mathcal{K}^{(fin)}$.

Consider an arbitrary scattering ray γ in Ω_K and let X and Y be arbitrary cross-sections of the incoming and outgoing rays of γ. Define the cross-sectional map $\mathcal{P}_K : S^*_X(\mathbb{R}^n) \to S^*_Y(\mathbb{R}^n)$ by the shift along the flow $\mathcal{F}^{(K)}{}_t$. Now assume that the obstacle K and L have almost the same SLS. It then follows from Theorem 7.3 that $\mathcal{P}_K = \mathcal{P}_L$. In particular the singularities of $\mathcal{P}_K$ and $\mathcal{P}_L$ are the same, and this implies that for any $\sigma_0 = (x_0, \xi_0) \in S^*(\mathbb{R}^n \setminus U_0) \setminus \operatorname{Trap} \Omega_K$, the ray $\gamma_K(\sigma_0)$ contains a point of tangency to ∂K if and only if $\gamma_L(\sigma_0)$ contains a point of tangency to ∂L.

Next, suppose that $\sigma(s)$, $s \in [0, a]$, is a continuous curve in $S^*(\Omega_K)$ consisting of nontrapped points. Using an idea of Melrose and Sjöstrand [20] involving winding numbers, one shows that if $\gamma_K(\sigma(s))$ is simply reflecting for each s, then the number of reflection points of $\gamma_K(\sigma(s))$ is the same for all $s \in [0, a]$. Now assume that $\sigma = \sigma(0)$ generates a ray $\gamma_K(\sigma)$ containing a gliding segment on ∂K. If $s_k \searrow 0$ are such that each $\gamma_K(\sigma(s_k))$ is simply reflecting, it follows from [20] that the number of reflection points of $\gamma_K(\sigma(s_k))$ tends to ∞. Hence there must be infinitely many $s \in (0, a]$ such that $\mathcal{F}^{(K)}{}_t(\sigma(s)) \in S^*(\partial K)$ for some $t = t(s)$. On the other hand if $\gamma_K(\sigma)$ is tangent to ∂K but does not contain a gliding segment, then it is not difficult to construct a continuous curve $\sigma(s)$ $(0 \le s \le a,\ a > 0)$ in $S^*(\Omega_K)$ with $\sigma(0) = \sigma$ such that $\gamma(\sigma(s))$ is a simply reflecting ray for all $s \in (0, a]$.

These observation yield that from the SLS of an obstacle one can determine which points $\sigma \in S^*(\Omega_K) \setminus \operatorname{Trap} \Omega_K$ generate rays containing gliding segments on ∂K.

Corollary 7.10. ([29]) *Let K, L have almost the same SLS. If there exists a scattering ray containing a gliding segment in Ω_K, then Ω_L has the same property. Consequently, if K is a finite disjoint union of convex domains in $\mathbb{R}^n$ and* $\dim \operatorname{Trap} \Omega_L \cap S^*(\partial L) < 2n-3$, *then L is also a finite disjoint union of convex domains, moreover K and L must have the same number of connected components and are therefore diffeomorphic.*

A point $\sigma \in S^*_C(\Omega_K)$ will be called *accessible* if it belongs to a connected component of $S^*_C(\Omega_K) \setminus \operatorname{Trap} \Omega_K$ containing a point that generates a free ray. Presumably the SLS provides more substantial information about the behavior of the flow $\mathcal{F}^{(K)}{}_t$ near accessible points $\rho \in S^*_C(\Omega_K)$ and correspondingly about parts of ∂K that can be reached by rays generated by accessible points. The following result shows for example that the SLS determines uniquely the number of reflection points of simply reflecting rays $\gamma_K(\sigma)$ generated by accessible points σ.

Proposition 7.11 [29]. *Let K, L have almost the same SLS. For every connected component W of $S^*_C(\Omega_K) \setminus \operatorname{Trap} \Omega_K$ there exists an integer $m = m(K, L, W)$ such that*

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L) + m$$

for all $\sigma \in W \cap U^{(K)}$. Whenever W is accessible, $m = 0$; that is,

$$\#(\gamma_K(\sigma) \cap \partial K) = \#(\gamma_L(\sigma) \cap \partial L)$$

for any accessible point σ.

See [29] for further results concerning relationship between obstacles having almost the same SLS.

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