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Geometry and Analysis in Many-Body Scattering

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Abstract. This chapter explains in relatively nontechnical terms recent results in many-body scattering and related topics. Many results in the many-body setting should be understood as new results on the propagation of singularities, here understood as lack of decay of wave functions at infinity, with much in common with real principal type propagation (wave phenomena). Classical mechanics plays the role that geometric optics has in the study of the wave equation, but even at this point quantum phenomena emerge. Propagation of singularities has immediate applications to the structure of scattering matrices and to inverse scattering; these topics are addressed here. The final section studies a problem very closely related to many-body scattering, namely scattering on higher rank noncompact symmetric spaces.

1. Introduction

This chapter is an effort to explain in relatively nontechnical terms recent results in many-body scattering and related topics. Thus, many results in the many-body setting should be understood as new results on the propagation of singularities, here understood as lack of decay of wave functions at infinity, with much in common with real principal type propagation, i.e. wave phenomena. Motivated by this, I first briefly describe propagation of singularities for the wave equation. This is a remarkable relationship between geometric optics (the particle view of light) and the solutions of the wave equation (the wave view).

Next, in Section 3, I explain the geometry of many-body scattering, which includes both that of the configuration space and phase space. This geometry is closely related to classical mechanics, playing the role of geometric optics, but even at this point quantum phenomena emerge. This leads to the analytic results, namely the propagation of singularities connecting classical and quantum mechanics.

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Much as for the wave equation, such a result has immediate applications, including the description of the scattering matrices and of the scattering phase. Slightly stronger versions can even lead to inverse results, a topic covered in the following section.

After so explaining the results, in Sections 5-6, I will try to at least give a flavor of how they are proved. This uses a many-body pseudodifferential algebra and positive commutator estimates, so these are discussed. We remark that these techniques are closely related to the proofs of the propagation of singularities for the wave equation, but there are significant differences as well, mostly arising from bound states of particles, which have no analogues for the wave equation. The pseudodifferential algebra itself is very interesting from the viewpoint of noncommutative geometry: there is a hierarchy of operator valued symbols at infinity.

Asymptotic completeness was the main focus of work in many-body scattering for a long period. In Section 7, I briefly explain how it relates to the microlocal estimates.

There is another area that is very closely related to many-body scattering, namely scattering on higher rank noncompact symmetric spaces. Here, in Section 8, we only discuss rank two, which corresponds to three-body scattering, since this is the only part that has been properly written up, but it is expected that very soon these results will extend to all higher rank spaces.

I hope that these notes will make many of these results more accessible, the connections more transparent, and explain the motivation behind them. Many-body scattering has a long history, and here I can only talk about the most recent developments. An excellent overview of results known in the early 1990s can be found in Hiroshi Isozaki's lecture notes [30]. Indeed, in some sense, the current notes continue where [30] left off. I introduce a fully microlocal picture, motivated by the geometric approach of Richard Melrose [42], and emphasize the results these give, but the basic spectral and scattering results follow from a simpler 'partial' microlocalization, which is one of the subjects of [30].

Acknowledgements. The notes were originally prepared for a mini-course at the Université de Nantes at the invitation of Professor Xue-Ping Wang, whose hospitality I gratefully acknowledge. The analytic continuation of the resolvent on symmetric spaces is a more recent development, but it was fueled by a discussion during the visit of Rafe Mazzeo, my collaborator, to Nantes. Over an espresso, Gilles Carrón mentioned that the existence of the analytic continuation was not known, something that was hard to believe, but we immediately realized that our methods should yield such a continuation rather directly. I also thank Gunther Uhlmann for urging me to write up these notes: without him, they may never have been written up, and Rafe Mazzeo for a careful reading of the manuscript.

2. Geometric Optics and the Wave Equation

According to the rules of geometric optics, light propagates in straight lines, and reflects/refracts from surfaces according to the Snell–Descartes law. That is to say, considering light as a stream of billiard balls, the energy as well as the tangential component of the momentum (tangential to the surface hit) is conserved upon hitting the surface.

But light satisfies the wave equation, i.e. if $u = u(x,t)$ is the electromagnetic field on $\Omega_x \times \mathbb{R}_t$, $\Omega \subset \mathbb{R}^n$, then $Pu = 0$ where P is the wave operator $c^2\Delta - D_t^2$, and a boundary condition also holds (say, Dirichlet), if Ω is not the whole space. (Here $D_t = \frac{1}{i}\frac{\partial}{\partial t}$ and $\Delta = \sum_j D_{x_j}^2$ is the *positive* Laplacian.) How are these two viewpoints related?

One can phrase the connection in different ways. The most usual one in physics is that the billiard ball picture is accurate in the high frequency, i.e. low wave length, limit. That is to say, for high frequency light, geometric optics is accurate up to a 'small' error. A slightly different way of looking at this, which however does not involve approximations, is that the location of singularities of the solution of the wave equation is *exactly* predicted by geometric optics. Here singularities are understood as lack of smoothness, or possibly lack of analyticity.

Indeed, it is convenient at this point to generalize the setting somewhat. So let (Ω, g) be a Riemannian manifold with corners, $P = c^2\Delta_g - D_t^2$, $c > 0$. The speed of light, c, may be absorbed in the metric g, of course, we keep the notation in analogy with the usual wave equation.

For simplicity of notation in this paragraph we assume that $M_z = \Omega_x \times \mathbb{R}_t$ is boundaryless; in general, the same definitions hold in the interior of M. Thus, we associate a homogeneous real function on T^*M to P, namely its principal symbol: $p = c^2|\xi|_g^2 - \tau^2$, where we write $\zeta = (\xi, \tau)$ as the dual variable of $z = (x,t)$. Now T^*M is a symplectic manifold with symplectic form $\omega = \sum d\zeta_j \wedge dz_j$. Thus, p gives rise to a vector field H_p, called the Hamilton vector field, by requiring that $\omega(V, H_p) = Vp$ for any vector field V on T^*M. Hence H_p is a smooth vector field on T^*M explicitly given by

$$H_p = \frac{\partial p}{\partial \zeta}\frac{\partial}{\partial z} - \frac{\partial p}{\partial z}\frac{\partial}{\partial \zeta}.$$

Note that p is constant along the integral curves of H_p since taking $V = H_p$, $0 = \omega(H_p, H_p) = H_p p$. Null bicharacteristics are the integral curves of H_p inside its characteristic set $\Sigma = p^{-1}(\{0\})$. Thus, if $\gamma : I \to T^*M$ is a null bicharacteristic (here I is an interval), and $z(s) = z(\gamma(s))$, $\zeta(s) = \zeta(\gamma(s))$, then these solve the ODE's $dz/ds = \partial p/\partial\zeta$, $d\zeta/ds = -\partial p/\partial z$. Hence, when $M = \Omega \times \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $p = c^2|\xi|^2 - \tau^2$ as above, we deduce that ξ and τ are constant along the integral curves of H_p, hence their projection to M consists of straight line segments. More generally, the projection of null-bicharacteristics to Ω are geodesics of g.

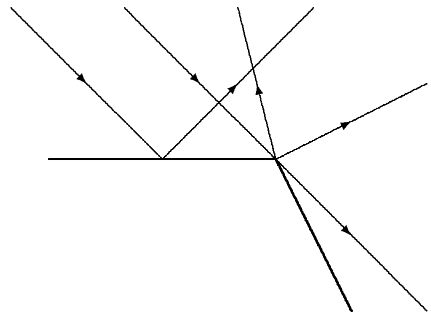

Figure 1. Projection of broken bicharacteristics to Ω. When rays hit the boundary hypersurfaces, the tangential component of the momentum and the kinetic energy are conserved, but the normal component may change. At the corner, there is no tangential component (though there would be if the time variable were not projected out), so the only constraint is the conservation of kinetic energy.

There is an appropriate extension of this at boundary surfaces and even at corners, called generalized broken bicharacteristics, see [43; 36], which I will not explain in full generality, though I remark that many-body scattering, discussed in the next section in detail, is rather similar. However, a somewhat typical example is that of broken bicharacteristics. These are piecewise bicharacteristics, i.e. there is a sequence s_j, j in a subset of integers, such that for each j, $\gamma|_{(s_j,s_{j+1})}$ is a null bicharacteristic in the sense described above, the projection $z\circ\gamma$ of γ to M is continuous, and $\gamma(s_j+)-\gamma(s_j-)$ is conormal to the smallest dimensional boundary face containing $z(\gamma(s_j))$. Thus, the tangent vectors to $z\circ\gamma|_{(s_j,s_{j+1})}$ and $z\circ\gamma|_{(s_{j-1},s_j)}$ differ by a vector normal to the smallest boundary face containing $z(\gamma(s_j))$. This expresses that the normal component of the momentum may change, while the tangential component is conserved, when a light ray hits a boundary.

Now one can describe the singularities of u using null bicharacteristics. Let o be the zero section of T^*M. The location of the singularities is described by an object

$$\mathrm{WF}(u)\subset T^*M\setminus o=\{(z,\zeta):\zeta\neq 0\}$$

that is conic in ζ, i.e. $(z,\zeta)\in\mathrm{WF}(u)$ if and only if $(z,r\zeta)\in\mathrm{WF}(u)$ for every $r>0$. $\mathrm{WF}(u)$ is called the wave front set of u, and it describes where (in z) and in which codirection ζ is the distribution u not $\mathcal{C}^\infty$. More precisely, the definition of $\mathrm{WF}(u)$ is that $(z_0,\zeta_0)\notin\mathrm{WF}(u)$ if and only if there exists $\phi\in\mathcal{C}^\infty_c(M)$, $\phi(z_0)\neq 0$ such that the Fourier transform $\mathcal{F}(\phi u)$ of ϕu is rapidly decreasing in an open cone around ζ_0. Here we assume that M is boundaryless; otherwise we need to require that ϕ is supported in the interior of M. Again, there is a natural definition at ∂M which we do not give here. (There are more natural versions of this definition using pseudodifferential operators that I will describe later.) As an example, consider the step function: write $z = (z_1, z'')$, $u(z) = 1$ if $z_1 > 0$, $u(z) = 0$ if $z_1 < 0$. Then

$$\mathrm{WF}(u) = N^*\{z_1 = 0\} \setminus o = \{(0, z'', \zeta_1, 0) : \zeta_1 \neq 0\},$$

the conormal bundle of the hypersurface $z_1 = 0$, with its zero section removed. The same statement holds, with $=$ possibly replaced by $\subset$, if we take any $\mathcal{C}^\infty$ function u_0 on M, and then define $u = u_0$ in $z_1 > 0$ and $u = 0$ in $z_1 < 0$. Informally, one might say that u is singular in z_1 at $z_1 = 0$, but it depends smoothly on z''. The wave front set thus pinpoints not only the locations z of singularities (lack of smoothness) in M, but it refines it by also giving the frequencies (or rather direction of frequencies) at which these appear at z.

The theorem we are after is the following. In early versions it goes back to Lax [35], its boundaryless version is due to Hörmander [28], the smooth boundary versions are due to Melrose, Sjöstrand, Taylor and Ivrii [33; 43; 44; 61], and the corner version in the analytic category is due to Lebeau [36] (the $\mathcal{C}^\infty$ version is still not known in the corner setting) while a different extension, to conic points, is due to Melrose and Wunsch [45].

THEOREM 2.1. *Suppose* $Pu \in \mathcal{C}^\infty(M)$, *and if* $\partial M \neq 0$ *then* $u|_{\partial M} = 0$. *Then* $\mathrm{WF}(u) \subset \Sigma = p^{-1}(\{0\})$ (*microlocal elliptic regularity*). *Moreover*, $\mathrm{WF}(u)$ *is a union of maximally extended generalized broken bicharacteristics inside* Σ (*propagation of singularities*).

This theorem states that if a point $(z, \zeta) \in T^*M \setminus o$ is in $\mathrm{WF}(u)$ and u solves $Pu \in \mathcal{C}^\infty(M)$, and satisfies a boundary condition if appropriate, then there is at least one maximally extended generalized broken bicharacteristic through (z, ζ) that is completely contained in $\mathrm{WF}(u)$. Of course, in the absence of boundaries, and often even in their presence, there is a unique maximally extended generalized broken bicharacteristic through (z, ζ), so the statement is that this bicharacteristic is completely in $\mathrm{WF}(u)$. However, as soon as codimension two or higher corners appear, there is no hope for such uniqueness, and this theorem is the optimal statement.

At least in the nicest settings (no boundaries, or nondegeneracy assumption at the boundaries which are assumed to be smooth), this theorem can be improved significantly to predict not only the location, but also the amplitude of the singularities of u.

3. Propagation in Many-Body Scattering

There is an analogous setup for scattering. Now we want to understand how interacting particles behave. Again, there is a classical mechanical setup (the analogue of geometric optics) and a quantum mechanical setup (the analogue of the wave equation). To focus on the most relevant points, I formulate the prob-

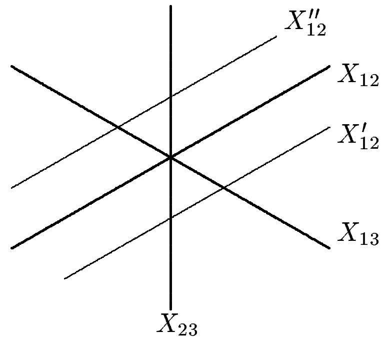

Figure 2. Collision planes X_{12}, X_{13} and X_{23} and translates X'_{12} and X''_{12} of X_{12}. V_{12} is constant along X_{12}, is a (typically different) constant along X'_{12}, etc., so it does not decay at infinity unless it is identically zero.

lem in a time-independent fashion, though it is easy to reformulate everything in a time dependent way. We only do this in a remark following Theorem 3.1.

Thus, we want to understand tempered distributional solutions u of $(H - \lambda)u = 0$; here $\lambda \in \mathbb{R}$ is the energy, and H is the Hamiltonian, i.e. the analogue of $H - \lambda$ is P above. Namely, if we have N particles, each of which is d-dimensional with positions $x_1, \dots, x_N \in \mathbb{R}^d$, mass $m_1, \dots, m_N$, and the interaction between particle i and j is given by a potential V_{ij} (which is a function on $\mathbb{R}^d$), then the Hamiltonian describing this system is

$$H = \sum_{i=1}^{N} \frac{1}{2m_i} \Delta_{x_i} + \sum_{i<j} V_{ij}(x_i - x_j) = \Delta + V,$$

which is an operator on (functions on) $\mathbb{R}^n = \mathbb{R}^{Nd}$. Planck's constant $\hbar$ is here taken to be 1; it could be absorbed in the x_i by a simple rescaling.

Now H is elliptic in the standard sense, namely its principal symbol is

$$\sum \frac{1}{2m_i} |\xi|^2,$$

which never vanishes outside the zero section o. Note that the potential is lower order than Δ in the standard sense, so it is not part of the principal symbol. So, by the previous theorem,

$$(H - \lambda)u = 0 \implies \mathrm{WF}(u) = \varnothing \implies u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

So the only possibility of interesting behavior for u is at infinity, and this is exactly what we want to understand.

The main feature of many-body problems is that even if V_{ij} decays at infinity on $\mathbb{R}^d$, it does *not* decay at infinity in $\mathbb{R}^n$ since it is a constant along $X_{ij} = \{x_i = x_j\}$, as well as along its translates X'_{ij}, X''_{ij}, so it does not decay if we go to infinity, say, along X_{ij}; see Figure 2. The X_{ij} are called collision planes (as are their intersections) since at X_{ij} particles i and j are at the same place.

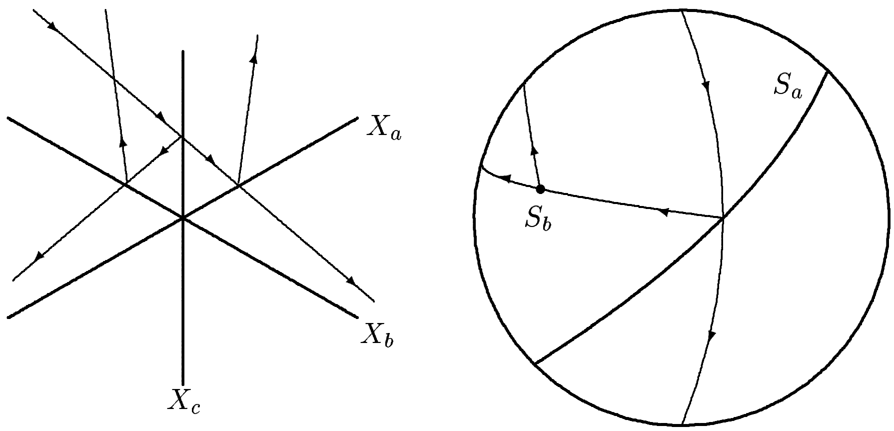

Figure 3. On the left, broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 2$, broken at the collision planes X_a, X_b and X_c. On the right, the projection of broken geodesics in $\mathbb{R}^n \setminus \{0\}$, $n = 3$, emanating from the north pole, to the unit sphere S_0, better understood as the sphere at infinity. The C_a, C_b are the intersection of the collision planes X_a, X_b with S_0; $\dim X_a = 2$, $\dim X_b = 1$.

In the two-body problem one actually has $H = \Delta_{x_1,x_2} + V_{12}(x_1 - x_2)$, i.e. V_{12} still does not decay at infinity, e.g. if one keeps $x_1 = x_2$ but lets $x_1 \to \infty$. However, one can easily remove the center of mass by performing a Fourier transform along X_{12}. This conjugates $H - \lambda$ to $H^{12} + |\xi_{12}|^2 - \lambda$, where ξ_{12} is the variable on X_{12}^*, and $H^{12} = \Delta_{X^{12}} + V_{12}$, X^{12} being the orthocomplement of X_{12}. Thus, one reduces the study of $H - \lambda$ to that of a Hamiltonian on X^{12}, namely $H^{12} - \lambda'$, $\lambda' = \lambda - |\xi_{12}|^2$ being a shifted spectral parameter. Now V_{12} decays at infinity (we are working on X^{12}!), so H^{12} can be considered as a perturbation of $\Delta_{X^{12}}$, hence its analysis is rather simple. Notice that the point spectrum of H^{12} gives rise to a branch of the continuous spectrum of H: this is a phenomenon that is very typical in many-body scattering. The center of mass can also be removed in any actual many-body problem, but one still obtains a Hamiltonian with nondecaying potentials as before.

One can still talk about classical mechanics, just as for the wave equation, using bicharacteristics. These are deterministic – if V is smooth enough (we usually assume that V is $\mathcal{C}^\infty$). But much like for corners, there is a compressed description of dynamics near infinity. This is somewhat more complicated than for the wave equation, but only because particles can be bound together. Thus, even the 'classical' description is partly quantum. These two facts, the presence of collision planes and the bound states, are the two crucial features of many-body scattering.

The compressed dynamics in the absence of bound states looks just like in the wave equation setting. One should think of this as a good description when a classical trajectory is uniformly near infinity.

More precisely, it is convenient to introduce Agmon's generalization of the many-body problem, which amounts to using the vector space structure of $\mathbb{R}^n$ as the setting. One can also give geometric generalizations (in the sense of differential geometry) that arose from the work of Melrose [41; 42], and I will do this later.

So we work on the vector space $X_0 = \mathbb{R}^n$, equipped with the Euclidean metric. We are also given a finite collection $\mathcal{X} = \{X_a : a \in I\}$ of linear subspaces X_a of $\mathbb{R}^n$, called the collision planes. We assume that $\mathcal{X}$ is closed under intersections, and $X_0 = \mathbb{R}^n \in \mathcal{X}$, $X_1 = \{0\} \in \mathcal{X}$. We let $X^a = X_a^\perp$ be the orthocomplement of X_a in $\mathbb{R}^n$, so $\mathbb{R}^n = X_a \oplus X^a$. (Agmon's generalization is thus that the X_a do not have to come from intersections of the planes $X_{ij} = \{x_i = x_j\}$.) We write the corresponding coordinates as (x_a, x^a), and denote the orthogonal projection to X^a by π^a. A many-body Hamiltonian in potential scattering is an operator of the form

$$H = \Delta + \sum_a (\pi^a)^* V_a,$$

where V_a is a real valued function on X^a in a certain class, for example V_a is a symbol on X^a of negative order: $V_a \in S^{-\rho}(X^a)$, $\rho > 0$. We also assume that $V_0 = 0$ for normalization; note that $X^0 = \{0\}$, so V_0 would simply play the role of the spectral parameter. We sometimes drop the pull-back notation from now on and write $H = \Delta + \sum_a V_a$.

Another useful piece of terminology is the following. We say that V_a is short range if $V_a \in S^{-\rho}(X^a)$ for some $\rho > 1$. We say that V_a is long-range if $V_a \in S^{-\rho}(X^a)$ for some $\rho \in (0, 1]$. The Coulomb potential is thus 'marginally long-range', at least if we ignore its singularity at 0 (which is not a serious problem anyway). Whether V_a is short- or long-range does not make any difference for the propagation phenomena we discuss in this section. However, it does make a major difference for the precise behavior of generalized eigenfunctions at the 'radial sets' which we discuss later. This also shows up in the related issue of asymptotic completeness.

Yet another notation we use on occasion is that of a k-cluster. Physically, a cluster describes particles that are close (or collide), and a k-cluster means that there are k clusters of particles, inside each of which the particles are close to each other. So in N-particle scattering, the N-cluster describes N asymptotically free particles (none is close to any other), hence we say that the collision plane $X_0 = \mathbb{R}^n$ is the N-cluster. On the other hand, if $X_a \neq \{0\}$ is such that $X_b \subsetneq X_a$ implies that $X_b = \{0\}$, then X_a, or rather a, is a 2-cluster. E.g. given five particles, a 2-cluster is where $x_1 = x_2$ and $x_3 = x_4 = x_5$, i.e. the particles 1 and 2, resp. 3, 4 and 5, are close to each other. In general, a k-cluster X_a can be defined by the length of nested chains of collision planes inside X_a.

One need not assume that all interactions between the particles are via potentials. Indeed, V_a may be allowed to be any first order differential operator on the vector space X^a with symbolic coefficients of negative order. Also, one may generalize the metric g in an analogous fashion, as discussed later, which in effect allows V_a to be second order provided that H remains elliptic. To simplify the notation, and due to the traditions, we mostly talk as if V_a were potentials, but the generalization to such higher order perturbations requires only occasional and minor modifications, which will be pointed out.

The subsystem Hamiltonians are defined by

$$H^a = \Delta_{X^a} + \sum_{X_a \subset X_b} V_b.$$

Note that $X_a \subset X_b$ if and only if $X^a \supset X^b$, so above V_b is really the pull-back of V_b from X^b to to X^a by the orthogonal projection. Thus, H^a is an operator on (functions on) X^a, and indeed it is a many-body Hamiltonian.

We also let

$$X_{a,\mathrm{sing}} = \bigcup\{X_b : X_b \subsetneq X_a\} \quad \text{and} \quad X_{a,\mathrm{reg}} = X_a \setminus X_{a,\mathrm{sing}}$$

be the singular and regular parts of X_a. Thus, if X_c is a collision plane and X_a is not a subset of X_c, then $X_a \cap X_c$ is a proper subset of X_a, and is a collision plane (since $\mathcal{X}$ is closed under intersections), so $X_a \cap X_c \subset X_{a,\mathrm{sing}}$. Correspondingly, V_c decays at $X_{a,\mathrm{reg}}$, so

$$H_a = \Delta_{X_a} + H^a,$$

which is an operator on (functions on) $X_0 = \mathbb{R}^n$, has the property that $H - H_a$ is a function that decreases at $X_{a,\mathrm{reg}}$. So H_a should be thought of as a good approximation of H at $X_{a,\mathrm{reg}}$. Note that $X_{a,\mathrm{sing}}$ is a finite union of codimension ≥ 1 submanifolds of X_a, so $X_{a,\mathrm{reg}}$ is in particular an open dense subset of X_a. Also, note that Δ_{X_a} plays a role analogous to the kinetic energy of the center of mass in the two-body setting, but now this description only valid locally, at $X_{a,\mathrm{reg}}$.

Having thus described the configuration space $X = X_0 = \mathbb{R}^n$, the next step is to describe the phase space, as was done first in [65] and [66]. The main goal in the process is to obtain a space on which broken bicharacteristics behave well. We remind the reader that we are concerned with singularities at infinity, hence with bicharacteristics that are uniformly close to infinity. Later we give a compactified description, but here for simplicity we give its homogeneous version, much as for the wave equation where bicharacteristics were integral curves of the homogeneous principal symbol. So we start with T^*X, but we wish to compress it at X_a in such a way that at $X_{a,\mathrm{reg}}$, $T^*_{X_{a,\mathrm{reg}}}X$ is replaced by $T^*_{X_{a,\mathrm{reg}}}X_a = T^*X_{a,\mathrm{reg}}$. For broken bicharacteristics this has the effect that only the X_a-tangential component of the momentum is preserved at $X_{a,\mathrm{reg}}$. So we define the compressed cotangent bundle as

$$\dot{T}^*X = \bigcup_{a \neq 1} T^*X_{a,\mathrm{reg}}.$$

Note that this is at first just a set, equipped with a projection $\dot{T}^*X \to X \setminus \{0\}$ induced by the bundle projections $T^*X_{a,\mathrm{reg}} \to X_{a,\mathrm{reg}}$. There is also a natural $\mathbb{R}^+$-action on $\dot{T}^*X$ via dilation in the configuration variables:

$$\mathbb{R}_r^+ \times T^*X_{a,\mathrm{reg}} \ni (r, x_a, \xi_a) \mapsto (rx_a, \xi_a) \in T^*X_{a,\mathrm{reg}}. \tag{3–1}$$

We topologize $\dot{T}^*X$ via the projection

$$\pi : T^*_{X\setminus\{0\}}X \to \dot{T}^*X,$$

whose restriction to $T^*_{X_{a,\mathrm{reg}}}X$ is the pull-back of one-forms by the inclusion map $X_{a,\mathrm{reg}} \hookrightarrow X$. Thus, writing (ξ_a, ξ^a) as the momenta dual to (x_a, x^a), π projects out the normal component of the momentum, ξ^a. The topology is then the weakest topology that makes π continuous, i.e. a set C in $\dot{T}^*X$ is closed if and only if $\pi^{-1}(C)$ is closed.

We can now describe the contribution of the bound states to the characteristic sets. As mentioned above, this is one of the most interesting features of many-body scattering that has no analogue for the wave equation. These are conic subsets of $\dot{T}^*X$ (conic with respect to the $\mathbb{R}_r^+$-action in (3–1)). The characteristic sets describe where certain operators are not elliptic, i.e. invertible, at infinity, in a precise sense described in the subsequent sections. They correspond to the 'energy shell', i.e. being on the characteristic set at energy λ means that the particles have total energy λ. We let

$$\mathrm{Char}_0(\lambda) = \{(x, \xi) \in T^*X : g(\xi) = \lambda\}$$

be the free characteristic variety, with g being the metric function on T^*X, and more generally we set

$$\mathrm{Char}_a(\lambda) = \{(x_a, \xi_a) \in T^*X_a : \lambda - g_a(\xi_a) \in \mathrm{spec}_{\mathrm{pp}} H^a\} \subset T^*X_a.$$

Notice that $\lambda = g_a(\xi_a) + \varepsilon_\alpha$, $\varepsilon_\alpha \in \mathrm{spec}_{\mathrm{pp}} H^a$, corresponds to the splitting of the total energy λ to the kinetic energy of the cluster, $g_a(\xi_a)$, plus the energy of the bound state, ε_α. Thus, $\mathrm{Char}_a(\lambda)$ describes that particles may exist in a bound state of H^a, of energy ε_α, along X_a, with kinetic energy $g_a(\xi_a) = \lambda - \varepsilon_\alpha$. Moreover, H^0 is the zero operator on $X^0 = \{0\}$, so if $a = 0$, these two definitions are consistent. If $X_a \subset X_b$, the pull-back of one-forms gives a projection $\pi_{ba} : T^*_{X_a}X_b \to T^*X_a$. Let

$$\dot{\mathrm{C}}\mathrm{har}(\lambda) = \bigcup \dot{\mathrm{C}}\mathrm{har}_a(\lambda) \subset \dot{T}^*X,$$

$$\dot{\mathrm{C}}\mathrm{har}_a(\lambda) = \bigcup_{X_b \supset X_a} \pi_{ba}(\mathrm{Char}_b(\lambda)) \cap T^*X_{a,\mathrm{reg}}.$$

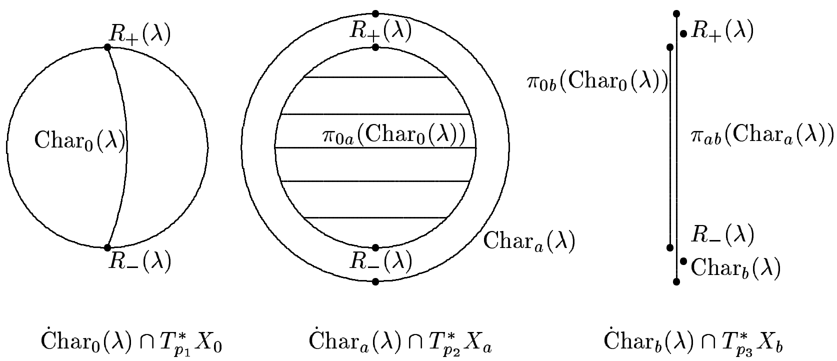

Figure 4. The characteristic set of many-body Hamiltonians. Here H is a 4-body Hamiltonian, a is a 3-cluster, b is a 2-cluster, $p_1 \in X_{0,\mathrm{reg}}$, $p_2 \in X_{a,\mathrm{reg}}$, $p_3 \in X_{b,\mathrm{reg}}$. The solid dots are the radial sets, defined below.

In order to understand $\dot{\mathrm{C}}\mathrm{har}(\lambda)$ it is important to keep in mind several results on the structure of the eigenvalues of the subsystems. So let

$$\Lambda_a = \bigcup_{b: X^b \subsetneq X^a} \operatorname{spec}_{\mathrm{pp}} H^b$$

be the set of thresholds of H^a. Fundamental results of Perry, Sigal and Simon [53] and of Froese and Herbst [16] show that Λ_a is closed, countable, and the countable set $\operatorname{spec}_{\mathrm{pp}} H^a$ can only accummulate at Λ_a, so

$$\Lambda'_a = \Lambda_a \cup \operatorname{spec}_{\mathrm{pp}} H^a = \bigcup_{b: X^b \subset X^a} \operatorname{spec}_{\mathrm{pp}} H^b$$

is also closed. Hence, $\dot{\mathrm{C}}\mathrm{har}(\lambda)$ is a closed subset of $\dot{T}^*X$. In fact, the quotient $\dot{\Sigma}(\lambda)$ of $\dot{\mathrm{C}}\mathrm{har}(\lambda)$ by the $\mathbb{R}^+$ action (which can be realized by restricting the various bundles to the unit sphere, $S_0 = \{x \in X_0 : |x| = 1\}$) is compact, and indeed it is metrizable, see [65]. Since compact topological spaces have better properties than noncompact ones, it is quite natural to work with $\dot{\Sigma}(\lambda)$, although we do not follow this route in this section. We also remark that it is much better to think of $\dot{\Sigma}(\lambda)$ lying at the sphere at infinity, rather than at S_0, since it is the dynamics at infinity that is described here. We will take up this approach in later sections.

We also recall another result of Froese and Herbst [15], namely that eigenfunctions ψ_α of H^a with eigenvalue ε_α decay exponentially on X^a, at a specified rate, if $\varepsilon_\alpha \notin \Lambda_a$. This generalizes to higher order perturbations, but requires a somewhat different approach, see [67]. In fact, this is the only place where second order perturbations behave differently from first or zeroth order ones. For the latter, there can be no positive energy bound states, while for the former this has been only proved for small metric perturbations in [67], and it is not clear whether it holds more generally, especially for trapping perturbations. (Note that if H^a, $a \neq 1$, is trapping then H is trapping at infinity!)

A generalized broken bicharacteristic (at energy λ) is then a continuous map $\gamma : I \to \dot{\mathrm{Char}}(\lambda)$, I an interval, such that a Hamilton vector field condition holds. To see what this is, we consider a subset of continuous functions on $\dot{T}^*X$, namely the class of π-invariant $\mathcal{C}^\infty$ functions on T^*X. π-invariance means that if $\zeta, \zeta' \in T^*X$ and $\pi(\zeta) = \pi(\zeta')$ then $f(\zeta) = f(\zeta')$. If f is π-invariant then it induces a function f_π on $\dot{T}^*X$ by $f_\pi(q) = f(\zeta)$ if $q = \pi(\zeta)$. Moreover, if f is smooth (or indeed just continuous) then f_π is continuous by the definition of the topology on $\dot{T}^*X$.

Now, if $\tilde{\gamma}$ is a curve in a manifold, one way to put that it is an integral curve of a vector field V is that

$$\frac{d}{ds}(f \circ \tilde{\gamma})|_{s=s_0} = (Vf)(\tilde{\gamma}(s_0))$$

for all smooth functions f. If f is a smooth π-invariant function on T^*X, then f defines a $\mathcal{C}^\infty$ function on T^*X_a for all a, so $H_{g_a}f$ makes sense. Here H_{g_a} is the Hamilton vector field of the metric function g_a on T^*X_a, so explicitly, $H_{g_a} = 2\xi_a \cdot \partial_{x_a}$. Now we would like to say that along a generalized broken bicharacteristic γ, $(d/ds)(f_\pi \circ \gamma)|_{s=s_0}$ should be given by $H_{g_b}f$ for some b and some ζ with $\pi(\zeta) = \gamma(s_0)$. The problem is that there are many such points ζ and clusters b, so this statement does not make any sense. However, we may replace the derivative by the lim inf of the difference quotients, i.e. by

$$D_\pm h(s_0) = \liminf_{s \to s_0} \frac{h(s) - h(s_0)}{s - s_0},$$

and demand an inequality instead of the equality. That is, we may demand that $D_\pm(f_\pi \circ \gamma)(s_0)$ may not be less than the worst possible scenario as we run over all such b and ζ. Thus, the condition for a continuous map $\gamma : I \to \dot{\mathrm{Char}}(\lambda)$ to be a generalized broken bicharacteristic is then that for any $s_0 \in I$, if $\gamma(s_0) \in T^*X_{a,\mathrm{reg}}$ then

$$D_\pm(f_\pi \circ \gamma)(s_0) \geq \inf\big\{(H_{g_b}f)(\zeta) : \zeta \in \mathrm{Char}_b(\lambda),\ \pi(\zeta) = \gamma(s_0),\ X_a \subset X_b\big\}.$$

If the set of bound states is discrete, then such a curve γ is piecewise an integral curve of the Hamilton vector field of g_b inside $\mathrm{Char}_b(\lambda)$, where b may of course vary. In particular, if there are no bound states in any proper subsystem, the picture is very similar to wave propagation: the definition can be reduced to the analogue of Lebeau's [36].

The structure of the generalized broken bicharacteristics, including the above claims, depends on having a large supply of π-invariant functions. But these exist, since the pull-backs of all functions on X to T^*X is π-invariant, so one can localize in X using smooth cutoffs. Moreover, near $X_{a,\mathrm{reg}}$, each component of ξ_a is π-invariant, as is $\xi^a \cdot x^a$. Note that the generalized broken bicharacteristics depend on V, but only via the characteristic set $\dot{\mathrm{Char}}(\lambda)$, i.e. only via the bound states of the subsystem Hamiltonians.

There is also a wave front set associated to many-body scattering which measures the microlocal decay of tempered distributions at infinity. For a tempered distribution u, $\mathrm{WF}_{\mathrm{sc}}(u)$ is a closed conic subset of $\dot{T}^*X$. Apart from u, it depends on $\mathcal{X}$, since $\dot{T}^*X$ depends on $\mathcal{X}$, but we suppress this in the notation, and write

$$\mathrm{WF}_{\mathrm{sc}}(u) = \mathrm{WF}_{\mathrm{sc},\mathcal{X}}(u).$$

Its definition is slightly complicated, and I only refer to [65] for the general definition, which uses the structure of the pseudodifferential algebra, in particular the operator-valued nature of symbols at infinity. However, for generalized eigenfunctions of many-body Hamiltonians it is simple. Namely, suppose that $(H-\lambda)u \in \mathcal{S}(X)$, where $\mathcal{S}(X)$ is the space of Schwartz functions. For $\bar{x} = \bar{x}_a \in X_{a,\mathrm{reg}}$ and $\bar{\xi}_a \in X_a^*$ we say that $(\bar{x}_a, \bar{\xi}_a) \notin \mathrm{WF}_{\mathrm{sc}}(u)$ if there exists $\phi \in \mathcal{C}_c^\infty(X_a^*)$ such that $\phi(\bar{\xi}_a) \neq 0$ and $\mathcal{F}^{-1}\phi\mathcal{F}u$ is rapidly decreasing in an open cone in X around $\bar{x}_a$. Two examples are:

$$\begin{aligned} &\mathrm{WF}_{\mathrm{sc}}(e^{ix\cdot\xi_0}) = \pi(\{(x,\xi_0) : x \neq 0\}), && \xi_0 \in X_0^*,\\ &\mathrm{WF}_{\mathrm{sc}}(e^{i\alpha|x|}) = \pi(\{(x,\, \alpha x/|x|) : x \neq 0\}). && \alpha \in \mathbb{R}. \end{aligned}$$

More generally, if v is a symbol of any order on X_0, say $v \in S^k(X_0)$, and $\phi \in \mathcal{C}^\infty(X_0)$ is homogeneous degree 1 for $|x| > 1$, then

$$\mathrm{WF}_{\mathrm{sc}}(e^{i\phi(x)}v(x)) \subset \pi(\mathrm{graph}\, d\phi) = \pi(\{(x, (d\phi)(x)) : |x| > 1\}).$$

The condition $|x| > 1$ is due to the requirement of the homogeneity of ϕ only for $|x| > 1$; technically we should add a subset of $|x| \leq 1$ to the right hand side to make it conic. The theorem on the propagation of singularities is the following.

Theorem 3.1. *Suppose that* $\lambda \in \mathbb{R}$ *and* H *is a many-body Hamiltonian. If* $u \in \mathcal{S}'(\mathbb{R}^n)$ *and* $(H-\lambda)u \in \mathcal{S}(\mathbb{R}^n)$ *then* $\mathrm{WF}_{\mathrm{sc}}(u) \subset \dot{\mathrm{Char}}(\lambda)$ (*microlocal elliptic regularity*). *Moreover*, $\mathrm{WF}_{\mathrm{sc}}(u)$ *is a union of maximally extended generalized broken bicharacteristics of* $H-\lambda$ (*propagation of singularities*).

Remark. In the time dependent version, one considers tempered distributional solutions of $(D_t + H)u = 0$ on $X_0 \times \mathbb{R}_t$. Then $D_t + H$ still has the structure of a many-body Hamiltonian, with $D_t + \Delta$ in place of Δ, with collision planes given by $X_a \times \mathbb{R}$, with $\{0\}$ added for the sake of completeness. Thus, t is always a variable along the collision planes, so in particular, its dual variable τ, is π-invariant. Moreover, $\mathrm{Char}_a(\lambda)$ is replaced by

$$\mathrm{Char}_a = \{(x_a, t, \xi_a, \tau) \in T^*(X_a \times \mathbb{R}) : -\tau - g_a(\xi_a) \in \mathrm{spec}_{\mathrm{pp}}\, H^a\} \subset T^*X_a,$$

so effectively $-\tau$ plays the role of the energy λ. Generalized broken bicharacteristics can be defined as before with H_{g_b} replaced by $H_{\tau+g_b} = \partial_t + 2\xi_b \cdot \partial_{x_b}$. The main additional issue is that they can only be expected to give a good description of propagation at finite energies since $D_t + H$ is not elliptic in the usual sense. So the analogue of Theorem 3.1 is that if $u \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R})$, $(D_t + H)u = 0$, and $u = \psi(H)u$ for some $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, then $\mathrm{WF}_{\mathrm{sc}}(u)$ is a subset of the characteristic set, and in fact $\mathrm{WF}_{\mathrm{sc}}(u)$ is a union of maximally extended generalized broken bicharacteristics of $D_t + H$ inside it. The proof of this statement only requires simple modifications of the proof of the theorem.

The interpretation of the theorem is much analogous to that for the wave equation. However, there is a difference which also occurs in the traditional microlocal setting for more general operators (i.e. for operators other than the wave operator), see [21]. Namely, the orbits of the $\mathbb{R}^+$-action may be bicharacteristics, and then the statement of the theorem is empty at the points lying on these orbits since the wave front set is *a priori* conic. This happens for $(x_a, \xi_a) \in T^*X_{a,\mathrm{reg}} \subset \dot{T}^*X$ if and only if there exists some cluster b with $X_b \supset X_a$, and $\zeta \in \mathrm{Char}_b(\lambda)$ such that H_{g_b} at ζ is tangent to the orbits of the $\mathbb{R}^+$-action. This happens, in turn, if and only if ξ_a is parallel to x_a and $\lambda - |\xi_a|^2 \in \operatorname{spec}_{\mathrm{pp}} H^b$. Such points are called radial points, and their collection is denoted by

$$\mathcal{R}(\lambda) = \bigcup_{a \neq 1} \big\{(x_a, \xi_a) \in T^*X_{a,\mathrm{reg}} : \exists c \in \mathbb{R},\ \xi_a = c x_a,\\ \exists b,\ X_b \supset X_a,\ \lambda - |\xi_a|^2 \in \operatorname{spec}_{\mathrm{pp}} H^b\big\}.$$

As we discuss in Section 7, $\mathcal{R}(\lambda)$ plays an important role in asymptotic completeness. In many-body scattering it appeared in the work of Sigal and Soffer [55] and was called 'propagation set' because in the time-dependent picture this is where particles end up as time goes to infinity. (In the stationary semiclassical picture, this is where nontrapped classical trajectories starting in a compact region end up.) It is thus unfortunate, in terms of terminology, that this is also the region where there is no real principal type propagation.

Remark. In the time-dependent problem, the set of radial points is

$$\mathcal{R} = \bigcup_a \big\{(x_a, t, \xi_a, \tau) \in T^*(X_{a,\mathrm{reg}} \times \mathbb{R}) : x_a = 2t\xi_a,\\ \exists b,\ X_b \supset X_a,\ -\tau - |\xi_a|^2 \in \operatorname{spec}_{\mathrm{pp}} H^b\big\}.$$

In terms of radial points, the difference between threshold energies $\lambda \in \Lambda = \Lambda_0$ and nonthreshold energies is that if $\lambda \in \Lambda$, then there are *constant* generalized broken bicharacteristics, i.e. bicharacteristics whose image is a single point. Namely, if $x_a \in X_{a,\mathrm{reg}}$ and $\lambda \in \operatorname{spec}_{\mathrm{pp}} H^a$, then

$$(x_a, 0) \in \mathrm{Char}_a(\lambda) \cap T^*X_{a,\mathrm{reg}} \subset \dot{\mathrm{C}}\mathrm{har}(\lambda),$$

and $H_{g_a} = 2\xi_a \cdot \partial_{x_a}$ vanishes there, so $(x_a, 0)$ is indeed the image of a constant bicharacteristic. While this does not make any difference for the propagation of singularities, it does for the related limiting absorption principle, which in this generality is due to Perry, Sigal and Simon [53].

THEOREM 3.2. *If $\lambda \notin \Lambda$, then the limits $R(\lambda \pm i0) = (H - (\lambda \pm i0))^{-1}$ exists as bounded operators between L^2_s and H^2_{-s} for $s > \frac{1}{2}$. Here H^m_l is the weighted Sobolev space $\langle x \rangle^{-l} H^m(\mathbb{R}^n)$, $L^2_s = H^0_s$.*

In fact, the proofs of the limiting absorption principle and the propagation of singularities are related. Indeed, the statement on propagation of singularities can be strengthened for $R(\lambda+i0)f$, $f \in \mathcal{S}(\mathbb{R}^n)$, by saying that $\mathrm{WF}_{\mathrm{sc}}(R(\lambda+i0)f)$ is not only a union of maximally extended generalized broken bicharacteristics, as follows from Theorem 3.1, but in fact it is a union of generalized broken bicharacteristics $\gamma : \mathbb{R}_s \to \dot{T}^*X$ which go to

$$\mathcal{R}_+(\lambda) = \mathcal{R}(\lambda) \cap \bigcup_{a \neq 1} \big\{(x_a, \xi_a) \in T^*X_{a,\mathrm{reg}} : x_a \cdot \xi_a > 0\big\}$$

as $s \to -\infty$. That is, the singularities at $\mathcal{R}_+(\lambda)$ (where the statement of Theorem 3.1 is empty) can only leave $\mathcal{R}_+(\lambda)$ in the *forward* direction. The limiting absorption principle is thus strengthened to:

THEOREM 3.3. *If $\lambda \notin \Lambda$, then for $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathrm{WF}_{\mathrm{sc}}(R(\lambda+i0)f)$ is a subset of the image of $\mathcal{R}_+(\lambda)$ under the* forward *generalized broken bicharacteristic relation. A similar statement holds for $R(\lambda - i0)f$ with $\mathcal{R}_+(\lambda)$ replaced by*

$$\mathcal{R}_-(\lambda) = \mathcal{R}(\lambda) \cap \bigcup_{a \neq 1} \big\{(x_a, \xi_a) \in T^*X_{a,\mathrm{reg}} : x_a \cdot \xi_a < 0\big\},$$

and the forward relation by the backward relation.

In fact, if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathrm{WF}_{\mathrm{sc}}(u)$ is disjoint from the image of $\mathcal{R}_-(\lambda)$ under the backward generalized broken bicharacteristic relation, then $R(\lambda+i0)u$ is defined by duality and $\mathrm{WF}_{\mathrm{sc}}(R(\lambda+i0)u)$ is a subset of the image of $\mathcal{R}_+(\lambda) \cup \mathrm{WF}_{\mathrm{sc}}(u)$ under the forward relation.

REMARK. $\lambda \notin \Lambda$ can be also characterized by $\mathcal{R}(\lambda) = \mathcal{R}_+(\lambda) \cup \mathcal{R}_-(\lambda)$, i.e. that $x_a \cdot \xi_a$ never vanishes on $\mathcal{R}(\lambda) \cap T^*X_{a,\mathrm{reg}}$ for any a.

In the time-dependent setting, $x_a = 2t\xi_a$ on $\mathcal{R}$, so $x \cdot \xi_a = 2t$. So $\mathcal{R}_+$, defined in $\mathcal{R}$ by $x_a \cdot \xi_a > 0$, is the subset of $\mathcal{R}$ where $t > 0$. Hence the 'outgoing' terminology for $R(\lambda + i0)$ and 'incoming' for $R(\lambda - i0)$. In fact, the solution of $(D_t + H)u = 0$ with $u|_{t=0} = \phi$, $\phi \in \mathcal{S}(X_0)$, say, is $u(\,\cdot\,, t) = e^{-iHt}\phi$. The time-dependent propagation of singularities shows that $\mathrm{WF}_{\mathrm{sc}}(u)$ is a subset of the union of the image of $\mathcal{R}_+$ under the forward broken bicharacteristic relation and the image of $\mathcal{R}_-$ under the backward bicharacteristic relation. Using the spectral measure and Stone's theorem,

$$u(\,\cdot\,, t) = \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-i\lambda t} (R(\lambda + i0) - R(\lambda - i0))\phi \, d\lambda$$

Fixing some $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$, for ϕ in the range of $\psi(H)$ we thus deduce that in $t > 0$, $\mathrm{WF}_{\mathrm{sc}}(u)$ arises from the $R(\lambda+i0)$ term, and in $t < 0$ from the $R(\lambda-i0)$ term. So the time-dependent and stationary settings are very close: the only difference is that in the latter, λ is a parameter, while in the former, it is a variable, $\lambda = -\tau$.

Again, one can make more precise propagation statements in some circumstances, such as three-body scattering, where the precise nature of the singularities can be analyzed, see [22; 63]. Here we only state the stronger implication for the structure of the scattering matrices, which we proceed to analyze.

4. Scattering Matrices

Physically, the scattering matrices relate incoming and outgoing data in an experiment. In the time independent framework (where $-\lambda$ is the dual variable of time), for short-range potentials an incoming wave of energy λ in channel α (a channel is the choice of a cluster a and an L^2-eigenfunction ψ_α of H^a of energy ε_α) takes the following form in $|x| > 1$:

$$u_{\alpha,-} = e^{-i\sqrt{\lambda-\varepsilon_\alpha}|x_a|}|x_a|^{-(\dim X_a-1)/2} g_{\alpha,-}\Big(\frac{x_a}{|x_a|}\Big)\psi_\alpha(x^a) + u'_-$$

Similarly, an outgoing wave has the form

$$u_{\alpha,+} = e^{i\sqrt{\lambda-\varepsilon_\alpha}|x_a|}|x_a|^{-(\dim X_a-1)/2} g_{\alpha,+}\Big(\frac{x_a}{|x_a|}\Big)\psi_\alpha(x^a) + u'_+$$

i.e. the sign of the phase has changed. Here $g_{\alpha,\pm}$ may be taken e.g. L^2 functions on S_a, the unit sphere in X_a, or ideally, at least one of them may be taken $\mathcal{C}^\infty$. In either case, $u'_\pm$ are 'lower order terms', namely they must be in $L^2_{-1/2}$. (Note that $\langle x_a\rangle^{-(\dim X_a-1)/2} \in L^2_s(X_a)$ for $s < -\frac{1}{2}$ but not for $s = -\frac{1}{2}$.) In fact, for $g_{\alpha,\pm} \in \mathcal{C}^\infty_c(S_{a,\mathrm{reg}})$ we may take them to be of the form $e^{-i\sqrt{\lambda-\varepsilon_\alpha}|x_a|}|x_a|^{-(\dim X_a+1)/2}v$ where v is a 0th order symbol, with $S_{a,\mathrm{reg}}$ denoting $X_{a,\mathrm{reg}} \cap S_0$.

One can now produce tempered distribution with given incoming, or alternatively of given outgoing, asymptotics. A typical example is of the form

$$P_{\alpha,+}(\lambda)g_{\alpha,-} = u_{\alpha,-} - (H - (\lambda + i0))^{-1}((H-\lambda)u_{\alpha,-}); \tag{4–1}$$

here the lower order terms may be dropped from $u_{\alpha,-}$ without affecting $u = P_{\alpha,+}(\lambda)g_{\alpha,-}$ and $g_{\alpha,-}$ can be specified to be any smooth function on S_a. In general, even if the incoming data are in a single channel α, as in (4–1), the corresponding generalized eigenfunction u of H will have outgoing waves in all channels. The S-matrix $S_{\alpha\beta}(\lambda)$ picks out the component in channel β by projection in a certain sense, see [64]. Thus, $S_{\alpha\beta}(\lambda)$ maps functions on S_a, the unit sphere in X_a, to functions on S_b, by

$$S_{\alpha\beta}(\lambda)g_{\alpha,-} = g_{\beta,+}$$

for u as in (4–1). For example, the free-to-free (i.e. N-cluster to N-cluster in N-body scattering) S-matrix $S_{00}(\lambda)$ maps functions on S_0, the unit sphere in $\mathbb{R}^n$, to functions on S_0, more precisely $S_{00}(\lambda) : L^2(S_0) \to L^2(S_0)$ is bounded.

More precisely, let T_+ be a pseudodifferential operator that is identically 1 on the outgoing radial set and identically 0 on the incoming radial set; see the paragraph of (5–6) for a precise statement. Then

$$S_{\alpha\beta}(\lambda) = \frac{1}{2i\sqrt{\lambda - \varepsilon_\beta}}((H-\lambda)T_+P_{\beta,-}(\lambda))^* P_{\alpha,+}(\lambda), \tag{4–2}$$

i.e. for any $g \in \mathcal{C}^\infty(S_{a,\mathrm{reg}})$, $h \in \mathcal{C}^\infty(S_{b,\mathrm{reg}})$,

$$\langle h, S_{\alpha\beta}(\lambda)g\rangle = \Big\langle (H-\lambda)T_+P_{\beta,-}(\lambda)h,\ \frac{1}{2i\sqrt{\lambda-\varepsilon_\beta}}P_{\alpha,+}(\lambda)g\Big\rangle.$$

This is equivalent to the usual wave operator definition in the time-dependent setting, see [64]. An immediate consequence of the propagation of singularities and the definition of the scattering matrices is the following:

Theorem 4.1. *The wave front relation of $S_{\alpha\beta}(\lambda)$ is given by the broken bicharacteristic relation. In particular, if no proper subsystem of H has bound states, the wave front relation of $S_{00}(\lambda)$ is given by the broken geodesic flow on S_0 at distance π.*

While typically broken bicharacteristics can be continued in many ways when they hit a collision plane, it is important to keep in mind that under suitable assumptions (which rule out geometric complications) the broken bicharacteristic relation is Lagrangian, hence its dimension is the same as if there were no collision planes. The reason is that only a low dimensional family of broken bicharacteristics hits any specified collision plane, with the dimension of the possible continuations of each of these these bicharacteristics compensating to yield the correct dimension for Lagrangian submanifolds.

The significant improvement in the three-body case, as shown by Hassell and the author [63; 22], is that one can pinpoint not only the location of the singularities, but also their precise form. This theorem was motivated by the geometric result of Melrose and Zworski [46], showing that the scattering matrix on asymptotically Euclidean manifolds is a Fourier integral operator.

Theorem 4.2. *Suppose that H is a three-body Hamiltonian and the V_a are Schwartz on X^a for all a. Then $S_{00}(\lambda)$ is a finite sum of Fourier integral operators (FIOs) associated to the broken geodesic relation on S_0 to distance π. Its canonical relation corresponds to the various collision patterns. The principal symbol of the term corresponding to a single collision at X_a is given by, and in turn determines, the 2-body S-matrix of H^a at energies $\lambda' \in (0, \lambda)$.*

Remark. This result presumably extends to short range symbolic potentials, using the same methods, though it is technically more complicated to write down the argument in that case, and it has not been done. In fact, it should also extend to the N-body problem, provided that there are no bound states in any proper subsystem. Some assumption on the bound states is necessary, for otherwise the generalized broken bicharacteristic relation can become fairly complicated; see [66]. The reason why one does not need any assumption on bound states in three-body scattering is that for any 2-cluster a, $\mathrm{Char}_a(\lambda) \cap \pi_{0a}(\mathrm{Char}_0(\lambda))$ is either empty (if 0 is not an eigenvalue of H^a) or consists of the boundary of $\pi_{0a}(\mathrm{Char}_0(\lambda))$. In the former case there is no interaction (modulo smoothing terms) between the 0-cluster and the a-cluster dynamics, while in the latter case in the only place they interact, the two dynamics give the same propagation.

It should also be noted that the normalization of $S_{\alpha\beta}(\lambda)$ is not the standard one in many-body scattering (which is based on wave operators), but rather follows the geometric conventions [41]. The difference is that in the wave operator approach, free motion is factored out, so the free scattering matrix is the identity operator. On the other hand, in the geometric approach we describe the asymptotics of generalized eigenfunctions, or alternatively of the Schrödinger equation. Since free particles move to infinity in the opposite direction from which they came, it is reasonable that the two should differ by (a constant multiple of) pull-back by the antipodal map, and this is indeed the case, see [64]. The distance π propagation along (not broken!) geodesics on the sphere indeed takes particles to the antipodal point.

An immediate corollary, when combined with two-body results (e.g. analyticity of the S-matrix in λ' and the Born approximation) is the following inverse result.

Corollary. *If the V_a decay exponentially and* $\dim X_a \geq 2$ *for all a then $S_{00}(\lambda)$ for a single value of λ determines all interactions.*

This result is analogous to the recovery of cracks in a material by directing sound waves at it and observing the singularities of the reflected waves, except the last step which uses two-body results to get the potentials from the two-body S-matrices.

The other extremal scattering matrices are the 2-cluster to 2-cluster ones, and they describe the physically most interesting events. Indeed, it is hard to make more than two particles collide in an accelerator, so the initial state in a physical experiment tends to be a 2-cluster. The following result is due to Skibsted [59], and it also follows from the propagation of singularities and the definition of the S-matrices.

Theorem 4.3. *Let α and β be two-clusters, and suppose that either $\varepsilon_\alpha \in \mathrm{spec}_d H^a$ and $\varepsilon_\beta \in \mathrm{spec}_d H^b$, or V_c is Schwartz for all c. Then the two-cluster to two-cluster S-matrix $S_{\alpha\beta}(\lambda)$ has $\mathcal{C}^\infty$ Schwartz kernel, except if $\alpha = \beta$ in which case the Scwartz kernel of $S_{\alpha\alpha}(\lambda)$ is conormal to the graph of the antipodal map on S_a, corresponding to free motion.*

Thus, principal symbol calculations do not help in this inverse problem. Note that if H is a 3-body Hamiltonian, then $\varepsilon_\alpha \in \mathrm{spec}_d H^a$ and $\varepsilon_\beta \in \mathrm{spec}_d H^b$ holds for any nonthreshold bound state energies. The new result, in a joint project with Gunther Uhlmann, is the following [62].

THEOREM 4.4. *Suppose that H is a 3-body Hamiltonian, a is a 2-cluster, α is a channel of energy $\varepsilon_\alpha < 0$, V_a is a symbol of negative order (i.e. may be long range). For any $\mu > \dim X_a$ there exists $\delta > 0$ such that the following holds.*

Suppose that $\sup |(1 + |x^b|)^\mu V_b(x^b)| < \delta$ for all $b \neq a$. Suppose also that $I \subset (\varepsilon_\alpha, 0)$ is a nonempty open set, and let

$$R = 2\sqrt{\sup I - \varepsilon_\alpha}.$$

Then $S_{\alpha'\alpha''}(\lambda)$ given for all $\lambda \in I$ and for all bound states α', α'' of H^a with $\varepsilon_{\alpha'}, \varepsilon_{\alpha''} < \sup I$, determines the Fourier transform of the effective interaction $V_{\alpha,\mathrm{eff}}$ in the ball of radius R centered at 0.

The effective interaction is the interaction that arises if we consider the 3-body problem as a 2-body problem, i.e. if we regard the two particles forming the cluster a as a single particle. Mathematically, this amounts to projecting to the state ψ_α in X^a and obtaing a new Hamiltonian $\Delta_{X_a} + V_{\alpha,\mathrm{eff}}$ on X_a. Thus, the effective interaction is physically relevant. Moreover, there is no hope for recovering anything better than $V_{\alpha,\mathrm{eff}}$ as shown by the high-energy inverse results of Enss and Weder [11; 13], Novikov [49] and Wang [70; 71].

This theorem says that if the unknown interactions are small then the effective interaction can be determined from the knowledge of all S-matrices with incoming and outgoing data in the cluster a in the relevant energy range. In fact, near-forward information suffices as in two-body scattering, where this was observed recently by Novikov [50]. Also, if one is willing to take small R and α is the ground state of H^a, it suffices to know $S_{\alpha\alpha}(\lambda)$ to recover $\hat{V}_{\alpha,\mathrm{eff}}$ in a small ball.

In case V_b decay exponentially on X^b for all $b \neq a$, then $V_{\alpha,\mathrm{eff}}$ decays exponentially on X_a, hence its Fourier transform is analytic, so $V_{\alpha,\mathrm{eff}}$ itself can be recovered from these S-matrices.

REMARK. It is clear from the proof in [62] that there is a natural extension of this theorem to many-body scattering at low energies.

This result should extend to higher energies, i.e. $\sup I \leq 0$ is *not* expected to be essential. But it is hard to make R greater than $2\sqrt{-\varepsilon_\alpha}$ even then. The reason is that our method relies on the construction of exponential solutions following Faddeev [14], Calderón [6], Sylvester and Uhlmann [60] and Novikov and Khenkin [48], but in the three-body setting. One thus allows complex momenta $\rho \in \mathbb{C}(X_a)$, the complexification of X_a, and one wants to construct solutions of $(H - \lambda)u = 0$ of the form

$$e^{i\rho\cdot x_a}(\psi_\alpha(x^a) + v),$$

where $v = v_\rho$ is supposed to be 'small' in the sense that it goes to 0 as $\rho \to \infty$ in an appropriate fashion. Note that with $v = 0$ these complex plane waves solve $(H_a - \lambda)u = 0$ with

$$\lambda = \rho \cdot \rho + \varepsilon_\alpha; \tag{4–3}$$

this expresses that the total energy λ is the sum of the kinetic energy, $\rho \cdot \rho$, and the potential energy ε_α.

To construct u, we need to find v, and its study reduces to that of the conjugated Hamiltonian

$$e^{-i\rho \cdot x_a}(H-\lambda)e^{i\rho \cdot x_a} = H^a + \Delta_{X_a} + 2\rho \cdot D_{X_a} + I_a - \varepsilon_\alpha$$

with $\rho \in \mathbb{C}(X_a)$ the complex frequency. Here we used (4–3). Now, I_a is considered as a perturbation (this is the reason for the smallness assumption in the theorem), so we really study the model operator,

$$H^a + \Delta_{X_a} + 2\rho \cdot D_{X_a} - \varepsilon_\alpha.$$

Taking the Fourier transform in the X_a variables, one obtains

$$H^a + |\xi_a|^2 + 2\rho \cdot \xi_a - \varepsilon_\alpha.$$

Writing $\rho = z\nu + \rho_\perp$ with $|\nu| = 1$, $\rho_\perp \cdot \nu = 0$, ρ, ν real, $z \in \mathbb{C}$, this operator becomes

$$H^a + |\xi_a|^2 + 2\rho_\perp \cdot \xi_a + 2z\nu \cdot \xi_a - \varepsilon_\alpha = H^a + (\xi_a + \rho_\perp)^2 + 2z\nu \cdot \xi_a - |\rho_\perp|^2 - \varepsilon_\alpha.$$

If ρ is not real, then neither is z, so this operator is invertible if $\nu \cdot \xi_a \neq 0$ since H^a is self-adjoint. On the other hand, if $\nu \cdot \xi_a = 0$, this operator becomes

$$H^a + (\xi_a + \rho_\perp)^2 - |\rho_\perp|^2 - \varepsilon_\alpha,$$

i.e. its invertibility properties correspond to the behavior of the boundary values of the resolvent of H^a at the real axis. If $|\rho_\perp|^2 + \varepsilon_\alpha < 0$, i.e. if $|\rho_\perp| < \sqrt{-\varepsilon_\alpha}$, then the spectral parameter $|\rho_\perp|^2 + \varepsilon_\alpha - (\xi_a + \rho_\perp)^2$ is negative, so only the bound states of H^a contribute to the characteristic variety, i.e. the two-cluster a may not break up. On the other hand, if $|\rho| \geq \sqrt{-\varepsilon_\alpha}$, such a break-up is possible *even if* $\lambda < 0$, i.e. where the break up may not happen for *real* frequencies. The break-up greatly influences analyticity properties, hence one cannot easily use large $\rho_\perp$. On the other hand, one needs such large $\rho_\perp$ to recover $V_{\alpha,\mathrm{eff}}$ on larger balls, hence the limitation in the theorem. This also suggests that the fixed energy problem would be hard, since then one always needs to let $\rho_\perp \to \infty$ to keep $\rho \cdot \rho = |\rho_\perp|^2 + z^2$ fixed and yet have $\rho \to \infty$.

5. Many-body Scattering Pseudo-Differential Operators

I will present the calculus from the compactified point of view. Both the one-step polyhomogeneous (i.e. 'classical') and the nonpolyhomogeneous calculus can be described in noncompact terms, i.e. directly on X_0, but this is more complicated and less natural. Indeed, one of the beauties of compactification is that it exactly captures the structure of many-body Hamiltonians. *We warn the reader here that from now on the Euclidean variable is written as z, rather than*

x in the preceeding sections, for compatibility with previous papers espousing this approach, such as [41; 42].

To see how the compactification should go, recall first that a classical symbol of order 0 on $\mathbb{R}^n_z$ has an asymptotic expansion

$$a(r\omega) \sim \sum_{j=0}^{\infty} r^{-j} a_j(\omega),\ a_j \in \mathcal{C}^\infty(\mathbb{S}^{n-1}),$$

in the polar coordinates (r, ω): $z = r\omega$. The meaning of such an expansion is that, for any k, the difference of a and the sum of the first k terms on the right hand side is a symbol of order $-k$. This expansion is just a Taylor series at $r = \infty$, or rather at '$r^{-1} = 0$'. So we compactify $\mathbb{R}^n$ into a ball $\overline{\mathbb{B}^n}$ by adding points $(0, \omega)$, $\omega \in \mathbb{S}^{n-1}$, and making $(r^{-1}, \omega) = (x, \omega)$ coordinates near these points. The resulting space is called the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$. Thus, a classical symbol of order 0 is simply a smooth function of $\overline{\mathbb{R}^n}$; the asymptotic expansion at infinity is its Taylor series around the boundary, $x = 0$.

This compactification, whose utility in this context was emphasized by Melrose [41], can also be realized as the closed unit upper hemisphere via a modified stereographic projection. So let $\mathrm{RC} : \mathbb{R}^n \to \mathbb{S}^n_+$ be given by

$$\mathrm{RC}(z) = \left(\frac{1}{\langle z\rangle}, \frac{z}{\langle z\rangle}\right), \qquad \text{where } \langle z\rangle = (1 + |z|^2)^{1/2}, \quad z \in \mathbb{R}^n.$$

Then n of the $n+1$ variables $\big(1/\langle z\rangle, z/\langle z\rangle\big)$ give local coordinates on various regions of $\mathbb{S}^n_+$. In particular, in coordinate patches near the equator, which is $\partial\mathbb{S}^n_+$, $1/\langle z\rangle$ (or indeed $x = |z|^{-1}$) and $n-1$ of $z_j/\langle z\rangle$ (or indeed $\omega_j = z_j/|z|$) can be taken as coordinates, showing that $\mathbb{S}^n_+$ can be identified with the radial compactification $\overline{\mathbb{R}^n}$. A slightly modified version of x (it needs to be smoothed at $z = 0$, where '$x = \infty$'), or $\langle z\rangle^{-1}$, can be taken as a boundary defining function. We will usually write x for this, so $x = |z|^{-1}$ for $|z| \geq 1$, say. (A boundary defining function is a nonnegative function whose zero set is exactly the boundary, and whose differential does not vanish there.)

How can we adapt this to many-body scattering? Let $\bar{X}_a$ denote the closure of X_a in the compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, and let $C_a = \partial\bar{X}_a \subset \partial\mathbb{S}^n_+ = C_0$. The closure of any translate of X_a intersects C_0 in the same submanifold (a sphere) as X_a itself. Indeed, writing the coordinates as (z_a, z^a) on $X_0 = X_a \oplus X^a$, local coordinates near C_a are given by $Z^a = z^a/|z|$, $|z|^{-1}$ and $\dim X_a - 1$ of $(z_a)_j/|z|$. Thus, $Z^a \to 0$ as $x \to 0$ along any translate, since z^a is constant along these. So V_a is not even continuous on $\bar{X}_0$, as it takes different values on the different translates of X_a. However, it *is* a negative order symbol (in particular continuous with boundary value 0) on $\bar{X}_0 \setminus C_a$, if V_a is such on X^a; see Figure 5.

So the compactification works for V_a, except at C_a. To remedy this, we blow up C_a. This is an invariant way of introducing polar coordinates about it (i.e. projective coordinates in various charts). That is, curves approaching C_a from various normal directions will correspond to different points on the blown-up

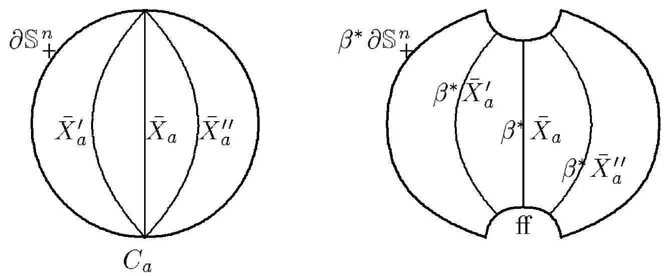

Figure 5. Translates of X_a on $[\bar X_0; C_a]$.

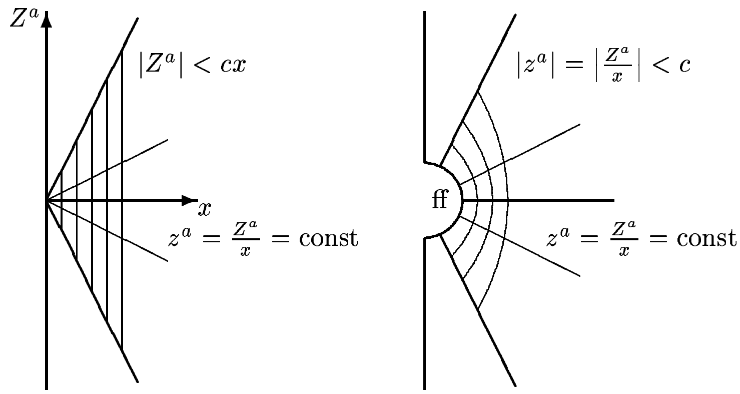

Figure 6. The blow up of C_a, given by $Z^a = 0$, $x = 0$.

space $[\bar X_0; C_a]$. Since C_a is given by $x = 0$, $Z^a = 0$, in local coordinates, this means concretely that the components of Z^a/x become coordinate functions on the part of $[\bar X_0; C_a]$ where this quotient is finite. (For the sake of completeness, a complete set of coordinates in this region is given by x, the components of Z^a as well as the $\dim X_a - 1$ coordinates on the sphere $y_a = z_a/|z_a|$; see Figure 6.) But $Z^a/x = z^a$, so it is now easy to see that for classical symbols V_a on X^a (of negative integer order), V_a is a $\mathcal{C}^\infty$ function on $[\bar X_0; C_a]$.

In general, there are many collision planes, and we blow them up recursively, starting with ones of the largest codimension, to get $[\bar X_0; \mathcal{C}]$,

$$\mathcal{C} = \{C_a : X_a \in \mathcal{X},\ a \neq 1\}.$$

We refer to [65] for details.

There is no reason at all to take $\bar X_0$ as the space we start with. Given any compact manifold with boundary, $\bar X$, and a cleanly intersecting family of closed embedded submanifolds $\mathcal{C}$ of $\partial \bar X$, we can define $[\bar X; \mathcal{C}]$ analogously. For instance, one can start with $\bar X = \overline{\mathbb{R}^n} \times \mathbb{S}^k$. The space $[\bar X; \mathcal{C}]$ is equipped with boundary fibrations given by the blow-down maps, see [38] for a simpler case where these first appeared explicitly.

Having described the configuration space, we turn to differential operators. X_0 has a nice algebra of differential operators, consisting of operators with symbolic coefficients: $\sum_{|\alpha|\le m} a_\alpha(z) D_z^\alpha$, $a_\alpha \in S^0(X_0)$. We may require instead that a_α is 'classical', i.e. that $a_\alpha \in \mathcal{C}^\infty(\bar{X}_0)$. The resulting algebras were denoted $\mathrm{Diff}_{\mathrm{sccl}}(\bar{X}_0)$ and $\mathrm{Diff}_{\mathrm{sc}}(\bar{X}_0)$ by Melrose; he called them 'scattering differential operators'.

This setup generalizes to the geometric set-up as follows. Let (x, y), with $y = (y_1, \dots, y_{n-1})$, be local coordinates near $\partial\bar{X}_0$. Then the vector fields in $\mathrm{Diff}_{\mathrm{sc}}(\bar{X}_0)$ are linear combinations of $x^2 D_x$ and the $x D_{y_j}$ with coefficients in $\mathcal{C}^\infty(\bar{X}_0)$, as can be seen easily by an explicit calculation.

Now, if $\bar{X}$ is a manifold with boundary, $\mathcal{V}_{\mathrm{b}}(\bar{X})$ is the Lie algebra of vector fields tangent to $\partial\bar{X}$, and $\mathcal{V}_{\mathrm{sc}}(\bar{X}) = x\mathcal{V}_{\mathrm{b}}(\bar{X})$, where x is a defining function of $\partial\bar{X}$. $\mathcal{V}_{\mathrm{sc}}(\bar{X})$ is independent of the choice of x. Then $\mathcal{V}_{\mathrm{b}}(\bar{X})$ is spanned by $x\partial_x$ and ∂_y over $\mathcal{C}^\infty(\bar{X})$, so $\mathcal{V}_{\mathrm{sc}}(\bar{X})$ is spanned by $x^2\partial_x$ and $x\partial_y$ over $\mathcal{C}^\infty(\bar{X})$. By definition, these generate $\mathrm{Diff}_{\mathrm{sc}}(\bar{X})$. Also, $\mathcal{V}_{\mathrm{sc}}(\bar{X})$ is the set of all smooth sections of a vector bundle over $\bar{X}$, this is denoted by ${}^{\mathrm{sc}}T\bar{X}$. Its dual bundle is the scattering cotangent bundle, denoted by ${}^{\mathrm{sc}}T^*\bar{X}$. In the Euclidean setting,

$$ {}^{\mathrm{sc}}T\bar{X}_0 = \bar{X}_0 \times X_0, \ \ {}^{\mathrm{sc}}T^*\bar{X}_0 = \bar{X}_0 \times X_0^*. $$

The way to generalize this differential operator algebra to one that includes many-body potentials is to allow singular coefficients $a_\alpha \in \mathcal{C}^\infty([\bar{X};\mathcal{C}])$. Thus,

$$ \mathrm{Diff}_{\mathrm{sc}}(\bar{X};\mathcal{C}) = \mathcal{C}^\infty([\bar{X};\mathcal{C}]) \otimes_{\mathcal{C}^\infty(\bar{X})} \mathrm{Diff}_{\mathrm{sc}}(\bar{X}). $$

In particular, if H is a many-body Hamiltonian, with either potential or higher order interactions, then $H \in \mathrm{Diff}^2_{\mathrm{sc}}(\bar{X}_0;\mathcal{C})$.

We let ${}^{\mathrm{sc}}T[\bar{X};\mathcal{C}] = \beta^{*\mathrm{sc}}T\bar{X}$ and ${}^{\mathrm{sc}}T^*[\bar{X};\mathcal{C}] = \beta^{*\mathrm{sc}}T^*\bar{X}$, where $\beta : [\bar{X};\mathcal{C}] \to \bar{X}$ is the blow-down map, and we are pulling back the vector bundles by it. Hence in the Euclidean setting,

$$ {}^{\mathrm{sc}}T[\bar{X}_0;\mathcal{C}] = [\bar{X}_0;\mathcal{C}] \times X_0, \quad {}^{\mathrm{sc}}T^*[\bar{X}_0;\mathcal{C}] = [\bar{X}_0;\mathcal{C}] \times X_0^*. $$

Now it is natural to define pseudodifferential operators using these bundles. Although I restrict the discussion to the Euclidean setting, the construction generalizes to any $\bar{X}$ via localization.

So we consider symbols

$$ a \in \langle z\rangle^{-l}\langle\zeta\rangle^m \mathcal{C}^\infty([\bar{X}_0;\mathcal{C}] \times \bar{X}_0^*), \tag{5–1} $$

$X_0 = \mathbb{R}^n$, where ζ is the dual variable of z, i.e. the variable on X_0^*. Note that this means that a is a classical symbol of order m in ζ. As usual, we define the Schwartz kernel of the left quantization of a by

$$ A = q_L(a) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} a(z,\zeta)\, d\zeta, \tag{5–2} $$

understood as an oscillatory integral. In particular, for any $f \in \mathcal{S}(\mathbb{R}^n)$,

$$Af(z) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} a(z,\zeta) f(z')\, d\zeta\, dz',$$

again understood as an oscillatory integral. We write $A \in \Psi_{\mathrm{sc}}(\bar{X}_0; \mathcal{C})$ for this class of operators.

Note that $\langle z \rangle^l a \in S^m_\infty(X_0; X_0^*)$, Hörmander's uniform symbol space [27, Section 18.1], so $A = \langle z \rangle^l \tilde{A}$, $\tilde{A} \in \Psi^m_\infty(X_0)$, the uniform ps.d.o.-algebra arising by quantizing $S^m_\infty(X_0; X_0^*)$ as in (5–2). In particular, since the mapping properties of $\Psi^m_\infty(X_0)$ between weighted Sobolev spaces $H^{r,s}$ are well known, the corresponding properties of A follow. Namely, $A : H^{r,s} \to H^{r-m,s+l}$ for all r, s, where

$$H^{r,s} = \langle z \rangle^{-s} H^r = \langle f \in \mathcal{S}'(\mathbb{R}^n) : \langle z \rangle^s f \in H^r \}.$$

Now, $\Psi_{\mathrm{sc}}(\bar{X}_0; \mathcal{C})$ is a $*$-algebra, in particular is closed under composition. Indeed, since $\Psi_{\mathrm{sc}}(\bar{X}_0; \mathcal{C}) \subset \Psi_\infty(X_0)$, and the latter is closed under composition, it suffices to follow the usual proof and make sure that the product is in $\Psi_{\mathrm{sc}}(\bar{X}_0; \mathcal{C})$, rather than merely in $\Psi_\infty(X_0)$. Thus, the key fact is that for any

$$b \in \langle z \rangle^{-l} \langle \zeta \rangle^m \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}]_z \times [\bar{X}_0; \mathcal{C}]_{z'} \times (\bar{X}_0^*)_\zeta)$$

there exists a as in (5–1) such that the induced operators

$$B = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} b(z, z', \zeta)\, d\zeta, \tag{5–3}$$

and A as in (5–2) are the same. The proof of this claim is standard. Indeed, we can expand b in Taylor series in z' around $z = z'$ to finite order k. The finite order terms depend on z' only via $(z'-z)^\alpha$, $|\alpha| \le k$. We rewrite $(z'-z)^\alpha e^{i(z-z')\cdot\zeta}$ as $(-1)^{|\alpha|} D^\alpha_\zeta e^{i(z-z')\cdot\zeta}$, and integrate by parts with respect to ζ. Thus, the α-term is the left quantization of

$$\frac{1}{\alpha!} D^\alpha_{z'} D^\alpha_\zeta b(z, z', \zeta)|_{z'=z}, \tag{5–4}$$

which is of the desired form, i.e. is in $\langle z \rangle^{-l} \langle \zeta \rangle^m \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}] \times \bar{X}_0^*)$. In fact, the weight $\langle \zeta \rangle^m$ can be replaced by $\langle \zeta \rangle^{m-|\alpha|}$ due to the symbolic properties of b in $(X_0^*)_\zeta$, but no corresponding change may be made for the z weight. Similarly, the remainder term is of the form

$$\begin{aligned} K_k(z, z') &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(z-z')\cdot\zeta} b_k(z, z', \zeta)\, d\zeta, \\ b_k &\in \langle z \rangle^{-l} \langle \zeta \rangle^{m-k-1} \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}]_z \times [\bar{X}_0; \mathcal{C}]_{z'} \times (\bar{X}_0^*)_\zeta). \end{aligned} \tag{5–5}$$

Now we can asymptotically sum the b_α to get a new symbol

$$c \in \langle z \rangle^{-l} \langle \zeta \rangle^m \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}] \times \bar{X}_0^*).$$

Let C be the left quantization of c. Then $B - C$ is of the form (5–5) for all k, with b_k replaced by some b'_k with the same properties. It is then straightforward to show that the Schwartz kernel K' of $B - C$ is $\mathcal{C}^\infty$, decays rapidly with all derivatives as $\langle z - z'\rangle \to \infty$, and more precisely it is of the form

$$K' \in \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}]_z \times (\bar{X}_0)_{z-z'})$$

with infinite order vanishing at the boundary of the second factor. Taking its Fourier transform b' in $z - z'$, K' is thus the left quantization of $a = c + b'$, proving the claim, hence in turn that $\Psi_{\mathrm{sc}}(\bar{X}_0; \mathcal{C})$ is closed under composition.

In the two-body setting, where $\mathcal{C} = \varnothing$, there is a principal symbol at infinity. Namely, if $A \in \Psi^{m,l}_{\mathrm{sc}}(\bar{X})$, $A = q_L(a)$, then $\sigma_{m,l}(A)$ is given by the restriction of $\langle z\rangle^l \langle\zeta\rangle^{-m} a \in \mathcal{C}^\infty(\bar{X}_0 \times \bar{X}_0^*)$ to $\partial(\bar{X}_0 \times \bar{X}_0^*) = (\partial\bar{X}_0 \times \bar{X}_0^*) \cup (\bar{X}_0 \times \partial\bar{X}_0^*)$. Of the two boundary hypersurfaces, the restriction to $\bar{X}_0 \times \partial\bar{X}_0^*$ yields the usual principal symbol, while the restriction to $\partial\bar{X}_0 \times \bar{X}_0^*$ is the principal symbol at infinity. More precisely, if $l = 0$, we can indeed define the part of $\sigma_{m,0}(A)$ at infinity to be the restriction of a to $(\partial\bar{X}_0) \times \bar{X}_0^*$. The principal symbol is multiplicative, i.e. $\sigma_{m+m',l+l'}(AB) = \sigma_{m,l}(A)\sigma_{m',l'}(B)$. Thus, $[A, B] \in \Psi^{m+m'-1,l+l'+1}_{\mathrm{sc}}(\bar{X})$, and its principal symbol is given by the Poisson bracket of their symbols, see Section 6.

Since in the many-body setting we do not gain decay in z in (5–4), we cannot expect to have a commutative principal symbol at infinity, i.e. at $\partial[\bar{X}_0; \mathcal{C}]$. For $C \in \Psi^{m,0}_{\mathrm{sc}}(\bar{X}, \mathcal{C})$, $y_a \in C_{a,\mathrm{reg}}$, $\zeta_a \in X_a^*$, we let

$$\hat{C}_a(y_a, \zeta_a) = (2\pi)^{-\dim X^a} \int e^{i(z^a - (z')^a)\cdot\zeta^a} c(y_a, z^a, \zeta)\, d\zeta \in \mathcal{S}'(X^a \times X^a)$$

be the operator valued principal symbol of C at (y_a, ζ_a). Thus, $\hat{C}_a(y_a, \zeta_a)$ is a tempered distribution on $X^a \times X^a$ (denoted by the variables $(z^a, (z^a)')$), and it is in fact a many-body ps.d.o. itself: $\hat{C}_a(y_a, \zeta_) \in \Psi^{m,0}_{\mathrm{sc}}(\bar{X}^a, \mathcal{C}^a)$ corresponding to the collision planes $X^a \cap X_b$, with b satisfying $X_b \supset X_a$. We also call it the indicial operator of C to make it clear we are not talking about the standard principal symbol. We also write $\hat{C}_a(z_a, \zeta_a)$ in the same setting, where we extend $\hat{C}_a(y_a, \zeta_a)$ to be homogeneous degree 0 in z_a. It can be easily seen to satisfy

$$\hat{A}_a \hat{B}_a = \widehat{(AB)}_a,$$

where on the left hand side we compose the operators $\hat{A}_a(z_a, \zeta_a)$ and $\hat{B}_a(z_a, \zeta_a)$. Thus, multiplication of operators is only partially commutative, even to top order. This can be observed already from $[D_{z_a}, V_a] = 0$, hence certainly lower order at infinity, while $[D_{z^a}, V_a] \in \mathcal{C}^\infty([\bar{X}_0; \mathcal{C}])$ without *any* decay at C_a.

This observation has important implications for the positive commutator estimates that we take up in the next section. Namely, H must commute to leading order with the operators we want to microlocalize with. This means that these operators A must have $\hat{A}_a$ commute with $\hat{H}_a$, and the most reasonable way of achieving this is to have $\hat{A}_a$ be a scalar multiple of $\psi(\hat{H}_a)$, where e.g. $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$. This multiple defines a function on $\dot{T}^*\bar{X}_0$; we want this to arise from a smooth π-invariant function for our estimates. On the other hand, $\psi(\hat{H}_a)$ provides localization at the characteristic set.

Here, however, I would like to talk about pseudodifferential constructions first. Namely, if $\lambda \notin \mathbb{R}$, or indeed if $\lambda \in \mathbb{C} \setminus [\inf \Lambda, +\infty)$ then there exists a parametrix $G(\lambda) \in \Psi_{\mathrm{sc}}^{-2,0}(\bar{X}_0;\mathcal{C})$ for $H-\lambda$, i.e. such that

$$(H-\lambda)G(\lambda) - \mathrm{Id},\ G(\lambda)(H-\lambda) - \mathrm{Id} \in \Psi_{\mathrm{sc}}^{-\infty,\infty}(\bar{X}_0;\mathcal{C}).$$

Then the parametrix identities show that

$$\lambda \in \mathbb{C} \setminus \operatorname{spec} H \implies (H-\lambda)^{-1} \in \Psi_{\mathrm{sc}}^{-2,0}(\bar{X}_0;\mathcal{C}).$$

The parametrix construction proceeds inductively by constructing $(\hat{H}_a - \lambda)^{-1}$ in $\Psi_{\mathrm{sc}}^{-2,0}(\bar{X}_a, \mathcal{C}^a)$ for every $a \neq 1$ and then combining these: there exists a $G_0(\lambda) \in \Psi_{\mathrm{sc}}^{-2,0}(\bar{X}_0;\mathcal{C})$ with specified indicial operators $(\hat{H}_a - \lambda)^{-1}$, hence satisfying

$$(H-\lambda)G_0(\lambda) - \mathrm{Id},\ G_0(\lambda)(H-\lambda) - \mathrm{Id} \in \Psi_{\mathrm{sc}}^{-1,1}(\bar{X}_0;\mathcal{C}).$$

Then the standard Neumann series argument yields $G(\lambda)$.

The Helffer-Sjöstrand argument [24] then shows that for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\phi(H) = \frac{-1}{2\pi i} \int_{\mathbb{C}} \overline{\partial}_\lambda \tilde{\phi}(\lambda)(H-\lambda)^{-1}\, d\lambda \wedge d\bar{\lambda},$$

where $\tilde{\phi}$ is an almost analytic extension of ϕ: $\tilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{C})$, $|\overline{\partial}_\lambda \tilde{\phi}| \leq C_k |\operatorname{Im}\lambda|^k$ for all k. We can control $(H-\lambda)^{-1}$ in $\Psi_{\mathrm{sc}}^{-2,0}(\bar{X},\mathcal{C})$ as $\lambda \to \mathbb{R}$ with semi-norm estimates $\mathcal{O}(|\operatorname{Im}\lambda|^{-j})$ (j depends on the norm), so we conclude that $\phi(H) \in \Psi_{\mathrm{sc}}^{-\infty,0}(\bar{X};\mathcal{C})$.

We can now explain the precise specifications on T_+ in (4–2). Namely, we require that on a neighborhood of $\mathcal{R}_+(\lambda)$ in $\dot{T}^*X$, the indicial operators $\widehat{T_+}$ should equal $\widehat{\phi(H)}$ for some $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ identically 1 near λ, and on a neighborhood of $\mathcal{R}_-(\lambda)$ they should vanish. Explicitly this can be arranged by taking any ϕ as above, and any $\chi \in \mathcal{C}^\infty(\mathbb{R})$ identically 1 on $\big(\sqrt{\lambda}/2, +\infty\big)$, identically 0 on $(-\infty, -\frac{\sqrt{\lambda}}{2})$. Then let $T_+ = \phi(H) q_R(\chi((z\cdot\zeta)/\langle z\rangle))$, with q_R denoting the 'right quantization' (i.e. where we take $b = \chi((z'\cdot\zeta)/\langle z'\rangle)$ in (5–3)). Although $q_R(\chi((z\cdot\zeta)/\langle z\rangle))$ is not in $\Psi_{\mathrm{sc}}(\bar{X};\mathcal{C})$, due to the nonsymbolic behavior of b as $\zeta \to \infty$, T_+ is, namely $T_+ \in \Psi_{\mathrm{sc}}^{-\infty,0}(\bar{X};\mathcal{C})$, since $\phi(H)$ is smoothing: see [65]. Moreover,

$$\widehat{T_+}_a(z_a,\zeta_a) = \chi\Big(\frac{z_a\cdot\zeta_a}{|z_a|}\Big)\phi(\hat{H}_a(z_a,\zeta_a)), \tag{5–6}$$

hence has the desired properties.

Our construction of $\phi(H)$ in fact shows that if all potentials are in $S^{-\rho}(X_a)$, $\rho > 0$, and $\chi_a \in \mathcal{C}^\infty(\bar{X}_0)$ is supported away from C_b such that $C_b \supset C_a$ does *not* hold, then $\chi_a(\phi(H) - \phi(H_a)) \in \Psi_{\mathrm{sc}}^{-\infty,\rho}(\bar{X};\mathcal{C})$, hence trace class if $\rho > \dim X_0$. In the three-body setting this shows that

$$\phi(H) - \phi(H_0) - \sum_{\#b=2} (\phi(H_b) - \phi(H_0))$$

is trace class. Indeed, near C_a this can be written as

$$(\phi(H)-\phi(H_a))-\sum_{\#b=2,\ b\neq a}(\phi(H_b)-\phi(H_0)),$$

and now all terms in parantheses are in $\Psi_{\mathrm{sc}}^{-\infty,\rho}(\bar{X};\mathcal{C})$ near C_a. So we conclude, with a proof that shows much more, a result of Buslaev and Merkureev:

$$\sigma(\phi)=\operatorname{tr}((\phi(H)-\phi(H_0))-\sum_{\#b=2}(\phi(H_b)-\phi(H_0)))$$

defines a distribution $\sigma\in\mathcal{C}^{-\infty}(\mathbb{R})$. Writing $\sigma=\xi'$ defines the spectral shift function, up to a constant, which in turn, in two-body scattering, is the well-known generalization of the eigenvalue counting function on compact manifolds. These statements, as well as the following theorem, which is joint work with Xue-Ping Wang [68], generalize to arbitrary many-body Hamiltonians (with short-range interactions as indicated).

THEOREM 5.1. *Suppose H is a three-body Hamiltonian with Schwartz interactions*: $V_a\in\mathcal{S}(X^a)$, *and that all interactions are pair interactions* (*i.e.* $V_a\neq 0$ *implies that a is a* 2*-cluster*.) *Then the spectral shift function σ is $\mathcal{C}^\infty$ in* $\mathbb{R}\setminus(\Lambda\cup\operatorname{spec}_{\mathrm{pp}}H)$, *and it is a classical symbol at infinity* (*i.e. outside a compact set*) *with a complete asymptotic expansion*:

$$\sigma(\lambda)\sim\lambda^{(n/2)-3}\sum_{j=0}^{\infty}c_j\lambda^{-j},\ c_0=C_0(n)\sum_{a,b:a\neq b}\int_{\mathbb{R}^n}V_aV_b\,dg.$$

Note that σ decays one order faster than in 2-body scattering, and two orders faster than Weyl's law on compact manifolds. This is because $\phi(H_0)+\sum_{\#b=2}(\phi(H_b)-\phi(H_0))$ is, in a high-energy sense, closer to $\phi(H)$ than $\phi(H_0)$ is to $\phi(H)$ in two body scattering. If not all interactions are pair interactions, the order of the leading term changes, namely becomes $\lambda^{(n/2)-2}$ as in 2-body scattering.

The proof of this theorem relies on the propagation of singularities, applied to the Schwartz kernel of the resolvent, $R(\lambda+i0)$. (In fact, the theorem should generalize to symbolic potentials, but the proof would require a more precise microlocalization than provided by $\mathrm{WF}_{\mathrm{sc}}$.) So we now turn to the positive commutator methods that prove this.

6. Microlocal Positive Commutator Estimates

First I sketch, somewhat vaguely, the idea of positive commutator estimates. So suppose that we want to obtain estimates on the solutions of $Pu=f$, where f is known, and is 'nice', and P is self-adjoint. Suppose that we can construct an operator A which is self-adjoint and is such that

$$i[A,P]=B^*B+E. \tag{6–1}$$

Here B^*B is the positive term, giving the name to the estimate. The point is that we can estimate Bu in terms of Eu. Indeed, at least formally,

$$\langle u, i[A,P]u\rangle = \langle u, B^*Bu\rangle + \langle u, Eu\rangle = \|Bu\|^2 + \langle u, Eu\rangle.$$

On the other hand,

$$\langle u, i[A,P]u\rangle = \langle u, iAPu\rangle - \langle u, iPAu\rangle = \langle u, iAPu\rangle + \langle iAPu, u\rangle = 2\operatorname{Re}\langle u, iAPu\rangle.$$

Combining these yields

$$\|Bu\|^2 \le 2|\operatorname{Re}\langle u, iAPu\rangle| + |\langle u, Eu\rangle|. \tag{6–2}$$

This means that Bu can be estimated in terms of Pu, which is known from the PDE, and Eu, on which we need to make assumptions. The typical application is that E is supported in one region of phase space and B in another, in which case we can *propagate* estimates of u.

In fact, one can also apply this estimate if one does not know a priori that $Bu \in L^2$. Namely, an approximation argument gives that if Pu and Eu are in appropriate spaces so that the right hand side of (6–2) makes sense, then $Bu \in L^2$, and (6–2) holds. Considering pseudodifferential operators A of various orders, this means that we obtain microlocal weighted Sobolev estimates for u. Also, typically one has an error term F, i.e. $i[A,P] = B^*B + E + F$, but F is 'lower order' in some sense. Thus, $|\langle u, Fu\rangle|$ is added to the right hand side of (6–2), but being 'lower order' means that $|\langle u, Fu\rangle|$ automatically makes sense, hence is irrelevant when proving that $Bu \in L^2$.

In fact, this method also yields estimates for the resolvent very directly. Since for $t \in \mathbb{R}$, $i[A, P - it] = i[A,P]$, and

$$\begin{aligned}\langle u, i[A, P-it]u\rangle &= \langle u, iA(P-it)u\rangle - \langle u, i(P-it)Au\rangle\\ &= \langle u, iA(P-it)u\rangle + \langle iA(P+it)u, u\rangle\\ &= 2\operatorname{Re}\langle u, iA(P-it)u\rangle - 2t\langle Au, u\rangle.\end{aligned}$$

Thus, we deduce

$$\|Bu\|^2 + 2t\langle Au, u\rangle \le 2|\operatorname{Re}\langle u, iAPu\rangle| + |\langle u, Eu\rangle|. \tag{6–3}$$

This is in particular an estimate for $\|Bu\|$ provided that $t \ge 0$ and A is positive. Here we may take $u = u_t = (P - it)^{-1} f$, defined for $t > 0$, say, and we find a uniform estimate for Bu_t as $t \to 0$.

The question is thus how one can produce operators A which have a positive commutator with P as above. First, we recall how this happens in the scattering calculus. Namely, if $A \in \Psi_{\mathrm{sc}}^{m,l}(\bar X)$, $P \in \Psi_{\mathrm{sc}}^{m',l'}(\bar X)$ then $[A,P] \in \Psi_{\mathrm{sc}}^{m+m'-1,l+l'+1}(\bar X)$ and

$$\sigma_{m+m'-1,l+l'+1}(i[A,P]) = H_a p = -H_p a,\ a = \sigma_{m,l}(A),\ p = \sigma_{m',l'}(P),$$

where H_a is the Hamilton vector field of a, H_p the Hamilton vector field of p. So modulo lower terms, which I ignore here and which are easy to deal with, we need to arrange that

$$H_p a = -b^2 + e, \tag{6–4}$$

and then take B, E with $\sigma(B) = b$, $\sigma(E) = e$. Indeed, under these assumptions (6–2) shows that $\|Bu\|$ can be estimated in terms of Pu and Eu. That is, u microlocally on $\operatorname{supp} b$ is estimated by u on $\operatorname{supp} e$ (and Pu) in the precise sense described in the next paragraph, so we can *propagate* estimates of u from $\operatorname{supp} e$ to $\operatorname{supp} b$. (Incidentally, this is a good example of the F term: only the principal symbols of E and B are specified. Take any E and B with these principal symbols, $F = i[A, P] - B^*B - E$ is lower order.)

This can be used in a very straightforward manner to obtain bounds on $\mathrm{WF}_{\mathrm{sc}}(u)$. Namely, one works with 'relative wave front sets', relative to $x^s H^r = H^{r,s}$, that is. Thus, for $\bar{X} = \overline{\mathbb{R}^n}$, $(z, \zeta) \notin \mathrm{WF}^{r,s}_{\mathrm{sc}}(u)$ means that there is a cutoff function $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\zeta) \neq 0$ such that $\mathcal{F}^{-1}\phi\mathcal{F}u$ is in $H^{r,s}$ in an open cone around z. But this is equivalent to the existence of some $Q \in \Psi^{r,-s}_{\mathrm{sc}}(\bar{X})$ such that $\sigma(Q)(z, \zeta) \neq 0$ and $Qu \in L^2$. Note that $\sigma(Q)(z, \zeta) \neq 0$ means that Q is elliptic at (z, ζ). So if we find $A \in \Psi^{m,l}_{\mathrm{sc}}(\bar{X})$, and consequently $B \in \Psi^{(m-1)/2,(l+1)/2}_{\mathrm{sc}}(\bar{X})$, $E \in \Psi^{m-1,l+1}_{\mathrm{sc}}(\bar{X})$ as above, then the conclusion is that (if $Pu \in \dot{\mathcal{C}}^\infty(\bar{X})$)

$$\mathrm{WF}^{(m-1)/2,-(l+1)/2}_{\mathrm{sc}}(u) \cap \operatorname{supp} e = \varnothing \implies \mathrm{WF}^{(m-1)/2,-(l+1)/2}_{\mathrm{sc}}(u) \cap \operatorname{supp} b = \varnothing.$$

In scattering theory m is usually irrelevant by standard elliptic regularity. Thus, one iteratively reduces l, proving that $\operatorname{supp} b$ is disjoint from the wave front set with respect to more and more decaying Sobolev spaces. (In fact, b is shrunk slightly during the iterative procedure for technical reasons.)

I now illustrate how to prove the propagation of singularities at $\partial\bar{X}$ for real principal type $P \in \Psi^{m,0}_{\mathrm{sc}}(\bar{X})$. For example, we may take $P = H - \lambda$, $\lambda > 0$, and H is a two-body Hamiltonian. (Note that microlocal elliptic regularity is the consequence of the standard microlocal parametrix construction.) We thus want to prove that if $Pu \in \dot{\mathcal{C}}^\infty(\bar{X})$ (or a microlocal version of it holds), $\bar{\xi} \in {}^{\mathrm{sc}}T^*_{\partial\bar{X}}\bar{X}$ and there is a point on the backward bicharacteristic through $\bar{\xi}$ which is not in $\mathrm{WF}_{\mathrm{sc}}(u)$, then $\bar{\xi} \notin \mathrm{WF}_{\mathrm{sc}}(u)$. In fact, by a simple argument it suffices to prove that there exists a neighborhood U of $\bar{\xi}$ such that if there is a point $\tilde{\xi}$ in U which is also on the backward bicharacteristic through $\bar{\xi}$ and which is not in $\mathrm{WF}_{\mathrm{sc}}(u)$, then $\bar{\xi} \notin \mathrm{WF}_{\mathrm{sc}}(u)$.

The standard proof proceeds via linearization of H_p, see [28]. Thus, first note that $x^{-1}H_p$ is a smooth vector field on ${}^{\mathrm{sc}}T^*\bar{X}$ which is tangent to the boundary. (For example, for Euclidean two-body Hamiltonians,

$$x^{-1}H_p = 2|z|\zeta \cdot \partial_z = 2\sum_j \zeta_j |z| \partial_{z_j},$$

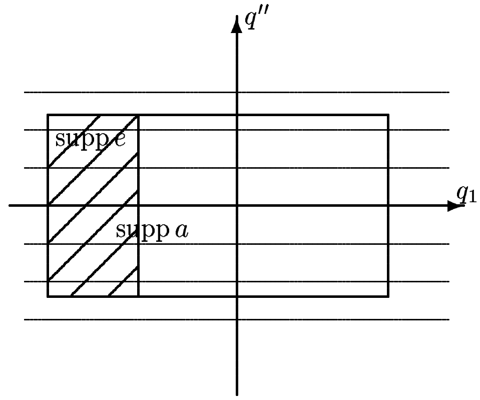

Figure 7. The region $\operatorname{supp} a$ superimposed on the linearized Hamilton flow. $\operatorname{supp} e$ is the shaded region on the left.

and $|z|\partial_{z_j}$ is a smooth vector field tangent to $\partial\bar{X}$, i.e. it is in $\mathcal{V}_{\mathrm{b}}(\bar{X})$.) Thus, given any point $\bar{\xi} \in {}^{\mathrm{sc}}T^*_{\partial\bar{X}}\bar{X}$ one can introduce local coordinates $(q_1, \ldots, q_{2n-1}) = (q_1, q'')$ on ${}^{\mathrm{sc}}T^*_{\partial\bar{X}}\bar{X}$ centered at $\bar{\xi}$ such that $x^{-1}H_p = \partial_{q_1}$ at ${}^{\mathrm{sc}}T^*_{\partial\bar{X}}\bar{X}$. Thus, bicharacteristics at $\partial\bar{X}$ are curves $q'' =$ constant. Now let $\chi_1 \in \mathcal{C}^\infty_c(\mathbb{R}_{q_1})$ and $\chi_2 \in \mathcal{C}^\infty_c(\mathbb{R}^{2n-2}_{q''})$ be smooth functions supported near 0 with the property that

$$\chi_1' = -b_1^2 + e_1,$$

$b_1, e_1 \in \mathcal{C}^\infty_c(\mathbb{R})$, and $\operatorname{supp} e_1 \subset (-\infty, 0)$. Let

$$a = \chi_1\chi_2^2, \quad b = b_1\chi_2, \quad e = e_1\chi_2^2.$$

Then (6–4) holds. In fact, we can even allow weights and take

$$a_s = x^s\chi_1\chi_2^2, \quad s \in \mathbb{R},$$

since $(x^{-1}H_p x^s)\chi_1$ can be absorbed in $x^s(x^{-1}H_p\chi_1)$ by choosing χ_1' large compared to χ_1. This gives microlocal weighted L^2 estimates in $x^{-s-1/2}L^2$. The iterative argument, in which we gradually let $s \to -\infty$, then allows one to conclude that

$$\operatorname{supp} e \cap \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing \wedge \operatorname{supp} a \cap \mathrm{WF}_{\mathrm{sc}}(Pu) = \varnothing \implies \{b > 0\} \cap \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing.$$

By choosing $\operatorname{supp}\chi_1$ and $\operatorname{supp}\chi_2$ appropriately, we may arrange that e is supported near $\tilde{\xi}$ so that $\operatorname{supp} e \cap \mathrm{WF}_{\mathrm{sc}}(u) = \varnothing$, and so that $b(\bar{\xi}) > 0$, as shown below.

There are several directions from here. One can use finer notion of regularity, such as Lagrangian regularity, which would correspond to using χ_2 that vanishes simply on a Lagrangian submanifold, or such as regularity at radial points, which is the subject of a joint paper with Andrew Hassell and Richard Melrose [23].

Here I will talk about a rougher version, namely what happens if the bicharacteristic 'flow' is more complicated, e.g. in the presence of boundaries or corners for the wave equation [43; 36] or many-body scattering. In fact, here I will not explain the detailed behavior of bicharacteristics, rather just show how to microlocalize positive commutator estimates in a versatile fashion. This method goes back to the work of Melrose and Sjöstrand [43].

The main point is that if we cannot put the operator P, or at least its Hamilton vector field H_p in a model form, the previous construction will not work. Indeed, unless $H_p\chi_2 = 0$, $H_p(\chi_1\chi_2)$ will always yield a term $\chi_1 H_p\chi_2$, which cannot be controlled by $(H_p\chi_1)\chi_2$: the problem being near the boundary of $\operatorname{supp}\chi_2$. So instead use a different form of localization. First let $\eta \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*\bar{X})$ be a function with

$$\eta(\bar{\xi}) = 0, \quad H_p\eta(\bar{\xi}) > 0.$$

Thus, η measures propagation along bicharacteristics, e.g. $\eta = q_1$ in the above example would work, but so would many other choices. We will use a function ω to localize near putative bicharacteristics. This statement is deliberately vague; at first we only assume that $\omega \in \mathcal{C}^\infty({}^{\mathrm{sc}}T^*\bar{X})$ is the sum of the squares of $\mathcal{C}^\infty$ functions σ_j, $j = 1, \dots, l$, with nonzero differentials at $\bar{\xi}$ such that $d\eta$ and $d\sigma_j$, $j = 1, \dots, l$, span $T_{\bar{\xi}}{}^{\mathrm{sc}}T^*_{\partial\bar{X}}\bar{X}$. Such a function ω is nonnegative and it vanishes quadratically at $\bar{\xi}$, i.e. $\omega(\bar{\xi}) = 0$ and $d\omega(\bar{\xi}) = 0$. An example is $\omega = q_2^2 + \dots + q_{2n-1}^2$ with the notation from before, but again there are many other possible choices. We now consider a family symbols, parameterized by constants $\delta \in (0,1)$, $\varepsilon \in (0,\delta]$, of the form

$$a = \chi_0\big(2 - \frac{\phi}{\varepsilon}\big)\chi_1\Big(\frac{\eta+\delta}{\varepsilon\delta} + 1\Big),$$

where

$$\phi = \eta + \frac{1}{\varepsilon}\omega, \qquad \chi_0(t) = \begin{cases} 0 & \text{if } t \le 0, \\ e^{-1/t} & \text{if } t > 0, \end{cases}$$

and $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ with $\operatorname{supp}\chi_1 \subset [0, +\infty)$ and $\operatorname{supp}\chi_1' \subset [0,1]$. Although we do not do it explicitly here, weights such as x^s can be accommodated for any $s \in \mathbb{R}$, by replacing the factor $\chi_0(2 - \frac{\phi}{\varepsilon})$ by $\chi_0(A_0^{-1}(2 - \frac{\phi}{\varepsilon}))$ and taking $A_0 > 0$ large.

We analyze the properties of a step by step. First, note that $\phi(\bar{\xi}) = 0$, $H_p\phi(\bar{\xi}) = H_p\eta(\bar{\xi}) > 0$, and $\chi_1((\eta+\delta)/(\varepsilon\delta) + 1)$ is identically 1 near $\bar{\xi}$, so $H_p a(\bar{\xi}) < 0$. Thus, $H_p a$ has the correct sign, and is in particular nonzero, at $\bar{\xi}$.

Next,

$$\xi \in \operatorname{supp} a \implies \phi(\xi) \le 2\varepsilon \quad \text{and} \quad \eta(\xi) \ge -\delta - \varepsilon\delta.$$

Since $\varepsilon < 1$, we deduce that in fact $\eta = \eta(\xi) \ge -2\delta$. But $\omega \ge 0$, so $\phi = \phi(\xi) \le 2\varepsilon$ implies that $\eta = \phi - \varepsilon^{-1}\omega \le \phi \le 2\varepsilon$. Hence, $\omega = \omega(\xi) = \varepsilon(\phi - \eta) \le 4\varepsilon\delta$. Since ω vanishes quadratically at $\bar{\xi}$, it is useful to rewrite the estimate as $\omega^{1/2} \le 2(\varepsilon\delta)^{1/2}$. Combining these, we have seen that on $\operatorname{supp} a$,

$$-\delta - \varepsilon\delta \le \eta \le 2\varepsilon \quad \text{and} \quad \omega^{1/2} \le 2(\varepsilon\delta)^{1/2}. \tag{6–5}$$

Moreover, on $\operatorname{supp} a \cap \operatorname{supp}\chi_1'$,

$$-\delta - \varepsilon\delta \le \eta \le -\delta \quad \text{and} \quad \omega^{1/2} \le 2(\varepsilon\delta)^{1/2}.$$

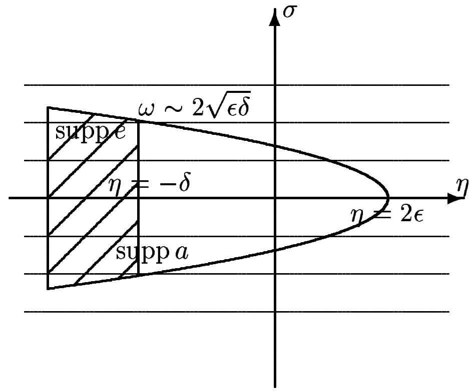

Figure 8. The region $\operatorname{supp} a$ in (η, σ) coordinates. Again, $\operatorname{supp} e$ is the shaded region on the left.

Note that given any neighborhood U of $\bar{\xi}$, we can thus make a supported in U by choosing ε and δ sufficiently small. Below we illustrate the parabola shaped region given by $\operatorname{supp} a$.

Note that as $\varepsilon \to 0$, but δ fixed, the parabola becomes very sharply localized at $\omega = 0$. In particular, for very small $\varepsilon > 0$ we obtain a picture quite analogous to letting $\operatorname{supp} \chi_2 \to \{0\}$ in Figure 7.

So we have shown that a is supported near $\bar{\xi}$. We define

$$e = \chi_0(2 - \frac{\phi}{\varepsilon}) H_p \chi_1((\eta + \delta)/(\varepsilon\delta) + 1),$$

so the crucial question is whether $H_p\phi \geq 0$ on $\operatorname{supp} a$. Note that choosing $\delta_0 \in (0, 1)$ and $\varepsilon_0 \in (0, \delta_0)$ sufficiently small, one has $H_p\eta \geq c_0 > 0$ where $|\eta| \leq 2\delta_0$, $\omega^{1/2} \leq 2(\varepsilon_0\delta_0)^{1/2}$. So $H_p\phi \geq 0$ on $\operatorname{supp} a$, provided that $|H_p\omega| \leq \frac{c_0}{2}\varepsilon$ there.

But being a sum of squares of functions with nonzero differentials, $H_p\omega$ vanishes at $\omega = 0$ and satisfies $|H_p\omega| \leq C\omega^{1/2}$. Due to (6–5), we deduce that $|H_p\omega| \leq 2C(\varepsilon\delta)^{1/2}$. So $|H_p\omega| \leq \frac{c_0}{2}\varepsilon$ holds if $\frac{c_0}{2}\varepsilon \geq 2C(\varepsilon\delta)^{1/2}$, i.e. if $\varepsilon \geq C'\delta$ for some constant $C' > 0$ independent of ε, δ. Note that this constraint on ε, i.e. that it cannot be too small, gives very rough localization: the width of the parabola at $\eta = -\delta$ is roughly $\omega^{1/2} \sim \delta$, i.e. it is very wide, and in particular insufficient to prove the propagation of singularities along the bicharacteristics. The reason is simple: our localizing function ω has no relation to H_p, so we cannot expect a more precise estimate. Note, however, that the estimate is still nontrivial! Indeed, it shows that singularities propagate in the sense that $\bar{\xi}$ cannot be an isolated point of $\mathrm{WF}_{\mathrm{sc}}(u)$. (We required $\varepsilon \in (0, \delta]$ beforehand, but in fact we could have dealt with $\varepsilon \leq \mu\delta$, even if $\mu > 1$, if we localized slightly differently.)

We need to adapt ω to H_p to get a better estimate. If we linearize H_p as above, and take $\omega = q_2^2 + \ldots + q_{2n-1}^2$, then $H_p\omega = 0$ and any $\varepsilon > 0$ works. Thus,

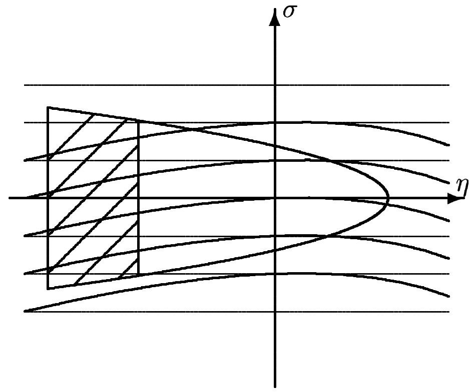

Figure 9. Bicharacteristics and $\operatorname{supp} a$. The labels from Figure 8 have been removed to make the picture less cluttered. The straight horizontal lines are the $\sigma = \text{constant}$ lines, while the nearby parabolae are the bicharacteristics.

in this case, we can prove propagation of singularities much like by the previous, simpler, construction.

However, we do not need such a strong relationship to H_p. Suppose instead that we merely get ω 'right' at $\bar{\xi}$, in the sense that $\omega = \sum \sigma_j^2$ and $H_p\sigma_j(\bar{\xi}) = 0$. Then $|H_p\sigma_j| \leq C_0(\omega^{1/2} + |\eta|)$, so $|H_p\omega| \leq C\omega^{1/2}(\omega^{1/2} + |\eta|)$. Using (6–5), we deduce that $|H_p\omega| \leq (c_0/2)\varepsilon$ provided that $(c_0/2)\varepsilon \geq C''(\varepsilon\delta)^{1/2}\delta$, i.e. that $\varepsilon \geq C'\delta^3$ for some constant C' independent of ε, δ. Now the size of the parabola at $\eta = -\delta$ is roughly $\omega^{1/2} \sim \delta^2$, i.e. we have localized along a single direction, namely the direction of H_p at $\bar{\xi}$. By a relatively simple argument, one can piece together such estimates (i.e. where the direction is correct 'to first order') and deduce the propagation of singularities. We emphasize that the lower bound for ε is natural. Indeed, with q_j as above, we may take σ_j e.g. to be $\sigma_j = q_j + q_1^2$, $j \geq 2$. The bicharacteristics are $q_j = \text{constant}$, but we are localizing near $\sigma_j = \text{constant}$, and at $\eta = -\delta$ these differ by δ^2. So any localization better than $\omega^{1/2} \sim \delta^2$ would in fact contradict the propagation of singularities!

The microlocal positive commutator estimates in many-body scattering arise by this method. In particular, one can take $\eta = \frac{z\cdot\zeta}{|z|}$, which is the radial component of the momentum. The function ω needs to be π-invariant, so if $\bar{\xi} \in {}^{\mathrm{sc}}T^*\bar{X}_a$, it involves functions on ${}^{\mathrm{sc}}T^*\bar{X}_a$ as well as $Z^a = \frac{z^a}{|z|}$ and η. The only additional argument needed is to show that the commutator is indeed positive, which has to be understood in an operator sense. Thus, the key point is that the commutator of $H - \lambda$ with a quantization B of η is positive, modulo compact terms, when localized at λ in the spectrum of H and microlocalized away from the radial set $\mathcal{R}(\lambda)$. Note that, as usual, there is nothing to prove at $\mathcal{R}(\lambda)$, since each point in it is the image of a maximally extended generalized broken bicharacteristic.

This positivity can be proved directly by showing that the indicial operators of the commutator are positive away from $\mathcal{R}(\lambda)$, which follows from an iterative argument. However, it also reduces to the Mourre estimate, involving the generator of dilations $A = \frac{1}{2}(\cdot D_z + D_z \cdot z)$, which has principal symbol at $\zeta \cdot z$. The Mourre estimate states the following. Suppose that $\lambda \notin \Lambda$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that if $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ is supported in $(\lambda - \delta,\ \lambda + \delta)$ there exists $K' \in \Psi_{\mathrm{sc}}^{-\infty,1}(\bar{X}, \mathcal{C})$ such that

$$\phi(H) i[A, H]\phi(H)) \geq 2(d(\lambda) - \varepsilon)\phi(H)^2 + K', \tag{6–6}$$

where

$$d(\lambda) = \inf\{\lambda - \lambda' : \lambda' \leq \lambda,\ \lambda' \in \Lambda\} \geq 0$$

is the distance of λ to the next threshold below it if $\lambda \geq \inf \Lambda$, and $d(\lambda)$ an arbitrary positive number if $\lambda < d(\lambda)$. Since $d(\lambda) > 0$ if $\lambda \notin \Lambda$, (6–6) is indeed a positive commutator estimate, which does not even have a 'negative' term E, unlike (6–1). The Mourre estimate, originating in [47], has been well understood since the work of Perry, Sigal and Simon [53] and Froese and Herbst [16]. Here I just outline the argument in the simplest case, namely if no proper subsystem has any L^2-eigenvalues.

In this simplest case, the argument of [16] (see also [8] and [67]) proceeds as follows. In order to prove (6–6), one only needs to show that for all b, the corresponding indicial operators satisfy the corresponding inequality, i.e. that

$$\phi(\hat{H}_b) i\widehat{[A, H]}_b \phi(\hat{H}_b) \geq 2(d(\lambda) - \varepsilon)\phi(\hat{H}_b)^2. \tag{6–7}$$

(This means that the operators on the two sides, which are families of operators on X^b, depending on $(y_b, \zeta_b) \in {}^{\mathrm{sc}}T^*_{C_b}\bar{X}$, satisfy the inequality for all $(y_b, \zeta_b) \in {}^{\mathrm{sc}}T^*_{C_b}\bar{X}$.) It is convenient to assume that ϕ is identically 1 near λ; if (6–7) holds for such ϕ, it holds for any ϕ_0 with slightly smaller support, as follows by multiplication by $\phi_0(\hat{H}_b)$ from the left and right.

Note that for $b = 0$ the estimate certainly holds: it comes from the Poisson bracket formula in the scattering calculus, or from a direct computation yielding $i\widehat{[A, H]}_0 = 2\Delta_{g_0}$. Hence, if the the localizing factor

$$\phi(\hat{H}_0) = \phi(|\zeta|^2)$$

is supported in $(\lambda - \delta,\ \lambda + \delta)$ and $\lambda > 0$, (6–7) holds even with $d(\lambda) - \varepsilon$ replaced with $\lambda - \delta$. Note that $\lambda \geq d(\lambda)$, if $\lambda > 0$, since 0 is a threshold of H. On the other hand, if $\lambda < 0$, both sides of (6–7) vanish for ϕ supported near λ, so the inequality holds trivially.

In general, we may assume inductively that at all clusters c with $C_c \subsetneq C_b$, i.e. $X^b \subsetneq X^c$, (6–7) has been proved with ϕ replaced by a cutoff $\tilde{\phi}$ and ε replaced by ε', i.e. we may assume that for all $\varepsilon' > 0$ there exists $\delta' > 0$ such that for all c with $C_c \subsetneq C_b$, and for all $\tilde{\phi} \in \mathcal{C}_c^\infty(\mathbb{R}; [0, 1])$ supported in $(\lambda - \delta',\ \lambda + \delta')$,

$$\tilde{\phi}(\hat{H}_c) i\widehat{[A, H]}_c \tilde{\phi}(\hat{H}_c) \geq 2(d(\lambda) - \varepsilon')\tilde{\phi}(\hat{H}_c)^2. \tag{6–8}$$

But these are exactly the indicial operators of $\tilde{\phi}(\hat{H}_b)i\widehat{[A,H]}_b\tilde{\phi}(\hat{H}_b)$, so, as discussed in [65, Proposition 8.2], (6–7) implies that

$$\tilde{\phi}(\hat{H}_b)i\widehat{[A,H]}_b\tilde{\phi}(\hat{H}_b) \geq 2(d(\lambda)-\varepsilon')\tilde{\phi}(\hat{H}_b)^2 + K_b, \quad K_b \in \Psi_{\mathrm{sc}}^{-\infty,1}(X^b,\mathcal{C}^b). \tag{6–9}$$

This implication relies on a square root construction in the many-body calculus, which is particularly simple in this case.

Now, we first multiply (6–9) through by $\phi(H)$ from both the left and the right. Recall that we use coordinates (z_b, z^b) on $X_b \oplus X^b$ and (ζ_b, ζ^b) are the dual coordinates. We remark that $\hat{H}_b = |\zeta_b|^2 + H^b$, so if $\lambda - |\zeta_b|^2$ is not an eigenvalue of H^b, then as $\operatorname{supp}\phi \to \{\lambda\}$, $\phi(H^b + |\zeta_b|^2) \to 0$ strongly, so as K_b is compact, $\phi(H^b + |\zeta_b|^2)K_b \to 0$ in norm; in particular it can be made to have norm smaller than $\varepsilon' - \varepsilon > 0$. After multiplication from both sides by $\phi_1(\hat{H}_b)$, with ϕ_1 having even smaller support, (6–7) follows (with ϕ_1 in place of ϕ), with the size of $\operatorname{supp}\phi_1$ a priori depending on ζ_b. However, $i\phi_1(\hat{H}_b)\widehat{[A,H]}_b\phi_1(\hat{H}_b)$ is continuous in ζ_b with values in bounded operators on $L^2(X^b)$, so if (6–7) holds at one value of ζ_b, then it holds nearby. Moreover, for large $|\zeta_b|$ both sides vanish as $\hat{H}_b = H^b + |\zeta_b|^2$, with H^b bounded below, so the estimate is in fact uniform if we slightly increase $\varepsilon > 0$.

In general, the proof requires to treat the range of E, the spectral projection of H^b to $\{\lambda\}$, separately. Roughly, the positivity estimate on the range of E comes from the virial theorem, $iE[z^b D_{z^b}, H^b]E = 0$, which is formally clear, and is easy to prove. Thus,

$$iE[A, H_b]E = iE[z^b D_{z^b}, H^b]E + iE[z_b D_{z_b}, \Delta_{X_b}]E = i[z_b D_{z_b}, \Delta_{X_b}]E,$$

and the commutator $i\phi(H_b)[z_b D_{z_b}, \Delta_{X_b}]\phi(H_b)$ is easily computed to be positive. Of course, there are also cross-terms that need to be considered, but they can be estimated by Cauchy-Schwartz estimates, see [16] or [67].

I refer to [66] and [65] for the detailed arguments proving propagation of singularities in the many-body setting, and to [68, Appendix] for weaker estimates with simplified proofs.

7. Asymptotic Completeness

Asymptotic completeness (AC) is an L^2-based statement describing the long-term behavior of solutions of the Schrödinger equation. In the short-range setting it says that for any $\phi \in L^2(X_0)$ in the range of $\mathrm{Id} - E_{\mathrm{pp}}$, E_{pp} being the projection onto the bound states of H (i.e. onto its L^2-eigenfunctions), there exist $\phi_\alpha \in L^2(X_a)$ such that

$$\|e^{-iHt}\phi - \sum_\alpha e^{-iH_a t}(\phi_\alpha \otimes \psi_\alpha)\| \to 0 \text{ as } t \to +\infty.$$

In the long-range setting, $e^{-iH_a t}$ must be somewhat modified. After the ground-breaking work of Enss [12; 10], AC was first proved by Sigal and Soffer [55] in the short-range setting (see Graf's paper [20] for a different proof), and later by Dereziński [7], and also by Sigal and Soffer [56; 57], in the long-range setting. In the short range case the main ingredient is equivalent to certain estimates of the resolvent at the radial sets in a sense that I now describe. In the long-range setting, as already in two-body scattering, additional constructive steps are needed. The estimates, in a different language, appeared first in the work of Sigal and Soffer [55]. I hope that the following discussion makes it clearer how they relate to the propagation of singularities.

While asymptotic completeness gives a complete long-term L^2-description of solutions of the Schrödinger equation, the question remains whether an analogous description exists on other spaces, such as weighted L^2-spaces. For example, if ϕ is Schwartz, are the ϕ_α Schwartz? Or dually, starting with a tempered distribution ϕ, are there tempered distributions α such that the convergence holds, as $t \to +\infty$, in a suitable sense? A different point of view is the parameterization of generalized eigenfunctions of H using the Poisson operators $P_{\alpha,+}(\lambda)$, and the analogues of these questions can be asked there as well. The answer is affirmative in the two-body setting (even in the geometric setting, see [41; 46]). However, as indicated by the related issue of the mapping properties of the scattering matrices, discussed at the end of this section, it is unlikely that the same holds in the many-body setting. One can then ask weaker question, e.g. whether it holds in weighted spaces L^2_s, s near 0. Or, one may ask whether one can give a precise description of the map $\phi \mapsto \phi_\alpha$ e.g. as some sort of Fourier integral operator.

As a starting point of relating the propagation of singularities to AC, we note that the propagation of singularities is proved by showing its 'relative' versions, i.e. that for any l, $\mathrm{WF}^{*,l}_{\mathrm{sc}}(u)$ is also a union of maximally extended generalized broken bicharacteristics. When considering the resolvent, first recall that for $f \in \dot{\mathcal{C}}^\infty(\bar{X})$, $R(\lambda + i0)f \in H^{\infty,l}$ for all $l < -\frac{1}{2}$, so we only need to find

$$\mathrm{WF}^{*,l}_{\mathrm{sc}}(R(\lambda + i0)f)$$

for $l \geq -\frac{1}{2}$. Theorem 3.3 is also valid for $\mathrm{WF}^{*,l}_{\mathrm{sc}}$, i.e. the following holds.

THEOREM 7.1. *If $\lambda \notin \Lambda$, then for $f \in \mathcal{S}(\mathbb{R}^n)$, $l \geq -\frac{1}{2}$, $\mathrm{WF}_{\mathrm{sc}}(R(\lambda + i0)f)$ is a subset of the image of $\mathcal{R}_+(\lambda)$ under the* forward *generalized broken bicharacteristic relation.*

This result allows $u = R(\lambda + i0)f$ *not* to lie in $H^{*,-1/2}$ on the image of $\mathcal{R}_+(\lambda)$ under the forward generalized broken bicharacteristic relation. This is a small set, but it is important to know whether $\mathrm{WF}^{*,l}_{\mathrm{sc}}(u)$ may indeed intersect the forward image of $\mathcal{R}_+(\lambda)$. Of course, we cannot expect an improvement at $\mathcal{R}_+(\lambda)$, as shown already by the example of the free Euclidean Laplacian. The crucial improvement is the following estimate, due to Sigal and Soffer [55].

THEOREM 7.2. *If $\lambda \notin \Lambda$, then for $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathrm{WF}^{*,-1/2}_{\mathrm{sc}}(R(\lambda + i0)f) \subset \mathcal{R}_+(\lambda)$.*

Remark. This theorem also has a time-dependent analogue. If u is a solution of the Schrödinger equation $(D_t + H)u = 0$ with $u|_{t=0} \in \mathcal{S}(X_0)$ then on the one hand $u \in H^{\infty,l}(X_0 \times \mathbb{R})$ for $l < -\frac{1}{2}$, on the other hand $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(u) \subset \mathcal{R}$.

In fact, this theorem can be improved along the lines of the distributional statement in Theorem 3.3:

Corollary 7.3. *Suppose that* $\lambda \notin \Lambda$, $f \in H^{*,1/2}$ *and* $\mathrm{WF}_{\mathrm{sc}}^{*,1/2+\varepsilon}(f) \cap \mathcal{R}_-(\lambda) = \varnothing$ *for some* $\varepsilon > 0$. *Then* $R(\lambda+i0)f = \lim_{t\to 0} R(\lambda+it)f$ *exists in* $H^{*,-1/2-\varepsilon'}(\bar{X})$, $\varepsilon' > 0$, *and* $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(R(\lambda+i0)f) \subset \mathcal{R}_+(\lambda)$.

Theorem 7.2 can be proved rather simply. The main issue is how to obtain a positive commutator at the radial point. Away from the radial sets arbitrary weights can be accommodated by suitable construction, as pointed out in the previous section. At radial points only the weights can give positive commutators. Now, one has to use weights x^{-2l-1} to obtain estimates for $\mathrm{WF}_{\mathrm{sc}}^{*,l}$, and these weights will give a commutator whose sign depends on that of $-2l-1$, hence on whether $l > -\frac{1}{2}$, $l < -\frac{1}{2}$ or $l = -\frac{1}{2}$. It turns out that the sign is correct for (6–3) to be of use if $l < -\frac{1}{2}$; this yields the limiting absorption principle. The sign is wrong if $l > -\frac{1}{2}$, so no results can be expected then. In the borderline case $l = -\frac{1}{2}$, the weight vanishes. The way to obtain a positive commutator is thus to consider operators A which are microlocally (a multiple of) the identity near $\mathcal{R}_+(\lambda)$. The commutator then vanishes microlocally near $\mathcal{R}_+(\lambda)$, which is reasonable since no estimate on $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(u)$ can be expected there.

It is then straightfoward to construct A so that (6–3) can be used to prove Theorem 7.2. Indeed, it suffices to show that on $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(u)$, $\eta = \frac{z\cdot\zeta}{|z|}$ must satisfy $\lambda - \eta^2 \in \Lambda$, for then the full statement of the theorem follows by the propagation of singularities for $\mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(u)$. So we proceed to prove this simpler result, namely that if $\lambda - \bar{\eta}^2 \notin \Lambda$ then for any point ξ, $\eta(\xi) = \bar{\eta}$ implies $\xi \notin \mathrm{WF}_{\mathrm{sc}}^{*,-1/2}(u)$.

To do so, we let $a = \chi(\eta)$ where $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi \geq 0$, is chosen so that $\chi \equiv 1$ on $[0, \bar{\eta} - \delta]$ for some $\delta > 0$, $\chi' \leq 0$ on $(0, \infty)$, $\chi'(\bar{\eta}) < 0$, and $t \in \operatorname{supp} \chi'$ implies that $\lambda - t^2 \notin \Lambda$. This can be arranged as Λ is closed. We can further make sure that $\sqrt{-\chi'}$ is $\mathcal{C}^\infty$ on $(0, \infty)$. Then the positive commutator methods outlined show the commutator of A, a quantization of a, with $H - \lambda$ is positive, in the region $\eta > 0$, yielding the estimate that proves the theorem. We remark that partial microlocalization, using functions of η, hes been used extensively in many-body scattering, especially by Gérard, Isozaki and Skibsted [18; 19] and Wang [69], to obtain partially microlocal statements such as radiation conditions and uniqueness statements [32; 31], and indeed to prove the smoothness of 2-cluster to 2-cluster scattering matrices [59].

It turns out that there is an even simpler way of proving Theorem 7.2, or indeed a stronger statement, which is due to Yafaev [73]. His estimate states that in a neighborhood of $C_{a,\mathrm{reg}}$, where we write y_a for the coordinates $z_a/|z_a|$ along

$C_{a,\mathrm{reg}}$, $xD_{y_a}R(\lambda+i0)f$ is in $H^{*,-1/2}(\bar{X})$. Since the principal symbol of xD_{y_a} is invertible on $(T^*X_{a,\mathrm{reg}}\cap\dot{\mathrm{C}}\mathrm{har}(\lambda))\setminus\mathcal{R}(\lambda)$, this result implies Theorem 7.2. Yafaev's proof relies on a simple and explicit commutator calculation, which allows one to deal with various error terms that one may, a priori, expect. However, exactly because of its explicit nature, it is presumably hard to generalize to more geometric settings, while the argument we sketched does not face this difficulty.

As discussed by Yafaev [73] in the usual time-dependent version, short-range asymptotic clustering, hence asymptotic completeness, are relatively easy consequences of Corollary 7.3, and we refer to [73] and [8] for more details. However, it is worth pointing out that the reason why Coulomb-type potentials (i.e. those in S^{-1}) are not 'short-range' is that the Hamilton vector field in some subsystem vanishes at radial points. This degeneracy makes even the subprincipal term important in describing the precise behavior of generalized eigenfunctions microlocally near this point.

Before turning to scattering theory on symmetric spaces, we note the implications of Theorem 7.2 for the scattering matrices. Previously, $S_{\alpha\beta}(\lambda)$ was only defined as a map $S_{\alpha\beta}(\lambda):\mathcal{C}_c^\infty(S_{a,\mathrm{reg}})\to\mathcal{C}^{-\infty}(S_{b,\mathrm{reg}})$. Indeed, part of the broken bicharacteristic relation connects $\mathcal{R}_+(\lambda)$ with its image, and this can a priori give a singularity in the kernel of $S_{\alpha\beta}(\lambda)$ of the kind that does not even allow one to conclude that $S_{\alpha\beta}(\lambda):\mathcal{C}_c^\infty(S_{a,\mathrm{reg}})\to\mathcal{C}^{\infty}(S_{b,\mathrm{reg}})$. The pairing formula, (4–2), combined with Theorem 7.2, show that in fact

$$S_{\alpha\beta}(\lambda):L^2(S_a)\to L^2(S_b). \tag{7–1}$$

It is an interesting question whether this can be improved if we restrict $S_{\alpha\beta}(\lambda)$ to $\mathcal{C}_c^\infty(S_{a,\mathrm{reg}})$. Namely, except in special cases such as N-clusters and two-clusters, the best known result is the trivial consequence of (7–1):

$$S_{\alpha\beta}(\lambda):\mathcal{C}_c^\infty(S_{a,\mathrm{reg}})\to L^2(S_b).$$

(In the case of N-clusters and 2-clusters, the geometry of generalized broken bicharacteristics gives $S_{\alpha\beta}(\lambda):\mathcal{C}_c^\infty(S_{a,\mathrm{reg}})\to\mathcal{C}^{\infty}(S_{b,\mathrm{reg}})$.) The putative improvement would have to be connected to an improvement of Theorem 7.2, namely to the existence of *some* $l>-\frac{1}{2}$ such that $\mathrm{WF}_{\mathrm{sc}}^{*,l}(R(\lambda+i0)f)\subset\mathcal{R}_+(\lambda)$. It would also be connected to a better understanding of $R(\lambda\pm i0)$ at the thresholds, in which direction Wang's paper [72] is the only one I am aware of.

8. Scattering on Higher Rank Symmetric Spaces

In this section I discuss $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N,\mathbb{R})$ — indeed, mostly I will discuss $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$. The books [25], [34] and [9] are good general references. $N=2$ yields the hyperbolic plane $\mathbb{H}^2$, which is a rank one symmetric space on which many aspects of analysis, such as the asymptotic behavior of the resolvent kernel and the analytic continuation of the resolvent are well understood. Indeed, these have been described on asymptotically hyperbolic spaces by Mazzeo and Melrose [37] and Perry [51; 52].

Higher rank symmetric spaces, such as $\mathrm{SL}(N)/\mathrm{SO}(N)$, $N \geq 3$, are much less understood. For example, using results of Harish-Chandra, and Trombi and Varadarajan (see [25]), Anker and Ji only recently obtained the leading order behavior of the Green's function [2; 3; 4]. Also, while spherical functions, which are most analogous to partial plane-partial spherical waves in the Euclidean setting, have been analyzed by Harish-Chandra, Trombi and Varadarajan, and in particular their analytic continuation is understood, the same cannot be said about the Green's function. The analysis of spherical functions relies on perturbation series expansions, much like in the proof of the Cauchy-Kovalevskaya theorem, and it does not work well at the walls of the Weyl chambers. Here I only illustrate some recent joint results with Rafe Mazzeo [40; 39], that illuminate the connections with many-body scattering, and in particular give rather direct results for the resolvent.

First I describe the space $\mathrm{SL}(3)/\mathrm{SO}(3)$. The polar decomposition states that any $C \in \mathrm{SL}(3)$ can be written uniquely as $C = VR$, $V = (CC^t)^{1/2}$ is positive definite and has determinant 1, $R \in \mathrm{SO}(3)$. Thus, $\mathrm{SL}(3)/\mathrm{SO}(3)$ can be identified with the set M of positive definite matrices of determinant 1; this is a five-dimensional real analytic manifold. The Killing form provides a Riemannian metric g. The associated Laplacian $\Delta = \Delta_g$ gives a self-adjoint unbounded operator on $L^2(M, dg)$, with spectrum $[\lambda_0, +\infty)$, $\lambda_0 = \frac{1}{3}$. Let $R(\lambda) = (\Delta - \lambda)^{-1}$ be the resolvent of Δ_g, $\lambda \notin [\lambda_0, +\infty)$.

Fix a point $o \in M$, which we may as well assume is the image of the identity matrix I in the identification above. The stabilizer subgroup K_o (in the natural $\mathrm{SL}(3)$ action on M) is isomorphic to $\mathrm{SO}(3)$. The Green function $G_o(\lambda)$ with pole at o and at eigenvalue λ is, by definition $R(\lambda)\delta_o$. It is standard that G_o lies in the space of K_o-invariant distributions on M. It is thus natural to study Δ on K_o-invariant functions.

Perhaps the most interesting property is the analytic continuation of the resolvent, which I state before indicating how it, and other results, relate to many-body scattering.

Fix the branch of the square root function $\sqrt{\ }$ on $\mathbb{C} \setminus [0, +\infty)$ which has negative imaginary part when $w \in \mathbb{C} \setminus [0, +\infty)$. Let S denote that part of the Riemann surface for $\lambda \mapsto \sqrt{\lambda - \lambda_0}$ where we continue from $\lambda - \lambda_0 \notin [0, +\infty)$ and allow $\arg(\lambda - \lambda_0)$ to change by any amount less than π. In other words, starting in the region $\operatorname{Im}\sqrt{\lambda - \lambda_0} < 0$, we continue across either of the rays where $\operatorname{Im}\sqrt{\lambda - \lambda_0} = 0$ and $\operatorname{Re}\sqrt{\lambda - \lambda_0} > 0$, respectively < 0, allowing the argument of $\sqrt{\lambda - \lambda_0}$ to change by any amount less than $\pi/2$ (so that only the positive imaginary axis is not reached).

Theorem 8.1. *With all notation as above, the Green function $G_o(\lambda)$ continues meromorphically to S as a distribution. Similarly, as an operator between* *appropriate spaces of K_o-invariant functions, the resolvent $R(\lambda)$ itself has a meromorphic continuation in this region, with all poles of finite rank.*

Having stated the theorem, I indicate how it relates to many-body scattering. To do so, fix the point o – we may as well take it to be the identity matrix I. Now, M is a perfectly nice real analytic manifold and Δ is an elliptic operator on it in the usual sense, so the only question is its behavior at infinity. In order to describe this, we remark that any matrix $A \in M$ can be diagonalized, i.e. written as $A = O\Lambda O^t$, with $O \in \mathrm{SO}(3)$ and Λ diagonal, with entries given by the eigenvalues of A. If $\mathfrak{a}$ is the set of diagonal matrices of trace 0, then $\Lambda \in \exp(\mathfrak{a})$. If all eigenvalues of A are distinct, then Λ is determined except for the ordering of the eigenvalues, and there are only finitely many possibilities for O as well. However, at the walls, which are defined to be the places where two eigenvalues coincide, there is much more indeterminacy. For example, if two eigenvalues coincide, only their joint eigenspace is well-defined. Correspondingly, we may replace O by $O'O$ for any $O' \in \mathrm{SO}(3)$ preserving the eigenspace decomposition and still obtain the desired diagonalization.

This is closely reflected in the structure of the Laplacian at infinity. In fact, it turns out that on SO(3)-invariant functions, Δ is essentially a three-body Hamiltonian on $\mathfrak{a}$ with first order interactions and with collision 'planes' given by the walls (they are lines), see e.g. [25, Chapter II, Proposition 3.9]. So rather than particles, eigenvalues scatter in this case! Consequently, many-body results can be adapted to this setting.

We indicate how this is done. The most succint way of describing the geometry of M at infinity is to compactify it to a manifold $\bar{M}$ with codimension two corners. It has two boundary hypersurfaces, $H_\sharp$ and $H^\sharp$, which are perhaps easiest to describe in terms of a natural system of local coordinates derived from the matrix representation of elements in M. As above, we write $A \in M$ as $A = O\Lambda O^t$, with $O \in \mathrm{SO}(3)$ and Λ diagonal. The ordering of the diagonal entries of Λ is undetermined, but in the region where no two of them are equal, we denote them as $0 < \lambda_1 < \lambda_2 < \lambda_3$ (but recall also that $\lambda_1\lambda_2\lambda_3 = 1$). In this region the ratios

$$\mu = \frac{\lambda_1}{\lambda_2}, \qquad \nu = \frac{\lambda_2}{\lambda_3}$$

are independent functions, and near the submanifold $\exp(\mathfrak{a})$ in M we can complete them to a full coordinate system by adding the above-diagonal entries c_{12}, c_{13}, c_{23} in the skew-symmetric matrix $T = \log O$. On M we have $\mu, \nu > 0$, and locally the compactification consists of replacing $(\mu, \nu) \in (0,1) \times (0,1)$ by $(\mu, \nu) \in [0,1) \times [0,1)$. Then $H^\sharp = \{\mu = 0\}$ and $H_\sharp = \{\nu = 0\}$, and this coordinate system gives the $\mathcal{C}^\infty$ structure near the corner $H_\sharp \cap H^\sharp$.

On the other hand, in a neighborhood of the interior of $H_\sharp$, for example, we obtain the compactification and its $\mathcal{C}^\infty$ structure as follows. Write the eigenvalues of $A \in M$, i.e. the diagonal entries of Λ in the decomposition for A above,

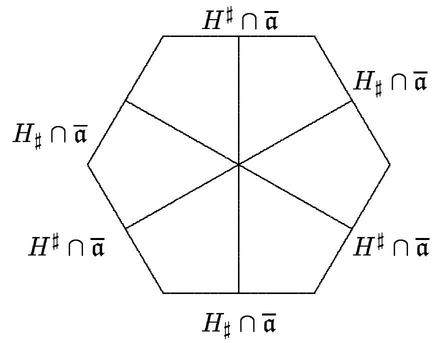

Figure 10. The closure of $\mathfrak{a}$, or rather $\exp(\mathfrak{a})$, in the compactification $\bar{M}$ of M. The lines in the interior are the Weyl chamber walls, playing the role of collision planes in many-body scattering. The side faces $H^\sharp \cap \bar{\mathfrak{a}}$ and $H_\sharp \cap \bar{\mathfrak{a}}$ correspond to the front faces on Figure 5. The main face on Figure 5 would only show up if we did a logarithmic blow-up of all boundary hypersurfaces of $\bar{M}$ and then blew up the corner.

as λ_1, λ_2 and λ_3. Suppose that A lies in a small neighbourhood $\mathcal{U}$ where

$$c < \frac{\lambda_1}{\lambda_2} < \frac{1}{c} < \lambda_3,$$

for some fixed $c \in (0,1)$. Recall also that $\lambda_3 = 1/\lambda_1\lambda_2$. These inequalities imply that $\lambda_1 = (\lambda_1/\lambda_2)^{1/2}\lambda_3^{-1/2} < 1$ and $\lambda_2 = (\lambda_2/\lambda_1)^{1/2}\lambda_3^{-1/2} < 1$, and $\lambda_3 > 1$ in $\mathcal{U}$. Hence there is a well-defined decomposition $\mathbb{R}^3 = E_{12} \oplus E_3$ for any $A \in \mathcal{U}$, where E_{12} is the sum of the first two eigenspaces and E_3 is the eigenspace corresponding to λ_3, regardless of whether or not λ_1 and λ_2 coincide. We could write equivalently $A = OCO^t$, where C is block-diagonal, preserving the splitting $\mathbb{R}^2 \oplus \mathbb{R}$ of $\mathbb{R}^3$. The ambiguity in this factorization is that C can be conjugated by an element of $\mathrm{O}(2)$ (acting in the upper left block), and $\mathrm{O}(2)$ can be included in the top left corner of $\mathrm{SO}(3)$ (the bottom right entry being set equal to ± 1 appropriately). Let C' denote the upper-left block of C; the bottom right entry of C is just λ_3, and so $\lambda_3 \det C' = 1$. In other words, $C' = \lambda_3^{-1/2} C''$ where C'' is positive definite and symmetric with determinant 1, hence represents an element of $\mathrm{SL}(2)/\mathrm{SO}(2) \equiv \mathbb{H}^2$. Hence for an appropriate neighbourhood $\mathcal{V}$ of I in $\mathrm{SL}(2)/\mathrm{SO}(2)$, the neighbourhood $\mathcal{U}$ is identified with $(\mathcal{V} \times \mathrm{SO}(3))/\mathrm{O}(2) \times (0, c^{3/2})$, where the variable on the last factor is $s = \lambda_3^{-3/2}$. The compactification then simply replaces $(0, c^{3/2})_s$ by $[0, c^{3/2})$. Note that although the action of $\mathrm{O}(2)$ on $\mathcal{V}$ has a fixed point (namely I), its action on $\mathrm{SO}(3)$, and hence on the product, is free. The neighbourhood $\mathcal{V}$ can be chosen larger when λ_3 is larger, and the limiting 'cross-section' $\lambda_3 = \text{const}$ has the form $(\mathbb{H}^2 \times \mathrm{SO}(3))/\mathrm{O}(2)$. This space is a fibre bundle over $\mathrm{SO}(3)/\mathrm{O}(2)$ $(= \mathbb{R}P^2)$ with fibre $\mathbb{H}^2$. Notice that the Weyl chamber wall corresponds to the origin (i.e. the point fixed by the $\mathrm{SO}(2)$ action) in $\mathbb{H}^2$. I refer to [40] for a more thorough description of $\bar{M}$.

On each boundary hypersurface of M, it is now easy to describe model operators for Δ acting on SO(3)-invariant functions. For instance, at $H_\sharp$ this model can be considered as an operator $L_\sharp$ on $\mathbb{R}_s \times \mathbb{H}^2$, acting on SO(2)-invariant functions. Explicitly,

$$L_\sharp = \tfrac{1}{4}(sD_s)^2 + i\tfrac{1}{2}(sD_s) + \tfrac{1}{3}\Delta_{\mathbb{H}^2}.$$

This is tensor product type, so its resolvent can be obtained from an integral of the resolvents of $\frac{1}{4}(sD_s)^2 + i\frac{1}{2}(sD_s)$ and $\frac{1}{3}\Delta_{\mathbb{H}^2}$. (Note that I am ignoring the weights of the L^2 spaces on which we are working, hence the appearance of the perhaps strange first order terms.)

This framework allows one to develop the elliptic theory, for example to analyze $(\Delta-\lambda)^{-1}$ for $\lambda \in \mathbb{C}\setminus[\lambda_0, +\infty)$. In particular, one can construct a parametrix for Δ on $\bar{M}$ that has a smoothing error. Since this error has no decay at infinity, it is not compact. However, the error can be improved by pasting together the resolvents of $L_\sharp$ and $L^\sharp$, and applying the result to the error term to remove it modulo a decaying, hence compact, new error term. One of the consequences is then the description of the asymptotic behavior of the Green's function, see [40].

The point of complex scaling is to rotate the essential spectrum of the operator being studied, in this case the Laplacian. To give the reader a rough idea how this works, consider the hyperbolic space $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$, which may be identified with the set of two-by-two positive definite matrices A of determinant 1. In terms of geodesic normal coordinates (r,ω) about $o = I$, the Laplacian is

$$\Delta_{\mathbb{H}^2} = D_r^2 - i\coth r\, D_r + (\sinh r)^{-2}D_\omega^2.$$

Now consider the diffeomorphism $\Phi_\theta : A \mapsto A^w$, $w = e^\theta$, on $\mathbb{H}^2$, $\theta \in \mathbb{R}$. This corresponds to dilation along the geodesics through o, since these have the form $\gamma_A : s \mapsto A^{cs}$, $c > 0$. Thus, in geodesic normal coordinates, $\Phi_\theta : (r,\omega) \mapsto (e^\theta r, \omega)$. Φ_θ defines a group of unitary operators on $L^2(\mathbb{H}^2)$ via

$$(U_\theta f)(A) = (\det D_A\Phi_\theta)^{1/2}(\Phi_\theta^* f)(A),\ J = \det D_A\Phi_\theta = w\frac{\sinh wr}{\sinh r},\ w = e^\theta.$$

Now, for θ real, consider the scaled Laplacian

$$\begin{aligned}(\Delta_{\mathbb{H}^2})_\theta &= U_\theta\Delta_{\mathbb{H}^2}U_\theta^{-1} = J^{1/2}\Phi_\theta^*\Delta_{\mathbb{H}^2}\Phi_{-\theta}^*J^{-1/2}\\ &= J^{1/2}(w^{-2}D_r^2 - iw^{-1}\coth(wr)D_r + (\sinh(wr))^{-2}D_\omega^2)J^{-1/2}.\end{aligned}$$

This is an operator on $\mathbb{H}^2$, with coefficients which extend analytically in the strip $|\mathrm{Im}\,\theta| < \pi/2$. The square root is continued from the standard branch near $w > 0$. (The singularity of the coefficients at $r = 0$ is only an artifact of the polar coordinate representation.) Note that $(\Delta_{\mathbb{H}^2})_\theta$ and $(\Delta_{\mathbb{H}^2})_{\theta'}$ are unitary equivalent if $\mathrm{Im}\,\theta = \mathrm{Im}\,\theta'$ because of the group properties of U_θ. The scaled operator, $(\Delta_{\mathbb{H}^2})_\theta$, is not elliptic on all of $\mathbb{H}^2$ when $0 < |\mathrm{Im}\,\theta| < \frac{\pi}{2}$ because for

r large enough, $w^2 \sinh(wr)^{-2}$ can lie in $\mathbb{R}^-$. However, it is elliptic in some uniform neighbourhood of o in $\mathbb{H}^2$, and its radial part

$$(\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}} = J^{1/2}(w^{-2}D_r^2 - iw^{-1}\coth(wr)D_r)J^{-1/2},$$

which corresponds to its action on SO(2)-invariant functions, is elliptic on the entire half-line $r > 0$. The model operator for $(\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}} - \lambda$ at infinity,

$$\begin{aligned} e^{(w-1)r/2}&(w^{-2}D_r^2 - iw^{-1}D_r - \lambda)e^{-(w-1)r/2} \\ &= e^{(w-1)r/2}((w^{-1}D_r - \tfrac{i}{2})^2 - (\lambda - \tfrac{1}{4}))e^{-(w-1)r/2}, \end{aligned}$$

is also invertible on the model space at infinity, $L^2(\mathbb{R}; e^r\, dr)$, since this is equivalent to the invertibility of

$$e^{wr/2}((w^{-1}D_r - \tfrac{i}{2})^2 - (\lambda - \tfrac{1}{4}))e^{-wr/2} = w^{-2}D_r^2 - (\lambda - \tfrac{1}{4})$$

on $L^2(\mathbb{R}; dr)$. Thus, a parametrix with compact remainder can be constructed for $(\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}}$, and this show that its essential spectrum lies in $\frac{1}{4} + e^{-2i\,\mathrm{Im}\,\theta}[0, +\infty)$. Hence $((\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}} - \lambda)^{-1}$ is meromorphic outside this set. In fact, it is well known that there are no poles in this entire strip (although there are an infinite number on $\arg\sqrt{\lambda - \lambda_0} = \pi/2$).

Combining this with some more standard technical facts, we are in a position to apply the theory of Aguilar-Balslev-Combes to prove that $((\Delta_{\mathbb{H}^2})_{\mathrm{rad}} - \lambda)^{-1}$, and hence $(\Delta_{\mathbb{H}^2} - \lambda)^{-1}$, has an analytic continuation in λ across $(\frac{1}{4}, +\infty)$. This is done by noting that for SO(2)-invariant functions $f, g \in L^2(\mathbb{H}^2)$ and $\theta \in \mathbb{R}$,

$$\langle f, ((\Delta_{\mathbb{H}^2})_{\mathrm{rad}} - \lambda)^{-1} g\rangle = \big\langle U_{\bar\theta} f,\, ((\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}} - \lambda)^{-1} U_\theta g\big\rangle,$$

by the unitarity of U_θ. Now if f, g lie in a smaller (dense) class of functions such that $U_\theta f$ and $U_\theta g$ continue analytically from $\theta \in \mathbb{R}$, then the meromorphic continuation in λ of the right hand side is obtained by first making θ complex with imaginary part of the appropriate sign, and then allowing λ to cross the continuous spectrum of $\Delta_{\mathbb{H}^2}$ without encountering the essential spectrum of $(\Delta_{\mathbb{H}^2})_{\theta,\mathrm{rad}}$. Hence the left hand side continues meromorphically also. With some additional care, one can even allow g to be the delta distribution at o, yielding the meromorphic continuation of the Green's function.

The argument on the higher rank symmetric space $M = \mathrm{SL}(3)/\mathrm{SO}(3)$ is similar. We still use the same scaling $\Phi_\theta : A \mapsto A^w$, $w = e^\theta$ with $\theta \in \mathbb{R}$, on M. Again, the first concern is that, allowing θ to become complex, the scaled operator Δ_θ is not elliptic. However, it is elliptic near $o = \mathrm{Id}$, and the scaled models for it near the walls, such as $(L_\sharp)_\theta$, remain elliptic at the walls. After all, for the latter, this is just the ellipticity of $(\Delta_{\mathbb{H}^2})_\theta$ near the origin, which we have already observed. This again allows the elliptic parametrix construction to proceed, supplying the results we needed in order to reach the framework of complex scaling. This in turn finishes the proof of Theorem 8.1.

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