Inside Out: Inverse Problems
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Preface

Inverse problems arise in practical situations such as medical imaging, geophysical exploration, and nondestructive evaluation where measurements made on the exterior of a body are used to determine properties of the inaccessible interior. In this book leading experts in the theoretical and applied aspects of inverse problems have written extended surveys on some of the main topics of the inverse problems semester held at MSRI during the Fall of 2001. We describe here briefly the chapters of the book.

The chapter by Faridani is an introduction to *computed tomography* (CT), which is probably the inverse problem best known to the general public. In this imaging method the attenuation in intensity of an X-ray beam is measured, and the information from many X-rays from different sources is assembled and analyzed on a computer. Mathematically it is a problem of recovering a function from the set of its line integrals (or the set of its plane integrals). Radon found in the early part of the twentieth century a formula to recover a function from this information. The application to diagnostic radiology did not happen until the late 1960s with the aid of the increasing calculating power of the computer. In 1970 the first computer tomograph that could be used in clinical work was developed by G. N. Hounsfield. He and Allan M. Cormack, who independently proposed some of the algorithms, were jointly awarded the 1979 Nobel prize in medicine. In practice only integrals from finitely many lines can be measured, and the distribution of these lines is sometimes restricted. The focus of Faridani's chapter is on what features of a function f can be stably recovered from a given collection of line integrals of f. If a full reconstruction is not possible then one tries to detect the location of boundaries (jump discontinuities of f) from local tomographic data.

In a related topic, Finch writes about the *attenuated X-ray transform.* Mathematically the problem is again to recover a function from its line integrals, but this time with an exponential weight. This problem arises in Single Photon Emission Computed Tomography (SPECT), where one would like to find the distribution of a radiopharmaceutical f in a cross section of the body from measuring the radiation outside the body. During the past few years there have been substantial developments in the mathematical theory of the inverse problem of recovering f from its attenuated X-ray transform, assuming the attenuation is known. For instance, very simple reconstruction formulas have been obtained, thanks to the work of Boman, Bukhgeim, Natterer, Novikov, Strömberg, and others. Finch describes these developments, which are expected to have significant applications.

An important class of inverse problems are *inverse scattering* problems. The chapter by Colton is devoted to this topic; in particular to the developments deriving from the sampling method developed by Colton, Kirsch, Kress and others. In inverse scattering a wave field is generated far away from a target having unknown physical properties and propagates through the region containing the target. The scattered field is measured and from this one attempts to determine the properties of the scatterer. A classical example of this type of problem is the determination of an obstacle or an acoustic medium by measuring the response to time harmonic waves. The success of radar and sonar soon caused scientists to ask if more could be determined about a scattering object than simply its location. However, due to the lack of a mathematical theory of inverse problems, together with limited computational capabilities, further progress was not possible at the time. This situation was dramatically changed in the mid 1960s with Tikhonov's introduction of regularization methods for ill-posed problems and subsequently, starting in the 1980s, the mathematical basis for the inverse scattering problem together with numerical algorithms for its solution began to be developed. Colton describes the tremendous progress made on this subject in the last 20 years or so.

Optical tomography is a relatively new imaging technique with several potential applications in diagnostic imaging. The goal in this imaging method is to determine the optical absorption and diffusion coefficients in a highly absorbing body by making boundary measurements of near-infrared light transmitted through the body. The possibility of performing imaging with infra-red radiation opens up numerous new possibilities compared to traditional tomography, but requires handling much more complex mathematical problems. This difficulty is due to the fact that infra-red radiation does not travel along straight lines as, for example, X-rays. Rather, due to multiple scattering interactions, it travels along essentially random paths inside the interior of tissues and objects. As a consequence, the forward problem becomes a highly nonlinear function of the model parameters, and the inverse problem becomes much harder. In spite of this, there have been substantial developments in both the theory and applications of optical tomography. The chapter by Stefanov describes some recent work on the inverse problem for the linear Boltzmann equation, relevant to optical tomography. This equation is also known as the radiative transport equation and is used in neutron transport and other fields.

Carney and Schotland consider *near-field tomography*, which is, roughly speaking, the use of inverse scattering methods to reconstruct tomographic images in near-field optics. In particular, they study near-field scanning optical microscopy, total internal reflection microscopy, and photon scanning tunneling microscopy.

The presence of evanescent fields makes near-field tomography very ill-posed. Carney and Schotland show how to use the overdeterminacy of the problem to develop a singular value decomposition of the relevant scattering operators. The chapter by Ola, Päivärinta, and Somersalo deals with inverse boundary problems associated to time harmonic electromagnetic fields at fixed frequency. It is shown that the electromagnetic parameters like the electrical permittivity, electrical conductivity and magnetic permeability can be reconstructed if one measures the tangential component of the magnetic and electric fields at the boundary of the medium. This information is equivalent to the near-field at fixed energy. The reconstruction method relies on the construction of exponentially growing solutions for Maxwell's system (also called complex geometrical optics solutions) pioneered by Calderón and Faddeev and further developed by Sylvester and Uhlmann for the Schrödinger equation. They are related to the evanescent fields in the chapter of Carney and Schotland, and perhaps the connection should be explored further.

One of the fascinating aspects of inverse problems is the continuous interplay between pure and applied mathematics. This interplay has been particularly noticeable in the applications of microlocal analysis (MA) to inverse problems. MA, which is, roughly speaking, local analysis in phase space, was developed about 30 years ago by Hörmander, Maslov, Sato, and many others in order to understand the propagation of singularities of solutions of partial differential equations. The early roots of MA were in the theory of geometrical optics. Microlocal analysis has been used successfully in determining the singularities of medium parameters in several inverse problems ranging from X-ray tomography to reflection seismology and electrical impedance tomography. The chapter by Finch, Lan and Uhlmann considers several applications of the theory of paired Lagrangian distributions to inverse problems. A close study is made of the microlocal analysis in three dimensions of the *X-ray transform with sources on a curve*, a topic also mentioned in Faridani's chapter. Other applications include the inverse backscattering problem for conormal potentials, electrical impedance tomography, and the microlocal characterization of the range of generalized Radon transforms.

De Hoop gives an extensive review of applications of MA to *reflection seismology*. In this inverse method one attempts to estimate the index of refraction of waves in the earth from seismic data measured at the Earth's surface. The techniques used in reflection seismology are very relevant to imaging using ultrasound. Seismic imaging creates images of the Earth's upper crust using seismic waves generated by artificial sources and recorded into extensive arrays of sensors (geophones or hydrophones). The technology is based on a complex, and rapidly evolving, mathematical theory that employs advanced solutions to a wave equation as tools to solve approximately the general seismic inverse problem. The heterogeneity and anisotropy of the Earth's crust require advanced mathematics to generate wave-equation solutions suitable for seismic imaging. In his chapter, de Hoop describes several important developments using MA to generate these wave-solutions by manipulating the wavefields directly on their phase space. De Hoop also considers some recent applications of MA to global seismology.

The study of propagation of singularities of solutions of partial differential equations, a fundamental question in MA, connects classical and quantum mechanics. The next three chapters make explicit and use that connection in several contexts.

The chapter by Petkov and Stoyanov considers, as does Colton's, *inverse scattering by an obstacle*, but with a different emphasis. The authors, using MA, study the information obtained from the singularities of the scattering kernel, which is the Fourier transform in frequency of the scattering amplitude. This determines the sojourn or travel times of rays incoming in a given direction and outgoing in another. The authors also consider at length the inverse problem of recovering geometric information about the obstacle from the set of sojourn times, also called the scattering length spectrum.

Vasy gives an extensive survey of *many body quantum scattering*, which is the analysis of motion of several interacting particles. In particular the author studies in detail the singularities of the scattering operator and propagation of singularities in many-body scattering. Vasy makes the analogy between the studies of these singularities and propagation of singularities for the wave equation for manifolds with corners. Vasy uses the information so obtained to prove inverse scattering results for single cluster to single cluster scattering. He also considers some inverse results for the three-body problem at low energies for the case of two-cluster to two-cluster scattering, by using complex geometrical optics solutions similar to the two-body problem. Finally he describes some recent results on scattering on locally symmetric spaces which has several features analogous to the many-body problem.

Time reversal mirrors have attracted a lot of attention in recent years because of potential applications in medical imaging, underwater acoustics, and other areas. It has been seen experimentally that inhomogeneities, randomness, and ergodicity contribute to much better refocusing of waves at the source, which is the desired feature of the time reversal mirror. The chapter by Bardos gives a rigorous analysis of this phenomenon for the case of cavities for which the associated classical flow is ergodic. MA provides the tools for the study of the high-frequency asymptotics or propagation of singularities of solutions in this case.

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