Heegner Points and Rankin L-Series
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Gross–Zagier Formula for GL(2), II

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1. Introduction and Notation

Let A be an abelian variety defined over a number field F and let

$$\rho : \mathrm{Gal}(\bar{F}/F) \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

be a finite dimensional representation of the Galois group of F. Then the Birch and Swinnerton-Dyer conjecture predicts the identity

$$\mathrm{ord}_{s=1} L(s, \rho, A) = \dim(A(\bar{F}) \otimes \rho)^{\mathrm{Gal}(\bar{F}/F)}.$$

Here $L(s,\rho,A)$ denotes an Euler product over all places of F:

$$L(s,\rho,A) := \prod_v L_v(s,\rho,A), \qquad (\operatorname{Re} s \gg 0)$$

with good local factors given by

$$L_v(s,\rho,A) = \det(1 - q_v^{-s}\operatorname{Frob}_v |_{\mathrm{T}_\ell(A)\otimes\rho})^{-1},$$

where ℓ is a prime different than the residue characteristic of v, and $\mathbb{Z}_\ell$ has been embedded into $\mathbb{C}$. More precisely, the Birch and Swinnerton-Dyer conjecture predicts that the leading term of $L(s,\rho,A)$ in the Taylor expansion in $(s-1)$ is given in terms of periods, Tate–Shafarevich groups, and Mordell–Weil group. We refer to Tate's Bourbaki talk [9] for the details of the formulation.

In this paper, we will restrict ourself to the following very special situation:

- A/F is an abelian variety associated to a Hilbert newform ϕ over a totally real field F with trivial central character;
- ρ is a representation induced from a ring class character χ of $\operatorname{Gal}(\bar{K}/K)$ where K/F is a totally imaginary quadratic extension;
- the conductor N of ϕ, the conductor c of χ, and the discriminant $d_{K/F}$ of K/F are coprime to each other.

In this case, $L(s+\frac{1}{2},\rho,A)$ is a product of the Rankin L-series $L(s,\chi^\sigma,\phi^\sigma)$, where χ^σ and ϕ^σ are the Galois conjugates of χ and ϕ. Moreover, $L(s,\chi,\phi)$ has a *symmetric functional equation*:

$$L(s,\chi,\phi) = \varepsilon(\chi,\phi)\cdot \mathrm{N}_{F/\mathbb{Q}}(ND)^{1-2s}\cdot L(1-s,\chi,\phi)$$

where

$$\varepsilon(\chi,\phi) = \pm 1, \qquad D = c^2 d_{K/F}.$$

The main result in our *Asian Journal* and *Annals* papers [16; 17] is to express $L'(1,\chi,\phi)$ when $\varepsilon(\chi,\phi) = -1$ and $L(1,\chi,\phi)$ when $\varepsilon(\chi,\phi) = +1$ in terms of Heegner cycles in certain Shimura varieties of dimension 1 and 0, respectively, of level ND. This result is a generalization of the landmark work of Gross and Zagier in their *Inventiones* paper [6] on Heegner points on $X_0(N)/\mathbb{Q}$ with squarefree discriminant D.

The aim of this paper is to review the proofs in our previous papers [16; 17]. We also take this opportunity to deduce a new formula for Shimura varieties of level N. In odd case, the formula reads as

$$L'(\tfrac{1}{2},\chi,\phi) = \frac{2^{g+1}}{\sqrt{\mathrm{N}(D)}}\|\phi\|^2\|x_\phi\|^2$$

where x_ϕ is a Heegner point in the Jacobian of a Shimura curve. See Theorem 6.1 for details. In even case, the formula reads as

$$L(\tfrac{1}{2},\chi,\phi) = \frac{2^{g}}{\sqrt{\mathrm{N}(D)}}\|\phi\|^2|(\widetilde{\phi},P_\chi)|^2$$

where $(\widetilde{\phi}, P_\chi)$ is the evaluation of certain test form $\widetilde{\phi}$ on a CM-cycle P_χ on a Shimura variety of dimension 0. See Theorem 7.1 for details. These results have more direct applications to the Birch and Swinnerton-Dyer conjecture and p-adic L-series and Iwasawa theory. See the papers [1; 11] of Bertolini–Darmon and Vatsal for details.

To do so, we need to compute various constants arising in the comparisons of normalizations of newforms or test vectors. This will follow from a comparison of two different ways to compute the periods of Eisenstein series. One is an extension of the method for cusp form in our *Asian Journal* paper [16], and another one is a direct evaluation by unfolding the integrals. Notice that the residue and constant term of Dedekind zeta function can be computed by the periods formula for Eisenstein series. Thus, the Gross–Zagier formula can be considered as an extension of the class number formula and the Kronecker limit formula not only in its statement but also in its method of proof.

Notice that Waldspurger has obtained a formula (when χ is trivial [12]) and a criterion (when χ is non trivial [13]) in the general situation where

- K/F is any quadratic extension of number fields, and
- ϕ is any cusp form for $\mathrm{GL}_2(\mathbb{A}_F)$, and
- χ is any automorphic character of $\mathrm{GL}_1(\mathbb{A}_K)$ such that the central character is reciprocal to $\chi|_{\mathbb{A}_F^\times}$.

We refer to the papers of Gross and Vatsal in this volume [5; 10] for the explanation of connections between our formula and his work. There seems to be a lot of rooms left to generalize our formula to the case considered by Waldspurger. In this direction, Hui Xue in his thesis [15] has obtained a formula for the central values for L-series attached to a holomorphic Hilbert modular form of parallel weight $2k$.

This paper is organized as follows. In the first part (Sections 2–7), we will give the basic definitions of forms, L-series, Shimura varieties, CM-points, and state our main formula (Theorem 6.1 and Theorem 7.1) in level N. The definitions here are more or less standard and can be found from our previous work as well as the work of Jacquet, Langlands, Waldspurger, Deligne, Carayol, Gross, and Prasad. Forms have been normalized as *newforms or test vectors* according to the action of unipotent or torus subgroup.

In the second part (Sections 8–10), we will review the original ideas of Gross–Zagier in their *Inventiones* paper [6] on $X_0(N)$ with squarefree D and its generalization to Shimura curves of (N, K)-type in our *Annals* paper [17]. The central idea is to compare the Fourier coefficients of certain *natural* kernel functions of level N with certain *natural* CM-points on Shimura curves $X(N, K)$ of (N, K)-type. This idea only works perfectly when D is squarefree and when $X(N, D)$ has regular integral model but has essential difficulty for the general case.

In the third part (Sections 11–16), we review the basic construction and the proof in our *Asian Journal* paper [16] for formulas in level ND. The kernel function and CM-points we pick are good for computation but have level ND. Their correspondence is given by the local Gross–Zagier formula, which is of course the key of the whole proof. The final formulas involve the notion of *quasi-newforms or toric newforms* as variations of newforms or test vectors.

In the last part (Sections 17–19) which is our new contribution in addition to our previous papers, we will deduce the formula in level N from level ND. The plan of proof is stated in the beginning of Section 17 in three steps. The central idea is to use Eisenstein series to compute certain local constants. This is one more example in number theory of the possibility of solving local questions by a global method, as in the early development of local class field theory and in the current work of Harris and Taylor on the local Langlands conjecture.

The first three parts (Sections 2–18) simply review ideas used in our previous papers. For details one may need to go to the original papers. For an elementary exposition of the Gross–Zagier formula (with variants) and its applications to Birch and Swinnerton–Dyer conjecture, see our paper in *Current Developments in Mathematics* [18].

Notation. The notations of this note are mainly adopted from our *Asian Journal* paper [16] with some simplifications.

1. Let F denote a totally real field of degree g with ring of integers $\mathcal{O}_F$, and adeles $\mathbb{A}$. For each place v of F, let F_v denote the completion of of F at v. When v is finite, let $\mathcal{O}_v$ denote the ring of integers and let π_v denote a uniformizer of $\mathcal{O}_v$. We write $\widehat{\mathcal{O}}_F$ for the product of $\mathcal{O}_v$ in $\mathbb{A}$.

2. Let ψ denote a fixed nontrivial additive character of $F\backslash\mathbb{A}$. For each place v, let ψ_v denote the component of ψ and let $\delta_v \in F_v^\times$ denote the conductor of ψ_v. When v is finite, $\delta_v^{-1}\mathcal{O}_v$ is the maximal fractional ideal of F_v over which ψ_v is trivial. When v is infinite, $\psi_v(x) = e^{2\pi\delta_v x}$. Let δ denote $\prod \delta_v \in \mathbb{A}^\times$. Then the norm $|\delta|^{-1} = d_F$ is the discriminant of F.

3. Let dx denote a Haar measure on $\mathbb{A}$ such that the volume of $F\backslash\mathbb{A}$ is one. This measure has a decomposition $dx = \bigotimes dx_v$ into local measures dx_v on F_v which are self-dual with respect to characters ψ_v. Let $d^\times x$ denote a Haar measure on $\mathbb{A}^\times$ which has a decomposition $d^\times x = \bigotimes d^\times x_v$ such that $d^\times x_v = dx_v/x_v$ on $F_v^\times = \mathbb{R}^\times$ when v is infinite, and such that the volume of $\mathcal{O}_v^\times$ is one when v is finite. Notice that our choice of multiplicative measures is different than that in Tate's thesis, where the volume of $\mathcal{O}_v^\times$ is $|\delta_v|^{1/2}$.

4. Let K denote a totally imaginary quadratic extension of F and T denote the algebraic group $K^\times/F^\times$ over F. We will fix a Haar measure dt and its decomposition $dt = \bigotimes dt_v$ such that $T(F_v)$ has volume 1 when v is infinite.

5. Let B denote a quaternion algebra over F and let G denote the algebraic group $B^\times/F^\times$ over F. We will fix a Haar measure dg on $G(\mathbb{A})$ and a decomposition $dg = \bigotimes dg_v$ such that at an infinite place $G(F_v)$ has volume one if it is

compact, and that when $G(F_v) \simeq \mathrm{PGL}_2(\mathbb{R})$,

$$dg_v = \frac{|dx\,dy|}{2\pi y^2} d\theta,$$

with respect to the decomposition

$$g_v = z \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

In this way, the volume $|U|$ of the compact open subgroup U of $G(\mathbb{A}_f)$ (or $G(F_v)$ for some $v \nmid \infty$) is well defined. We write $(f_1, f_2)_U$ for the hermitian product

$$(f_1, f_2)_U = |U|^{-1} \int_{G(\mathbb{A})} f_1 \bar{f}_2 \, dg$$

for functions f_1, f_2 on $G(\mathbb{A})$ (or $G(\mathbb{A}_f)$, or $G(F_v)$). This product depends only on the choice of U but not on dg.

2. Automorphic Forms

Let F be a totally real field of degree g, with ring of adeles $\mathbb{A}$, and discriminant d_F. Let ω be a (unitary) character of $F^\times \backslash \mathbb{A}^\times$. By an *automorphic form* on $\mathrm{GL}_2(\mathbb{A})$ with central character ω we mean a continuous function ϕ on $\mathrm{GL}_2(\mathbb{A})$ such that the following properties hold:

- $\phi(z\gamma g) = \omega(z)\phi(g)$ for $z \in Z(\mathbb{A}), \gamma \in \mathrm{GL}_2(F)$;
- ϕ is invariant under right action of some open subgroup of $\mathrm{GL}_2(\mathbb{A}_f)$;
- for a place $v \mid \infty$, ϕ is smooth in $g_v \in \mathrm{GL}_2(F_v)$, and the vector space generated by
$$\phi(gr_v), \quad r_v \in \mathrm{SO}_2(F_v) \subset \mathrm{GL}_2(\mathbb{A})$$
is finite dimensional;
- for any compact subset Ω there are positive numbers C, t such that
$$\left|\phi\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right)\right| \le C(|a| + |a^{-1}|)^t$$
for all $g \in \Omega$.

Let $\mathcal{A}(\omega)$ denote the space of automorphic forms with central character ω. Then $\mathcal{A}(\omega)$ admits an *admissible representation* ρ by $\mathrm{GL}_2(\mathbb{A})$. This is a combination of a representation ρ_f of $\mathrm{GL}_2(\mathbb{A}_f)$ via right action:

$$\rho_f(h)\phi(g) = \phi(gh), \qquad h \in \mathrm{GL}_2(\mathbb{A}_f),\ \phi \in \mathbb{A}(\omega),\ g \in \mathrm{GL}_2(\mathbb{A}),$$

and an action ρ_∞ by pairs

$$(M_2(F_v), O_2(F_v)), \qquad v \mid \infty.$$

Here the action of $O_2(F_v)$ is the same as above while the action of $M_2(F_v)$ is given by

$$\rho_\infty(x)\phi(g) = \frac{d\phi}{dt}(ge^{tx})|_{t=0}, \qquad x \in M_2(F_v),\ \phi \in \mathcal{A}(\omega),\ g \in \mathrm{GL}_2(\mathbb{A}),\ v \mid \infty.$$

An admissible and irreducible representation Π of $\mathrm{GL}_2(\mathbb{A})$ is called *automorphic* if it is isomorphic to a sub-representation of $\mathcal{A}(\omega)$. It is well-known that the multiplicity of any irreducible representation in $\mathcal{A}(\omega)$ is at most 1. Moreover, if we decompose such a representation into local representations $\Pi = \bigotimes \Pi_v$ then *the strong multiplicity one* says that Π is determined by all but finitely many Π_v.

Fix an additive character ψ on $F\backslash\mathbb{A}$. Then any automorphic form will have a Fourier expansion:

$$\phi(g) = C_\phi(g) + \sum_{\alpha \in F^\times} W_\phi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right), \tag{2.1}$$

where C_ϕ is the constant term:

$$C_\phi(g) := \int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx, \tag{2.2}$$

and $W_\phi(g)$ is the Whittaker function:

$$W_\phi(g) = \int_{F\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)\, dx. \tag{2.3}$$

It is not difficult to show that a form with vanishing Whittaker function will have the form $\alpha(\det g)$ where α is a function on $F^\times\backslash\mathbb{A}^\times$. Every automorphic representation of dimension 1 appears in this space and corresponds to a characters μ of $F^\times\backslash\mathbb{A}^\times$ such that $\mu^2 = \omega$.

We say that an automorphic form ϕ is *cuspidal* if the constant term $C_\phi(g) = 0$. The space of cuspidal forms is denoted by $\mathcal{A}_0(\omega)$. We call an automorphic representation *cuspidal* if it appears in $\mathcal{A}_0(\omega)$.

An irreducible automorphic representation which is neither one dimensional nor cuspidal must be isomorphic to the space $\Pi(\mu_1, \mu_2)$ of Eisenstein series associated to two quasi characters μ_1, μ_2 of $F^\times\backslash\mathbb{A}^\times$ such that $\mu_1\mu_2 = \omega$. To construct an Eisenstein series, let Φ be a Schwartz–Bruhat function on $\mathbb{A}^2$. For s a complex number, define

$$f_\Phi(s, g) := \mu_1(\det g)|\det g|^{s+1/2} \int_{\mathbb{A}^\times} \Phi[(0,t)g]\mu_1\mu_2^{-1}(t)|t|^{1+2s} d^\times t. \tag{2.4}$$

Then $f_\Phi(s, g)$ belongs to the space $\mathcal{B}(\mu_1 \cdot |\cdot|^s, \mu_2 \cdot |\cdot|^{-s})$ of functions on $\mathrm{GL}_2(\mathbb{A})$ satisfying

$$f_\Phi\left(s, \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right) = \mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{1/2+s} f_\Phi(s, g). \tag{2.5}$$

The Eisenstein series $E(s,g,\Phi)$ is defined as follows:

$$E(s,g,\Phi) = \sum_{\gamma\in P(F)\backslash \mathrm{GL}_2(F)} f_\Phi(s,\gamma g). \tag{2.6}$$

One can show that $E(s,g,\Phi)$ is absolutely convergent when $\mathrm{Re}\, s$ is sufficiently large, and has a meromorphic continuation to the whole complex plane. The so-defined meromorphic function $E(s,g,\Phi)$ has at most simple poles with constant residue. The space $\Pi(\mu_1,\mu_2)$ consists of the Eisenstein series

$$E(g,\Phi) := \lim_{s\longrightarrow 0}\left(E(s,g,\Phi) - (\text{residue})s^{-1}\right). \tag{2.7}$$

3. Weights and Levels

Let N be an ideal of $\mathcal{O}_F$ and let $U_0(N)$ and $U_1(N)$ be the following subgroups of $\mathrm{GL}_2(\mathbb{A}_f)$:

$$U_0(N) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) : c\equiv 0 \pmod N\right\}, \tag{3.1}$$

$$U_1(N) := \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in U_0(N) : \quad d\equiv 1 \pmod N\right\}. \tag{3.2}$$

For each infinite place v of F, let k_v be an integer such that $\omega_v(-1) = (-1)^{k_v}$.

An automorphic form $\phi\in\mathcal{A}(\omega)$ is said to have *level* N *and weight* $k = (k_v : v \mid \infty)$ if the following conditions are satisfied:

- $\phi(gu) = \phi(g)$ for $u\in U_1(N)$;
- for a place $v\mid\infty$,
$$\phi(gr_v(\theta)) = \phi(g)e^{2\pi i k_v\theta}$$
where $r_v(\theta)$ is an element in $\mathrm{SO}_2(F_v)\subset \mathrm{GL}_2(\mathbb{A})$ of the form
$$r_v(\theta) = \begin{pmatrix} \cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}.$$

Let $\mathcal{A}_k(N,\omega)$ denote the space of forms of weight k, level N, and central character ω. For any level $N'\mid N$ of N and weight $k'\le k$ by which we mean that $k-k'$ has nonnegative components, we may define embeddings

$$\mathcal{A}_{k'}(N',\omega)\longrightarrow \mathcal{A}_k(N,\omega)$$

by applying some of the operators

$$\phi\mapsto \rho_v\begin{pmatrix} \pi_v^{-1} & 0\\ 0 & 1\end{pmatrix}\phi \qquad (v\nmid\infty),$$

$$\phi\mapsto \rho_v\begin{pmatrix} 1 & i\\ i & -1\end{pmatrix}\phi \qquad (v\mid\infty).$$

The first operator increases level by order 1 at a finite place v; while the second operator increases weight by 2 at an infinite place v. Let $\mathcal{A}_k^{\mathrm{old}}(N,\omega)$ denote the subspace of forms obtained from lower level N' or lower weight k' by applying *at least one of the above operators.*

For any ideal a prime to N, the *Hecke operator* T_a on $\mathcal{A}_k(N,\omega)$ is defined as follows:

$$\mathrm{T}_a\phi(g) = \sum_{\substack{\alpha\beta=a \\ x \mod \alpha}} \phi\left(g\begin{pmatrix}\alpha & x \\ 0 & \beta\end{pmatrix}\right) \tag{3.3}$$

where α and β runs through representatives of integral ideles modulo $\widehat{\mathcal{O}}_F^\times$ with trivial component at the place dividing N such that $\alpha\beta$ generates a. One has for the Whittaker function the formula

$$W_\phi\left(g\begin{pmatrix}a\delta^{-1} & 0 \\ 0 & 1\end{pmatrix}\right) = |a| W_{\mathrm{T}_a\phi}(g), \tag{3.4}$$

where $g \in \mathrm{GL}_2(\mathbb{A})$ with component 1 at places $v \nmid N\cdot\infty$.

We say that ϕ is an *eigenform* if for any ideal a prime to N, ϕ is an eigenform under the Hecke operator T_a. We say an eigenform ϕ is *new* if all $k_v \geq 0$, and if there is no old eigenform with the same eigenvalues as ϕ. One can show that two new eigenforms are proportional if and only if they share the same eigenvalues for all but finitely many T_v.

For $\phi \in \mathcal{A}_k(N,\omega)$, let's write $\Pi(\phi)$ for the space of forms in

$$\mathcal{A}(\omega) = \cup_{k,N}\mathcal{A}_k(N,\omega)$$

generated by ϕ by right action of $\mathrm{GL}_2(\mathbb{A})$. Then one can show that $\Pi(\phi)$ is irreducible if and only if ϕ is an eigenform. Conversely, any irreducible representation Π of $\mathrm{GL}_2(\mathbb{A})$ in $\mathcal{A}(\omega)$ contains a unique line of new eigenform. An eigenform ϕ with $\dim \Pi(\phi) < \infty$ will have vanishing Whittaker function and is a multiple of a character.

It can be shown that an eigen newform ϕ with $\dim \Pi(\phi) = \infty$ will have Whittaker function nonvanishing and decomposable:

$$W_\phi(g) = \bigotimes W_v(g_v) \tag{3.5}$$

where $W_v(g_v)$ at finite places can be normalized such that

$$W_v\begin{pmatrix}\delta_v^{-1} & 0 \\ 0 & 1\end{pmatrix} = 1. \tag{3.6}$$

Each local component Π_v is realized in the subspace

$$\mathcal{W}(\Pi_v,\psi_v) = \Pi(W_v)$$

generated by W_v under the right action of $\mathrm{GL}_2(F_v)$ (or $(M_2(F_v), O_2(F_v))$ when v is infinite.)

4. Automorphic L-Series

For an automorphic form ϕ, we define its L-series by

$$\begin{aligned} L(s,\phi) : &= d_F^{1/2-s} \int_{F^\times \backslash \mathbb{A}^\times} (\phi - C_\phi) \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a \\ &= d_F^{1/2-s} \int_{\mathbb{A}^\times} W_\phi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a, \end{aligned} \tag{4.1}$$

which is absolutely convergent for $\operatorname{Re} s \gg 0$ and has a meromorphic continuation to the entire complex plane, and satisfies a functional equation.

Assume that ϕ is an eigen newform. Then its Whittaker function is decomposable. The L-series $L(s,\phi)$ is then an Euler product

$$L(s,\phi) = \prod_v L_v(s,\phi) \tag{4.2}$$

where

$$L_v(s,\phi) = |\delta_v|^{s-1/2} \int_{F_v^\times} W_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^\times a. \tag{4.3}$$

For a finite place v, the L-factor has the usual expression:

$$L_v(s,\phi) = \begin{cases} (1 - \lambda_v |\pi_v|^s + \omega(\pi_v)|\pi_v|^{2s})^{-1}, & \text{if } v \nmid N, \\ (1 - \lambda_v |\pi_v|^s)^{-1}, & \text{if } v \mid N, \end{cases} \tag{4.4}$$

where $\lambda_v \in \mathbb{C}$ is such that $\lambda_v |\pi_v|^{-1/2}$ is the eigenvalue of T_v if $v \nmid N$.

For an archimedean place v, the local factor $L_v(s,\phi)$ is a certain product of gamma functions and is determined by analytic properties of ϕ at v. For the purposes of this paper, we will only consider newforms that at an infinite place are either *holomorphic* or *even of weight* 0, that is, invariant under $O_2(F_v)$ rather than $\mathrm{SO}_2(F_v)$. More precisely, at an infinite place v let's consider the function on $\mathcal{H} \times \mathrm{GL}_2(\mathbb{A}^v)$ defined by

$$f(z, g^v) := |y|^{-(k_v+w_v)/2} \phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}, g^v\right), \quad z = x + yi, \tag{4.5}$$

where $w_v = 0$ or 1 is such that $\omega_v(-1) = (-1)^{w_v}$. Then we require that $f(z, g^v)$ is holomorphic in z if $k_v \geq 1$, and that $f(z, g^v) = f(-\bar{z}, g^v)$ if $k_v = 0$. If ϕ is of weight 0, then ϕ is an eigenform for the Laplacian

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right). \tag{4.6}$$

We write eigenvalues as $\frac{1}{4} + t_v^2$ and call t_v the *parameter* of ϕ at v. Let's define *the standard Whittaker function* at archimedean places v of weight k_v in the following way: if $k_v > 0$,

$$W_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} 2a^{(w_v+k_v)/2} e^{-2\pi a} & \text{if } a > 0, \\ 0 & \text{if } a < 0, \end{cases} \tag{4.7}$$

and if $k_v = 0$,

$$W_v \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = |a|^{1/2} \int_0^\infty e^{-\pi|a|(y+y^{-1})} y^{it_v} d^\times y. \tag{4.8}$$

In this manner (up to a constant $c \neq 0$) ϕ will have a Whittaker function decomposable as in (3.5) with local function W_v normalized as in (3.6), (4.7), (4.8). We say that ϕ is a *newform* if $c = 1$. Equivalently, ϕ is a newform if and only if $L_v(s, \phi)$ has decomposition (4.2) with local factors given by (4.4) when $v \nmid \infty$, and the following when $v \mid \infty$:

$$L_v(s, \phi) = \begin{cases} G_2(s + k_v + w_v), & \text{if } k_v > 0, \\ G_1(s + it_v) G_1(s - it_v), & \text{if } k_v = 0, \end{cases} \tag{4.9}$$

where

$$\begin{aligned} G_1(s) &= \pi^{-s/2} \Gamma(s/2), \\ G_2(s) &= 2(2\pi)^{-s} \Gamma(s) = G_1(s) G_1(s+1). \end{aligned} \tag{4.10}$$

If Π is an automorphic representation generated by a newform ϕ, we write $L(s, \Pi)$ and $L(s, \Pi_v)$ for $L(s, \phi)$ and $L_v(s, \phi)$, respectively.

5. Rankin–Selberg L-Series

Let K be a totally imaginary quadratic extension of F, and let ω be the non-trivial quadratic character of $\mathbb{A}^\times / F^\times \mathrm{N}\mathbb{A}_K^\times$. The conductor $c(\omega)$ is the relative discriminant of K/F. Let χ be a character of finite order of $\mathbb{A}_K^\times / K^\times \mathbb{A}^\times$. The conductor $c(\chi)$ is an ideal of $\mathcal{O}_F$ which is maximal such that χ is factorized through

$$\mathbb{A}_K^\times / K^\times \mathbb{A}^\times \widehat{\mathcal{O}}_{c(\chi)}^\times K_\infty^\times = \mathrm{Gal}(H_{c(\chi)}/K),$$

where $\mathcal{O}_c = \mathcal{O}_F + c(\chi)\mathcal{O}_K$ and H_c is the ring class filed of conductor $c(\chi)$. We define the ideal $D = c(\chi)^2 c(\omega)$, and call χ a *ring class character of conductor* $c(\chi)$.

Let ϕ be a newform with trivial central character and of level N. The Rankin–Selberg convolution L-function $L(s, \chi, \phi)$ is defined by an Euler product over primes v of F:

$$L(s, \chi, \phi) := \prod_v L_v(s, \chi, \phi) \tag{5.1}$$

where the factors have degree ≤ 4 in $|\pi_v|^s$. This function has an analytic continuation to the entire complex plane, and satisfies a functional equation. We will assume that the ideals $c(\omega)$, $c(\chi)$, N are coprime each other. Then the local factors can be defined explicitly as follows.

For v a finite place, let's write

$$\begin{aligned} L_v(s, \phi) &= (1 - \alpha_1 |\pi_v|^s)^{-1} (1 - \alpha_2 |\pi_v|^s)^{-1}, \\ \prod_{w|v} L(s, \chi_w) &= (1 - \beta_1 |\pi_v|^s)^{-1} (1 - \beta_2 |\pi_v|^s)^{-1}. \end{aligned}$$

Then

$$L_v(s,\chi,\phi) = \prod_{i,j}(1-\alpha_i\beta_j|\pi_v|^s)^{-1}. \tag{5.2}$$

Here, for a place w of K, the local factor $L(s,\chi_w)$ is defined by

$$L(s,\chi_w) = \begin{cases} (1-\chi(\pi_w)|\pi_w|^s)^{-1}, & \text{if } w \nmid c(\chi)\cdot\infty, \\ G_2(s), & \text{if } v \mid \infty, \\ 1, & \text{if } v \mid c(\chi). \end{cases} \tag{5.3}$$

At an infinite place v, using formula $G_2(s) = G_1(s)G_1(s+1)$ we may write

$$L_v(s,\phi) = G_1(s+\sigma_1)G_1(s+\sigma_2),$$
$$L_v(s,\chi) = G_1(s+\tau_1)G_1(s+\tau_2).$$

Then the L-factor $L_v(s,\chi,\phi)$ is defined by

$$\begin{aligned} L_v(s,\chi,\phi) &= \prod_{i,j} G_1(s+\sigma_i+\tau_j) \\ &= \begin{cases} G_2(s+\frac{1}{2}(k_v-1))^2, & \text{if } k_v \geq 2, \\ G_2(s+it_v)G_2(s-it_v), & \text{if } k_v = 0, \end{cases} \end{aligned} \tag{5.4}$$

where t_v is the parameter associated to ϕ at a place v where the weight is 0.

The functional equation is then

$$L(1-s,\chi,\phi) = (-1)^{\#\Sigma}\mathrm{N}_{F/\mathbb{Q}}(ND)^{1-2s}L(s,\chi,\phi), \tag{5.5}$$

where $\Sigma = \Sigma(N,K)$ is the following set of places of F:

$$\Sigma(N,K) = \left\{ v \;\middle|\; \begin{array}{l} v \text{ is infinite and } \phi \text{ has weight } k_v > 0 \text{ at } v, \\ \text{or } v \text{ is finite and } \omega_v(N) = -1. \end{array} \right\} \tag{5.6}$$

6. The Odd Case

Now we assume that all $k_v = 2$ and that the sign of the functional equation (5.5) is -1, so $\#\Sigma$ is odd. Our main formula expresses the central derivative $L'(\frac{1}{2},\chi,\phi)$ in terms of the heights of CM-points on a Shimura curve. Let τ be any real place of F, and let B be the quaternion algebra over F ramified exactly at the places in $\Sigma - \{\tau\}$. Let G be the algebraic group over F, which is an inner form of PGL_2, and has $G(F) = B^\times/F^\times$.

The group $G(F_\tau) \simeq \mathrm{PGL}_2(\mathbb{R})$ acts on $\mathcal{H}^\pm = \mathbb{C} - \mathbb{R}$. If $U \subset G(\mathbb{A}_f)$ is open and compact, we get an analytic space

$$M_U(\mathbb{C}) = G(F)_+\backslash\mathcal{H}\times G(\mathbb{A}_f)/U \tag{6.1}$$

where $G(F)_+$ denote the subgroup of elements of $G(F)$ with totally positive determinants. Shimura proved these were the complex points of an algebraic curve M_U, which descends canonically to F (embedded in $\mathbb{C}$, by the place τ). The curve M_U over F is independent of the choice of τ in Σ.

To specify M_U, we must define $U \subset G(\mathbb{A}_f)$. To do this, we fix an embedding $K \longrightarrow B$, which exists, as all places in Σ are either inert or ramified in K. One can show that there is an order R of B containing $\mathcal{O}_K$ with reduced discriminant N. For an explicit description of such an order, we fix a maximal ideal $\mathcal{O}_B$ of B containing $\mathcal{O}_K$ and an ideal $\mathcal{N}$ of $\mathcal{O}_K$ such that

$$\mathrm{N}_{K/F}\mathcal{N} \cdot \mathrm{disc}_{B/F} = N, \tag{6.2}$$

where $\mathrm{disc}_{B/F}$ is the reduced discriminant of $\mathcal{O}_B$ over $\mathcal{O}_F$. Then we take

$$R = \mathcal{O}_K + \mathcal{N} \cdot \mathcal{O}_B. \tag{6.3}$$

We call R *an order of* (N, K)*-type*. Define an open compact subgroup U_v of $G(F_v)$ by

$$U_v = R_v^\times / \mathcal{O}_v^\times. \tag{6.4}$$

Let $U = \prod_v U_v$. This defines the curve M_U up to F-isomorphism. Let X be its compactification over F, so $X = M_U$ unless $F = \mathbb{Q}$ and $\Sigma = \{\infty\}$, where X is obtained by adding many cusps. We call X *a Shimura curve of* (N, K)*-type*. We write $R(N, K)$, $U(N, K)$, $X(N, K)$ when types need to be specified.

We will now construct points in $\mathrm{Jac}(X)$, the connected component of $\mathrm{Pic}(X)$, from CM-points on the curve X. The CM-points corresponding to K on $M_U(\mathbb{C})$ form a set

$$G(F)_+ \backslash G(F)_+ \cdot h_0 \times G(\mathbb{A}_f)/U = T(F)\backslash G(\mathbb{A}_f)/U, \tag{6.5}$$

where $h_0 \in \mathcal{H}$ is the unique fixed point of the torus points $T(F) = K^\times/F^\times$. Let P_c denote a point in X represented by (h_0, i_c) where $i_c \in G(\mathbb{A}_f)$ such that

$$U_T := i_c U i_c^{-1} \cap T(\mathbb{A}_f) \simeq \widehat{\mathcal{O}}_c^\times / \widehat{\mathcal{O}}_F^\times. \tag{6.6}$$

By Shimura's theory, P_c is defined over the ring class field H_c of conductor c corresponding to the Artin map

$$\mathrm{Gal}(H_c/K) \simeq T(F)\backslash T(\mathbb{A}_f)/T(F_\infty)U_T.$$

Let P_χ be a divisor on X with complex coefficients defined by

$$P_\chi = \sum_{\sigma \in \mathrm{Gal}(H_c/K)} \chi^{-1}(\sigma)[P_c^\sigma]. \tag{6.7}$$

If χ is not of form $\chi = \nu \cdot \mathrm{N}_{K/F}$ with ν a quadratic character of $F^\times \backslash \mathbb{A}^\times$, then P_χ has degree 0 on each connected component of X. Thus P_χ defines a class x in $\mathrm{Jac}(X) \otimes \mathbb{C}$. Otherwise we need a reference divisor to send P_χ to $\mathrm{Jac}(X)$. In the modular curve case, one uses cusps. In the general case, we use the *Hodge class* $\xi \in \mathrm{Pic}(X) \otimes \mathbb{Q}$: the unique class whose degree is 1 on each connected component and such that

$$\mathrm{T}_m \xi = \deg(\mathrm{T}_m)\xi$$

for all integral nonzero ideals m of $\mathcal{O}_F$ prime to ND. The Heegner class we want now is the class difference

$$x := [P_\chi - \deg(P_\chi)\xi] \in \mathrm{Jac}(X)(H_c) \otimes \mathbb{C}, \tag{6.8}$$

where $\deg(P_\chi)$ is the multi-degree of P_χ on geometric components.

Notice that the curve X and its Jacobian have an action by the ring of good Hecke operators. Thus x is a sum of eigen vectors of the Hecke operators.

THEOREM 6.1. *Let x_ϕ denote the ϕ-typical component of x. Then*

$$L'(\tfrac{1}{2}, \chi, \phi) = \frac{2^{g+1}}{\sqrt{\mathrm{N}(D)}} \cdot \|\phi\|^2 \cdot \|x_\phi\|^2,$$

where $\|\phi\|^2$ is computed using the invariant measure on

$$\mathrm{PGL}_2(F)\backslash\mathcal{H}^g \times \mathrm{PGL}_2(\mathbb{A}_f)/U_0(N)$$

induced by $dx\,dy/y^2$ on $\mathcal{H}$, and $\|x_\phi\|^2$ is the Néron–Tate pairing of x_ϕ with itself.

To see the application to the Birch and Swinnerton-Dyer conjecture, we just notice that x_ϕ actually lives in a unique abelian subvariety A_ϕ of the Jacobian $\mathrm{Jac}(X)$ such that

$$L(s, A_\phi) = \prod_{\sigma:\mathbb{Z}[\phi]\to\mathbb{C}} L(s, \phi^\sigma). \tag{6.9}$$

Y. Tian [14] has recently generalized the work of Kolyvagin and Bertolini–Darmon to our setting and showed that the rank conjecture of Birch and Swinnerton-Dyer for A in the case $\mathrm{ord}_{s=1/2}L(s, \chi, \phi) \le 1$.

Notice that $\|\phi\|^2$ is not exactly the periods of A_ϕ appearing in the Birch and Swinnerton-Dyer conjecture, but it has an expression in L-series:

$$\|\phi\|^2 = 2\mathrm{N}(N) \cdot d_F \cdot L(1, \mathrm{Sym}^2\phi) \tag{6.10}$$

where $L(s, \mathrm{Sym}^2\phi)$ is the L-series defined by an Euler product with local factors $L_v(s, \mathrm{Sym}^2\phi)$ given by

$$L_v(s, \mathrm{Sym}^2\phi) = G_2(s + \tfrac{1}{2})^2 G_1(s)^{-1}, \tag{6.11}$$

if $v \mid \infty$, and by

$$L_v(s, \mathrm{Sym}^2\phi) = (1 - \alpha^2|\pi_v|^s)^{-1}(1 - \beta^2|\pi_v|^s)^{-1}(1 - \alpha\beta|\pi_v|^s)^{-1}, \tag{6.12}$$

if $v \nmid \infty$, and α and β are given as follows:

$$L_v(s, \phi) = (1 - \alpha|\pi_v|^s)^{-1}(1 - \beta|\pi_v|^s)^{-1}.$$

It will be an interesting question to see how this relates the periods in A_ϕ.

7. The Even Case

We now return to the case where ϕ has possible nonholomorphic components, but we assume that all weights be either 0 or 2 and that the sign of the functional equation of $L(s,\chi,\phi)$ is $+1$, or equivalently, Σ is even. In this case, we have an explicit formula for $L(\frac{1}{2},\chi,\phi)$ in terms of CM-points on locally symmetric varieties covered by $\mathcal{H}^n$ where n is the number of real places of F where ϕ has weight 0.

More precisely, let B be the quaternion algebra over F ramified at Σ, and G the algebraic group associated to $B^\times/F^\times$. Then

$$G(F\otimes\mathbb{R})\simeq \mathrm{PGL}_2(\mathbb{R})^n\times \mathrm{SO}_3^{g-n} \tag{7.1}$$

acts on $(\mathcal{H}^\pm)^n$. The locally symmetric variety we will consider is

$$M_U=G(F)_+\backslash\mathcal{H}^n\times G(\mathbb{A}_f)/U, \tag{7.2}$$

where $U=\prod U_v$ was defined in the previous section. Again we call M_U or its compactification X a *quaternion Shimura variety of* (N,K)*-type*. We will also have a CM-point P_c and a CM-cycle P_χ defined as in (6.6) and (6.7) but with $\mathrm{Gal}(H_c/K)$ replaced by $T(F)\backslash T(\mathbb{A}_f)/T(F_\infty)U_T$.

By results of Waldspurger, Tunnel, and Gross–Prasad [17, Theorem 3.2.2], there is a unique line of cuspidal functions $\widetilde{\phi}$ on M_U such that for each finite place v not dividing $N\cdot D$, $\widetilde{\phi}$ is the eigenform for Hecke operators T_v with the same eigenvalues as ϕ. We call any such a form a *test form* of (N,K)-type.

Theorem 7.1. *Let $\widetilde{\phi}$ be a test form of norm* 1 *with respect to the measure on X induced by $dx\,dy/y^2$ on $\mathcal{H}$. Then*

$$L(\tfrac{1}{2},\chi,\phi)=\frac{2^{g+n}}{\sqrt{\mathrm{N}(D)}}\cdot\|\phi\|^2\cdot|(\widetilde{\phi},P_\chi)|^2.$$

Here

$$(\widetilde{\phi},P_\chi)=\sum_{t\in T(F)\backslash T(\mathbb{A}_f)/U_T}\chi^{-1}(t)\widetilde{\phi}(tP_c).$$

Remark. There is a naive analogue between even and odd cases via Hodge theory which is actually a starting point to believe that there will be a simultaneous proof for both cases. To see this, let's consider the space $Z(\Omega^1_X)$ of closed smooth 1-forms on a Shimura curve X of (N,K)-type with hermitian product defined by

$$(\alpha,\beta)=\frac{i}{2}\int\alpha\bar{\beta}.$$

The Hodge theory gives a decomposition of this space into a direct sum

$$Z(\Omega^1_X)=\Big(\bigoplus\nolimits_\alpha\mathbb{C}\alpha\Big)\oplus(\text{continuous spectrum})$$

where α runs through eigenforms under the Hecke operators and the Laplacian. Each α is either holomorphic, anti-holomorphic or exact. In either case, α corresponds to a test form $\widetilde{\phi}$ of weight 2, -2, or 0 on X, in the sense that

$$\alpha = \begin{cases} \widetilde{\phi}\, dz, & \text{if } \alpha \text{ is holomorphic,} \\ \widetilde{\phi}\, d\bar{z}, & \text{if } \alpha \text{ is anti-holomorphic,} \\ d\widetilde{\phi}, & \text{if } \alpha \text{ is exact.} \end{cases}$$

We may take integration $c \mapsto \int_{\widetilde{c}} \alpha$ to define a map

$$\pi_\alpha : \mathrm{Div}^0(X) \otimes \mathbb{C} \longrightarrow \begin{cases} A_\phi \otimes \mathbb{C}, & \text{if } \alpha \text{ is holomorphic,} \\ \mathbb{C}, & \text{if } \alpha \text{ is exact.} \end{cases}$$

Here $\widetilde{c}$ is an 1-cycle on X with boundary c. In this manner, we have

$$\pi_\alpha(P_{\bar{\chi}}) = \begin{cases} z_\phi, & \text{if } \alpha \text{ is holomorphic,} \\ (\phi, P_\chi), & \text{if } \alpha \text{ is exact.} \end{cases}$$

Thus we can think of $\mathbb{C}$ as an abelian variety corresponding to ϕ in the even case with Néron–Tate heights given by absolute value. This gives a complete analogue of the right-hand side of the Gross–Zagier formulas in the even and odd case.

On the other hand, in the even case, one can define L-series $L(s, \chi, \partial\phi/\partial z_v)$ by (4.1) where v is the only archimedean place where $k_v = 0$. It is not difficult to see that this L-series is essentially $(s - \frac{1}{2})L(s, \chi, \phi)$. Its derivative is given by $L(\frac{1}{2}, \chi, \phi)$. Thus we have an analogue of the left-hand side as well!

8. The Idea of Gross and Zagier

We now describe the original idea of Gross and Zagier in the proof of a central derivative formula (for Heegner points on $X_0(N)/\mathbb{Q}$ with squarefree discriminant D) in their famous *Inventiones* paper. For simplicity, we fix N, χ and assume that $\Sigma(N, K)$ is odd. For a holomorphic form ϕ of weight 2, we define its *Fourier coefficient* $\widehat{\phi}(a)$ *at an integral idele* a by the equation

$$W_\phi \begin{pmatrix} a y_\infty \delta^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \widehat{\phi}(a) W_\infty \begin{pmatrix} y_\infty & 0 \\ 0 & 1 \end{pmatrix}, \tag{8.1}$$

where $W_\infty = \prod_{v \nmid \infty} W_v$ is the standard Whittaker function for weight 2 defined in (4.7).

With the notation of Section 6, there is a cusp form Ψ of level N whose Fourier coefficient is given by

$$\widehat{\Psi}(a) = |a| \langle x, \mathrm{T}_a x \rangle. \tag{8.2}$$

This follows from two facts:

- the subalgebra $\mathbb{T}'$ of $\mathrm{Jac}(X)\otimes\mathbb{C}$ generated by Hecke operators T_a is a quotient of the subalgebra $\mathbb{T}$ in $\mathrm{End}(S_2(N))$ generated by Hecke operators T_a. Here $S_2(N)$ is the space of holomorphic cusp forms of weight $(2,\dots,2)$, level N, with trivial central character;
- any linear functional ℓ of $\mathbb{T}$ is represented by a cusp form $f \in S_2(N)$ in the sense that $|a|\ell(\mathrm{T}_a) = \widehat{f}(a)$.

(This form Ψ is not unique in general. But it is if we can normalize it to be a sum of newforms.)

It is then easy to see that

$$(\phi, \Psi) = \langle x_\phi, x_\phi\rangle(\phi, \phi). \tag{8.3}$$

Here (ϕ, Ψ) denotes the inner product as in Theorem 6.1, which is the same as $(\phi, \Psi)_{U_0(N)}$ in our notations in Introduction.

Thus, the question is reduced to showing that

$$L'(\tfrac{1}{2}, \chi, \phi) = \frac{2^{g+1}}{\sqrt{\mathrm{N}(D)}}(\phi, \Psi). \tag{8.4}$$

On the other hand, one can express $L(s, \chi, \phi)$ using a method of Rankin and Selberg:

$$L(s, \chi, \phi) = \frac{d_F^{1/2-s}}{|U_0(ND)|}\int_{\mathrm{PGL}_2(F)\backslash \mathrm{PGL}_2(\mathbb{A})} \phi(g)\theta(g)E(s, g)\, dg. \tag{8.5}$$

We need to explain various term in this integration.

First of all, θ is a theta series associated to χ. More precisely, θ is an eigen form of weight $(-1,\dots,-1)$, level D, and central character ω such that its local Whittaker functions $W_v(g)$ produces the local L-functions for χ:

$$|\delta_v|^{s-1/2}\int_{F_v^\times} W_v\begin{pmatrix} -a & 0\\ 0 & 1\end{pmatrix}|a|^{s-1/2}d^\times a = \prod_{w|v} L(s, \chi_w). \tag{8.6}$$

Here $L(s, \chi_w)$ is defined in (5.3). It follows that the automorphic representation $\Pi(\chi) := \Pi(\theta)$ generated by θ is irreducible with newform $\theta_\chi(g) = \theta(g\varepsilon)$, where $\varepsilon = \left(\begin{smallmatrix} -1 & \\ -0 & 1\end{smallmatrix}\right)$.

Secondly, $E(s, g) = E(s, g, \mathcal{F})$ is an Eisenstein series (2.6) for the quasi-characters $|\cdot|^{s-1/2}$ and $|\cdot|^{1/2-s}\omega$ with decomposition $\mathcal{F} = \bigotimes \mathcal{F}_v \in \mathcal{S}(\mathbb{A}^2)$. If v is a finite place, then

$$\mathcal{F}_v(x, y) = \begin{cases} 1, & \text{if } v \nmid c(\omega),\ |x| \le |ND|_v,\ |y| \le 1,\\ \omega_v^{-1}(y), & \text{if } v \mid c(\omega),\ |x| \le |ND|_v,\ |y| = 1,\\ 0, & \text{otherwise.}\end{cases} \tag{8.7}$$

If v is an infinite place, then

$$\mathcal{F}_v(x, y) = (\pm ix + y)e^{-\pi(x^2+y^2)} \tag{8.8}$$

where we take the + sign if $k_v = 2$ and the − sign if $k_v = 0$.

Taking a trace, we obtain a form of level N:

$$\Phi_s(g) = \mathrm{tr}_D \Phi_s(g) = \sum_{\gamma \in U_0(D)/U_0(ND)} \overline{d_F^{1/2-s}\theta(g\gamma)E(s, g\gamma)}. \tag{8.9}$$

This form has the property:

$$L(s, \chi, \phi) = (\phi, \Phi_s)_{U_0(N)}. \tag{8.10}$$

The idea of Gross and Zagier (in the odd case) is to compute the derivative $\Phi'_{1/2}$ of Φ_s with respect to s at $s = \frac{1}{2}$ and take a holomorphic projection to obtain a holomorphic form Φ so that

$$L'(\tfrac{1}{2}, \chi, \phi) = (\phi, \Phi)_{U_0(N)}. \tag{8.11}$$

(See [4] for a direct construction of the kernel using Poincaré series instead of the Rankin–Selberg method and holomorphic projection.) Now the problem is reduced to proving that

$$\Phi - \frac{2^{g+1}}{\sqrt{\mathrm{N}(D)}}\Psi$$

is an old form. In other words, we need to show that the Fourier coefficients of Φ are given by height pairings of Heegner points on $\mathrm{Jac}(X)$:

$$\widehat{\Phi}(a) = |a|\langle x, \mathrm{T}_a x\rangle \tag{8.12}$$

for any finite integral ideles a prime to ND.

One expects to prove the above equality by explicit computations for both sides respectively. These computations have been successfully carried out by Gross and Zagier [6] when $F = \mathbb{Q}$, D is squarefree, and $X(N, K) = X_0(N)$. The computation of Fourier coefficients of Φ is essentially straightforward and has been carried out for totally real fields [17]. For the computation of $\langle x, \mathrm{T}_a x\rangle$, Gross and Zagier represented the Hodge class ξ by cusps 0 and ∞ on $X_0(N)$:

$$\langle x, \mathrm{T}_a x\rangle = \big\langle P_\chi - h_\chi 0,\ \mathrm{T}_a(P_\chi) - h_{\chi,a}\infty\big\rangle \tag{8.13}$$

where h_χ and $h_{\chi,a}$ are integers that make both divisors to have degree 0. The right-hand side can be further decomposed into local height pairings by deforming self-intersections using Dedekind η-functions. These local height pairings can be finally computed by a modular interpretation in terms of deformation of formal groups.

When F is arbitrary (even when D is squarefree), the computation of heights has a lot of problems as there is no canonical representatives for the Hodge class, and no canonical modular form for self-intersections. In Sections 9–10, we will see how Arakelov theory been used to compute the heights.

When D is arbitrary (even when $F = \mathbb{Q}$), the computations of both kernels and heights for Theorem 6.1 seem impossible to carry out directly because of *singularities* in both analysis and geometry. Alternatively, we will actually prove a Gross–Zagier formula for level ND (Sections 11–16) and try to reduce the level by using continuous spectrum (Sections 17–19).

9. Calculus on Arithmetic Surfaces

The new idea in our *Annals* paper [17] is to use Arakelov theory to decompose the heights of Heegner points as locally as possible, and to show that the contribution of those terms that we don't know how to compute is *negligible*.

Let F be a number field. By an arithmetic surface over $\mathrm{Spec}\mathcal{O}_F$, we mean a projective and flat morphism $\mathcal{X} \longrightarrow \mathrm{Spec}\mathcal{O}_F$ such that that $\mathcal{X}$ is a regular scheme of dimension 2. Let $\widehat{\mathrm{Div}}(\mathcal{X})$ denote the group of *arithmetic divisors* on $\mathcal{X}$. Recall that an arithmetic divisor on $\mathcal{X}$ is a pair $\widehat{D} := (D, g)$ where D is a divisor on $\mathcal{X}$ and g is a function on

$$X(\mathbb{C}) = \coprod X_\tau(\mathbb{C})$$

with some logarithmic singularities on $|D|$. The form $-(\partial\bar{\partial}/\pi i)g$ on $X(\mathbb{C}) - |D|$ can be extended to a smooth form $c_1(\widehat{D})$ on $X(\mathbb{C})$ which is called the *curvature* of the divisor $\widehat{D}$. If f is a nonzero rational function on $\mathcal{X}$ then we can define the corresponding *principal arithmetic divisor* by

$$\widehat{\mathrm{div}} f = (\mathrm{div} f, -\log|f|). \tag{9.1}$$

An arithmetic divisor (D, g) is called *vertical* if D is supported in the special fibers, and *horizontal* if D does not have component supported in the special fiber).

The group of arithmetic divisors is denoted by $\widehat{\mathrm{Div}}(\mathcal{X})$ while the subgroup of principal divisor is denoted by $\widehat{\mathrm{Pr}}(\mathcal{X})$. The quotient $\widehat{\mathrm{Cl}}(\mathcal{X})$ of these two groups is called the *arithmetic divisor class group* which is actually isomorphic to the group $\widehat{\mathrm{Pic}}(\mathcal{X})$ of hermitian line bundles on $\mathcal{X}$. Recall that a hermitian line bundle on $\mathcal{X}$ is a pair $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, where $\mathcal{L}$ is a line bundle on $\mathcal{X}$ and $\|\cdot\|$ is hermitian metric on $\mathcal{L}(\mathbb{C})$ over $X(\mathbb{C})$. For a rational section ℓ of $\mathcal{L}$, we can define the corresponding divisor by

$$\widehat{\mathrm{div}}(\ell) = (\mathrm{div}\ell, -\log\|\ell\|). \tag{9.2}$$

It is easy to see that the divisor class of $\widehat{\mathrm{div}}(\ell)$ does not depend on the choice of ℓ. Thus one has a well defined map from $\widehat{\mathrm{Pic}}(\mathcal{X})$ to $\widehat{\mathrm{Cl}}(\mathcal{X})$. This map is actually an isomorphism.

Let $\widehat{D}_i = (D_i, g_i)$ $(i = 1, 2)$ be two arithmetic divisors on $\mathcal{X}$ with disjoint support in the generic fiber:

$$|D_{1F}| \cap |D_{2F}| = \varnothing.$$

Then one can define an *arithmetic intersection pairing*

$$\widehat{D}_1 \cdot \widehat{D}_2 = \sum_v (\widehat{D}_1 \cdot \widehat{D}_2)_v, \tag{9.3}$$

where v runs through the set of places of F. The intersection pairing only depends on the divisor class. It follows that we have a well defined pairing on $\widehat{\mathrm{Pic}}(\mathcal{X})$:

$$(\bar{\mathcal{L}}, \overline{\mathcal{M}}) \longrightarrow \widehat{c}_1(\bar{\mathcal{L}}) \cdot \widehat{c}_1(\overline{\mathcal{M}}) \in \mathbb{R}. \tag{9.4}$$

Let $V(\mathcal{X})$ be the group of *vertical metrized line bundles*: namely $\bar{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})$ with $\mathcal{L} \simeq \mathcal{O}_X$. Then we have an exact sequence

$$0 \longrightarrow V(\mathcal{X}) \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{X}) \longrightarrow \mathrm{Pic}(\mathcal{X}_F) \longrightarrow 0.$$

Define the group of *flat* bundles $\widehat{\mathrm{Pic}}^0(\mathcal{X})$ as the orthogonal complement of $V(\mathcal{X})$. Then we have an exact sequence

$$0 \longrightarrow \widehat{\mathrm{Pic}}(\mathcal{O}_F) \longrightarrow \widehat{\mathrm{Pic}}^0(\mathcal{X}) \longrightarrow \mathrm{Pic}^0(X_F) \longrightarrow 0.$$

The following formula, called the *Hodge index theorem*, gives a relation between intersection pairing and height pairing: for $\bar{\mathcal{L}}, \overline{\mathcal{M}} \in \widehat{\mathrm{Pic}}^0(\mathcal{X})$,

$$\langle \mathcal{L}_F, \mathcal{M}_F \rangle = -\widehat{c}_1(\bar{\mathcal{L}}) \cdot \widehat{c}_1(\overline{\mathcal{M}}), \tag{9.5}$$

where the left-hand side denotes the Néron–Tate height pairing on $\mathrm{Pic}^0(X) = \mathrm{Jac}(X)(F)$.

For X a curve over F, let $\widehat{\mathrm{Pic}}(X)$ denote the direct limit of $\widehat{\mathrm{Pic}}(\mathcal{X})$ over all models over X. Then the intersection pairing can be extended to $\widehat{\mathrm{Pic}}(X)$. Let $\bar{F}$ be an algebraic closure of F and let $\widehat{\mathrm{Pic}}(X_{\bar{F}})$ be the direct limit of $\mathrm{Pic}(X_L)$ for all finite extensions L of F, then the intersection pairing on $\widehat{\mathrm{Pic}}(X_L)$ times $[L : F]^{-1}$ can be extended to an intersection pairing on $\widehat{\mathrm{Pic}}(X_{\bar{F}})$.

Let $\bar{\mathcal{L}} \in \widehat{\mathrm{Pic}}(\mathcal{X})_{\mathbb{Q}}$ be a fixed class with degree 1 at the generic fiber. Let $x \in X(F)$ be a rational point and let $\bar{x}$ be the corresponding section $\mathcal{X}(\mathcal{O}_F)$. Then $\bar{x}$ can be extended to a unique element $\widehat{x} = (x+D, g)$ in $\widehat{\mathrm{Div}}(\mathcal{X})_{\mathbb{Q}}$ satisfying the following conditions:

- the bundle $\mathcal{O}(\widehat{x}) \otimes \bar{\mathcal{L}}^{-1}$ is flat;
- for any finite place v of F, the component D_v of D on the special fiber of $\mathcal{X}$ over v satisfies
$$D_v \cdot c_1(\mathcal{L}) = 0;$$
- for any infinite place v,
$$\int_{X_v(\mathbb{C})} g c_1(\bar{\mathcal{L}}) = 0.$$

We define now *Green's function* $g_v(x, y)$ on $(X(F) \times X(F) - \text{diagonal})$ by

$$g_v(x, y) = (\widehat{x} \cdot \widehat{y})_v / \log q_v, \tag{9.6}$$

where $\log q_v = 1$ or 2 if v is real or complex. It is easy to see that $g_v(x, y)$ is symmetric, does not depend on the model $\mathcal{X}$ of X, and is stable under base change. Thus we have a well-defined Green's function on $X(\bar{F})$ for each place v of F.

10. Decomposition of Heights

We now want to apply the general theory of the previous section to intersections of CM-points to Shimura curves $X = X(N, K)$ over a totally real field F as defined in Section 6. Recall that X has the form

$$X = G(F)_+ \backslash \mathcal{H} \times G(\mathbb{A}_f)/U(N, K) \cup \{\text{cusps}\} \tag{10.1}$$

which is a smooth and projective curve over F but may not be connected.

To define Green's function we need to extend the Hodge class ξ in $\mathrm{Pic}(X)_{\mathbb{Q}}$ to a class in $\widehat{\mathrm{Pic}}(X) \otimes \mathbb{Q}$. Notice that $\xi \in \mathrm{Pic}(X)_{\mathbb{Q}}$ is *Eisenstein* under the action of Hecke operators:

$$\mathrm{T}_a\xi = \sigma_1(a) \cdot \xi, \qquad \sigma_1(a) := \deg \mathrm{T}_a = \sum_{b|a} \mathrm{N}(b), \tag{10.2}$$

for any integral idele a prime to the level of X.

It is an interesting question to construct a class $\widehat{\xi}$ to extend ξ such that the above equation holds for $\widehat{\xi}$. But in [17], Corollary 4.3.3, we have constructed an extension $\widehat{\xi}$ of ξ such that

$$\mathrm{T}_a\widehat{\xi} = \sigma_1(a)\widehat{\xi} + \phi(a) \tag{10.3}$$

where $\phi(a) \in \widehat{\mathrm{Pic}}(F)$ is a σ_1-*derivation*, i.e., for any coprime a', a''

$$\phi(a'a'') = \sigma(a')\phi(a'') + \sigma(a'')\phi(a').$$

We have the following general definition.

Definition 10.1. Let $\mathbb{N}_F$ denote the semigroup of nonzero ideals of $\mathcal{O}_F$. For each $a \in \mathbb{N}_F$, let $|a|$ denote the *inverse norm* of a:

$$|a|^{-1} = \#\mathcal{O}_F/a.$$

For a fixed ideal M, let $\mathbb{N}_F(M)$ denote the sub-semigroup of ideals prime to M.

A function f on $\mathbb{N}_F(M)$ is called *quasi-multiplicative* if

$$f(a_1a_2) = f(a_1) \cdot f(a_2)$$

for all coprime $a_1, a_2 \in \mathbb{N}_F(M)$. For a quasi-multiplicative function f, let $\mathcal{D}(f)$ denote the set of all f-*derivations*, that is the set of all linear combinations

$$g = cf + h,$$

where c is a constant and h satisfies

$$h(a_1a_2) = h(a_1)f(a_2) + h(a_2)f(a_1)$$

for all $a_1, a_2 \in \mathbb{N}_F(M)$ with $(a_1, a_2) = 1$.

For a representation Π, the Fourier coefficient $\widehat{\Pi}(a)$ is defined to be

$$\widehat{\Pi}(a) := W_{\Pi,f}\begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $W_{\Pi,f}$ is the product of Whittaker newvectors at finite places. In other words, $\widehat{\Pi}(a)$ is defined such that the finite part of L-series has expansion

$$L_f(s,\Pi)=\sum \widehat{\Pi}(a)|a|^{s-1/2}.$$

Then $\widehat{\Pi}(a)$ is quasi-multiplicative.

We can now define Green's functions g_v on divisors on $X(\bar{F})$ which are disjoint at the generic fiber for each place v of F. Let's try to decompose the heights of our Heegner points. The linear functional

$$a\longrightarrow |a|\langle x,\mathrm{T}_a x\rangle$$

is now the Fourier coefficient of a cuspform Ψ of weight 2:

$$\widehat{\Psi}(a)=|a|\langle x,\mathrm{T}_a x\rangle. \tag{10.4}$$

In the following we want to express this height in terms of intersections modulo some Eisenstein series and theta series.

Let $\widehat{P}_\chi$ be the arithmetic closure of P_χ with respect to $\widehat{\xi}$. Then the Hodge index formula (9.5) gives

$$\begin{aligned}|a|\langle x,\mathrm{T}_a x\rangle &= -|a|\big(\widehat{P}_\chi-\deg(P_\chi)\widehat{\xi},\ \mathrm{T}_a\widehat{P}_\chi-\deg(\mathrm{T}_a\widehat{P}_\chi)\widehat{\xi}\big)\\ &= -|a|(\widehat{P}_\chi,\mathrm{T}_a\widehat{P}_\chi)+\widehat{E}(a),\end{aligned} \tag{10.5}$$

where $\widehat{E}$ is a certain derivation of Eisenstein series.

The divisor P_χ and $\mathrm{T}_a P_\chi$ have some common components. We want to compute its contribution in the intersections. Let $r_\chi(a)$ denote the Fourier coefficients of the theta series associated to χ:

$$r_\chi(a)=\sum_{b|a}\chi(b). \tag{10.6}$$

Then the divisor

$$\mathrm{T}_a^0 P_\chi := \mathrm{T}_a P_\chi - r_\chi(a)P_\chi \tag{10.7}$$

is disjoint with P_χ.

It follows that $\widehat{\Psi}(a)$ is essentially given by a sum of local intersections

$$-\frac{1}{[L:F]}\sum_v\sum_{\iota\in\mathrm{Gal}(H_c/F)} g_v(\mathrm{T}_a^0 P_\chi^\iota, P_\chi^\iota)|a|\log q_v$$

modulo some derivations of Eisenstein series, and theta series of weight 1. We can further simplify this sum by using the fact that the Galois action of $\mathrm{Gal}(K^{\mathrm{ab}}/F)$ is given by the composition of the class field theory map

$$\nu:\mathrm{Gal}(K^{\mathrm{ab}}/F)\longrightarrow N_T(F)\backslash N_T(\mathbb{A}_f),$$

and the left multiplication of the group $N_T(\mathbb{A}_f)$. Finally we obtain:

$$\widehat{\Psi}(a) = -|a| \sum_v g_v(P_\chi, \mathrm{T}_a^0 P_\chi) \log q_v \pmod{\mathcal{D}(\sigma_1) + \mathcal{D}(r_\chi)}. \tag{10.8}$$

Assume that D is squarefree, and that $X(N, K)$ has a regular canonical integral model over $\mathcal{O}_K$, which is the case when $\mathrm{ord}_v(N) = 1$ if v is not split in K or $v \mid 2$. We can use the theory of Gross on canonical or quasi-canonical lifting to compute $g_v(P_\chi, \mathrm{T}_a^0 P_\chi)$ and to prove that the functional

$$\widehat{\Phi} - \frac{2^{g+1}}{\sqrt{\mathrm{N}(D)}} \widehat{\Psi}$$

vanishes modulo derivations of Eisenstein series or theta series. It then follows that this functional is actually zero by the following lemma:

LEMMA 10.2. *Let $f_1, \dots, f_r$ be quasi-multiplicative functions on $\mathbb{N}_F(ND)$, and assume they are all distinct. Then the sum*

$$\mathcal{D}(f_1) + \mathcal{D}(f_2) + \cdots + \mathcal{D}(f_r)$$

is a direct sum.

Thus we can prove the Gross–Zagier formula in the case D is squarefree and $X(N, K)$ has regular canonical model over $\mathcal{O}_K$. This is the main result in our *Annals* paper [17].

11. Construction of the Kernels

From this section to the end, we want to explain how to prove the Gross–Zagier formula of Sections 6 and 7 for the general case. We will start with a kernel construction. As explained earlier, there is no good construction of kernels in level N. The best we can do is to construct some *nice kernel* in level ND in the sense that the Fourier coefficients are *symmetric* and easy to compute, and that the projection of this form in $\Pi(\phi)$ is *recognizable.*

Recall from (8.5) that we have an integral expression of the Rankin–Selberg convolution:

$$L(s, \chi, \phi) = \frac{|\delta|^{s-1/2}}{|U_0(ND)|} \int \phi(g)\theta(g)E(s, g)\, dg \tag{11.1}$$

To obtain a more symmetric kernel we have to apply Atkin–Lehner operators to $\theta(g)E(s, g)$. Let S be the set of finite places ramified in K. For each such set T of S, let h_T be an element in $\mathrm{GL}_2(\mathbb{A})$ which has component 1 outside T and has component

$$\begin{pmatrix} 0 & 1 \\ -t_v & 0 \end{pmatrix},$$

where t_v has the same order as $c(\omega_v)$ and such that $\omega_v(t_v) = 1$. Now one can show that

$$L(s, \chi, \phi) = \frac{\gamma_T(s)}{|U_0(ND)|} \int \phi(g)\theta(gh_T^{-1}\varepsilon)E(s, gh_T^{-1})\, dg \tag{11.2}$$

where γ_T is a certain exponential function of s. Finally we define the kernel function

$$\Theta(s, g) = 2^{-|S|} \sum_{T \subset S} \gamma_T(s)\theta(gh_T^{-1})E(s, gh_T^{-1}).$$

By our construction,

$$L(s, \chi, \phi) = (\phi, \bar{\Theta}(s, -))_{U_0(ND)}.$$

Now the functional equation of $L(s, \chi, \phi)$ follows from the following equation of the kernel function which can be proved by a careful analysis of Atkin–Lehner operators:

$$\Theta(s, g) = \varepsilon(s, \chi, \phi)\Theta(1 - s, g). \tag{11.3}$$

Assume that ϕ is cuspidal, then we may define the projection of $\bar{\Theta}$ in $\Pi(\phi)$ as a form $\varphi \in \Pi(\phi)$ such that

$$\int f\Theta\, dg = \int f\bar{\varphi}\, dg, \qquad \text{for all } f \in \Pi.$$

Since the kernel $\Theta(s, g)$ constructed above has level ND, its projection onto Π will have level DN and thus is a linear combination of the forms

$$\phi_a := \rho \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} \phi, \qquad (a \mid D). \tag{11.4}$$

Proposition 11.1. *The projection of $\bar{\Theta}(s, g)$ on Π is given by*

$$\frac{L(s, \chi, \phi)}{(\phi_s^\sharp, \phi_s^\sharp)_{U_0(ND)}} \cdot \phi_s^\sharp,$$

where $\phi_s^\sharp$ is the unique nonzero form in the space of $\Pi(\phi)$ of level ND satisfying the identities

$$(\phi_s^\sharp, \phi_a) = \nu^*(a)_s(\phi_s^\sharp, \phi_s^\sharp) \qquad (a \mid D),$$

where

$$\nu^*(a)_s = \prod_{v|S} \frac{|a|_v^{s-1/2} + |a|_v^{1/2-s}}{2} \begin{cases} \nu(a), & \textit{if } a|c(\omega), \\ 0, & \textit{otherwise.} \end{cases}$$

Write $\phi^\sharp = \phi^\sharp_{1/2}$ and call it the *quasi-newform* with respect to χ.

The function $\Theta(s, g)$ has a Fourier expansion

$$\Theta(s, g) = C(s, g) + \sum_{\alpha \in F^\times} W\left(s, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right). \tag{11.5}$$

Since $\Theta(s,g)$ is a linear combination of the form

$$\Theta(s,g)=\sum_i \theta_i(g)E_i(g)$$

with $\theta_i \in \Pi(\chi)$ and $E_i(g)\in \Pi(|\cdot|^{s-1/2},|\cdot|^{1/2-s}\omega)$, the constant and Whittaker function of $\Theta(s,g)$ can be expressed precisely in terms of Fourier expansions of θ_i and $E_i(g)$.

More precisely, let

$$\theta_i(g)=\sum_{\xi\in F} W_{\theta_i}(\xi,g)\,, \qquad E_i(g)=\sum_{\xi\in F} W_{E_i}(\xi,g)\,, \tag{11.6}$$

be Fourier expansions of θ_i and E_i respectively into characters $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\mapsto \psi(\xi x)$ on $N(\mathbb{A})$. Then

$$C(s,g)=\sum_{\xi\in F} C(s,\xi,g), \tag{11.7}$$

$$W(s,g)=\sum_{\xi\in F} W(s,\xi,g), \tag{11.8}$$

where

$$C(s,\xi,g)=\sum_i W_{\theta_i}(-\xi,g)\,W_{E_i}(\xi,g)\,, \tag{11.9}$$

$$W(s,\xi,g)=\sum_i W_{\theta_i}(1-\xi,g)\,W_{E_i}(\xi,g)\,. \tag{11.10}$$

The behavior of the degenerate term $C(s,\xi,g)$ can be understood very well. The computation shows that the complex conjugation of $\Theta(s,g)$ is finite at each cusp unless χ is a form $\nu\circ \mathrm{N}_{K/F}$ in which case, we need to remove two Eisenstein series in the space

$$E_1\in \Pi(\|\cdot\|^s,\|\cdot\|^{-s})\otimes\nu, \qquad E_2\in\Pi(\|\cdot\|^{1-s},\|\cdot\|^{s-1})\otimes\nu\omega.$$

We let $\Phi(s,g)$ denote $\bar\Theta(\frac12,g)$ if χ is not of form $\nu\circ \mathrm{N}_{K/F}$, or

$$\bar\Theta(s,g)-E_1-E_2$$

if it is. Then $\Phi(s,g)$ is a form with following growth:

$$\Phi\left(s,\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\right)=c_1(g)|a|^{s-1/2}+c_2(g)|a|^{1/2-s}+O_g(e^{-\varepsilon|a|})$$

where $c_1(g)$, $c_2(g)$, and O_g term are all smooth functions of g and s. It follows that the value or all derivatives of $\Phi(s,g)$ at $s=\frac12$ are L^2-forms.

With our very definition of $\Theta(s,g)$ in the last section, we are able to decompose the nondegenerate term:

$$W(s,\xi,g)=\bigotimes W_v(s,\xi_v,g_v). \tag{11.11}$$

An explicit computation gives the local functional equation

$$W_v(s,\xi_v,g_v)=\omega_v(1-\xi_v^{-1})(-1)^{\#\Sigma\cap\{v\}}W_v(1-s,\xi_v,g). \tag{11.12}$$

If Σ is even, we can compute the Fourier coefficients of $\Theta(\frac{1}{2},g)$ for

$$g=\begin{pmatrix} a\delta^{-1} & 0\\ 0 & 1\end{pmatrix} \tag{11.13}$$

very explicitly. The computation of the nondegenerate term $W(s,\xi,g)$ is reduced to local terms $W_v(s,\xi,g)$. By the functional equation, we need only consider those ξ such that

$$1-\xi^{-1}\in N(K_v^\times) \iff v\notin\Sigma.$$

The form Φ is holomorphic of weight 2 at infinite places where Π is of weight 2.

If Σ is odd, $\Theta(\frac{1}{2},g)$ vanishes by the functional equation. We want to compute its derivative $\Theta'(\frac{1}{2},g)$ at $s=\frac{1}{2}$. Let's now describe the central derivative for $W(s,\xi,g)$ for g of the form (11.13). Recall that $W(s,\xi,g)$ is a product of $W_v(s,\xi,g)$, and that $W_v(s,\xi,g)$ satisfies the functional equation (11.12). It follows that

$$W'(\tfrac{1}{2},g)=\sum_v W'(\tfrac{1}{2},g)_v, \tag{11.14}$$

where v runs through the places which are not split in K with

$$W'(\tfrac{1}{2},g)_v=\sum_\xi W^v(\tfrac{1}{2},\xi,g^v)\cdot W_v'(\tfrac{1}{2},\xi,g). \tag{11.15}$$

Here W^v is the product of W_ℓ over places $\ell\neq v$, W_v' is the derivative for the variable s, and $\xi\in F-\{0,1\}$ satisfies

$$1-\xi^{-1}\in N(K_w^\times) \iff w\notin {}_v\Sigma, \tag{11.16}$$

with ${}_v\Sigma$ given by

$${}_v\Sigma=\begin{cases}\Sigma\cup\{v\}, & \text{if } v\notin\Sigma,\\ \Sigma-\{v\}, & \text{if } v\in\Sigma.\end{cases}$$

All these terms can be computed explicitly. We need to find the holomorphic projection of $\bar\Theta'(\frac{1}{2},g)$. That is a holomorphic cusp form Φ of weight 2 such that $\bar\Theta'(\frac{1}{2},g)-\Phi$ is perpendicular to any holomorphic form.

Proposition 11.2. *With respect to the standard Whittaker function for holomorphic weight 2 forms, the a-th Fourier coefficients $\widehat\Phi(a)$ of the holomorphic projection Φ of $\bar\Theta'(\frac{1}{2},g)$ is a sum*

$$\widehat\Phi(a)=A(a)+B(a)+\sum_v\widehat\Phi_v(a)$$

where

$$A\in\mathcal{D}(\widehat\Pi(\chi)\otimes\alpha^{1/2}),$$
$$B\in\mathcal{D}\big(\widehat\Pi(\alpha^{1/2}\nu,\alpha^{-1/2}\nu)\big)+\mathcal{D}\big(\widehat\Pi\big(\alpha^{1/2}\nu\omega,\alpha^{-1/2}\nu\omega\big)\big),$$

and the sum is over places of F which are not split in K, with $\widehat{\Phi}_v(a)$ given by the following formulas:

(i) *If v is a finite place, then $\widehat{\Phi}_v(a)$ is a sum over $\xi \in F$ with $0 < \xi < 1$ of the following terms*:

$$(2i)^g |(1-\xi)\xi|_\infty^{1/2} \cdot \bar{W}_f^v \left(\tfrac{1}{2}, \xi, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \bar{W}_v' \left(\tfrac{1}{2}, \xi, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right).$$

(ii) *If v is an infinite place, then $\widehat{\Phi}_v(a)$ is the constant term at $s = 0$ of a sum over $\xi \in F$ such that $0 < \xi_w < 1$ for all infinite places $w \neq v$ and $\xi_v < 0$ of the following terms*:

$$(2i)^g |\xi(1-\xi)|_\infty^{1/2} \cdot \bar{W}_f \left(\tfrac{1}{2}, \xi, \begin{pmatrix} a\delta_f^{-1} & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot \int_1^\infty \frac{-dx}{x(1+|\xi|_v x)^{1+s}}.$$

12. Geometric Pairing

The key to prove the Gross–Zagier formula is to compare the Fourier coefficients of the kernel functions and the local heights of CM-points. These local heights are naturally grouped by definite quaternion algebras which are the endomorphism rings of the supersingular points in the reductions of modular or Shimura curves. Furthermore, the intersection of two CM-points at supersingular points is given by a *multiplicity function* depending only the *relative position* of these two CM-points. In this section, we would like to abstractly define this kind of pairing with respect to an arbitrary multiplicity function. We will describe the relative position of two CM-points by a certain parameter ξ which will relate the same parameter in the last section by a *local Gross–Zagier formula*.

Let G be an inner form of PGL_2 over F. This means that $G = B^\times/F^\times$ with B a quaternion algebra over F. Let K be a totally imaginary quadratic extension of F which is embedded into B. Let T denote the subgroup of G given by $K^\times/F^\times$. Then the set

$$C := T(F)\backslash G(\mathbb{A}_f) \tag{12.1}$$

is called the *set of CM-points*. This set admits a natural action by $T(\mathbb{A}_f)$ (resp. $G(\mathbb{A}_f)$) by left (resp. right) multiplication.

As in Sections 6 and 7, there is a map

$$\iota : C \longrightarrow M := G(F)_+\backslash \mathcal{H}^n \times G(\mathbb{A}_f) \tag{12.2}$$

from C to the Shimura variety defined by G sending the class of $g \in G(\mathbb{A}_f)$ to the class of (h_0, g), where $h_0 \in \mathcal{H}^n$ is fixed by T. This map is an embedding if G is not totally definite.

The set of CM-points has a topology induced from $G(\mathbb{A}_f)$ and has a unique $G(\mathbb{A}_f)$-invariant measure dx induced from the one on $G(\mathbb{A}_f)$. The space

$$\mathcal{S}(C) = \mathcal{S}(T(F)\backslash G(\mathbb{A}_f))$$

of locally constant functions with compact support is called the space of *CM-cycles* which admits a natural action by $T(\mathbb{A}_f) \times G(\mathbb{A}_f)$. There is a natural pairing between functions f on Shimura variety M and CM-cycles α by

$$(f, \alpha) = \int_C \bar{\alpha}(x) f(\iota x)\, dx. \tag{12.3}$$

Thus CM-cycles may serve as distributions or functionals on the space of functions on M. Of course this pairing is invariant under the action by $G(\mathbb{A}_f)$.

Since $T(F)\backslash T(\mathbb{A}_f)$ is compact, one has a natural decomposition

$$\mathcal{S}(C) = \oplus_\chi \mathcal{S}(\chi, C)$$

where the sum is over the characters of $T(F)\backslash T(\mathbb{A}_f)$. There is also a local decomposition for each character χ:

$$\mathcal{S}(\chi, C) = \bigotimes_v \mathcal{S}(\chi_v, G(F_v)). \tag{12.4}$$

In the following we will define a class of pairings on CM-cycles which are *geometric* since it appears naturally in the local intersection pairing of CM-points on Shimura curves . To do this, let's write CM-points in a slightly different way,

$$C = G(F)\backslash(G(F)/T(F)) \times G(\mathbb{A}_f), \tag{12.5}$$

then the topology and measure of C is still induced by those of $G(\mathbb{A}_f)$ and the *discrete* ones of $G(F)/T(F)$.

Let m be a *real valued* function on $G(F)$ which is $T(F)$-invariant and such that $m(\gamma) = m(\gamma^{-1})$. Then m can be extended to $G(F)/T(F) \times G(\mathbb{A}_f)$ such that

$$m(\gamma, g_f) = \begin{cases} m(\gamma), & \text{if } g_f = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{12.6}$$

We now have a kernel function

$$k(x, y) = \sum_{\gamma \in G(F)} m(x^{-1}\gamma y) \tag{12.7}$$

on $C \times C$. Then we can define a pairing on $\mathcal{S}(C)$ by

$$\langle \alpha, \beta \rangle = \int_{C^2} \alpha(x) k(x, y) \bar{\beta}(y)\, dx\, dy := \lim_{U \to 1} \int_{C^2} \alpha(x) k_U(x, y) \bar{\beta}(y)\, dx\, dy, \tag{12.8}$$

where U runs through the open subgroup of $G(\mathbb{A}_f)$ and

$$k_U(x, y) = \mathrm{vol}(U)^{-2} \int_{U^2} k(xu, yv)\, du\, dv.$$

This pairing is called a *geometric pairing with multiplicity function* m. For two function α and β in $\mathcal{S}(T(F)\backslash G(\mathbb{A}_f))$, one has

$$\langle \alpha, \beta \rangle = \sum_{\gamma \in T(F)\backslash G(F)/T(F)} m(\gamma) \langle \alpha, \beta \rangle_\gamma \tag{12.9}$$

where

$$\langle \alpha, \beta \rangle_\gamma = \int_{T_\gamma(F)\backslash G(\mathbb{A}_f)} \alpha(\gamma y)\bar{\beta}(y)\, dy, \tag{12.10}$$

and where

$$T_\gamma := \gamma^{-1} T \gamma \cap T = \begin{cases} T & \text{if } \gamma \in N_T, \\ 1 & \text{otherwise,} \end{cases} \tag{12.11}$$

and where N_T is the normalizer of T in G. The integral $\langle \alpha, \beta \rangle_\gamma$ is called the *linking number* of α and β at γ.

Since both α and β are invariant under left-translation by $T(F)$, the linking number at γ depends only on the class of γ in $T(F)\backslash G(F)/T(F)$. Let's define a parameterization of this set by by writing $B = K + K\varepsilon$ where $\varepsilon \in B$ is an element such that $\varepsilon^2 \in F^\times$ and $\varepsilon x = \bar{x}\varepsilon$. Then the function

$$\xi(a + b\varepsilon) = \frac{\mathrm{N}(b\varepsilon)}{\mathrm{N}(a + b\varepsilon)}$$

defines an embedding

$$\xi : T(F)\backslash G(F)/T(F) \longrightarrow F. \tag{12.12}$$

Write

$$\langle \alpha, \beta \rangle_\gamma = \langle \alpha, \beta \rangle_\xi. \tag{12.13}$$

Notice that $\xi(\gamma) = 0$ (resp. 1) if and only if $\xi \in T$ (resp. $\xi \in N_T - T$). The image of $G(F) - N_T$ is the set of $\xi \in F$ such that $\xi \neq 0, 1$ and where for any place v of F,

$$1 - \xi^{-1} \in \begin{cases} \mathrm{N}(K_v^\times), & \text{if } B_v \text{ is split,} \\ F_v^\times - \mathrm{N}(K_v^\times) & \text{if } B_v \text{ is not split.} \end{cases} \tag{12.14}$$

We may write $m(\xi)$ for $m(\gamma)$ when $\xi(\gamma) = \xi$, and extend $m(\xi)$ to all F by setting $m(\xi) = 0$ if ξ is not in the image of the map in (12.12). Then

$$\langle \alpha, \beta \rangle = \sum_{\xi \in F} m(\xi) \langle \alpha, \beta \rangle_\xi. \tag{12.15}$$

Let χ be a character of $T(F)\backslash T(\mathbb{A}_f)$. The linking number is easy to compute if $\xi = 0$ or 1. The difficult problem is to compute $\langle \alpha, \beta \rangle_\xi$ when $\xi \neq 0, 1$.

If both α and β are decomposable,

$$\alpha = \bigotimes \alpha_v, \qquad \beta = \bigotimes \beta_v,$$

then we have a decomposition of linking numbers into local *linking numbers* when $\xi \neq 0, 1$:

$$\langle \alpha, \beta \rangle_\xi = \prod \langle \alpha_v, \beta_v \rangle_\xi, \tag{12.16}$$

where

$$\langle \alpha_v, \beta_v \rangle_\xi = \int_{G(F_v)} \alpha_v(\gamma y)\bar{\beta}_v(y)\, dy. \tag{12.17}$$

Notice that when $\gamma \notin N_T$, these local linking numbers depend on the choice of γ in its class in $T(F)\backslash G(F)/T(F)$ while their product does not. This problem can be solved by taking γ to be a *trace-free element* in its class which is unique up to conjugation by $T(F)$.

Notation. For a compact open compact subgroup U of $G(\mathbb{A}_f)$ (or $G(F_v)$) and two CM-cycles α and β, we write $\langle \alpha, \beta\rangle_U$ and $\langle \alpha, \beta\rangle_{\xi,U}$ for

$$\langle \alpha, \beta\rangle_U = |U|^{-1}\langle \alpha, \beta\rangle, \qquad \langle \alpha, \beta\rangle_{\xi,U} = |U|^{-1}\langle \alpha, \beta\rangle_\xi.$$

Similarly, for a CM-cycle α and a function f on M, we write

$$(f, \alpha)_U = |U|^{-1}(f, \alpha).$$

13. Local Gross–Zagier Formula

In this section, we would like to compute the linking numbers for some special CM-cycles and then compare with Fourier coefficients of the kernel functions. The construction of CM-cycles is actually quite simple and is given as follows.

We will fix one order A of B such that, for each finite place v,

$$A_v = \mathcal{O}_{K,v} + \mathcal{O}_{K,v}\lambda_v c(\chi_v), \tag{13.1}$$

where $\lambda_v \in B_v^\times$ is such that $\lambda_v x = \bar{x}\lambda_v$ for all $x \in K$, and $\mathrm{ord}(\det \lambda_v) = \mathrm{ord}_v(N)$.

Let Δ be a subgroup of $G(\mathbb{A}_f)$ generated by images of $\widehat{A}^\times$ and $K_v^\times$ for v ramified in K:

$$\Delta = \prod_{v \nmid c(\omega_v)} A_v^\times F_v^\times / F_v^\times \cdot \prod_{v \mid c(\omega_v)} A_v^\times K_v^\times / F_v^\times. \tag{13.2}$$

The character can be naturally extended to a character of Δ. The CM-cycle we need is defined by the function

$$\eta = \prod \eta_v, \tag{13.3}$$

with η_v supported on $T(F_v)\cdot\Delta_v$ and such that

$$\eta_v(tu) = \chi_v(t)\chi_v(u), \qquad t \in T(F_v), u \in \Delta_v. \tag{13.4}$$

Take $a \in \mathbb{A}_f^\times$ integral and prime to ND. We would like to compute the pairing $\langle \mathrm{T}_a\eta, \eta\rangle$. The Hecke operator here is defined as

$$\mathrm{T}_a\eta = \prod_{v \mid a} \mathrm{T}_{a_v}\eta_v, \qquad \mathrm{T}_{a_v}\eta_v(x) = \int_{H(a_v)} \eta_v(xg)\, dg, \tag{13.5}$$

where

$$H(a_v) := \{g \in M_2(\mathcal{O}_v) : |\det g| = |a_v|\}, \tag{13.6}$$

and dg is a measure such that $\mathrm{GL}_2(\mathcal{O}_v)$ has volume 1. Then we have the decomposition

$$\langle \mathrm{T}_a\eta, \eta\rangle_\Delta = \mathrm{vol}(T(F)\backslash T(\mathbb{A}_f)\Delta)\bigl(m(0)\mathrm{T}_a\eta(e) + m(1)\mathrm{T}_a\eta(\varepsilon)\delta_{\chi^2=1}\bigr) + \sum_{\xi\neq 0,1} m(\xi)\prod_v \langle \mathrm{T}_a\eta_v, \eta_v\rangle_{\xi,\Delta_v}, \quad (13.7)$$

where $\varepsilon \in N_T(F) - T(F)$.

Let v be a fixed finite place of F. We want to compute all terms in the right-hand side involving η_v. Notice that we have extended the definition to all $\xi \in F - \{0,1\}$ by insisting that $\langle \mathrm{T}_a\eta_v, \eta_v\rangle_{\xi,\Delta_v} = 0$ when ξ is not in the image of (12.12).

The computation of degenerate terms is easy. The nondegenerate term is given by the following local Gross–Zagier formula:

PROPOSITION 13.1. *Let* $g = \begin{pmatrix} a\delta_v^{-1} & 0 \\ 0 & 1\end{pmatrix}$. *Then*

$$\bar W_v\bigl(\tfrac12, \xi, g\bigr) = |c(\omega_v)|^{1/2}\cdot \varepsilon(\omega_v,\psi_v)\chi_v(u)\cdot |(1-\xi)\xi|_v^{1/2}|a|\cdot\langle \mathrm{T}_a\eta_v,\eta_v\rangle_{\xi,\Delta_v},$$

where u is any trace-free element in $K^\times$.

COROLLARY 13.2. *Let $\langle\,\cdot\,,\cdot\,\rangle$ be the geometric pairing on the CM-cycle with multiplicity function m on F such that $m(\xi) = 0$ if ξ is not in the image of (12.12). Assume that $\delta_v = 1$ for $v \mid \infty$. Then there are constants c_1, c_2 such that for an integral idele a prime to ND,*

$$|c(\omega)|^{1/2}|a|\langle \mathrm{T}_a\eta,\eta\rangle_\Delta = (c_1 m(0) + c_1 m(1))|a|^{1/2}W_f(g) + i^{[F:\mathbb{Q}]}\sum_{\xi\in F-\{0,1\}} |\xi(1-\xi)|_\infty^{1/2}\bar W_f(\tfrac12,\xi,g)m(\xi),$$

where $g = \begin{pmatrix} a\delta_f^{-1} & 0\\ 0 & 1\end{pmatrix}$.

Remarks. If the kernel Θ had level N as in the original approach in Section 8, the CM-cycle we should consider is P_χ as in Section 6 corresponding to the function ζ supported in $T(\mathbb{A}_f)i_cU(N,K)$ such that $\zeta(ti_cu) = \chi(t)$. The computation of linking numbers for this divisor seems very difficult!

Our local formula is the key to the proof of the Gross–Zagier formula. But the formula is only proved under the condition that $c(\chi)$, $c(\omega)$, and N are coprime to each other. One may still expect that this local formula is still true in the general case considered by Waldspurger but with more than one term on the right-hand side. The main problem is to construct elements in

$$\mathcal{S}(\chi_v, G(F_v)) \quad\text{and}\quad \mathcal{W}(\Pi(\chi_v),\psi_v)\otimes\mathcal{W}(\Pi(|\cdot|^{s-1/2}, |\cdot|^{1/2-s}\omega).$$

More precisely, we need to find an element

$$W = \sum W_{i1}\otimes W_{2i} \in \mathcal{W}(\Pi(\chi_v),\psi_v)\otimes\mathcal{W}(\Pi(|\cdot|^{s-1/2}, |\cdot|^{1/2-s}\omega)$$

satisfying the following properties:

- Let Φ_i be the element in $\mathcal{S}(F_v^2)$ such that $W_{2i} = W_{\Phi_i}$. Then for any representation Π_v of $\mathrm{GL}_2(F_v)$ with a newform $W_v \in \mathcal{W}(\Pi, \psi_v)$,

$$L(s, \chi_v, \phi_v) = \sum \Psi(s, W_v, W_{1i}, \Phi_i)$$

  with notation in [16, § 2.5].
- Let's define

$$W(s, \xi, g) = \sum_i W_{1i}\left(\begin{pmatrix} 1-\xi & 0 \\ 0 & 1 \end{pmatrix} g\right) W_{2i}\left(\begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

  Then $W(s, \xi, g)$ satisfies the following functional equation

$$W(s, \xi, g) = \omega_v(1-\xi^{-1})\varepsilon_v(\Pi_v \otimes \chi_v, s) W(1-s, \xi, g).$$

The next step is to find elements $q_j \in \mathcal{S}(\chi_v, G(F_v))$ such that the above local Gross–Zagier formula is true with $\langle \mathrm{T}_a \eta_v, \eta_v \rangle_\xi$ replaced by

$$\sum_j \langle \mathrm{T}_a q_j, q_j \rangle_\xi .$$

We may even assume that $a = 1$ in time. Thus, what really varies is the parameter ξ.

14. Gross–Zagier Formula in Level ND

The study of kernel functions, geometric pairing, and local Gross–Zagier formula in the last three sections suggests that it may be easier to prove a Gross–Zagier formula in level ND instead of level N directly. This is the main result in our *Asian Journal* paper [16].

Let's start with the case where Σ is odd. Thus, we are in the situation of Section 6. Let U be any open compact subgroup of Δ over which χ is trivial. Let X_U be corresponding Shimura curve or its compactification over F.

Recall that the CM-points corresponding to K on $X_U(\mathbb{C})$ form a set

$$C_U := G(F)_+ \backslash G(F)_+ \cdot h_0 \times G(\mathbb{A}_f)/U = T(F)\backslash G(\mathbb{A}_f)/U,$$

where $h_0 \in \mathcal{H}^+$ is the unique fixed point of the torus points $K^\times/F^\times$. Let η_U be a divisor on X_U with complex coefficient defined by

$$\eta_U = \sum_{x \in C_U} \eta(x)[x].$$

The Heegner class we want now is the class difference

$$y := [\eta_U - \deg(\eta_U)\xi] \in \mathrm{Jac}(X_U)(H_c) \otimes \mathbb{C}.$$

Notice that this class has character χ_Δ under the action by Δ on $\mathrm{Jac}(H_c)$. Let y_ϕ denote the ϕ-typical component of y. The main theorem in our *Asian Journal* paper [16] is this:

THEOREM 14.1. *Let $\phi^\sharp$ be the quasi-newform as in Section* 11. *Then*

$$\widehat{\phi^\sharp}(1)L'(1,\chi,\phi) = 2^{g+1} d_{K/F}^{-1/2} \cdot \|\phi^\sharp\|^2_{U_0(ND)} \cdot \|y_\phi\|^2_\Delta,$$

where

- $d_{K/F}$ *is the relative discriminant of* K *over* F;
- $\|\phi^\sharp\|^2_{U_0(ND)}$ *is the* L^2*-norm with respect to the Haar measure* dg *normalized* (*as in the Introduction*) *so that* $\mathrm{vol}(U_0(ND)) = 1$;
- $\|y_\phi\|_\Delta$ *is the Néron–Tate height of* y_ϕ *on* X_U *times* $[\Delta : U]^{-1}$, *which is independent of the choice of* U;
- $\widehat{\phi^\sharp}(1)$ *is the first Fourier coefficient of* $\phi^\sharp$ *as defined in* (8.1).

We now move to the situation in Section 7 where ϕ has possible nonholomorphic components, but we assume that the sign of the functional equation of $L(s,\chi,\phi)$ is $+1$, or equivalently, Σ is even. We have the variety X_U which is defined in the same way as in the odd case. Then we have a unique line of cuspidal functions ϕ_χ on X_U with the following properties:

- ϕ_χ has character χ_Δ under the action of Δ;
- for each finite place v not dividing $N \cdot D$, ϕ_χ is the eigenform for Hecke operators T_v with the same eigenvalues as ϕ.

We call ϕ_χ *a toric newform* associated to ϕ. See [16, § 2.3] for more details.

The CM-points on X_U, associated to the embedding $K \longrightarrow B$, form the infinite set

$$C_U := G(F)_+ \backslash G(F)_+ h_0 \times G(\mathbb{A}_f)/U \simeq H\backslash G(\mathbb{A}_f)/U,$$

where h_0 is a point in $\mathcal{H}^n$ fixed by T and $H \subset G$ is the stabilizer of z in G. Notice that H is either isomorphic to T if $n \neq 0$ or $H = G$ if $n = 0$. In any case there is a finite map

$$\iota : C_U = T(F)\backslash G(\mathbb{A}_f)/U \longrightarrow M_U.$$

The Gross–Zagier formula for central value in level ND is the following:

THEOREM 14.2. *Let ϕ_χ be a toric newform such that $\|\phi_\chi\|_\Delta = 1$. Then*

$$\widehat{\phi^\sharp}(1)L(1,\chi,\phi) = 2^{g+n} d_{K/F}^{-1/2} \cdot \|\phi^\sharp\|^2_{U_0(ND)} \cdot |(\phi,\eta)_\Delta|^2,$$

where $\widehat{\phi^\sharp}(1)$ *is the first Fourier coefficient by the same formula as* (8.1) *with respect the standard Whittaker function defined in* (4.7) *and* (4.8), *and where*

$$(\phi_\chi,\eta)_\Delta := [\Delta : U]^{-1} \sum_{x\in C_U} \bar\eta(x)\phi_\chi(\iota(x)).$$

15. Green's Functions of Heegner Points

In this section we explain the proof of the central derivative formula for level ND stated in the last section. Just as explained in Section 8, the question is reduced to a comparison of the Fourier coefficients of the kernel and heights of CM-points. We need to show that, up to a constant and modulo some *negligible forms*, the newform Ψ with Fourier coefficient

$$\widehat{\Psi}(a) := |a|\langle \eta, \mathrm{T}_a \eta\rangle \tag{15.1}$$

is equal to the holomorphic cusp form Φ defined in Section 11 which represents the derivative of Rankin L-function $L'(\frac{1}{2}, \chi, \phi)$. Thus we need to show that the functional $\widehat{\Psi}$ on $\mathbb{N}_F(ND)$ is equal to the Fourier coefficient $\widehat{\Phi}(a)$.

As in Section 9, we would like to decompose the height pairing to Green's functions. It is more convenient work on the tower of Shimura curves than a single one. Let's first try to extend the theory of heights to the projective limit X_∞ of X_U. Let $\widehat{\mathrm{Pic}}(X_\infty)$ denote the direct limit of $\widehat{\mathrm{Pic}}(X_U)$ with respect to the pull-back maps. Then the intersection pairing can be extended to $\widehat{\mathrm{Pic}}(X_\infty)$ if we multiply the pairings on $\mathrm{Pic}(X_U)$ by the scale $\mathrm{vol}(U)$. Of course, this pairing depends on the choice of measure dg on $G(\mathbb{A})$ as in Introduction. For some fixed open compact subgroup U of $G(\mathbb{A}_f)$, we write $\langle z_1, z_2\rangle_U$ for the measure

$$\langle z_1, z_2\rangle_U = |U|^{-1}\langle z_1, z_2\rangle = [U : U']^{-1}\langle z_1, z_2\rangle_{X_{U'}}.$$

where z_1, z_2 are certain elements in $\widehat{\mathrm{Pic}}(X_\infty)$ realized on $X_{U'}$ for some $U' \subset U$. So defined pairing will depends only on the choice of U and gives the exact pairing on X_U.

Similarly, we can modify the local intersection pairing and extend the height pairing to $\mathrm{Jac}(X_\infty) = \mathrm{Pic}^0(X_\infty)$, which is the direct limit of $\mathrm{Pic}^0(X_U)$ where $\mathrm{Pic}^0(X_U)$ is the subgroup of $\mathrm{Pic}(X_U)$ of classes whose degrees are 0 on each connected component.

We can now define Green's functions g_v on divisors on $X_\infty(\bar{F})$ which are disjoint at the generic fiber for each place v of F by multiplying the Green's functions on X_U by $\mathrm{vol}(U)$. Notice that for two CM-divisors A and B on X_U with disjoint support represented by two functions α and β on $T(F)\backslash G(\mathbb{A}_f)$, the Green's function at a place v depends only on α and β. Thus, we may simply denote it as

$$g_v(A, B)_{U_v} = g_v(\alpha, \beta)_{U_v}.$$

Recall that η is a divisor on X_∞ defined by (13.3). As in Section 10, with P_χ replaced by η_U, we have

$$\widehat{\Psi} = \sum_v \widehat{\Psi}_v \mod \mathcal{D}(\sigma_1) + \mathcal{D}(r_\chi), \tag{15.2}$$

where v runs through the set of places of F, and

$$\widehat{\Psi}_v(a) := -|a| g_v(\eta, \mathrm{T}^0_a \eta)_{\Delta_v} \log q_v. \tag{15.3}$$

Thus, it suffices to compare these local terms for each place v of F. We need only consider v which is not split in K, since $\widehat{\Phi}_v = 0$ and $\widehat{\Psi}_v$ is a finite sum of derivations of Eisenstein series when v is split in K.

Our main tool is the local Gross–Zagier formula in Section 13 for quaternion algebra ${}_vB$ with the ramification set

$$ {}_v\Sigma = \begin{cases} \Sigma \cup \{v\}, & \text{if } v \notin \Sigma, \\ \Sigma - \{v\}, & \text{if } v \in \Sigma. \end{cases} \tag{15.4} $$

Let ${}_vG$ denote the algebraic group ${}_vB^\times/F^\times$.

Lemma 15.1. *For v an infinite place*,

$$ \widehat{\Phi}_v(a) = 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a). $$

The idea of proof is to use the local Gross–Zagier formula to write both sides as the constant terms at $s = 0$ of two geometric pairings of divisors $\mathrm{T}_a\eta$ and η with two multiplicity functions:

$$ m_s^v(\xi) = \int_1^\infty \frac{dx}{x(1+|\xi|_v x)^{1+s}}, \qquad 2Q_s(\xi) = \int_1^\infty \frac{(1-x)^s\,dx}{x^{1+s}(1+|\xi|_v x)^{1+s}}. $$

It follows that the difference of two sides will be the constant term of a geometric pairing on $T(F)\backslash{}_vG(\mathbb{A}_f)$ with multiplicity function

$$ m_s^v - 2Q_s $$

which has no singularity and converges to 0 as $s \longrightarrow 0$. Notice that the Legendre function Q_s appears here because an explicit construction of Green's function at archimedean place.

We now consider unramified cases.

Lemma 15.2. *Let v be a finite place prime to ND. Then there is a constant c such that*

$$ \widehat{\Phi}_v(a) - 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a) = c\log|a|_v \cdot |a|^{1/2}\widehat{\Pi}(\chi)(a). $$

The proof is similar to the archimedean case. Write $a = \pi_v^n a'$ ($\pi_v \nmid a'$). Since the Shimura curve and CM-points all have good reduction, using Gross' theory of canonical lifting, we can show that $\widehat{\Psi}_v(a)$ is the geometric pairing of $\mathrm{T}_{a'}\eta$ and η on on $T(F)\backslash{}_vG(\mathbb{A}_f)$ with multiplicity function

$$ m_n(\xi) = \begin{cases} \frac{1}{2}\mathrm{ord}_v(\xi\pi_v^{1+n}), & \text{if } \xi \neq 0 \text{ and } \mathrm{ord}_v(\xi\pi_v^n) \text{ is odd}, \\ n/2, & \text{if } \xi \neq 0 \text{ and } n \text{ is even}, \\ 0, & \text{otherwise}. \end{cases} $$

On other hand, by using the local Gross–Zagier formula, we may also write Φ, up to a multiple of $|a|^{1/2}\widehat{\Pi}(\chi)\log|a|_v$, as a geometric pairing $\langle \mathrm{T}_{a'}\eta, \eta\rangle$ with multiplicity

$$ -2m_n(\xi)\log q_v. $$

It remains to treat the case where v is place dividing ND. In this case we will not be able to prove the identity as in the archimedean case, or in the unramified case, since there is no explicit regular model of Shimura curves we can use. But we can classify these contributions:

LEMMA 15.3. *For v a finite place dividing ND, we have*

$$\widehat{\Phi}_v(a) - 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a) = c\,|a|^{1/2}\widehat{\Pi}(\chi)(a) +_v \widehat{f},$$

where c is a constant, and ${}_v\widehat{f}$ is a form on ${}_vG(F)\backslash {}_vG(\mathbb{A}_f)$. Moreover, the function ${}_vf$ has character χ under the right translation by $K_v^\times$.

Using the local Gross–Zagier formula, we still can show that $\widehat{\Phi}_v$ is equal the geometric local pairing

$$2^g|c(\omega))|^{1/2}|a|\langle\eta, \mathrm{T}_a\eta\rangle$$

for a multiplicity function $m(g)$ on ${}_vG(F)$ with singularity

$$\log|\xi|_v.$$

On other hand, it is not difficult to show that Green's function

$$\widehat{\Psi}_v(a) = -g_v(\eta, \mathrm{T}_a^0\eta)\log q_v$$

is also a geometric pairing for a multiplicity function with singularity

$$\frac{1}{2}\log|\xi|_v.$$

(This is equivalent to saying that $\xi^{1/2}$ is a local parameter in the v-adic space of CM-points). Thus the difference

$$\widehat{\Phi}_v(a) - 2^{g+1}|c(\omega)|^{1/2}\widehat{\Psi}_v(a)$$

is a geometric pairing without *singularity*. In other words, it is given by

$$\int_{[T(F)\backslash {}_vG(\mathbb{A}_f)]^2} \eta(x)k(x,y)\mathrm{T}_a\eta(y)\,dx\,dy,$$

for $k(x,y)$ a locally constant function of $({}_vG(F)\backslash {}_vG(\mathbb{A}_f))^2$ which has a decomposition

$$k(x,y) = \sum_i c_i(x)f_i(y)$$

into eigenfunctions f_j for Hecke operators on ${}_vG(F)\backslash {}_vG(\mathbb{A}_f)$. It follows that the difference of two sides in the lemma is given by

$$\sum_i \lambda_i(a)\int_{T(F)\backslash {}_vG(\mathbb{A}_f)} \eta(x)c_i(x)\,dx \cdot \int_{T(F)\backslash {}_vG(\mathbb{A}_f)} f_i(y)\bar{\eta}(y)\,dy,$$

where $\lambda_i(a)$ is the eigenvalue of T_a for f_i. Thus, we may take

$${}_vf = \sum_i \int_{T(F)\backslash {}_vG(\mathbb{A}_f)} \eta(x)c_i(x)\,dx \cdot \int_{T(F)\backslash {}_vG(\mathbb{A}_f)} f_i(y)\bar{\eta}(y)\,dy.$$

In summary, at this stage we have shown that the quasi-newform

$$\Phi - 2^{g+1}|c(\omega)|^{1/2}\Psi$$

has Fourier coefficients which are a sum of the following terms:

- derivations A of Eisenstein series,
- derivations B of theta series $\Pi(\chi)\otimes\alpha^{1/2}$,
- functions ${}_vf$ appearing in ${}_vG(F)\backslash{}_vG(\mathbb{A}_f)$ with character χ under the right translation of $K_v^\times$, where v are places dividing DN.

By the linear independence of Fourier coefficients of derivations of forms in Lemma 10.3, we conclude that $A = B = 0$.

Let Π be the representation generated by the form ϕ. All the projections of ${}_vf$'s in Π must vanish, by local results of Waldspurger and Gross–Prasad. See [16, § 2.3].

In summary we have shown that $\Phi - 2^{g+1}|c(\omega)|^{1/2}\Psi$ is an old form. By Proposition 11.1, the projection of this difference on $\Pi(\phi)$ is

$$\frac{L'(\frac{1}{2},\chi,\phi)}{(\phi^\sharp,\phi^\sharp)_{U_0(ND)}}\cdot\phi^\sharp - 2^{g+1}|c(\omega)|^{1/2}\cdot\langle y_\phi, y_\phi\rangle_\Delta\cdot\phi.$$

This is again an old form and has vanishing first Fourier coefficient. Theorem 14.1 follows by taking the first Fourier coefficient.

16. Spectral Decomposition

In this section we want to explain the proof for the central value formula, Theorem 14.2. The idea is copied from the odd case. Thus, we need to define a *height pairing of CM-cycles*. Since there is no natural arithmetic and geometric setting for heights corresponding to nonholomorphic forms, we would like to use the local Gross–Zagier formula to *suggest* a definition of height. Indeed, by corollary 13.2, modulo some Einstein series of type $\Pi(\|\cdot\|^{1/2},\|\cdot\|^{-1/2})\otimes\eta$ with η quadratic, the kernel $\Phi(g) := \Phi(\frac{1}{2}, g)$ has a Whittaker function satisfying

$$W_\Phi\left(g_\infty\cdot\begin{pmatrix} a\delta^{-1} & 0\\ 0 & 1\end{pmatrix}\right) = |c(\omega)|^{1/2}|a|\langle \mathrm{T}_a\eta,\eta\rangle_\Delta(g_\infty), \tag{16.1}$$

where $g_\infty\in\mathrm{GL}_2(F_\infty)$ is viewed as a parameter, and a is a finite integral idele which prime to ND, and the pairing $\langle\cdot,\cdot\rangle_\Delta$ is defined by the multiplicity function

$$m(\xi, g_\infty) := \prod_{v|\infty} m_v(\xi, g_v), \tag{16.2}$$

with each $m_v(\xi, g)$ the Whittaker function of weight k_v whose value at $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ is

$$m_v\left(\xi, \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{cases} 4|a|e^{-2\pi a}, & \text{if } 1 \geq \xi \geq 0,\ a > 0,\ k_v = 2, \\ 4|a|e^{2\pi a(2\xi-1)}, & \text{if } a\xi \leq \min(0, a),\ k_v = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{16.3}$$

This suggests a definition of the height pairing for CM-cycles by the above multiplicity function. This height pairing is no long valued in numbers but in Whittaker functions. In the case where all $k_v = 2$, then this Whittaker function is twice the standard one. Thus, we can get a pairing with values in $\mathbb{C}$. We would like to have a good understanding of decomposition of this height pairing according to eigenforms on X_U. By (12.8), we need only decompose the kernel k_U defined in (12.7). This decomposition is actually very simple: As Whittaker functions on $\mathrm{GL}_2(F_\infty)$,

$$\begin{aligned} k_U(x,y)(g_\infty) = 2^{[F:\mathbb{Q}]+n} \sum_{\phi_i} W_i(g_\infty) \cdot \phi_i(x)\bar{\phi}_i(y) \\ + 2^{[F:\mathbb{Q}]+n} \int_{\mathfrak{M}} W_{\mathfrak{m}}(g_\infty) E_{\mathfrak{m}}(x) \bar{E}_{\mathfrak{m}}(y)\, d\mathfrak{m}, \end{aligned} \tag{16.4}$$

where n is the number of places where $k_v = 0$, and the sum is over all cuspidal eigenforms ϕ_i of Laplacian and Hecke operators on $G(F)\backslash G(\mathbb{A})/U$ such that $\|\phi_i\|_\Delta = 1$, and W_i are standard Whittaker function for ϕ_i. Here the integration is nontrivial only when $n = g$ then $\mathfrak{M}$ is a measured space parameterizing an orthogonal basis of Eisenstein series of norm 1. (See Section 18 for more details.) Thus for a cuspidal eigenform ϕ,

$$\frac{1}{|\Delta|} \int_{G(F)\backslash G(\mathbb{A})} k(x,y)\phi(y)\, dy = 2^{[F:\mathbb{Q}]+n} W_\phi(g_\infty)\phi(x).$$

It follows that for any two CM-cycles α and β on X_U, the height pairing has a decomposition

$$\begin{aligned} \langle \alpha, \beta \rangle = 2^{[F:\mathbb{Q}]+n} \sum_{\phi_i} W_i(g_\infty) \cdot (\phi_i, \bar{\alpha})_\Delta (\bar{\phi}_i, \beta)_\Delta \\ + 2^{[F:\mathbb{Q}]+n} \int_{\mathfrak{M}} W_{\mathfrak{m}}(g_\infty)(E_{\mathfrak{m}}, \bar{\alpha})_\Delta (\bar{E}_{\mathfrak{m}}, \beta)_\Delta\, d\mathfrak{m}. \end{aligned} \tag{16.5}$$

This leads us to define the following form of $\mathrm{PGL}_2(\mathbb{A})$ of weight k_v at v:

$$\begin{aligned} H(\alpha, \beta) = 2^{[F:\mathbb{Q}]+n} \sum_{\phi_i} \phi_i^{\mathrm{new}}(g_\infty) \cdot (\phi_i, \bar{\alpha})_\Delta (\bar{\phi}_i, \beta)_\Delta \\ + 2^{[F:\mathbb{Q}]+n} \int_{\mathfrak{M}} E_{\mathfrak{m}}^{\mathrm{new}}(g_\infty)(E_{\mathfrak{m}}, \bar{\alpha})_\Delta (\bar{E}_{\mathfrak{m}}, \beta)_\Delta\, d\mathfrak{m}, \end{aligned} \tag{16.6}$$

where ϕ_i^{new} (resp. E_λ^{new}) is the *newform* of weight $(2, \ldots, 2, 0, \ldots, 0)$ in the representation Π_i of $\mathrm{PGL}_2(\mathbb{A})$ corresponding to the representation Π_i' of $G(\mathbb{A})$ generated by ϕ_i (resp. $E_{\mathfrak{m}}$) via Jacquet–Langlands theory. With α replaced by

$\mathrm{T}_a\alpha$ in (16.5), one obtains the usual relation between height pairing and Fourier coefficient:

$$|a|\langle \mathrm{T}_a\alpha, \beta\rangle_\Delta(g_\infty) = 2^{[F:\mathbb{Q}]+n} W_{H(\alpha,\beta)}\left(g_\infty \cdot \begin{pmatrix} a\delta^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{16.7}$$

Let Ψ denote the form $2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}H(\eta,\eta)$ having a decomposition

$$\Psi = 2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\sum_i \phi_i^{\mathrm{new}}\,|(\phi_i,\eta)_\Delta|^2 + 2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\int_{\mathfrak{M}} E_{\mathfrak{m}}^{\mathrm{new}}\,|(E_{\mathfrak{m}},\eta)|^2\, d\mathfrak{m}. \tag{16.8}$$

Since η has a character χ under the action by Δ, we may require that ϕ_i (resp. $E_{\mathfrak{m},\chi}$) has character χ under the action by Δ. For a given ϕ^{new} of level N, then ϕ_i with $\phi_i^{\mathrm{new}} = \phi^{\mathrm{new}}$ must be the toric newform as in Section 14.

Equations (16.1) and (16.7) shows that, modulo certain Eisenstein series in the space $\Pi(\|\cdot\|^{1/2}, \|\cdot\|^{-1/2})\otimes\eta$ with $\eta^2 = 1$, the form $\Phi - \Psi$ has vanishing Fourier coefficient at g such that

$$g_f = \begin{pmatrix} \delta^{-1}a & 0 \\ 0 & 1 \end{pmatrix}$$

with integral a prime to ND. Thus $\Phi - \Psi$ is an old form.

Let ϕ be the newform as in Theorem 14.2. By Proposition 11.1 and formula (16.6), the projection of $\Phi - \Psi$ in $\Pi(\phi)$ is given by

$$\frac{L(\frac{1}{2},\chi,\phi)}{\|\phi^\sharp\|^2_{U_0(ND)}}\phi^\sharp - 2^{[F:\mathbb{Q}]+n}|c(\omega)|^{1/2}\cdot|(\eta,\phi_\chi)_\Delta|^2\phi.$$

Thus, we have proven the Gross–Zagier formula, Theorem 14.2, by computing the first Fourier coefficient of the above form.

17. Lowering Levels

Now it is remains to deduce GZF(N) (the Gross–Zagier formulas for level N in Sections 6–7) from GZF(ND) (formulas in Section 14). Our plan is as follows:

1. Show GZF(N) up to a certain universal function of local parameters.
2. Prove GZF(ND) for Eisenstein series for level ND thus get GZF(N) for Eisenstein series with the same universal function.
3. Prove GZF(N) for Eisenstein series directly by evaluating the periods thus get the triviality of the universal function.

In this section we are doing the first step.

Proposition 17.1. *For each $v \mid D$, there is a rational function $Q_v(t) \in \mathbb{C}(t)$ depending only on χ_v which takes* 1 *at $t = 0$ and is regular for*

$$|t| < |\pi_v|^{1/2} + |\pi_v|^{-1/2},$$

such that both Gross–Zagier formulas in Sections 6–7 *are true after multiplying the left-hand side by*

$$C(\chi)\prod_{\mathrm{ord}_v(D)>0} Q_v(\lambda_v),$$

where $C(\chi)$ *is a constant depends only* χ, *and* λ_v *the parameter appeared in the* L*-function*:

$$L_v(s,\phi)=\frac{1}{1-\lambda_v|\pi_v|^s+|\pi_v|^{2s}}.$$

The idea of proof is to show that, in the comparison of GZF(N) and GZF(ND), all four quantities

$$\widehat{\phi^\sharp}(1), \qquad \frac{\|\phi^\sharp\|^2_{U_0(ND)}}{\|\phi\|^2_{U_0(N)}}, \qquad \frac{|(\eta,\phi_\chi)_\Delta|^2}{|i_\chi(\widetilde{\phi})|^2}, \qquad \frac{\|x_\phi\|^2_\Delta}{\|y_\phi\|^2_{U_0(N)}},$$

are universal functions described in the Proposition, and that the last two quantities have the same functions. Here the last two fractions are considered as ratios since the denominators may be 0.

Let's try to localize the definition of quasi-newform in Section 11. For each finite place v, let Π_v be the local component of $\Pi(\phi)$ at v. Then Π_v is a unitary representation as $\Pi=\bigotimes\Pi_v$ is. Fix a Hermitian form for the Whittaker model $\mathcal{W}(\Pi_v,\psi_v)$ such that the norm of the new vector is 1 for almost all v. The product of this norm induces a norm on Π which is proportional to the L^2-norm on Π. Now we can define the quasi-newform

$$W_v^\sharp\in\mathcal{W}(\Pi_v,\psi_v) \tag{17.1}$$

to be a certain form of level D_v. Recall that the space of forms of level D_v has a basis consisting of forms

$$W_{vi}(g)=W_v\left(g\begin{pmatrix}\pi_v^{-i} & 0\\ 0 & 1\end{pmatrix}\right), \qquad 0\le i\le \mathrm{ord}_v(D_v), \tag{17.2}$$

where $W_{v0}=W_v$ is the newform. Then $W_v^\sharp$ is the unique nonzero form of level D_v satisfying the equations

$$(W_v^\sharp, W_{vi}-\nu_v^i W_v^\sharp)=0, \qquad 0\le i\le \mathrm{ord}_v(D_v), \tag{17.3}$$

where $\nu_v=0$ if v is not ramified in K; otherwise $\nu_v=\chi_v(\pi_{K,v})$. It is not difficult to show that if we write

$$W_v^\sharp=\sum c_{vi}W_{v,i},$$

then c_{vi} is rational function of quantities

$$\alpha_{vi}:=(W_{vi},W_v)/(W_v,W_v).$$

The quantities α_{vi} do not depend on the choice of pairing $(\,\cdot\,,\cdot\,)$ on Whittaker models. On the other hand, it is easy to show that $\phi^\sharp$ has the Whittaker function as product of $W_v^\sharp$.

It follows that both $\widehat{\phi^\sharp}(1)$ and

$$\frac{\|\phi^\sharp\|^2_{U_0(ND)}}{\|\phi\|^2_{U_0(N)}} = \frac{\|\phi^\sharp\|^2_{U_0(N)}}{\|\phi\|^2_{U_0(N)}} \cdot [U_0(1) : U_0(D)]$$

are the products of rational functions at v of quantities α_{vi}. It remains to show that the α_{vi} are rational functions of λ_v. Let $U_v = \mathrm{GL}_2(\mathcal{O}_v)$. Then

$$\begin{aligned}\alpha_{vi} &= (W_v, W_v)^{-1}\mathrm{vol}(U_v)^{-1}\int_{U_v}(\rho(u)W_{vi}, \rho(u)W_v)\,du\\ &= (\rho(t_{vi})W_v, W_v)/(W_v, W_v),\end{aligned}$$

where t_{vi} is the Hecke operator corresponding the constant function $\mathrm{vol}(H_{vi})^{-1}$ on

$$H_{vi} = U_v\begin{pmatrix}\pi^{-i} & 0\\ 0 & 1\end{pmatrix}U_v.$$

It is well known that W_v is an eigenform under t_{vi} with eigenvalue a rational function of λ_v. This shows that the α_{vi} are rational functions of λ_v. Since we have used only the unitary property of the local representation Π_v for $v \mid D$, the so obtained rational functions as in theorem for these quantities are regular for any λ_v as long as Π_v is unitary. In other words, these functions are regular at λ_v satisfying

$$|\lambda_v| < |\pi_v|^{1/2} + |\pi_v|^{-1/2}.$$

It remains to compare the last two quantities in the odd and even cases, respectively. Obviously the ratio of normalizations of measures is given by

$$|U(N,K)|/|\Delta|,$$

which equals a product of constants at places dividing D. Thus we may take the same measure in the comparison. Let's define a function ζ on CM-points $T(F)\backslash G(\mathbb{A}_f)$ supported on $T(\mathbb{A})i_cU(N,K)$ such that

$$\zeta(ti_cu) = \chi(t), \qquad t \in T(\mathbb{A}_f), u \in U(N,K).$$

Then the CM-points in $\mathrm{GZF}(N)$ are defined by ζ, and those in $\mathrm{GZF}(ND)$ by η. The key to proving our result is to compare these CM-cycles.

Recall that for a finite place v and a compactly supported, locally constant function h on $G(F_v)$, one defines the Hecke operator $\rho(h)$ on CM-cycles by

$$\rho(h) = \int_{G(F_v)} h(g)\rho(g)\,dg.$$

Let U_1 and U_2 be the compact subgroups of $G(\mathbb{A}_f)$ defined by $U_i = \prod U_{iv}$ and

$$U_{1v} = (\mathcal{O}_{c_v} + c_v\mathcal{O}_{K,v}\lambda_v)^\times,$$

$$U_{2v} = U(N,K)_v^\times.$$

Lemma 17.2. *For each finite place v, let h_v denote the constant function* $\mathrm{vol}(U_{1v})^{-1}$ *on $G(F_v)$ supported on $U_{2,v}i_{c,v}^{-1}$. Then*

$$\zeta_v = \rho(h_v)\eta_v.$$

Before we prove this lemma, let us see how to use this lemma to finish the proof of our Proposition. First assume we are in the even case. Then we have

$$(\widetilde{\phi}, P_\chi) = (\widetilde{\phi}, \zeta)_{U(N,K)}.$$

Here the product is taken as pairings between CM-cycles and functions. Let Ψ_ζ be the form $H(\zeta, \zeta)$ defined in (16.6). Then Ψ_ζ has a decomposition

$$\Psi_\zeta = \sum_i \phi_i^{\mathrm{new}} |(\zeta, \phi_i)|^2 + \text{Eisenstein series}, \tag{17.4}$$

where ϕ_i is an orthonormal basis of eigenforms on $X(N, K)$. For $\phi_i^{\mathrm{new}} = \phi$ with level N, ϕ_i must be the test form $\widetilde{\phi}$ as in Section 7.

Now the equation (16.7) implies that the Hecke operator the adjoint of $\rho(h)$ is $\rho(h^\vee)$ with $h^\vee(g) = \bar{h}(g^{-1})$. It follows that,

$$H(\zeta, \zeta) = H(\rho(h)\eta, \rho(h)\eta) = H(\rho(h^\vee * h)\eta, \eta).$$

Since η has character χ under the action by Δ, we may replace $h^\vee * h$ by a function h_0 which has character (χ, χ^{-1}) by actions of Δ from both sides and invariant under conjugation. Now

$$\begin{aligned}\Psi_\zeta &= \sum \phi_i^{\mathrm{new}} (\rho(h_0)\eta, \phi_i)\overline{(\eta, \phi_i)} + \cdots \\ &= \sum \phi_i (\eta, \rho(h_0)\phi_i)\overline{(\eta, \phi_i)} + \cdots .\end{aligned} \tag{17.5}$$

Since η and $\rho(h_0)\eta$ both have character χ under the action by Δ, we may re-placing ϕ_i by functions $\phi_{i,\chi}$ which has character χ under χ. For $\phi_i^{\mathrm{new}} = \phi^{\mathrm{new}}$ with level N, $\phi_{i,\chi}$ must be the toric newform ϕ_χ as in Section 14. Of course,

$$\rho(h_0)\phi_\chi = \prod_{v|D} P_v(\lambda_v)\phi_\chi \tag{17.6}$$

as ϕ_χ with P_v some polynomial functions. From (17.4)–(17.6), we obtain

$$|(\zeta, \widetilde{\phi})|^2 = \prod_{v|D} P_v(\lambda_v) \cdot |(\eta, \phi_\chi)|^2.$$

In the odd case, the proof is same but simpler with $H(\zeta, \zeta)$ defined as a holomorphic cusp form of weight 2 with Fourier coefficients given by the height pairings of $\langle \mathrm{T}_a x, x\rangle$ for two CM-divisors after minus some multiple of Hodge class. Then we end up with the expression

$$\Psi_\zeta = \sum_i \phi_i \|x_{\phi_i}\|^2,$$

where ϕ_i are newforms of level dividing N. The same reasoning as above shows that

$$\Psi_\zeta = H(\rho(h_0)y, y) = \sum \phi_i \cdot \langle y_{\phi_i}, \rho(h_0) y_{\phi_i} \rangle.$$

Thus we have

$$\|x_\phi\|^2 = \langle y_\phi, \rho(h_0) y_\phi \rangle = \prod_{v|D} P_v(\lambda_v) \|y_\phi\|^2.$$

It remains to prove the lemma. By definition,

$$\rho(h_v)\eta_v(g) = \mathrm{vol}(U_{1v})^{-1} \int_{U_{2v}} \eta_v(gui_{c,v}^{-1})\, du.$$

If $\rho(h_v)\eta_v(g) \neq 0$, then

$$gU_{1,v} i_{c,v}^{-1} \in T(F_v)U_{1,v}$$

or equivalently,

$$g \in T(F_v)U_{1,v} i_{c,v} U_{2,v}.$$

By (iii) in the following lemma, we have $g \in T(F_v) i_{c,v} U_{2,v}$. Write $g = ti_{c,v}u_g$. It follows that

$$\rho(h_v)\eta_v(g) = \chi(t)\mathrm{vol}(U_{1v})^{-1} \int_{U_{2v}} \eta_v(i_{c,v} u i_{c,v}^{-1})\, du.$$

By (iii) in the following lemma again, the integral is the same as

$$\int_{U_{1v}} \eta_v(u)\, du.$$

Thus we have

$$\rho(h_v)\eta_v = \zeta_v.$$

Lemma 17.3. *For v split in B, there is an isomorphism*

$$\mu : M_2(F_v) \longrightarrow \mathrm{End}_{F_v}(K_v)$$

with compatible embedding of K_v and such that

$$\mu(M_2(\mathcal{O}_v)) = \mathrm{End}_{\mathcal{O}_v}(\mathcal{O}_{K,v}).$$

Moreover, let $i_{\pi^n} \in \mathrm{GL}_2(F_v)$ be such that

$$\mu(i_{\pi^n})(\mathcal{O}_{K,v}) = \mathcal{O}_{\pi^n} = \mathcal{O}_v + \pi_v^n \mathcal{O}_{K,v}.$$

Then

(i) $$\mathrm{GL}_2(F_v) = \coprod_{n \geq 0} K_v^\times i_{\pi^n} \mathrm{GL}_2(\mathcal{O}_v),$$

(ii) $$i_{\pi^n} \mathrm{GL}_2(\mathcal{O}_v) i_{\pi_v^n}^{-1} \cap K_v^\times = \mathcal{O}_{\pi^n}^\times,$$

(iii) $$i_{\pi^n} U_{2,v} i_{\pi^n}^{-1} \cap K_v^\times U_{1,v} = U_{1,v}.$$

Proof. Indeed for any given embedding $K_v \longrightarrow M_2(F_v)$ such that $\mathcal{O}_{K,v}$ maps to $M_2(\mathcal{O}_v)$, then F_v^2 becomes a K_v-module of rank 1 such that $\mathcal{O}_v^2$ is stable under $\mathcal{O}_{K,v}$. Then we find an isomorphism $\mathcal{O}_v^2 \simeq \mathcal{O}_{K,v}$ as $\mathcal{O}_{K,v}$ module. This induces the required isomorphism $\mu : B_v \longrightarrow \mathrm{End}_{F_v}(K_v)$.

Now, for any $g \in \mathrm{GL}_2(F_v)$, let $t \in g(\mathcal{O}_{K,v})$ be the elements with minimal order. Then $\mu(t^{-1}g)(\mathcal{O}_{K,v})$ will be an order of $\mathcal{O}_{K,v}$, say $\mathcal{O}_{\pi^n}$. Thus

$$\mu(t^{-1}g)(\mathcal{O}_{K,v}) = \mathcal{O}_{\pi^n} = \mu(i_{\pi^n})(\mathcal{O}_{K,v}).$$

It follows that

$$g \in t i_{\pi^n} \mathrm{GL}_2(\mathcal{O}_v).$$

The first equality follows.

For the second equality, let $t \in K_v$. Then

$$i_{\pi^n}^{-1} t i_{\pi_n} \in \mathrm{GL}_2(\mathcal{O}_v)$$

if and only if

$$\mu(i_{\pi^n}^{-1} t i_{\pi^n})\mathcal{O}_K = \mathcal{O}_K,$$

or equivalently

$$\mu(t i_{\pi^n})\mathcal{O}_K = \mu(i_{\pi^n})\mathcal{O}_K, \qquad t\mathcal{O}_{\pi^n} = \mathcal{O}_{\pi^n}.$$

This is equivalent to the fact that $t \in \mathcal{O}_{\pi^n}^\times$.

It remains to show the last equality. First we want to show

$$i_{c,v}^{-1} U_{1,v} i_{c,v} \subset U_{2,v}.$$

To see this, we need to show that

$$\mu(i_{c,v}^{-1} u i_{c,v})\mathcal{O}_{K,v} = \mathcal{O}_{K,v}$$

for each $u \in U_{1,v}$. This is equivalent to

$$\mu(u i_{c,v})\mathcal{O}_{K,v} = i_{c,v}\mathcal{O}_{K,v},$$

or equivalently,

$$\mu(u)\mathcal{O}_{c,v} = \mathcal{O}_{c,v}.$$

Thus is clear from the fact that $u = t(1 + cM_2(\mathcal{O}_v))$ for some $t \in \mathcal{O}_c^\times$.

Now the last equality follows easily: since

$$i_{\pi_v^n} U_{2,v} i_{\pi_v^n}^{-1} \cap K_v^\times = \mathcal{O}_{\pi_v^n}$$

and

$$i_{\pi_v^n} U_{2,v} i_{\pi_v^n}^{-1} \supset U_{1,v},$$

it follows that

$$i_{\pi_v^n} U_{2,v} i_{\pi_v^n}^{-1} \cap K_v^\times U_{1,v} = U_{1,v}. \qquad \square$$

18. Continuous Spectrum

In this section, we would like to extend GZF(ND) (the Gross–Zagier formula in level ND, Theorem 14.2) to Eisenstein series in the continuous spectrum. Recall that the space of L^2-forms on $\mathrm{PGL}_2(F)\backslash\mathrm{PGL}_2(\mathbb{A})$ is a direct sum of cusp forms, characters, and Eisenstein series corresponding to characters (μ, μ^{-1}). We say two characters μ_1, μ_2 are connected if $\mu_1 \cdot \mu_2^{\pm}$ is trivial on the subgroup $\mathbb{A}^1$ of norm 1. Thus each connected component is a homogeneous space of $\mathbb{R}$ or $\mathbb{R}/\pm 1$. See [3] for more details.

We now fix a component containing a character (μ, μ^{-1}). Without loss of generality, we assume that μ^2 is not of form $|\cdot|^t$ for some $t \neq 0$. Then the space $\mathrm{Eis}(\mu)$ of L^2-form corresponding to this component consists of the forms

$$E(g) = \int_{-\infty}^{\infty} E_t(g)\, dt \tag{18.1}$$

where $E_t(g)$ is the Eisenstein series corresponding to characters $(\mu|\cdot|^{it}, \mu^{-1}|\cdot|^{-it})$. For the uniqueness of this integration we assume that $E_t(g) = 0$ if $t < 0$ and $\mu^2 = 1$. Now the two elements $E_1(g)$ and $E_2(g)$ has inner product given by

$$(E_1, E_2) = \int_{-\infty}^{\infty} (E_{1t}, E_{2t})_t dt, \tag{18.2}$$

where $(\cdot, \cdot)_t$ is some Hermitian form on the space

$$\Pi_t := \Pi(\mu|\cdot|^{it}, \mu^{-1}|\cdot|^{-it}).$$

This Hermitian norm is unique up to constant multiple as the representation is irreducible. The precise definition of this norm is not important to us.

Now we want to compute the Rankin–Selberg convolution of $E \in \mathrm{Eis}(\mu)$ with θ as in Section 8. Assume that χ is not of form $\nu \cdot \mathrm{N}_{K/F}$. Then θ is a cusp form and the kernel function Θ is of L^2-form as its constant term has exponential decay near cusp. Thus it makes sense to compute $(E, \bar{\Theta})$.

For ϕ a function on $\mathbb{R}$ (or $\mathbb{R}_+$ when $\mu^2 = 1$), let's write E_ϕ for element in $\mathrm{Eis}(\mu)$ with form $E_t(g) = \phi(t)E_t^{\mathrm{new}}(g)$ with $E_t^{\mathrm{new}}(g)$ a newform in $\Pi(\mu|\cdot|^s, \mu^{-1}|\cdot|^{-s})$ and $\phi(s) \in \mathbb{C}$, then we still have

$$\begin{aligned}(E_\phi, \bar{\Theta}_s) &= \int_0^{\infty} L(s, \Pi_t \otimes \chi)\phi(t)\, dt \\ &= \int_0^{\infty} L(s + it, \mu \otimes \chi)L(s + it, \mu^{-1} \otimes \chi)\phi(t)\, dt.\end{aligned}$$

If $s = \frac{1}{2}$, we obtain

$$(E_\phi, \bar{\Theta}_{1/2}) = \int \bigl|L(\tfrac{1}{2} + it, \mu \otimes \chi)\bigr|^2 \phi(t)\, dt. \tag{18.3}$$

The form Θ has level D. For any a dividing D, let's define

$$E_{\phi,a} = \rho \begin{pmatrix} a^{-1} & 0 \\ 0 & 1 \end{pmatrix} E_\phi.$$

Then the space of Eisenstein series is generated by $E_{\phi,a}$. We can define so called *quasi-newforms* by the formula

$$E_\phi^\sharp = \int_0^\infty E_t^\sharp \phi(t)\, dt. \tag{18.4}$$

One can show that the projection of $\Phi := \bar{\Theta}_{1/2}$ on the continuous spectrum corresponding to μ is $E_\phi^\sharp$ with ϕ given by

$$\phi(t) = \frac{\left|L(\frac{1}{2} + it, \mu \otimes \chi)\right|^2}{\|E_t^\sharp\|^2}. \tag{18.5}$$

We now study the geometric pairing in Section 16. Formula (16.8) shows that the continuous contribution for representation Π_t in the form Ψ is given by

$$2^{2g}|c(\omega)|^2 E_\psi,$$

where

$$\psi(t) = (E_{t,\chi}, \eta). \tag{18.6}$$

Here $E_{t,\chi}$ is a form toric form of norm 1 with respect to Δ.

Again $E_\phi^\sharp - 2^{2g}|c(\omega)|^{1/2} E_\psi$ will be an old form. Its first Fourier coefficient vanishes. Thus the Gross–Zagier formula can be extended to Eisenstein series:

Proposition 18.1. *Assume that χ is not of form $\nu \circ \mathrm{N}_{K/F}$ with ν a character of $F^\times \backslash \mathbb{A}^\times$. Then*

$$\widehat{E_t^\sharp}(1)|L(\tfrac{1}{2} + it, \chi)|^2 = 2^{2g}|c(\omega)|^{1/2} \cdot \|E_t^\sharp\|^2 |(E_{\chi,t}, \eta)|^2.$$

Also the proof of Proposition 17.1 is purely local, and so can be extended to Eisenstein series:

Proposition 18.2. *Assume that χ is not of form $\nu \circ \mathrm{N}_{K/F}$ with ν a character of $F^\times \backslash \mathbb{A}^\times$. Let*

$$\lambda_v(t) = \mu_v(\pi_v)|\pi_v|^{it} + \mu_v(\pi_v)^{-1}|\pi_v|^{-it}, \quad \textit{and} \quad E_t^* := \|E_t\| \cdot \widetilde{E}_t,$$

then

$$c(\chi) \prod_{\mathrm{ord}_v(D)>0} Q_v(\lambda_v(t)) = \frac{2^{2g}}{\sqrt{\mathrm{N}(D)}} \left| \frac{(E_t^*, P_\chi)}{L(\frac{1}{2} + it, \chi)} \right|^2.$$

Notice that when μ is unramified $\widehat{R}$ is conjugate to $M_2(\widehat{\mathcal{O}}_F)$. The form E_t^* is obtained from E_t by $\rho(j)$ for a certain $j \in G(\mathbb{A})$ satisfying (19.1) and (19.2) below. Thus the formula does not involve the definition of hermitian forms on Π_t.

19. Periods of Eisenstein Series

In this section we want to compute the periods of Eisenstein series appearing in the GZF(N) (up to a universal function), Proposition 18.2. Our result shows that all the universal functions are trivial, and thus end up the proof of GZF(N).

First let's describe the main result. Let μ be a unramified quasi-character of $F^\times\backslash\mathbb{A}^\times$. Let R be a maximal order of $M_2(F)$ containing $\mathcal{O}_K$. Let E be the newform in $\Pi(\mu,\mu^{-1})$. Let $j\in G(\mathbb{A})$ such that

$$j_\infty \mathrm{SO}_2(\mathbb{R}) j_\infty^{-1} = T(\mathbb{R}) \tag{19.1}$$

and

$$j_f \mathrm{GL}_2(\widehat{\mathcal{O}}) j_f^{-1} = R. \tag{19.2}$$

Then the form $E^*(g) := E(gj)$ is invariant under $T(\mathbb{R})\cdot\widehat{R}^\times$. Let $\lambda\in K$ be a nonzero trace-free element. Then one can show that $\mathrm{ord}_v(\lambda/D)$ for all finite place v is always even. We thus assume that $4\lambda/D$ has a square root at a finite place and that $D_v=-1$ when $v\mid\infty$.

Proposition 19.1. *Assume that $\chi\neq\mu_K := \mu\circ \mathrm{N}_{K/F}$. Then*

$$(E^*, P_\chi) = 2^{-g}\mu\big(\delta^{-1}\sqrt{4\lambda/D}\big)|4\lambda/D|^{1/4}L(\tfrac12,\bar\chi\cdot\mu_K).$$

Before we go to the proof of this result, let's see how to use Proposition 19.1 to complete the proof of GZF(N). Combined Propositions 19.1 and 18.2 with $\mu(x)=|x|^{it}$ ($t\in\mathbb{R}$), we obtain

$$C(\chi)\prod_{\mathrm{ord}_v(D)>0} Q_v(\lambda_v(t)) = 1 \qquad \text{for all } t\in\mathbb{R}.$$

Notice that each $\lambda_v(t)$ is a rational function of p^{ti} where p is prime number divisible by v. Since functions p^{ti} for different primes p are rationally independent, we obtain that for each prime p

$$\prod_{\mathrm{ord}_v(D_p)>0} Q_v(\lambda_v(t)) = \mathrm{const}$$

where $D_p=\prod_{v|p} D_v$.

It is not difficult to show that for each χ_v we can find a finite character χ' of $\mathbb{A}_K^\times/K^\times\mathbb{A}^\times$ such that the following conditions are verified:

- $c(\chi')$ is prime to N, $c(\omega)$;
- χ' is unramified at all $w\mid p, w\neq v$;
- χ' is not of form $\nu\circ\mathrm{N}_{K/F}$.

If we apply the above result to χ' then we found that Q_v is constant thus is 1. Thus we have shown:

Proposition 19.2. *All polynomials $Q_v(t)$ and $C(\chi)$ are constant* 1.

Now GZF(N) follows from Proposition 17.1.

We now start the proof of Proposition 19.1 from the integral

$$(E^*, P_\chi) = \int_{T(F)\backslash T(\mathbb{A}_f)} \chi^{-1}(x) E^*(x_\infty x i_c)\, dx$$

where $x_\infty \in \mathcal{H}^g$ is fixed by $T(\mathbb{R})$ and $i_c \in G(\mathbb{A}_f)$ is an element such that

$$i_c R i_c^{-1} \cap K = \mathcal{O}_c.$$

Here we pick up a measure dt on $T(\mathbb{A})$ with local decomposition $dt = \bigotimes_v dt_v$ such that $T(\mathbb{R})$ and $T(\mathcal{O}_{c,v})$ all have volume 1. Write $h = i_c j$. It follows that

$$(E^*, P_\chi) = \int_{T(F)\backslash T(\mathbb{A})} \bar{\chi}(x) E(xh)\, dx.$$

Since E is obtained by analytic continuation from the newform in the Eisenstein series in $\Pi(\mu|\cdot|^s, \mu^{-1}|\cdot|^s)$ with $\operatorname{Re} s \gg 0$, we thus need only compute the periods for quasi-character μ with big exponent. In this case,

$$E(g) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma g)$$

with

$$f(g) = \mu^{-1}(\delta) f_\Phi,$$

where $\Phi = \bigotimes \Phi_v \in \mathcal{S}(\mathbb{A}^2)$ is the standard element: Φ_v is the characteristic function of $\mathcal{O}_v^2$ if $v \nmid \infty$, and $\Phi_v(x, y) = e^{-\pi(x^2+y^2)}$ if $v \mid \infty$. It is not difficult to show that the embedding $T \longrightarrow G$ defines an bijective map

$$T(F) \simeq P(F)\backslash G(F).$$

Thus

$$(E^*, P_\chi) = \int_{T(\mathbb{A})} \chi^{-1}(x) f(xh)\, dx.$$

This is of course the product of local integrals

$$i_{\chi_v}(f_v) = \int_{T(F_v)} \chi_v(x^{-1}) f_v(xh_v)\, dx.$$

Recall that f_v is defined as follows:

$$f_v(g) = \mu(\delta_v^{-1} \cdot \det g) |\det g|^{1/2} \int_{F_v^\times} \Phi[(0, t)g] \mu^2(t) |t| d^\times t.$$

It follows that

$$\begin{aligned}
i_{\chi_v}(f_v) &= \int_{T(F_v)} \chi_v(x^{-1}) \mu(\delta_v^{-1} \det xh_v) |\det xh_v|^{1/2} \int_{F_v^\times} \Phi[(0, t)xh_v] \mu^2(t) |t| d^\times t\, dx \\
&= \mu(\delta_v^{-1} \det h_v) |\det h_v|^{1/2} \int_{K_v^\times} \bar{\chi}_v \mu_K(x) |x|_K^{1/2} \Phi_{K_v}(x)\, dx \\
&= \mu(\delta_v^{-1} \det h_v) |\det h_v|^{1/2} Z(\tfrac{1}{2}, \bar{\chi} \cdot \mu_K, \Phi_{K_v}),
\end{aligned}$$

where for $x \in K_v^\times$,

$$\Phi_{K_v}(x) = \Phi_v[(0,1)xh_v].$$

Thus the period computation is reduced to the computation of local zeta functions.

Let v be a finite place. The map $x \longrightarrow (0,1)x$ defines an isomorphism between K and F^2 with compatible actions by K. Thus we have two lattices $\mathcal{O}_F^2$ and $\mathcal{O}_c$ in K. The element h_f as a class in

$$\widehat{K}^\times \backslash \mathrm{GL}_2(\mathbb{A}_f)/\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$$

is determined by the property that

$$h_f M_2(\widehat{\mathcal{O}}_F)h_f^{-1} \cap K = \mathcal{O}_c.$$

We may take h_f such that

$$(0,1)\mathcal{O}_c h_f = \mathcal{O}_F^2.$$

It follows that $\Phi_{K,v}$ is the characteristic function of $\widehat{\mathcal{O}}_c$. Now the zeta function is easy to compute:

$$Z(\tfrac{1}{2}, \bar{\chi}\cdot\mu_K, \Phi_{K,c}) = \int_{\mathcal{O}_{c_v}} \chi\cdot\mu_K(x)|x|_K^{1/2} d^\times x.$$

We get the standard L-function if $c = 0$.

We assume now that $c > 0$ thus K_v/F_v is an unramified extension. First we assume that K_v is a field. We decompose the set $\mathcal{O}_c$ into the disjoint union of $\mathcal{O}_{c,n}$ of subset of elements of order n. Then

$$Z(\tfrac{1}{2}, \bar{\chi}\cdot\mu_K, \Phi_{K,c}) = \sum_{n\geq 0} \mu(\pi_v)^{2n}|\pi_v|^n \int_{\mathcal{O}_{c,n}} \chi(x) d^\times x.$$

Write $\mathcal{O}_{K,v} = \mathcal{O}_v + \mathcal{O}_v\lambda$ then

$$\mathcal{O}_c = \mathcal{O}_v + \pi_v^c \mathcal{O}_v \lambda.$$

If $n \geq c$, then $\mathcal{O}_{c,n} = \pi^n \mathcal{O}_K^\times$. The integral vanishes as χ has conductor π^c. If $n < c$ then

$$\mathcal{O}_{c,n} = |\pi_v|^n \mathcal{O}_v^\times (1 + \pi_v^{c-n}\mathcal{O}_K).$$

The integration on $\mathcal{O}_{n,c}$ vanishes unless $n = 0$ as χ has conductor π^c. Thus the total contribution is

$$\mathrm{vol}(\mathcal{O}_{c_v}^\times) = 1.$$

We assume now that K_v/F_v is split. Then $K_v = F_v^2$ and $\mathcal{O}_c$ consists of integral elements (a,b) such that $a \equiv b \pmod{\pi_v^c}$. Write $\chi = (\nu, \nu^{-1})$ then ν has conductor π_v^c. It follows that

$$Z(\tfrac{1}{2}, \bar{\chi}\cdot\mu_K, \Phi_{K,c}) = \int_{(a,b)\in\mathcal{O}_c} \nu(a/b)\mu(ab)|ab|^{1/2} d^\times a d^\times b.$$

For a fixed $b \in \mathcal{O}_v$, the condition in a is as follows:

$$\begin{cases} a \in \pi^c \mathcal{O}_F, & \text{if } b \in \pi^c \mathcal{O}_v, \\ a \in b(1+\pi^{c-n}\mathcal{O}_v), & \text{if } b \in \pi^n \mathcal{O}_v^\times \text{ with } n < c. \end{cases}$$

Since ν has conductor c, the only case gives nontrivial contribution is when $b \in \mathcal{O}_v^\times$ and $a \in b(1+\pi^c\mathcal{O}_v)$. The contribution is given by

$$\mathrm{vol}(\mathcal{O}_{c_v}^\times) = 1.$$

To compute $\det h_v$, we write $K = F + F\sqrt{\lambda}$ and make the following embedding $K \longrightarrow M_2(F)$:

$$a + b\lambda \mapsto \begin{pmatrix} a & b\lambda \\ b & a \end{pmatrix}. \tag{19.3}$$

Then $\mathcal{O}_F^2$ corresponding to the lattice

$$\mathcal{O}_F + \mathcal{O}_F\sqrt{\lambda}.$$

Thus h_f satisfies

$$(0,1)(\mathcal{O}_v + \mathcal{O}_v\sqrt{\lambda}) = (0,1)\mathcal{O}_{c_v} h_f.$$

It follows that

$$\mathrm{disc}(\mathcal{O}_v + \mathcal{O}_v\sqrt{\lambda}) = \mathrm{disc}(\mathcal{O}_{c_v}) \det h_v^2.$$

Thus

$$\det h_v = \sqrt{4\lambda/D_v}$$

for a suitable D_v in its class modulo $\mathcal{O}_v^\times$ such that $4\lambda/D_v$ does have a square root in $F_v^\times$. In summary we have shown that

$$i_v(f_v) = L(\tfrac{1}{2}, \bar{\chi} \otimes \mu)\mu(\delta_v\sqrt{4\lambda/D_v})|4\lambda/D_v|^{1/4}. \tag{19.4}$$

It remains to compute the periods at archimedean places v. For equation (19.1), we may take

$$h_v = \begin{pmatrix} |\lambda_v|^{1/2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then it is easy to see that

$$\Phi_{K,v}(x) = e^{-\pi|x|^2}.$$

Assume that $\mu(x) = |x|^t$, $\chi = 1$, and notice that the measure on $K_v^\times = \mathbb{C}^\times$ is induced from the standard $d^\times x$ from $\mathbb{R}^\times$ and one from $\mathbb{C}^\times/\mathbb{R}^\times$ with volume one. Thus the measure has the form $dr\, d\theta/\pi r$ for polar coordinates $re^{i\theta}$. It follows that

$$\begin{aligned} Z(\tfrac{1}{2}, \bar{\chi}\cdot\mu_K, \Phi_{K_v}) &= \int_{\mathbb{C}^\times} e^{-\pi r^2} r^{2t+1} \frac{dr\, d\theta}{\pi r} = \pi^{-1/2-t}\Gamma(t+\tfrac{1}{2}) \\ &= \mu(2)2^{-1/2} L(\tfrac{1}{2}, \bar{\chi}\cdot\mu_K). \end{aligned}$$

The period at v is then given by

$$i_v(f_v) = 2^{-1}\mu_v(|4\lambda|^{1/2})|4\lambda_v|^{1/4} L(\tfrac{1}{2}, \bar{\chi}\mu_K). \tag{19.5}$$

Setting $D_v = -1$ for archimedean places, we obtain the same formula as (19.4). The proof of the proposition is completed.

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