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Inverse Problems in Transport Theory

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Abstract. We study an inverse problem for the transport equation in a bounded domain in $\mathbb{R}^n$. Given the incoming flux on the boundary, we measure the outgoing one. The inverse problem is to recover the absorption coefficient $\sigma_a(x)$ and the collision kernel $k(x, v', v)$ from this data. This paper is a survey of recent results about general k's without assuming that k depends on a reduced number of variables. We present uniqueness results in dimensions $n \geq 3$ for the time dependent and the stationary problem, and in the time dependent case we study the inverse scattering problem as well. The proofs are constructive and lead to direct procedures for recovering σ_a and k. For $n = 2$ the problem of recovering k is formally determined and we prove uniqueness for small k and a stability estimates.

1. Introduction

This paper is a review of the recent progress in the study of inverse problems for the transport equation in $\mathbb{R}^n$, $n \geq 2$ by the author and M. Choulli [CSt1], [CSt2], [CSt3], [CSt4] and the author and G. Uhlmann [StU]. We are focused here on the case when the collision kernel k introduced below depends on all of its variables x, v', v. There are a lot of works dealing with k's of the form $k(x, v' \cdot v)$ that is also physically important but we will not discuss those results here.

Define the transport operator T by

$$Tf = -v \cdot \nabla_x f(x, v) - \sigma_a(x, v) f(x, v) + \int_V k(x, v', v) f(x, v')\, dv', \qquad (1\text{–}1)$$

where $f = f(x, v)$ represents the density of a particle flow at the point $x \in X$ moving with velocity $v \in V$. Here $X \subset \mathbb{R}^n$ is a bounded domain with C^1-boundary, $V \subset \mathbb{R}^n$ is the velocity space. We assume that V is an open set in Sections 2 and 3, and that $V = S^{n-1}$ (and dv is replaced by dS_v) in Section 4. All of our results in Sections 2 and 3 hold in the case when $V = S^{n-1}$ with obvious modifications. In fact, the case $V = S^{n-1}$ leads to some simplifications,

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for example there is no need to work locally in the open set $V \setminus \{0\}$ in some cases and the measure $d\tilde{\xi}$ can be chosen to be $d\xi$ in Section 3. The coefficient $\sigma_a(x,v) \geq 0$ above measures the absorption of particles at the point (x,v) due to change of velocity or absorption by the surrounding media. The collision kernel $k(x,v',v) \geq 0$ is related to the number of particles that change their velocity from v' to v at the point x.

We study inverse problems for both the time-dependent transport equation

$$(\partial_t - T)f = 0 \tag{1–2}$$

and the stationary transport equation

$$Tf = 0. \tag{1–3}$$

Set $\Gamma_\pm = \{(x,v) \in \partial X \times V : \pm n(x) \cdot v > 0\}$, where $n(x)$ is the outer normal to ∂X at $x \in \partial X$. A typical boundary value problem for (1–2) or (1–3) is associated with boundary conditions

$$f|_{\Gamma_-} = f_-, \tag{1–4}$$

where f_- is a given function on Γ_-, that depends also on t in the first case. On the boundary we measure the outgoing flux f_+ generated by a given incoming flux f_-, i.e., we assume that we know that so called *albedo* operator $\mathcal{A}$ defined by

$$\mathcal{A} : f_- \longmapsto f_+ = f|_{\Gamma_+}. \tag{1–5}$$

The inverse problem that we study is this: Can we determine the functions σ_a and k from a knowledge of $\mathcal{A}$? In the time dependent case we study the inverse scattering problem as well — recovery of σ_a and k from the knowledge of the scattering operator S. In order to ensure uniqueness of recovery of σ_a, we assume that σ_a depends on x only, since otherwise there is clearly no uniqueness. Under this assumption and some natural assumptions in the stationary case that guarantee the solvability of the direct problem (1–2), we not only prove uniqueness but give an explicit solution for the inverse problem. Our approach is based on the study of the singularities of the Schwartz kernel of $\mathcal{A}$ and we show that all information about σ_a and k is contained in those singularities. For large σ_a, those singularities have very small amplitudes and are hard to measure in real applications, so then this approach is of less interest for the practical recovery of σ_a and k. In Section 4 we prove some stability estimates.

2. The Time Dependent Transport Equation

2.1. Main results In this section we present inverse problems results about the time dependent transport equation (1–2)

$$\frac{\partial}{\partial t} u(x,v,t) = -v \cdot \nabla_x u(x,v,t) - \sigma_a(x,v) u(x,v,t) + \int_V k(x,v',v) u(x,v',t)\, dv'. \tag{2–1}$$

We first introduce some terminology and notation. The production rate $\sigma_p(x, v')$ is defined by

$$\sigma_p(x, v') = \int_V k(x, v', v)\, dv.$$

Following [RS] we say that the pair (σ_a, k) is *admissible* if

(i) $0 \le \sigma_a \in L^\infty(\mathbb{R}^n \times V)$,
(ii) $0 \le k(x, v', \cdot) \in L^1(V)$ for a.e. $(x, v') \in \mathbb{R}^n \times V$ and $\sigma_p \in L^\infty(\mathbb{R}^n \times V)$.

Throughout this paper we assume that σ_a and k are extended as 0 for $x \notin X$.

Set $T_0 = -v \cdot \nabla_x$ with its maximal domain. It is well-known that T_0 is a generator of a strongly continuous group $U_0(t)f = f(x - tv, v)$ of isometries on $L^1(X \times V)$ preserving nonnegative functions. Following the notation in [RS], we introduce the operators

$$[A_1 f](x, v) = -\sigma_a(x, v) f(x, v), \qquad T_1 = T_0 + A_1, \qquad D(T_1) = D(T_0),$$

$$[A_2 f](x, v) = \int_V k(x, v', v) f(x, v')\, dv', \quad T = T_0 + A_1 + A_2 \quad D(T) = D(T_0),$$

and set $A = A_1 + A_2$. Operators A_1 and A_2 are bounded on $L^1(X \times V)$ and T_1, T are generators of strongly continuous groups $U_1(t) = e^{tT_1}$, $U(t) = e^{tT}$, respectively [RS]. For $U_1(t)$ we have an explicit formula

$$[U_1(t)f](x, v) = e^{-\int_0^t \sigma_a(x - sv, v)\, ds} f(x - tv, v), \tag{2–2}$$

while for $U(t)$ we have

$$\|U(t)\| \le e^{Ct}, \quad C = \|\sigma_p\|_{L^\infty}. \tag{2–3}$$

We work in the Banach space $L^1(X \times V)$, so here $\|U(t)\|$ is the operator norm of $U(t)$ in $\mathcal{L}(L^1(X \times V))$. It should be mentioned also that $U(t)$ preserves the cone of nonnegative functions for $t \ge 0$.

One can define the wave operators associated with T, T_0 by

$$W_- = \operatorname*{s-lim}_{t \to \infty} U(t) U_0(-t), \tag{2–4}$$

$$\tilde{W}_+ = \operatorname*{s-lim}_{t \to \infty} U_0(-t) U(t). \tag{2–5}$$

If W_- and $\tilde{W}_+$ exist, one can define the scattering operator

$$S = \tilde{W}_+ W_-$$

as a bounded operator in $L^1(X \times V)$. Scattering theory for (1–1) has been developed in [Hej], [Si], [V1] and we refer to these papers (see also [RS]) for sufficient conditions guaranteeing the existence of S. We would like to mention here also [P1], [U], [E], [St], [V2], [CMS]. An abstract approach based on the Limiting Absorption Principle has been proposed in [Mo]. We show below however that S can always be defined as an operator $S : L^1_c(\mathbb{R}^n \times V \backslash \{0\}) \to L^1_{\text{loc}}(\mathbb{R}^n \times V \backslash \{0\})$.

The inverse scattering problem for (2–1) is the following: Does S determine uniquely σ_a, k? The answer is affirmative if σ_a is independent of v.

THEOREM 2.1 ([CSt1], [CSt2]). *Let (σ_a, k), $(\hat\sigma_a, \hat k)$ be two admissible pairs such that σ_a, $\hat\sigma_a$ do not depend on v and denote by S, $\hat S$ the corresponding scattering operators. Then, if $S = \hat S$, we have $\sigma_a = \hat\sigma_a$, $k = \hat k$.*

The assumption that σ_a, $\hat\sigma_a$ depend on x only can be relaxed a little by assuming that they depend on x and $|v|$ only. In the general case however, there is no uniqueness. Assume, for example, that $k = 0$. Then

$$Sf = e^{-\int_{-\infty}^{\infty}\sigma_a(x-sv,v)\,ds} f, \quad (k=0) \tag{2–6}$$

and therefore S determines $\int_{-\infty}^{\infty}\sigma_a(x-sv,v)\,ds$ only for any $v \in V$. It is easy to see that those integrals do not determine σ_a uniquely. If σ_a is independent of $v/|v|$ however, then S determines the X-ray transform of σ_a and it therefore determines σ_a in this case.

Next we consider the albedo operator $\mathcal{A}$. Assume that X is convex and has C^1-smooth boundary ∂X. Consider the measure $d\xi = |n(x)\cdot v|d\mu(x)\,dv$ on $\Gamma_\pm$, where $d\mu(x)$ is the measure on ∂X. We will solve the problem

$$\begin{cases} (\partial_t - T)u = 0 & \text{in } \mathbb{R}\times X\times V, \\ u|_{\mathbb{R}\times\Gamma_-} = u_-, \\ u|_{t\ll 0} = 0 \end{cases} \tag{2–7}$$

for $u(t,x,v)$, where $u_- \in L^1_{\mathrm{c}}(\mathbb{R};\, L^1(\Gamma_-,\, d\xi))$. Problem (2–7) has a unique solution $u \in C(\mathbb{R}; L^1(X\times V))$ and one defines the albedo operator $\mathcal{A}$ by

$$\mathcal{A}g = u|_{\mathbb{R}\times\Gamma_+} \in L^1_{\mathrm{loc}}(\mathbb{R}; L^1(\Gamma_+, d\xi)). \tag{2–8}$$

Therefore, $\mathcal{A} : L^1_{\mathrm{c}}(\mathbb{R};\, L^1(\Gamma_-,\, d\xi)) \to L^1_{\mathrm{loc}}(\mathbb{R}; L^1(\Gamma_+, d\xi))$. It can be seen that $\mathcal{A}g$ can be defined more generally for $g \in L^1(\mathbb{R}\times\Gamma_-,\, dt\, d\xi)$ with $g = 0$ for $t \ll 0$. We show below that in fact $\mathcal{A}$ determines S uniquely and conversely, S determines $\mathcal{A}$ uniquely by means of explicit formulae in case when X is convex. This generalizes earlier results in [AE], [EP], [P2] showing that there is a relationship between S and $\mathcal{A}$. To this end, let us define the extension operators $J_\pm$ and the restriction (trace) operators $R_\pm$ as follows. Set

$$\Omega = \{(x,v) \in \mathbb{R}^n \times V\backslash\{0\} : x - tv \in X \text{ for some } t \in \mathbb{R}\}, \tag{2–9}$$

and define the functions

$$\tau_\pm(x,v) = \min\{t \in \mathbb{R} : (x \pm tv, v) \in \Gamma_\pm\}, \quad (x,v) \in \Omega. \tag{2–10}$$

Given $g \in L^1(\mathbb{R}\times\Gamma_\pm,\, dt\, d\xi)$, consider the following operators of extension:

$$(J_\pm g)(x,v) = \begin{cases} g\big(\pm\tau_\pm(x,v),\, x \pm \tau_\pm(x,v)v,\, v\big) & \text{if } (x,v) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $J_\pm : L^1(\mathbb{R}\times\Gamma_\pm,\, dt\, d\xi) \to L^1(X\times V)$ are isometric. Denote by $R_\pm$ the operator of restriction

$$R_\pm f = f|_{\Gamma_\pm}, \quad f \in C(\mathbb{R}^n \times V).$$

Although $R_\pm$ is not a bounded operator on $L^1(X\times V)$ (see [Ce1], [Ce2] and Theorem 3.2 below), we can show that $R_\pm U_0(t)f \in L^1(\mathbb{R}\times\Gamma_\pm,\, dt\, d\xi)$ is well defined for any $f\in L^1(X\times V)$. Denote by χ_Ω the characteristic function of Ω. We establish the following relationships between S and $\mathcal{A}$.

Theorem 2.2 ([CSt1], [CSt2]). *Assume that X is convex. Then*

(a) $\mathcal{A}g = R_+U_0(t)SJ_-g$ *for all* $g\in L^1_{\mathrm{c}}(\mathbb{R}\times\Gamma_-, dt\, d\xi)$,
(b) $Sf = J_+\mathcal{A}R_-U_0(t)f + (1-\chi_\Omega)f$, $f\in L^1_{\mathrm{c}}(\mathbb{R}^n\times V\backslash\{0\})$,
(c) $\mathcal{A}$ *extends to a bounded operator*

$$\mathcal{A} : L^1(\mathbb{R}\times\Gamma_-,\, dt\, d\xi) \to L^1(\mathbb{R}\times\Gamma_+,\, dt\, d\xi)$$

if and only if S extends to a bounded operator on $L^1(X\times V)$.

We decompose $L^1(\mathbb{R}^n\times V) = L^1(\Omega)\oplus L^1((\mathbb{R}^n\times V)\setminus\Omega)$. A similar decomposition holds for $L^1_{\mathrm{c}}(\mathbb{R}^n\times V\backslash\{0\})$. Then S leaves invariant both spaces, moreover $S|_{L^1((\mathbb{R}^n\times V)\setminus\Omega)} = \mathrm{Id}$, so S can be decomposed as a direct sum $S = S_\Omega\oplus\mathrm{Id}$. Denote $\mathcal{R}_\pm = R_\pm U_0(\,\cdot\,) : L^1(\Omega)\to L^1(\mathbb{R}\times\Gamma_\pm,\, dt\, d\xi)$. Then $\mathcal{R}_\pm$ are isometric and invertible and $\mathcal{R}_\pm^{-1} = \mathcal{J}_\pm$ with $\mathcal{J}_\pm : L^1(\mathbb{R}\times\Gamma_\pm,\, dt\, d\xi)\to L^1(\Omega)$, $\mathcal{J}_\pm f := J_\pm f|_{L^1(\Omega)}$. Then we can rewrite Theorem 2.2 (a), (b) as follows:

$$\begin{aligned} \mathcal{A} &= \mathcal{R}_+S_\Omega\mathcal{J}_- &&\text{on } L^1_{\mathrm{c}}(\mathbb{R}\times\Gamma_-, dt\, d\xi),\\ S_\Omega &= \mathcal{J}_+\mathcal{A}\mathcal{R}_- &&\text{on } L^1_{\mathrm{c}}(\mathbb{R}^n\times V\backslash\{0\}). \end{aligned}$$

Thus $\mathcal{A}$ can be obtained from S_Ω by a conjugation with invertible isometric operators and vice-versa.

An immediate consequence of Theorem 2.2 is that $\mathcal{A}$ determines σ_a and k uniquely for σ_a independent of v and X convex. In short, in this case the inverse boundary value problem is equivalent to the inverse scattering problem. However, we can prove uniqueness for the inverse boundary value problem for not necessarily convex domains as well independently of Theorems 2.1 and 2.2.

Theorem 2.3 [CSt1], [CSt2]. *Let (σ_a, k), $(\hat\sigma_a, \hat k)$ be two admissible pairs with σ_a, $\hat\sigma_a$ independent of v. Then, if the albedo operators $\mathcal{A}$, $\hat{\mathcal{A}}$ on ∂X coincide, we have $\sigma_a = \hat\sigma_a$, $k = \hat k$.*

2.2. Singular decomposition of the fundamental solution and the kernels of $\mathcal{S}$ and $\mathcal{A}$. Proof of the main results in section 2.1. The key to proving the uniqueness results above is to study the singularities of the Schwartz kernel of S and respectively $\mathcal{A}$. To this end we will study first the kernel of the solution operator of the problem (2–7). Given $(x', v') \in \mathbb{R}^n \times V\backslash\{0\}$, consider the problem

$$\begin{cases} (\partial_t - T)u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \times V, \\ u|_{t\ll 0} = \delta(x - x' - tv)\delta(v - v'), \end{cases} \tag{2–11}$$

δ being the Dirac delta function. Problem (2–1) has a unique solution

$$u^{\#}(t, x, v, x', v')$$

depending continuously on t with values in $\mathcal{D}'(\mathbb{R}^n_x \times V_v \times \mathbb{R}^n_{x'} \times V_{v'}\backslash\{0\})$. We have the following singular expansion of $u^{\#}$.

THEOREM 2.4 [CSt1], [CSt2]. *Problem* (2–11) *has the unique solution* $u^{\#} = u_0^{\#} + u_1^{\#} + u_2^{\#}$, *where*

$$u_0^{\#} = e^{-\int_0^\infty \sigma_a(x-sv,v)\,ds}\delta(x - x' - tv)\delta(v - v'),$$

$$u_1^{\#} = \int_0^\infty e^{-\int_0^s \sigma_a(x-\tau v,v)d\tau} e^{-\int_0^\infty \sigma_a(x-sv-\tau v',v')d\tau} \times k(x-sv, v', v)\delta(x - sv - (t-s)v' - x')\,ds,$$

$$u_2^{\#} \in C\big(\mathbb{R};\, L^\infty_{\mathrm{loc}}(\mathbb{R}^n_{x'} \times V_{v'};\, L^1(\mathbb{R}^n_x \times V_v))\big).$$

The proof of Theorem 2.4 is based on iterating twice Duhamel's formula

$$U(t-r) = U_1(t-r) + \int_r^t U(t-s)A_2U_1(s-r)\,ds$$

and on estimating the remainder term.

To build the scattering theory for the transport equation, we first show that the wave operators W_-, $\tilde{W}_+$ (see (2–4), (2–5)) exist as operators between the spaces

$$W_- : L^1_{\mathrm{c}}(\mathbb{R}^n \times V\backslash\{0\}) \longrightarrow L^1(X \times V),$$
$$\tilde{W}_+ : L^1(X \times V) \longrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n \times V\backslash\{0\}).$$

Then we define the scattering operator

$$S = \tilde{W}_+W_- : L^1_{\mathrm{c}}(\mathbb{R}^n \times V\backslash\{0\}) \longrightarrow L^1_{\mathrm{loc}}(\mathbb{R}^n \times V\backslash\{0\}). \tag{2–12}$$

It can be seen that S is well defined on the wider subset $\big\{f : \exists t_0$ such that $U_0(t)f = 0$ for $x \in X,\, t < -t_0\big\}$ (the incoming space).

The distribution $u^{\#}(t, x, v, x', v')$ is the Schwartz kernel of $U(t)W_-$. Let $\mathcal{S}(x, v, x', v')$ be the Schwartz kernel of the scattering operator S. Then

$$\mathcal{S}(x, v, x', v') = \lim_{t\to\infty} u^{\#}(t, x+tv, v, x', v').$$

This limit exists trivially, in fact for any $K \Subset \mathbb{R}^n \times V\backslash\{0\}$, the distribution $u^{\#}(t, x+tv, v, x', v')|_K$ is independent of t for $t \geq t_0(K)$. On the other hand, as mentioned in the Introduction, it is not trivial to show that under some condition, $\mathcal{S}$ is a kernel of a bounded operator in $L^1(\mathbb{R}^n \times V)$. One can also prove the following integral representation of the scattering kernel:

$$\mathcal{S}(x,v,x',v') = e^{-\int_{-\infty}^{\infty} \sigma_a(x-\tau v,v)d\tau}\delta(x-x')\delta(v-v') + \int_{-\infty}^{\infty} e^{-\int_s^{\infty} \sigma_a(x+\tau v,v)d\tau}(A_2 u^{\#})(s,x+sv,v,x',v')\,ds. \quad (2\text{–}13)$$

This formula is an analogue of the representation of the scattering amplitude (in our setting, that would be the second term in the right-hand side above) for the Schrödinger equation.

Now, combining Theorem 2.4 and the representation above, we get the following express for the kernel $\mathcal{S}$ of the scattering operator S.

THEOREM 2.5 [CSt1], [CSt2]. *We have $S = S_0 + S_1 + S_2$, where the Schwartz kernels $\mathcal{S}_j(x,v,x',v')$ of the operators S_j, $j=0,1,2$ satisfy*

$$\mathcal{S}_0 = e^{-\int_{-\infty}^{\infty} \sigma_a(x-\tau v,v)d\tau}\delta(x-x')\delta(v-v'),$$

$$\mathcal{S}_1 = \int_{-\infty}^{\infty} e^{-\int_s^{\infty} \sigma_a(x+\tau v,v)d\tau} e^{-\int_0^{\infty} \sigma_a(x+sv-\tau v',v')d\tau} k(x+sv,v',v)\delta(x-x'+s(v-v'))\,ds,$$

$$\mathcal{S}_2 \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^n_{x'} \times V_{v'}\backslash\{0\};\, L^1_{\mathrm{loc}}(\mathbb{R}^n_x \times V_v\backslash\{0\})).$$

We are ready now to complete the proof of Theorem 2.1. The idea of the proof is the following. Suppose we are given the scattering operator S corresponding to a unknown admissible pair (σ_a, k). Then we know the kernel $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$. It follows from Theorem 2.5 that $\mathcal{S}_0$ is a singular distribution supported on the hyperplane $x = x'$, $v = v'$ of dimension $2n$, $\mathcal{S}_1$ is supported on a $(3n+1)$-dimensional surface (for $v \neq v'$), while $\mathcal{S}_2$ is a function. Therefore, $\mathcal{S}_j$, $j=0,1,2$ have different degrees of singularities and given $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1 + \mathcal{S}_2$, one can always recover $\mathcal{S}_0$ and $\mathcal{S}_1$. From $\mathcal{S}_0$ one can recover the X-ray transform of σ_a and therefore σ_a itself, provided that σ_a is independent of v. Next, suppose for simplicity that σ_a and k are continuous. Then for fixed x, v, v' with $v \neq v'$, $\mathcal{S}_1$ is a delta-function supported on the line $x' = x + s(v-v')$, $s \in \mathbf{R}$ with density a multiple of $k(x+sv,v',v)$. Therefore, one can recover that density for each s and in particular setting $s=0$, we get $k(x,v',v)$. Moreover, based on this, we can write explicit formulae that extract σ_a and k from $\mathcal{S}$ by allowing $\mathcal{S}$ to act on a sequence of suitably chosen test functions that concentrate on one of the singular varieties described above, see [CSt2].

We will skip the proof of Theorem 2.2. We merely recall that it proves uniqueness for the inverse boundary value problem for convex X as a direct consequence of Theorem 2.5.

Now assume that X is not necessarily convex. We can still prove uniqueness for the inverse boundary value problem by showing that $\mathcal{A}$ determines uniquely $u^{\#}$ for x outside X by following arguments in [SyU], and then using (2–13) and Theorem 2.5. In order to give a constructive (in fact, explicit) reconstruction, we study next the Schwartz kernel of the operator $\mathcal{A}$ in the spirit of Theorem 2.5. A priori, this kernel is a distribution in $\mathcal{D}'(\mathbb{R}\times\Gamma_+\times\mathbb{R}\times\Gamma_-)$. Denote by δ_1 the Dirac delta function on $\mathbb{R}^1$ and by $\delta_y(x)$ the Delta function on ∂X defined by $\int_{\partial X}\delta_y\varphi\,d\mu(x)=\varphi(y)$. Set

$$\theta(x,y)=\begin{cases}1 & \text{if } px+(1-p)y\in X \text{ for each } p\in[0,1],\\ 0 & \text{otherwise.}\end{cases}$$

THEOREM 2.6 [CSt1], [CSt2]. *The Schwartz kernel of $\mathcal{A}$ has the form*

$$\alpha(t-t',x,v,x',v'),$$

that is, *formally*,

$$(\mathcal{A}g)(t,x,v)=\int_{\mathbb{R}\times\Gamma_-}\alpha(t-t',x,v,x',v')g(t',x',v')\,d\xi(x',v')$$

with $\alpha=\alpha_0+\alpha_1+\alpha_2$, *where the* $\alpha_j(\tau,x,v,x',v')$ (*with* $(x,v)\in\Gamma_+$, $(x',v')\in\Gamma_-$) *satisfy*

$$\alpha_0=|n(x')\cdot v'|^{-1}e^{-\int_0^{\tau_-(x,v)}\sigma_a(x-sv,v)\,ds}\delta_{\{x-\tau_-(x,v)v\}}(x')\delta(v-v')\delta_1(\tau-\tau_-(x,v)),$$

$$\alpha_1=|n(x')\cdot v'|^{-1}\int e^{-\int_0^s\sigma_a(x-pv,v)\,dp}e^{-\int_0^{\tau_-(x-sv,v')}\sigma_a(x-sv-pv',v')\,dp}\times$$
$$\delta_1(\tau-s-\tau_-(x-sv,v'))k(x-sv,v',v)\delta_{\{x-sv-\tau_-(x-sv,v')v'\}}(x')\theta(x-sv,x)\,ds,$$

$$\alpha_2\in L^\infty\big(\Gamma_-;\,L^1_{\mathrm{loc}}(\mathbb{R}_\tau;\,L^1(\Gamma_+,d\xi))\big).$$

The proof of Theorem 2.3 now follows from the theorem above by analyzing the singularities of α as above. In this case, we can also wite explicit formulae for σ_a and k as certain limits of the distribution α acting on a sequence of test functions concentrating near the singularities of α_0 and α_1, respectively; see [CSt2].

3. The Stationary Transport Equation

We turn our attention now to the boundary value problem (1–3), (1–4) for the stationary transport equation

$$\begin{cases}-v\cdot\nabla_x f(x,v)-\sigma_a(x,v)f(x,v)+\displaystyle\int_V k(x,v',v)f(x,v')\,dv'=0 \quad \text{in } X\times V,\\ f|_{\Gamma_-}=f_-.\end{cases}\tag{3–1}$$

Here X does not need to be convex. We impose some conditions in order to ensure the unique solvability of the direct problem (3–1). Recall the definition (2–10) of $\tau_\pm$. Set $\tau=\tau_-+\tau_+$. We will consider two cases. First we assume that

$$\|\tau\sigma_a\|_{L^\infty}<\infty,\qquad \|\tau\sigma_p\|_{L^\infty}<1.\tag{3–2}$$

This condition in particular guarantees that the dynamics $U(t)$ corresponding to the time-dependent transport equation $(\partial_t-T)u=0$ is subcritical [RS], that is, the L^1-norm of the solution is uniformly bounded for $t>0$; compare with (2–3).

Note that (3–2) holds if in particular $\big\||v|^{-1}\sigma_a\big\|_{L^\infty} < \infty$, diam $X\big\||v|^{-1}\sigma_p\big\|_{L^\infty} < 1$. The second situation we will consider is when [DL]

$$0 \le \nu \le \sigma_a(x,v) - \sigma_p(x,v) \quad \text{for a.e. } (x,v) \in X \times V \tag{3–3}$$

with some $\nu > 0$. This condition means that the absorption rate is greater than the production rate. This also implies that the corresponding dynamics is subcritical.

The main result in this section is this:

THEOREM 3.1 [CSt3], [CSt4]. *Assume that* (σ_a, k), $(\hat\sigma_a, \hat k)$ *are two admissible pairs with* $\sigma_a = \sigma_a(x,|v|)$, $\hat\sigma_a = \hat\sigma_a(x,|v|)$ *and assume that they satisfy either* (3–2) *or* (3–3). *Assume that the corresponding albedo operators* $\mathcal{A}$ *and* $\hat{\mathcal{A}}$ *coincide*. *Then*

(a) *if* $n \ge 3$, *then* $\sigma_a = \hat\sigma_a$, $k = \hat k$;
(b) *if* $n = 2$, *then* $\sigma_a = \hat\sigma_a$.

In the stationary case we have less data than in the time dependent one studied in Section 2 because the time variable is not present. The inverse boundary value problem is overdetermined in dimension $n \ge 3$ because the kernel of $\mathcal{A}$ depends on $4n-2$ variables while k is a function of $3n$ variables, $\sigma_a(x, v/|v|)$ is a function of $2n-1$ variables. In the 2D case however, the inverse problem for recovery of k is formally determined (but still overdetermined for the recovery of σ_a) and the theorem above does not provide uniqueness in this case. In Section 4 we formulate a uniqueness result in the 2D case for small k.

To prove Theorem 3.1, we again study the Schwartz kernel of the albedo operator $\mathcal{A}$. Note that in the stationary case, $\mathcal{A}$ acts on functions on Γ_- and maps them to functions on Γ_+ (see next subsection for details). It turns out that, similarly to the time dependent case, the kernel of $\mathcal{A}$ has a singular decomposition $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, where α_0 and α_1 are delta functions for $n \ge 3$ supported on varieties of different dimensions with densities that identify respectively σ_a and k. The third term α_2 is a locally L^1 function and can be distinguished form α_0 and α_1. If $n = 2$, α_0 is still a delta function, but α_1 is a locally L^1 function and cannot be distinguished from α_2 and this explains why this method does not work then. The 2D case is considered separately in next section.

3.1. Some estimates in the stationary case Similarly to the measure defined in Section 2.1, we define a measure on $\Gamma_\pm$ by:

$$d\tilde\xi = \min\{\tau(x,v), \lambda\}\big|n(x)\cdot v\big|\, d\mu(x)\, dv,$$

where $\lambda > 0$ is an arbitrary constant. Using the trace Theorem 3.2 below, we can show that $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$ is a bounded operator if (3–2) holds and $\mathcal{A} : L^1(\Gamma_-, d\tilde\xi) \to L^1(\Gamma_+, d\tilde\xi)$ is bounded if (3–3) holds; see also [DL]. Note that if a neighborhood of the origin is not included in V or in particular, if $V = S^{n-1}$, then we can choose $d\tilde\xi = \tau d\xi$.

Theorem 3.2 (Trace theorem). (a) [CSt3]

$$\big\|f|_{\Gamma_\pm}\big\|_{L^1(\Gamma_\pm,d\xi)} \le \|T_0 f\| + \|\tau^{-1} f\|.$$

(b) [Ce1], [Ce2]

$$\big\|f|_{\Gamma_\pm}\big\|_{L^1(\Gamma_\pm,d\tilde\xi)} \le \lambda\|T_0 f\| + \|f\|.$$

Note that (b) follows from (a) by setting $f = \min\{\lambda,\tau\}g$.

We introduce the spaces $\mathcal{W}$, $\tilde{\mathcal{W}}$ via the norms

$$\|f\|_{\mathcal{W}} = \|T_0 f\| + \|\tau^{-1} f\|, \qquad \|f\|_{\tilde{\mathcal{W}}} = \|T_0 f\| + \|f\|.$$

Then Theorem 3.2 says that taking the trace $f \mapsto f|_{\Gamma_\pm}$ is a continuous operator from $\mathcal{W}$ into $L^1(\Gamma_\pm, d\xi)$ and similarly from $\tilde{\mathcal{W}}$ into $L^1(\Gamma_\pm, d\tilde\xi)$.

Given $f_- \in L^1(\Gamma_-, d\xi)$, define Jf_- as the following prolongation of f_- inside $X \times V$:

$$Jf_- = e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-sv,v)\,ds} f_-(x - \tau_-(x,v)v, v), \qquad (x,v) \in X \times V.$$

Jf_- is defined so that $T_1 Jf_- = 0$, $Jf_-|_{\Gamma_-} = f_-$, therefore J is the solution operator of the problem $T_1 f = 0$, $f|_{\Gamma_-} = f_-$.

Proposition 3.1. (a) *Assume that* $\|\tau\sigma_a\|_{L^\infty} < \infty$. *Then*

$$\|Jf_-\|_{\mathcal{W}} \le C\|f_-\|_{L^1(\Gamma_-,d\xi)},$$

with $C = 1 + \|\tau\sigma_a\|_{L^\infty}$. *If* $\sigma_a = 0$, *then we have equality above* (*and* $C = 1$).
(b) *Assume* (3–3). *For any* $f_- \in L^1(\Gamma_-, d\tilde\xi)$ *we have*

$$\|Jf_-\|_{\tilde{\mathcal{W}}} \le C\|f_-\|_{L^1(\Gamma_-,d\tilde\xi)},$$

where $C = (1 + \|\sigma_a\|_{L^\infty}) \max\{1, (\nu\lambda)^{-1}\}$.

We reduce the boundary value problem (3–1) to an integral equation using standard arguments to get

$$(I + K)f = Jf_-, \tag{3–4}$$

where I stands for the identity and K is the integral operator

$$Kf = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)\,ds} (A_2 f)(x - tv, v)\,dt. \tag{3–5}$$

Formally, $K = T_1^{-1} A_2$, and for T_1^{-1} we have

$$T_1^{-1} f = -\int_0^{\tau_-(x,v)} e^{-\int_0^t \sigma_a(x-sv,v)\,ds} f(x - tv, v)\,dt.$$

Proposition 3.2. *Assume* (3–2). *Then*:

(a) $\tau^{-1}T_1^{-1}$, $\tau^{-1}T^{-1}$ *and* $A_2\tau$ *are bounded operators in* $L^1(X \times V)$ *and therefore* $K = T_1^{-1}A_2$ *is a bounded operator in* $L^1(X \times V; \tau^{-1}\,dx\,dv)$. *Moreover*, *the operator norm of* K *is not greater than* $\|\tau\sigma_p\|_{L^\infty} < 1$ *and therefore* $(I+K)^{-1}$ *exists in this space*.

(b) *The integral equation* (3–4) *and therefore the boundary value problem* (3–1) *are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\xi)$ *and then* $f \in \mathcal{W}$.
(c) *The albedo operator* $\mathcal{A}$ *is a bounded map* $\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi)$.

PROPOSITION 3.3. *Assume* (3–3). *Then*:

(a) K, T_1^{-1} *and* T^{-1} *are bounded operators in* $L^1(X \times V)$ *and* $K = T_1^{-1} A_2$. *Further*, $I + K$ *is invertible and* $(I+K)^{-1} = I - T^{-1}A_2$.
(b) *The integral equation* (3–4) *and therefore the boundary value problem* (3–1) *are uniquely solvable for any* $f_- \in L^1(\Gamma_-, d\tilde{\xi})$ *and then* $f \in \tilde{\mathcal{W}}$.
(c) $\mathcal{A}$ *is a bounded map* $\mathcal{A} : L^1(\Gamma_-, d\tilde{\xi}) \to L^1(\Gamma_+, d\tilde{\xi})$.

3.2. The fundamental solution of the stationary transport equation. We solve (3–1) with $f_- = \phi_-$, where

$$\phi_- = \frac{1}{|n(x') \cdot v'|} \delta_{\{x'\}}(x) \delta(v - v'),$$

where $(x', v') \in \Gamma_-$ are regarded as parameters, $n(x')$ is the outer normal, and $\delta_{\{x'\}}$ is a distribution on ∂X defined by $(\delta_{\{x'\}}, \varphi) = \int \delta_{\{x'\}}(x)\varphi(x) d\mu(x) = \varphi(x')$. On the other hand, we will denote by δ the ordinary Dirac delta function in $\mathbb{R}^n$. The integral above is to be considered in distribution sense. Let us denote by $\phi(x, v, x', v')$ the solution (in distribution sense) of

$$\begin{cases} T\phi = 0 & \text{in } X \times V, \\ \phi|_{\Gamma_-} = \phi_-. \end{cases} \tag{3–6}$$

To solve (3–6), we write

$$\varphi = J\varphi_- - KJ\varphi_- + (I+K)^{-1} K^2 J \varphi_-$$

and analyze each term. To treat the third term, we observe that

$$(I+K)^{-1} K^2 J\varphi_- = T^{-1} A_2 K J \varphi_-.$$

We are led to:

THEOREM 3.3. *Assume that* (σ_a, k) *is admissible and either* (3–2) *or* (3–3) *holds. Then the solution* $\phi(x, v, x', v')$ *of* (3–6) *can be written as* $\phi = \phi_0 + \phi_1 + \phi_2$, *where*

$$\phi_0 = \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x - pv, v)\, dp} \delta(x - x' - tv) \delta(v - v')\, dt,$$

$$\phi_1 = \int_0^{\tau_-(x,v)} \int_0^{\tau_+(x',v')} e^{-\int_0^{s} \sigma_a(x - pv, v)\, dp} e^{-\int_0^{\tau_-(x-sv,v')} \sigma_a(x - sv - pv', v')\, dp} \times k(x - sv, v', v) \delta(x - x' - sv - tv')\, dt\, ds,$$

and

$$\phi_2 \in L^\infty(\Gamma_-; \mathcal{W}) \qquad \textit{if } (3–2) \textit{ holds},$$
$$\min\{\tau, \lambda\}^{-1} \phi_2 \in L^\infty(\Gamma_-; \tilde{\mathcal{W}}) \qquad \textit{if } (3–3) \textit{ holds}.$$

The so constructed solution $\phi(x,v,x',v')$ is the distribution kernel of the solution operator $f_- \mapsto f$ of (3–1). In order to find the distribution kernel $\alpha(x,v,x',v')$ ($(x,v) \in \Gamma_+$, $(x',v') \in \Gamma_-$) of the albedo operator $\mathcal{A}$, it is enough to set

$$\alpha(x,v,x',v') := \phi(x,v,x',v')|_{(x,v)\in\Gamma_+}, \qquad (x',v') \in \Gamma_-.$$

Then, in the sense of distributions,

$$(\mathcal{A}f_-)(x,v) = \int_{\Gamma_-} \alpha(x,\theta,x',\theta') f_-(x',\theta')\, d\xi(x',\theta') \qquad \text{for all } f_- \in C_0^\infty(\Gamma_+).$$

Theorem 3.3 yields the following.

THEOREM 3.4. *Assume that* (σ_a, k) *is an admissible pair and that either* (3–2) *or* (3–3) *holds. Then the distribution kernel* $\alpha(x,v,x',v')$ *of* $\mathcal{A}$ *satisfies* $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ *with*

$$\alpha_0 = \frac{1}{n(x)\cdot v} e^{-\int_0^{\tau_-(x,v)} \sigma_a(x-pv,v)\,dp} \delta_{\{x'+\tau_+(x',v')v'\}}(x)\delta(v-v'),$$

$$\alpha_1 = \frac{1}{n(x)\cdot v} \int_0^{\tau_+(x',v')} e^{-\int_0^{\tau_+(x'+tv',v)} \sigma_a(x-pv,v)\,dp} e^{-\int_0^t \sigma_a(x+pv',v')\,dp} \times k(x+tv',v',v)\delta_{\{x'+tv'+\tau_+(x'+tv',v)\}}(x)\,dt,$$

and

$$\alpha_2 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\xi)) \qquad \textit{if } (3\text{–}2) \textit{ holds},$$
$$\min\{\tau(x',v'),\lambda\}^{-1}\alpha_2 \in L^\infty(\Gamma_-; L^1(\Gamma_+, d\tilde{\xi})) \qquad \textit{if } (3\text{–}3) \textit{ holds}.$$

Theorem 3.4 implies the following way for proving Theorem 3.1. Assume that we are given the albedo operator $\mathcal{A}$, corresponding to some admissible pair (σ_a, k), satisfying either (3–2) or (3–3). Then we also know the distribution $\alpha(x,v,x',v')$. By Theorem 3.4, $\alpha = \alpha_0+\alpha_1+\alpha_2$. Here α_0 is a delta-type distribution supported on a $(2n-1)$–dimensional variety in $\Gamma_+ \times \Gamma_-$. Next, α_1 is also a delta-type distribution (provided that $n \geq 3$) supported on a $3n$-dimensional variety in $\Gamma_+ \times \Gamma_-$, while α_2 is a (locally L^1) function on the $(4n-2)$-dimensional $\Gamma_+ \times \Gamma_-$. Notice that if $n = 2$, then α_1 is a function as well. Therefore, if $n \geq 3$, one can distinguish between $\alpha_0 + \alpha_1$ and α_2. Moreover, since α_0 and α_1 have different degrees of singularities, one can recover α_0 and α_1. Now, if $\sigma_a = \sigma_a(x,|v|)$, then α_0 determines the X-ray transform $\int \sigma_a(x+s\omega,|v|)\,ds$ of σ_a for all x, $|v|$ and ω in an open subset of S^{n-1} (for all $\omega \in S^{n-1}$ if V is spherically symmetric). This determines uniquely σ_a (see e.g. [H]). Next, once we know σ_a, from α_1 we can recover k. If $n = 2$, then we can recover α_0 and therefore σ_a, but we cannot (at least using those arguments) distinguish between α_1 and α_2 which are both functions and therefore this approach does not work for reconstructing k in two dimensions. Based on those arguments, we can write explicit formulas for recovering the X-ray transform of σ_a and for recovering k as limits of the action of α on certain sequences of test functions with supports shrinking to the singularities of α_0 and α_1, respectively; see [CSt4].

4. The Two-Dimensional Stationary Transport Equation

In this section we study the inverse problem for the stationary transport equation (1–3) in the 2D case. As explained in Section 2, the inverse problem of recovering k from the albedo operator $\mathcal{A}$ is formally determined in this case. We prove below that there exists unique solution provided that k is small enough in the L^∞ norm and we also derive a stability estimate. The results of this section are based on joint work by the author and G. Uhlmann [StU].

Let $X \subset \mathbb{R}^2$ be an open convex set with smooth boundary and let us write the 2D stationary transport equation (1–3) in the form

$$-v(\theta)\cdot\nabla_x f - \sigma_a(x,\theta)f + \int_{S^1} k(x,\theta',\theta)f(x,\theta')\,d\theta' = 0, \tag{4–1}$$

where $x \in X$, $\theta, \theta' \in S^1$ and

$$v(\theta) = (\cos\theta, \sin\theta).$$

We assume that σ_a and k are in L^∞. For simplicity, we consider the case $V = S^1$ here.

In this paper we work with k sufficiently small and under this assumption the direct problem (3–1) is always uniquely solvable with $f_- \in L^\infty(X \times S^1)$ and $\mathcal{A}f_- \in L^\infty(\Gamma_+)$. For the purpose of the inverse problem however, it is enough to think of $\mathcal{A}$ as an operator mapping $C_0^\infty(\Gamma_-)$ to $L^\infty(\Gamma_+)$.

Uniqueness and stability in the 2D case are proved for small k in the case when $k = k(x, \cos(\theta-\theta'))$ in [R1], and in the case $k = k(\theta,\theta')$ uniqueness for small k is proved in [T]. Note that in those cases the inverse problem is still formally overdetermined. In this section we prove a uniqueness results for general (but still small) $k(x,\theta',\theta)$. Our first result addresses the uniqueness of this inverse problem.

Theorem 4.1 [StU]. *Define the class*

$$\mathcal{U}_{\Sigma,\varepsilon} = \left\{(\sigma_a(x), k(x,\theta',\theta)) : \|\sigma_a\|_{L^\infty} \le \Sigma,\ \|k\|_{L^\infty} \le \varepsilon\right\}. \tag{4–2}$$

For any $\Sigma > 0$ there exists $\varepsilon > 0$ such that a pair $(\sigma_a, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ is uniquely determined by its albedo operator $\mathcal{A}$ in the class $\mathcal{U}_{\Sigma,\varepsilon}$.

To prove Theorem 4.1, we study again singularities of the distribution kernel $\alpha(x,\theta,x',\theta')$ of $\mathcal{A}$ as in Section 3. In the two dimensional case however $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ with α_0, α_1, α_2 as in Theorem 3.4, but α_1 is not a delta type distribution anymore, instead it is an L^1 function and cannot be distinguished from α_2 as in Section 3. We denote $b = \alpha_1 + \alpha_2$. The term α_1 does have a singularity (integrable) at $\theta = \pm\theta'$ and it is $L^\infty_{\rm loc}$ outside this set. Similarly, α_2 has a weaker, logarithmic singularity, as shown below. We will prove below that $\sin(\theta-\theta')b \in L^\infty$. We can therefore write

$$\alpha = \frac{\delta(\theta-\theta')\delta_{\{x'+\tau_+(x',\theta')v(\theta')\}}(x)}{n(x)\cdot v(\theta)} e^{-a(x',\theta')} + b(x,\theta,x',\theta'), \tag{4–3}$$

where

$$a(x',\theta') = \int_0^{\tau_+(x',\theta')} \sigma_a(x'+tv(\theta'),\theta')\,dt, \qquad \sin(\theta-\theta')b \in L^\infty. \tag{4–4}$$

In particular, knowing $\mathcal{A}$, we can uniquely determine a and b. Let we have two pairs (σ_a, k) and $(\tilde\sigma_a, \tilde k)$ with albedo operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. Set

$$\delta_1 = \|a-\tilde a\|_{H^1(\Gamma_-)}, \qquad \delta_2 = \|(b-\tilde b)\sin(\theta-\theta')\|_{L^\infty(\Gamma_-\times\Gamma_+)}. \tag{4–5}$$

Our second result is the following stability estimate.

THEOREM 4.2 [StU]. *Let*

$$\mathcal{V}^s_{\Sigma,\varepsilon} = \big\{(\sigma_a(x), k(x,\theta',\theta)) \in H^s(X)\times C(X\times S^1\times S^1) : \\ \|\sigma_a\|_{H^s} \le \Sigma,\ \|k\|_{L^\infty} \le \varepsilon\big\}. \tag{4–6}$$

Then, for any $s>1$, $\Sigma>0$, there exists $\varepsilon>0$ such that for any $(\sigma_a,k)\in\mathcal{V}^s_{\Sigma,\varepsilon}$ and $(\tilde\sigma_a,\tilde k)\in\mathcal{V}^s_{\Sigma,\varepsilon}$ and $0<\mu<1-1/s$, there exists $C>0$ such that

$$\|\sigma_a-\tilde\sigma_a\|_{L^\infty} \le C\delta_1^{1-1/s-\mu},$$
$$\|k-\tilde k\|_{L^\infty} \le C(\delta_1^{1-1/s-\mu}+\delta_2).$$

REMARK. It follows from (4–15) and (4–16) below that we can choose $\varepsilon = C(d)e^{-2d\Sigma}$, where $d = \operatorname{diam} D$.

IDEA OF THE PROOF OF THEOREM 4.1. Recall that σ_a and k are L^∞ in all variables. It is convenient to think later that σ_a and k are extended as 0 for $x\notin X$.

First we reduce the boundary value problem

$$\begin{cases} Tf = 0 & \text{in } X\times S^1, \\ f|_{\Gamma_-} = f_- \in C_0^\infty(\Gamma_+) \end{cases} \tag{4–7}$$

to the integral equation (3–4). Then f is given by

$$f = (I+K)^{-1}Jf_-, \tag{4–8}$$

provided that $I+K$ is invertible in a suitable space. The definition (3–5) of K implies immediately that

$$\|Kf\|_{L^\infty(X\times S^1)} \le C\|f\|_{L^\infty(X\times S^1)},$$

where $C = \operatorname{diam} X\|k\|_{L^\infty}$. Therefore, if $(\sigma_a,k)\in\mathcal{U}_{\Sigma,\varepsilon}$ and $\varepsilon < 1/\operatorname{diam} X$, then $I+K$ is invertible in $L^\infty(X\times S^1)$, and then the solution f to (4–7) is given by (4–8). By using Neumann series, it is not hard to see that the trace $(I+K)^{-1}f|_{\Gamma_+}$ is well defined in $L^\infty(\Gamma_+)$ for any $f\in L^\infty(X\times S^1)$. This proves in particular that $\mathcal{A}$ maps $C_0^\infty(\Gamma_-)$ into $L^\infty(\Gamma_+)$ under the smallness assumption on k above. The same arguments also show that $\mathcal{A}f_-$ can be defined for any $f_-\in L^\infty(\Gamma_-)$ as well but we will not need this since we work with the distribution kernel of $\mathcal{A}$.

Define the fundamental solution $\phi(x,\theta,x',\theta')$ of the boundary value problem (4–7) as in Section 3. For $(x',v')\in\Gamma_-$, let $\phi(x,v,x',v')$ solve

$$\begin{cases} T\phi = 0 & \text{in } X\times S^1,\\ \phi|_{\Gamma_-} = |n(x')\cdot v(\theta')|^{-1}\delta_{x'}(x)\delta(\theta-\theta'). \end{cases} \tag{4–9}$$

As before, the albedo operator $\mathcal{A}$ has distribution kernel

$$\alpha(x,\theta,x',\theta') = \phi(x,\theta,x',\theta')|_{(x,\theta)\in\Gamma_+},$$

with $(x',\theta')\in\Gamma_-$, $(x,\theta)\in\Gamma_+$.

As in Section 3, we construct a singular expansion $\phi=\phi_0+\phi_1+\phi_2$ as follows. Let

$$E(x,\theta,t) = e^{\mp\int_0^t \sigma_a(x+sv(\theta),\theta)\,ds}, \quad \pm t\ge 0$$

be the total absorption along the path $[x,\, x{+}tv(\theta)]$. Then

$$\phi_0 = J\phi_-, \qquad \phi_1 = KJ\phi_-, \qquad \phi_2 = (I+K)^{-1}K^2J\phi_-, \tag{4–10}$$

and $\phi_- = |n(x')\cdot v(\theta')|^{-1}\delta_{x'}(x)\delta(\theta-\theta')$ as in (4–9). Next,

$$\phi_0 = E(x,\theta,-\infty)\delta(\theta-\theta')\int_0^{\tau_+(x',v(\theta'))}\delta(x-x'-tv(\theta'))\,dt$$

and

$$\phi_1 = \chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty), \tag{4–11}$$

where $y=y(x',\theta',x,v)$ is the point of intersection of the rays $(0,\infty)\ni s\mapsto x'+sv(\theta')$ and $(-\infty,0)\ni t\mapsto x+tv(\theta)$ and $\chi=\chi(x,\theta,x',\theta')$ equals 1, if those two rays intersect in $\bar{X}$, otherwise $\chi=0$. Recall that X is convex.

To estimate ϕ_2, we need a lemma.

Lemma 4.1. *Let (σ_a,k) and $(\sigma_a,\tilde{k})$ be in L^∞. Let A_2, K and $\tilde{A}_2$, $\tilde{K}$ be related to k and $\tilde{k}$ (not necessarily nonnegative), respectively. Then there exists $C>0$ depending on* diam X *only such that*

$$|(\tilde{K}K\phi_0)(x,\theta,x',\theta')| \le C\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\Big(1+\log\frac{1}{\sin|\theta-\theta'|}\Big)$$

almost everywhere on $X\times S^1\times\Gamma_-$, and also almost everywhere on $\Gamma_+\times\Gamma_-$.

The proof of this lemma is based on the estimate

$$\big|(\tilde{A}_2K\phi_0)(x,\theta,x',\theta')\big| \le \int_{y\in l(x',\theta')}\frac{\big|\tilde{k}(x,\arg(x{-}y),\theta)k(y,\theta',\arg(x{-}y))\big|}{|x-y|}\,dl(y),$$

where $l(x',\theta')$ is the line through x' parallel to θ' and dl is the Euclidean measure on it. Using the following elementary estimate

$$\int_{-A}^{A}\frac{ds}{\sqrt{\nu^2+s^2}} \le 2\Big(1+\log\frac{A}{\nu}\Big), \qquad 0<\nu\le A,$$

we easily complete the proof of the lemma.

For ϕ_2 we therefore have $\phi_2 = (I+K)^{-1}\phi_2^{\#}$ with $\phi_2^{\#} = K^2\phi_0$ and by Lemma 4.1,

$$0 \le \phi_2^{\#}(x',\theta',x,\theta) \le C\|k\|_{L^\infty}^2 \left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right). \tag{4–12}$$

This implies a similar estimate for ϕ_2, because $\phi_2 = \phi_2^{\#} + (I-K)^{-1}K\phi_2^{\#}$. We summarize those estimates:

PROPOSITION 4.1. *For $\varepsilon > 0$ small enough, the fundamental solution ϕ of* (4–7) *defined by* (4–9) *admits the representation $\phi = \phi_0 + \phi_1 + \phi_2$ with*

$$\begin{aligned}
\phi_0 &= E(x,\theta,-\infty)\delta(\theta-\theta')\int_0^{\tau_+(x',v(\theta'))}\delta(x-x'-tv(\theta'))\,dt,\\
\phi_1 &= \chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty),\\
0 \le \phi_2 &\le C\|k\|_{L^\infty}^2\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right),
\end{aligned}$$

where y, χ are as in (4–11) *and $C = C(\operatorname{diam} X)$.*

Note that ϕ_1 is not a delta function but it is still singular with singularity at $v(\theta) = v(\theta')$ (forward scattering) and $v(\theta) = -v(\theta')$ (back-scattering). This singularity is integrable however, in fact it is easy to see that $\int_{\Gamma_+}\phi_1 d\xi \le \int \sigma_p(x'+tv(\theta'))\,dt$. The term ϕ_2 is also singular at $v(\theta) = \pm v(\theta')$ with a weaker, logarithmic singularity. Therefore, we can still distinguish between the singularities of the three terms as in the case $n \ge 3$. This analysis however can give us information about k only near forward and backward directions. It is interesting to see whether by studying subsequent lower order terms we can recover all derivatives of $k(x,\theta',\theta)$ at $\theta = \theta'$ and $\theta = \theta' + \pi$. If so, this would allow us to recover collision kernels analytic in θ, θ' (actually, analytic in $\theta - \theta'$ would be enough) and to approximate k near $\theta = \theta'$ and $\theta = \theta' + \pi$ for smooth k. This would not require smallness assumptions on k but we still have to be sure that the direct problem is solvable.

We are ready now to sketch the proof of Theorem 4.1. Fix $\Sigma > 0$ and assume that we have two pairs (σ_a, k), $(\tilde\sigma_a, \tilde k)$ in $\mathcal{U}_{\Sigma,\varepsilon}$ with the same albedo operator and σ_a, $\tilde\sigma_a$ depending on x only. Denote by ϕ_j and $\tilde\phi_j$, $j = 0,1,2$, the corresponding components of the fundamental solutions ϕ and $\tilde\phi$ as in Proposition 4.1. Then

$$\phi_0 + \phi_1 + \phi_2 = \tilde\phi_0 + \tilde\phi_1 + \tilde\phi_2 \qquad \text{for } (x,\theta) \in \Gamma_+. \tag{4–13}$$

As in Section 3, the most singular terms must agree; therefore,

$$\phi_0(x',\theta',x,\theta) = \tilde\phi_0(x',\theta',x,\theta) \qquad \text{for } (x',\theta') \in \Gamma_-,\ (x,\theta) \in \Gamma_+. \tag{4–14}$$

Thus the X-ray transform of $\sigma_a(x)$ and $\tilde{\sigma}_a(x)$ coincide. Therefore $\sigma_a(x) = \tilde{\sigma}_a(x)$. Next,

$$\chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)-\tilde{k}(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty) = \tilde{\phi}_2-\phi_2 \qquad \text{on } \Gamma_-\times\Gamma_+,$$

where y, χ are as in (4–11). This, together with (4–13) and (4–14), leads to the inequality

$$\begin{aligned}\chi|k(y,\theta',\theta)-\tilde{k}(y,\theta',\theta)| &\le C\big|\sin(\theta-\theta')\big||\tilde{\phi}_1-\phi_1|\\ &= C\big|\sin(\theta-\theta')\big||\tilde{\phi}_2-\phi_2| \qquad \text{on } \Gamma_-\times\Gamma_+, \end{aligned} \tag{4–15}$$

where $C = e^{2d\Sigma}$, $d = \operatorname{diam} D$. The rest of the proof is based on the estimate

$$\operatorname*{ess\,sup}_{\Gamma_-\times\Gamma_+}\big|\sin(\theta-\theta')\big||\tilde{\phi}_2-\phi_2| \le C\varepsilon\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)} \tag{4–16}$$

with $C>0$ depending on $\operatorname{diam} D$ only. An essential role in its proof is played by Lemma 4.1. Observe that in particular, the factor $\sin(\theta-\theta')$ cancels the weaker logarithmic singularity of ϕ_2 and $\tilde{\phi}_2$. Then by (4–15), (4–16),

$$\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)} \le C\varepsilon\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)},$$

and for $\varepsilon>0$ small enough this implies $k=\tilde{k}$. □

Sketch of the proof of Theorem 4.2. According to Proposition 4.1, for the distribution kernel $\alpha(x,\theta,x',\theta')$ of $\mathcal{A}$ we have the representation (4–3), where a is as in (4–4), while $b = (\phi_1+\phi_2)|_{(x,\theta)\in\Gamma_+}$. Then b is a function and an elementary calculation show that $b\in L^\infty(\Gamma_+, L^1(\Gamma_-))$. Proposition 4.1 shows that $b\sin(\theta-\theta')\in L^\infty$. Assume that we have two pairs of continuous functions (σ_a,k) and $(\tilde{\sigma}_a,\tilde{k})$ in $\mathcal{V}^s_{\Sigma,\varepsilon}$ with albedo operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. In what follows we will denote the quantities a, b, α, etc., related to the second pair by putting a tilde sign over it. Also, we will use the notation $\Delta\mathcal{A}$ to denote the difference $\Delta\mathcal{A} = \mathcal{A}-\tilde{\mathcal{A}}$, and similarly $\Delta a = a-\tilde{a}$, etc. In this paper, Δ never stands for the Laplacian.

Let δ_1 and δ_2 be as in (4–5). According to Proposition 4.1, $\delta_2<\infty$. Our goal is to estimate $\Delta\sigma_a$ and Δk in terms of δ_1 and δ_2. Observe that, as in the uniqueness proof, Δe^{-a} and $\Delta(\alpha_1+\alpha_2)$ can be recovered from $\Delta\alpha$ by separating the most singular part of $\Delta\alpha$ from the rest. Therefore, δ_1 measures the magnitude of the singular part of $\Delta\alpha$, while δ_2 measures the magnitude of the regular part.

We start by estimating $\Delta\sigma_a$. By a result of Mukhometov [Mu],

$$\|\Delta\sigma_a\|_{L^2} \le C\delta_1. \tag{4–17}$$

Next we estimate Δk in terms of δ_1 and δ_2. Set

$$E_1(y,\theta,\theta') = E(y,\theta',-\infty)E(y,\theta,\infty).$$

By (4–11), $\sin|\theta-\theta'|\alpha_1(x,\theta,x',\theta') = \chi(E_1k)(y,\theta',\theta)$ with y, χ as in (4–11).

Our starting point is the relation $\Delta(E_1 k) = k\Delta E_1 + \tilde{E}_1 \Delta k$. Note first that $|\Delta E_1| \le 2d|\Delta\sigma_a| \le 2d\delta_1'$, where $\delta_1' = \|\Delta\sigma_a\|_{L^\infty}$. Hence, on $\operatorname{supp}\chi$,

$$\begin{aligned}\tilde{E}_1|\Delta k|(y,\theta,\theta') &\le |\Delta(E_1 k)| + k|\Delta E_1| \\ &\le |\Delta(\alpha_1+\alpha_2)\sin|\theta-\theta'| - \Delta\alpha_2 \sin|\theta-\theta'|| + C\delta_1' \\ &\le \delta_2 + |\Delta\alpha_2|\sin|\theta-\theta'| + C\delta_1'. \end{aligned} \tag{4–18}$$

Therefore,

$$\|\Delta k\|_{L^\infty} \le C\big(\delta_1' + \delta_2 + \|\Delta\alpha_2 \sin|\theta-\theta'|\|_{L^\infty}\big) \tag{4–19}$$

The next step is to prove an estimate similar to (4–16) for the last term in the right-hand side above. Recall that in (4–16), we have $\sigma_a = \tilde{\sigma}_a$, while here we only know how to estimate the difference $\Delta\sigma_a = \sigma_a - \tilde{\sigma}_a$. Nevertheless, we can proceed along similar lines in order to get

$$|\Delta\alpha_2 \sin|\theta-\theta'|\|_{L^\infty} \le C\varepsilon\big(\|\Delta k\|_{L^\infty} + \delta_1'\big). \tag{4–20}$$

Therefore, for $\varepsilon > 0$ small enough, (4–19) implies the stability estimate

$$\|\Delta k\|_{L^\infty} \le C\left(\delta_1' + \delta_2\right). \tag{4–21}$$

Estimate (4–21) is the base of our stability estimate. Using an interpolation inequality, we conclude that, for any fixed $0 \le \mu \le 1 - 1/s$,

$$\|\Delta\sigma_a\|_{1+s\mu} \le C(\Sigma)\|\Delta\sigma_a\|_{L^2}^{1-1/s-\mu} \quad \text{for } \sigma_a, \tilde{\sigma}_a \in \mathcal{V}^s_{\Sigma,\varepsilon}.$$

By a standard Sobolev embedding theorem and (4–17),

$$\delta_1' = \|\Delta\sigma_a\|_{L^\infty} \le C(\Sigma)\|\Delta\sigma_a\|_{L^2}^{1-1/s-\mu} \le C'(\Sigma)\delta_1^{1-1/s-\mu}. \tag{4–22}$$

Therefore, (4–21) yields

$$\|\Delta k\|_{L^\infty} \le C\left(\delta_1^{1-1/s-\mu} + \delta_2\right). \tag{4–23}$$

Estimates (4–22) and (4–23) complete the proof of Theorem 4.2. □

5. Open Problems

Our choice of open problems is subjective and mainly reflects personal taste.

Uniqueness for $\boldsymbol{\sigma_a}$ depending on both $\boldsymbol{x}$ and $\boldsymbol{v}$. As we have demonstrated in the previous sections, if $k = 0$, then $\sigma_a(x,v)$ cannot be recovered from $\mathcal{A}$ (or from the scattering operator) because the line integrals $\int \sigma_a(x+tv,v)\,dt$ do not recover σ_a. Suppose however that $\sigma_p = \sigma_a$. Then the absorption is due only to the fact that particles may change velocity and each such event is interpreted as a particle instantly moving from the point (x,v') of the phase space (therefore absorption at (x,v')) to the point (x,v). Then the counter example above does not work. Can we recover $\sigma_a(x,v)$ in this case? If yes, then recovery of k goes along the same lines as above. More generally, one can assume that $\sigma_a(x,v) = \sigma_p(x,v) + a(x)$. Even more generally, the counter example above cannot be generalized in an obvious way if $k > 0$ in X. Is this condition alone enough for uniqueness?

Relaxing the smallness condition in the two-dimensional case. It would be interesting to prove uniqueness in the 2D case without smallness assumption on k. Some conditions on σ_a and k are needed even for the direct problem; see, for example, (3–1) and (3–3) — the first one does require k to be small (but with an explicit bound in general much larger than the one needed for the inverse problem) while the second one does not. As mentioned in Section 4, one can try to recover $k(x,v',v)$ near $v = \pm v'$ at infinite order by studying the singularities of the kernel of $\mathcal{A}$ which solves the problem for k analytic in v and v'. It would also be interesting to see whether one could recover the singularities of k from boundary measurements, at least if we assume that they are of jump type across some curve.

Stability for $n \geq 3$. Stability estimates in dimensions $n \geq 3$ have been proven by Romanov (see [R1], [R2], [R3], for example) and by Wang [W], under additional assumptions that k depend on fewer variables. In the general situation studied in Section 3, we know of no stability estimates, even for small k as in Theorem 4.2 (where $n = 2$). We believe that such an estimate should be possible to derive following the proof of Theorem 3.1. This is done in [W] under the additional assumption that $k = k(v', v)$.

Alternative recovery method for large σ_a and k. We do not impose smallness assumptions on the coefficients in dimensions $n \geq 3$, and our method gives in fact an explicit solution of the inverse problem which in particular implies a reconstruction method based on taking certain limits near the singularity of α. However, for large σ_a, the amplitude of the most singular part α_0 is exponentially small for large σ_a. For all practical purposes, measuring the leading singularity is hard or impossible in this case. Therefore, it would be important to develop a method for relatively large σ_a and k that does not rely on measuring the singularities of α. One possible way is to study the diffusion limit (replacing σ_a and k by $\lambda\sigma_a$ and λk and taking $\lambda \to \infty$) and an associated inverse problem.

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