Heegner Points and Rankin L-Series
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Preface

Modular curves and their close relatives, Shimura curves attached to multiplicative subgroups of quaternion algebras, are equipped with a distinguished collection of points defined over class fields of imaginary quadratic fields and arising from the theory of complex multiplication: the so-called *Heegner points*[1]. It is customary to use the same term to describe the images of degree zero divisors supported on these points in quotients of the jacobian of the underlying curve (a class of abelian varieties which we now know is rich enough to encompass all elliptic curves over the rationals). It was Birch who first undertook, in the late 70's and early 80's, a systematic study of Heegner points on elliptic curve quotients of jacobians of modular curves. Based on the numerical evidence that he gathered, he observed that the heights of these points seemed to be related to first derivatives at the central critical point of the Hasse–Weil L-series of the elliptic curve (twisted eventually by an appropriate Dirichlet character). The study initiated by Birch was to play an important role in the number theory of the next two decades, shedding light on such fundamental questions as the Gauss class number problem and the Birch and Swinnerton-Dyer conjecture.

The study of Heegner points took off in the mid-1980's thanks to two breakthroughs. The first of these was the Gross–Zagier formula, which, confirming Birch's observations, expressed the heights of Heegner points in the jacobian of the modular curve $X_0(N)$ in terms of the first derivative at the central point of an associated Rankin L-series. The second came a few years later, when Kolyvagin showed how the system of Heegner points on an elliptic curve controls the size and structure of its Selmer group. Taken together, these two insights led to a complete proof of the Birch and Swinnerton-Dyer conjecture (in its somewhat weaker form stipulating an equality between the rank of the elliptic curve, and the order of vanishing of its L-series at $s = 1$) for all modular elliptic curves over $\mathbb{Q}$ whose L-series has at most a simple zero at $s = 1$. The argument yielded a proof of the Shafarevich–Tate conjecture for these curves as well. The subsequent proof of the Shimura–Taniyama conjecture in 1994 showed that the result of Gross–Zagier and Kolyvagin applies unconditionally to all elliptic curves over the rationals.

[1]In the literature, Heegner points are often required to satisfy additional restrictions; in this preface the term is used more loosely, and synonymously with CM point.

Both the Gross–Zagier formula and the techniques introduced by Kolyvagin have proved fertile, and generalizations in a variety of different directions were actively explored throughout the 1990's. The organizers of the MSRI workshop on Special Values of Rankin L-series held in December 2001 felt the time was ripe to take stock of the new questions and insights that have emerged in the last decade of this study. The workshop focussed on several topics having Heegner points and Rankin L-series as a common unifying theme, grouped roughly under the following overlapping rubrics.

The original Gross–Zagier formula: Birch's partly historical article in this volume describes the study of Heegner points (both theoretical, and numerical) that preceded the work of Gross and Zagier. The editors are pleased to reproduce in this volume the correspondence between Birch and Gross in the months leading up to this work, which gives an enlightening snapshot of the subject at a time when it was about to undergo a major upheaval. Revisiting the height calculations of Gross and Zagier, the article by Conrad and Mann supplies further background on some of its more delicate aspects. Gross's contribution to this volume places the Gross–Zagier formula (together with its natural generalization for Heegner points on Shimura curves) in a broader conceptual setting in which the language of automorphic representations plays a key role. This larger perspective is useful in understanding Zhang's article which treats extensions of the Gross–Zagier formula for totally real fields.

Analytic applications: An immediate consequence of the Gross–Zagier formula is the first (and still, at present, the only) class of examples of L-series whose order of vanishing at $s = 1$ can be proved to be ≥ 3. These examples are produced by finding elliptic curves E over $\mathbb{Q}$ whose L-function $L(E/\mathbb{Q}, s)$ vanishes to odd order because of the sign in its functional equation, and whose associated Heegner point on $E(\mathbb{Q})$ is of finite order so that $L'(E/\mathbb{Q}, 1) = 0$. In his article, Goldfeld explains how the existence of such an L-function yields effective lower bounds for the growth of class numbers of imaginary quadratic fields. Before the work of Goldfeld, such bounds were only known ineffectively due to the possible existence of Siegel zeroes. Goldfeld's effective solution of the Gauss class number problem was one of the striking early applications of the Gross–Zagier formula.

Extensions to totally real fields: Experts were always aware of the potential generalizations of the Gross–Zagier formula to the context where the rational numbers are replaced by a totally real field F of degree d over $\mathbb{Q}$, although generalizations of this type gave rise to substantial technical difficulties. In this setting one is led to consider cuspidal automorphic forms on $\mathrm{GL}_2(F)$. A generalization of the Shimura–Taniyama conjecture predicts that any elliptic curve E over F (or, more generally, any two-dimensional ℓ-adic representation of $G_F := \mathrm{Gal}(\overline{F}/F)$ arising from the ℓ-adic Tate module of an abelian variety over F) corresponds to a particular type of automorphic form—a holomorphic Hilbert modular form of parallel weight 2. These forms can be described as differential d-forms on a d-dimensional Hilbert modular variety X. Such a variety admits an analytic description as a union of quotients of $\mathcal{H}^d$ by the action of appropriate subgroups of the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_F)$. Although these varieties are equipped with a supply of CM points, there is no direct generalization of the Eichler–Shimura construction which, when $d = 1$, realizes E as a quotient of $\mathrm{Jac}(X)$. Hence, CM points on X do not give rise to points on E.

There is nonetheless an extension of the Heegner point construction for totally real fields which is slightly less general and relies crucially on the notion of Shimura curves. These notions are discussed at length in the articles by Gross and Zhang in this volume. Let us briefly introduce the relevant definitions.

Given a set T of places of F of even cardinality, denote by B_T the quaternion algebra ramified exactly at the places $v \in T$. Let S be a finite set of places of F of odd cardinality containing all the archimedean places of F. One may attach to the data of S and an auxiliary choice of level structure a Shimura curve $X_{/F}$ which admits a v-adic analytic description, for each place $v \in S$, as a quotient of the v-adic upper half plane $\mathcal{H}_v$ by the action of an appropriate arithmetic subgroup of $B^\times_{S-\{v\}}$. It can also be described, following work of Shimura, in terms of the solution to a moduli problem, leading to a canonical model for X over F. Using the moduli description of the Shimura curve X, one can see that it admits a collection of CM points analogous in many respects to the Heegner points on $X_0(N)$ which provide the setting for the original Gross–Zagier formula.

If E is any elliptic over F having a suitable type of (bad) reduction at the non-archimedean places of S, a generalization of the Shimura–Taniyama conjecture predicts that E is isogenous to a factor of the jacobian of such a Shimura curve X. One thus has a direct analogue of the modular parametrization over any totally real field F, which should apply to many (albeit not all) elliptic curves over F.

The Heegner points arising from such Shimura curve parametrizations provide the setting for Zhang's extension of the Gross–Zagier formula to elliptic curves (and abelian varieties of "GL_2 type") over totally real fields that is the theme of his contribution in these proceedings.

Iwasawa theory: In late 70's Mazur formulated a program for studying the arithmetic of elliptic curves over a $\mathbb{Z}_p$-extension of the ground field along the lines pioneered by Iwasawa. The most natural setting is the one where E is defined over $\mathbb{Q}$ and one examines its Mordell–Weil group (and its p-power Selmer groups) over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. But Mazur also singled out the study of the Mordell–Weil groups of E over the anti-cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field because of the new phenomena manifesting themselves in this case through the possible presence of Heegner points. To give the flavor of the kind of results that Cornut and Vatsal obtain in a special case, let K be an imaginary quadratic field, let p be a prime which (for simplicity) we assume does not divide the conductor of E, and let K_∞ be the anticyclotomic $\mathbb{Z}_p$-extension of K, which is contained in the somewhat larger extension $\tilde{K}_\infty$, the union of all ring class fields of K of p-power conductor. An elementary argument shows that the collection of Heegner points of p-power conductor contained in $E(\tilde{K}_\infty)$ is either empty, or is infinite and generates a group of infinite rank, depending on the sign in the functional equation for $L(E/K, s)$. Mazur conjectures that the same is true for the traces of these points to $E(K_\infty)$. Cornut and Vatsal were able to prove Mazur's conjecture, as well as a non-triviality result for a p-adic L-function attached to E and K_∞ when the collection of Heegner points on $E(\tilde{K}_\infty)$ is empty.

The results of Cornut and Vatsal, when combined with those of Bertolini and Darmon on the anticyclotomic main conjecture of Iwasawa theory building on the methods of Kolyvagin, show that the Mordell–Weil group $E(K_\infty)$ is finitely generated when the sign in the functional equation for $L(E/K, s)$ is 1, and that, when this sign is -1, the rank of $E(L)$ grows like $[L : K]$ up to a bounded error term as L ranges over the finite subextensions of K_∞, the main contribution to the growth of the rank being accounted for by Heegner points. Thus the study of Heegner points over the anticyclotomic tower has yielded an almost complete understanding of Mazur's program in that setting.

Vatsal's article in this volume is motivated by the desire to extend his non-vanishing results for Heegner points and p-adic L-functions to the more general setting of totally real fields.

Higher dimensional analogues: Heegner points admit a number of higher-dimensional analogues, such as the arithmetic cycles on Shimura varieties "of orthogonal type" which are considered in Kudla's article. In his contribution Kudla surveys his far-reaching program of relating heights (in the Arakelov sense) of these cycles to the derivatives of associated Eisenstein series; while a tremendous amount of mathematics remains to be developed in this direction, substantial inroads have already been made, as is explained in Tonghai Yang's article in which one of the simplest cases of Kudla's program is worked out.

Function field analogues: Modular curves admit analogues in the function field setting: the so-called Drinfeld modular curves. Ulmer's article surveys recent work aimed at extending the ideas of Gross–Zagier and Kolyvagin to the function field setting, with the ultimate goal of proving the Birch and Swinnerton-Dyer conjecture for elliptic curves whose L-function has at most a simple zero. An interesting issue that is discussed in Ulmer's contribution is the possibility of constructing elliptic curves of arbitrarily large rank over the "ground field" $\mathbb{F}_p(T)$ with non-constant j-invariant. As is remarked in the note following Ulmer's contribution, Ulmer's Mordell–Weil groups of arbitrarily large rank might be accounted for, and generalized, via a suitable Heegner point construction. This stands in marked contract to the case of elliptic curves over $\mathbb{Q}$, where Heegner points are expected (and in many cases known) to yield only torsion points when the rank is strictly greater than one, and where the rank of elliptic curves is not even known to be unbounded.

Conjectural variants: Heegner points can be viewed as the elliptic curve analogue of special units such as circular or elliptic units, whose logarithms are related to first derivatives of Artin L-series at $s = 0$, just as the heights of Heegner points encode first derivatives of Rankin L-series via the Gross–Zagier formula. The article by Bertolini, Darmon and Green describes several largely *conjectural* analytic constructions of "Heegner-type" points which might be viewed as the elliptic curve analogue of Stark units. This is why the term "Stark–Heegner points" has been coined to describe them. Two cases of "Stark–Heegner" type constructions are considered in detail: one, p-adic analytic, of points over ring class fields of real quadratic fields, and the second, complex analytic, of points over ring class fields of so-called "ATR" extensions of a totally real field in terms of periods of Hilbert modular forms. Providing evidence (both experimental, and theoretical) for the conjectures on Stark–Heegner points, and understanding the relation between these points and special values of Rankin L-series, represent interesting challenges for the future.

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