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Faltings Heights and the Derivative of Zagier's Eisenstein Series

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D. Zagier discovered in 1975 [HZ] the following famous modular form of weight $\frac{3}{2}$ for $\Gamma_0(4)$:

$$E_{\text{Zagier}}(\tau) = -\frac{1}{12} + \frac{1}{8\pi\sqrt{v}} + \sum_{m=1}^{\infty} H_0(m) q^m + \sum_{n>0} 2g(n,v) q^{-n^2}. \tag{0.1}$$

Here $\tau = u + iv$ is in the upper half plane, $q = e^{2\pi i \tau}$, $H_0(m)$ is the Hurwitz class number of binary quadratic forms of discriminant $-m$, and

$$g(n,v) = \frac{1}{16\pi\sqrt{v}} \int_1^{\infty} e^{-4\pi n^2 v r} r^{-3/2} dr. \tag{0.2}$$

This function can be obtained, via analytic continuation, as a special value of an Eisenstein series $\mathcal{E}(\tau, s)$ at $s = \frac{1}{2}$. In this note, we will give an arithmetic interpretation to Zagier's Eisenstein series and its derivative at $s = \frac{1}{2}$, using Arakelov theory.

Let $\mathcal{M}$ be the Deligne–Rapoport compactification of the moduli stack over $\mathbb{Z}$ of elliptic curves [DR]. In Section 3 we will define a generating function of arithmetic Chow cycles of codimension 1 in $\mathcal{M}$ with real coefficients, in the sense of Bost and Kühn [Bos1,Kun]:

$$\hat{\phi}(\tau) = \sum_{m \in \mathbb{Z}} \hat{\mathcal{Z}}(m, v) q^m, \tag{0.3}$$

such that

$$2 \deg \hat{\phi}(\tau) = \mathcal{E}(\tau, \tfrac{1}{2}) = E_{\text{Zagier}}(\tau), \tag{0.4}$$

and

$$4\langle \hat{\phi}, \hat{\omega} \rangle = \mathcal{E}'(\tau, \tfrac{1}{2}). \tag{0.5}$$

Here $\hat{\omega}$ is a normalized metrized Hodge bundle on $\mathcal{M}$, to be defined in Section 3, and $\langle\, , \rangle$ is the Gillet–Soule intersection pairing ([GS]; see also section 2). Bost's arithmetic Chow cycles with *real* coefficients are crucial here since, for example,

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the negative Fourier coefficients of Zagier's Eisenstein series are clearly not rational numbers. This note is a slight variation of joint work with Stephen Kudla and Michael Rapoport [KRY2]. It confirms a conjecture of Kudla [Ku2].

1. The Chowla–Selberg Formula

Part of the formulas (0.4) and (0.5) can be viewed as a generalization of the Chowla–Selberg formula which we now describe.

For a positive integer $m \neq 0$, let $K_m = \mathbb{Q}(\sqrt{-m})$ and let $\mathcal{O}_m$ be the order in K_m of discriminant $-m$. Notice that $\mathcal{O}_m$ exists if and only if $m \equiv 0, -1 \bmod 4$. We can and will write $m = dn^2$ such that $-d$ is the fundamental discriminant of K_m and $n \geq 1$ is an integer.

Let $Z(m)$ be the set of isomorphic classes of elliptic curves E over $\mathbb{C}$ such that there is an embedding $\mathcal{O}_m \hookrightarrow \mathrm{End}(E)$. When $\mathcal{O}_m$ does not exist, we take $Z(m)$ to be empty. Let

$$\deg Z(m) = \sum_{E \in Z(m)} \frac{1}{\# \mathrm{Aut}\, E} \tag{1.1}$$

and

$$h_{\mathrm{Fal}}(Z(m)) = \sum_{E \in Z(m)} \frac{1}{\# \mathrm{Aut}\, E}\, h_{\mathrm{Fal}}(E), \tag{1.2}$$

where $h_{\mathrm{Fal}}(E)$ is the (renormalized) Faltings height, which measures, in some sense, the complexity of the elliptic curve E. It is defined as follows. Let L be a number field over which E is defined and has good reduction everywhere, and let ω be the Néron differential on E over $\mathcal{O}_L$. Then

$$h_{\mathrm{Fal}}(E) = -\frac{1}{2[L:\mathbb{Q}]} \sum_{\sigma: L \hookrightarrow \mathbb{C}} \log \left| \frac{i}{2\pi} e^{-C} \int_{E^\sigma(\mathbb{C})} \omega^\sigma \wedge \bar{\omega}^\sigma \right|, \tag{1.3}$$

where $C = \frac{1}{2}(\gamma + \log 4\pi)$ is a normalizing factor we have the liberty to add. Here γ is Euler's constant.

On the other hand, one can define a modified Dirichlet L-series for every integer $m \neq 0$

$$L(s, \chi_m) = L(s, \chi_d) \prod_{p \mid n} b_p(n, s) \tag{1.4}$$

where χ_d is the quadratic character associated to the quadratic field $K_m = K_d$, and $L(s, \chi_d)$ is the usual Dirichlet L-series of χ_d, and

$$b_p(n, s) = \frac{1 - \chi_d(p)X + \chi_d(p)p^k X^{1+2k} - (pX^2)^{1+k}}{1 - pX^2} \tag{1.5}$$

with $X = p^{-s}$ and $k = \mathrm{ord}_p\, n$. This L-series occurs in the Fourier coefficients of an Eisenstein series as we will see in Section 3. The complete L-series

$$\Lambda(s, \chi_m) = |m|^{s/2} \pi^{-(s+a)/2} \Gamma\Big(\frac{s+a}{2}\Big) L(s, \chi_m) \tag{1.6}$$

has an analytic continuation and the functional equation

$$\Lambda(s, \chi_m) = \Lambda(1 - s, \chi_m). \tag{1.7}$$

Here $a = (1 + \operatorname{sign} m)/2$. Now we can state part of our formula as

THEOREM 1.1. *With the notation as above, one has for* $m \geq 1$:

(1) $2 \deg Z(m) = \Lambda(0, \chi_m) = H_0(m)$.
(2) $4 h_{\mathrm{Fal}}(Z(m)) = -\Lambda'(0, \chi_m)$.

Only orders of the form $\mathcal{O}_{4m}$ are considered in [KRY2].

SKETCH OF PROOF. Part (1) is basically the analytic class number formula and is well-known. Part (2) is basically [KRY1, Corollary 10.12]. We give an outline for this special case for the reader's convenience.

Step 0: When $-m = -d$ is the fundamental discriminant of K_m, the formula is just the Chowla–Selberg formula (see [Gro1; Gro2; Col] for this interpretation and geometric proof): For an elliptic curve E with CM by the ring $\mathcal{O}_d$ of integers, one has

$$2 h_{\mathrm{Fal}}(E) = -\frac{\Lambda'(0, \chi_d)}{\Lambda(0, \chi_d)}.$$

In particular, the height $h_{\mathrm{Fal}}(E)$ is independent of the choice of the elliptic curves. Combining this with (1), one proves (2) for this case.

Step 1: In the general case of non-fundamental discriminants, the Chowla–Selberg formula was considered by Nakkajima and Taguchi [NT]. An elliptic curve E with CM by $K = K_d$ is of type c if $\mathrm{End} E \cong \mathcal{O}_{dc^2}$. Let E be an elliptic curve of type c. Choose a CM elliptic curve E_0 of type 1 together with an isogeny $u_L : E_0 \longrightarrow E$. Let L be a number field over which E, E_0 and u_L are defined and have good reduction everywhere. Extend u_L to an isogeny u on their Néron models over $\mathcal{O}_L$ with kernel N

$$0 \longrightarrow N \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then Raynaud's isogeny theorem [Ra, p. 205] asserts that

$$h_{\mathrm{Fal}}(E) = h_{\mathrm{Fal}}(E_0) + \tfrac{1}{2} \log(\deg u_L) - \frac{1}{[L : \mathbb{Q}]} \log |\varepsilon^*(\Omega_{N/\mathcal{O}_L})|.$$

The term $\log |\varepsilon^*(\Omega_{N/\mathcal{O}_L})|$ can be computed locally and was done in [KRY1, Section 10]. In this special case, we can choose E_0 so that u_L is of degree c by the theory of complex multiplication, and thus only [KRY1, Propositions 10.1 and 10.3] are needed. Indeed, if $E_c = \mathbb{C}/c^{-1}\mathcal{O}_{c^2 d}$, which can actually be defined over H_c, the ring class field of $\mathcal{O}_{c^2 d}$, then $E_1 = \mathbb{C}/\mathcal{O}_d$, and the desired map $E_c \longrightarrow E_1$ is induced by the identity map on $\mathbb{C}$. In general, every elliptic curve E of type c is a Galois conjugate E_c^σ of E_c by some $\sigma \in \mathrm{Gal}(H_c/K)$, and thus the desired map is $E_c^\sigma \longrightarrow E_1^\sigma$. The end result is the following formula [KRY1, Theorem 10.7]:

$$2h_{\mathrm{Fal}}(E) = 2h_{\mathrm{Fal}}(E_0) + \log c - \sum_p \frac{(1-p^{-\operatorname{ord}_p c})(1-\chi_d(p))}{(1-p^{-1})(p-\chi_d(p))} \log p.$$

In particular, the Faltings height of an elliptic curve E of type c depends only on c. In [KRY1], abelian surfaces are considered and an extra isogeny has to be studied.

Step 2: Now combining terms together with Step 0 produces

$$2h_{\mathrm{Fal}}(Z(m)) = 2\deg Z(m)\Bigg(-\frac{\Lambda'(0,\chi_d)}{\Lambda(0,\chi_d)} + \frac{\sum_{c|n} c\prod_{l|c}(1-\chi_d(l)l^{-1})\sum_{p|c}\eta_p(\operatorname{ord}_p c)\log p}{\prod_{p|n} b_p(n,0)}\Bigg).$$

Here

$$\eta_p(r) = r - \frac{(1-p^{-r})(1-\chi_d(p))}{(1-p^{-1})(p-\chi_d(p))}.$$

On the other hand,

$$-\frac{\Lambda'(0,\chi_m)}{\Lambda(0,\chi_m)} = -\frac{\Lambda'(0,\chi_d)}{\Lambda(0,\chi_d)} + \sum_{p|n}\left(\log|n|_p - \frac{b_p'(n,0)}{b_p(n,0)}\right).$$

Now what is needed is to verify an algebraic identity, which was done in [KRY1, Lemma 10.9] using induction on the number of prime factors of n. □

2. Bost's L_1^2-Arithmetic Divisors and Intersection Theory

As a background for section 3, we briefly review Bost's L_1^2-arithmetic divisors with real coefficients and the corresponding intersection theory for the convenience of the readers. We refer to [Bos1] for detail. A similar theory was also developed by Kühn [Kun].

Let $\mathcal{M}$ be an arithmetic surface over $\mathbb{Z}$, and let $M = \mathcal{M}(\mathbb{C})$ be the corresponding Riemann surface. A generalized function ϕ on M is (locally) L_1^2 if both ϕ and $\partial\phi$ are (locally) L^2, i.e., square integrable. For example, $\log\log|z|^{-1}$ is locally L_1^2 near $z=0$, but $\log|z|$ is not. Similarly, $\log \operatorname{Im} z$ is locally L_1^2 at $z = i\infty$. It is also known that if $\phi \in L_1^2$ then e^ϕ is L^p for every $p<\infty$. A current α on M is called (locally) L_{-1}^2 if it can be written (locally) as

$$\alpha = \partial\beta$$

for some (locally) L^2 1-form β.

Bost's arithmetic Chow groups. Let $\mathcal{Z}^1(\mathcal{M})$ be the free abelian group of Weil divisors of $\mathcal{M}$, and let $\mathcal{Z}^1_{\mathbb{R}}(\mathcal{M}) = \mathcal{Z}^1(\mathcal{M}) \otimes \mathbb{R}$. An L^2_1-arithmetic divisor on $\mathcal{M}$ is a pair $\hat{D} = (D, g)$ with $D \in \mathcal{Z}^1(\mathcal{M})$ and g is a L^2_1-Green function for D in the following sense: There is a usual C^∞-Green function g_0 for D and a L^2_1-generalized function $\phi \in L^2_1(M)$ such that

$$g = g_0 + \phi. \tag{2.1}$$

Equivalently, for any local holomorphic coordinate z on an open neighborhood of M, one has

$$g = \phi + \sum_{P \in U} n_P \log |z - z(P)|^{-2} \tag{2.2}$$

for some $\phi \in L^2_1(U)$ if $D = \sum n_P P$. In this case,

$$\omega_1(\hat{D}) = dd^c g + \delta_D$$

is a "L^2_{-1}"-current on M of degree 2 [Bos1, p. 255]. Let $\widehat{\mathcal{Z}}^1(\mathcal{M})$ be the abelian group of L^2_1-arithmetic divisors in $\mathcal{M}$. Notice that (2.2) makes sense even if $n_P \in \mathbb{R}$. In such a case, we call (D, g) a L^2_1-arithmetic divisor with real coefficients, and denote the abelian group of all L^2_1-arithmetic divisors with real coefficients by $\widehat{\mathcal{Z}}^1_{\mathbb{R}}(\mathcal{M})$. For a rational function $f \in \mathbb{Q}(\mathcal{M})$,

$$\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -\log |f|^2)$$

is certainly a L^2_1-arithmetic divisor—a principal arithmetic divisor. Let $\widehat{\mathrm{CH}}^1(\mathcal{M})$ and $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ be the quotient groups of $\widehat{\mathcal{Z}}^1(\mathcal{M})$ and $\widehat{\mathcal{Z}}^1_{\mathbb{R}}(\mathcal{M})$ respectively by the principal arithmetic divisors. There is a natural map

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) \longrightarrow \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$$

whose kernel is determined by [Bos1, Theorem 5.5], and the intersection pairing on $\widehat{\mathrm{CH}}^1(\mathcal{M})$ (to be reviewed below) factors through $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$.

The Arakelov–Gillet–Soule–Bost pairing. If $\hat{D}_1 = (D_1, g_1)$ and $\hat{D}_2 = (D_2, g_2)$ are L^2_1-arithmetic divisors on $\mathcal{M}$ such that $|D_1|$ and $|D_2|$ do not meet in the generic fiber $\mathcal{M}_{\mathbb{Q}}$, their intersection is defined as (see [Bos1, (5.8)])

$$\langle \hat{D}_1, \hat{D}_2 \rangle = \widehat{\deg}\, D_1.D_2 + \frac{1}{2} \int_M g_1 * g_2 \tag{2.3}$$

where the star product integral $\int_M g_1 * g_2$ is defined as follows [Bos1, (5.1) and (5.4)]. If both g_1 and g_2 are C^∞-Green functions, one has as usual

$$g_1 * g_2 = g_1 \omega_2 + g_2 \delta_{D_1}, \tag{2.4}$$

where

$$\omega_i = dd^c g_i + \delta_{D_i}$$

is a C^∞-form of degree 2 on M. If $\tilde{g}_i = g_i + \phi_i$ such that g_i are C^∞ and $\phi_i \in L_1^2(M)$, then

$$\int_M \tilde{g}_1 * \tilde{g}_2 = \int_M g_1 * g_2 + \int_M \phi_1\omega_2 + \int_M \phi_2\omega_1 + \frac{1}{2\pi i}\int_M \partial\phi_1 \wedge \bar{\partial}\phi_2. \tag{2.5}$$

It is checked in [Bos1, Section 5.1]) that this is well-defined, and $\langle \hat{D}, \widehat{\operatorname{div}}(f)\rangle = 0$. So (2.3) and (2.4) give an intersection pairing on $\widehat{\mathrm{CH}}^1(\mathcal{M})$ and $\widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ [Bos1, Theorem 5.5].

L_1^2-metrized line bundles and Bost's arithmetic Picard group. A L_1^2-metrized line bundle is a pair $\bar{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ where $\mathcal{L}$ is as usual a line bundle on $\mathcal{M}$ and $\|\cdot\|$ is a L_1^2-metric on $L = \mathcal{L}\otimes\mathbb{C}$ (invariant under complex conjugation) in the following sense: There is a C^∞ metric $\|\cdot\|_0$ on L together with an $\phi \in L_1^2(L)$ such that

$$\|\cdot\|^2 = \|\cdot\|_0^2 e^{-\phi}. \tag{2.6}$$

The natural map $\widehat{\mathrm{Pic}}(\mathcal{M}) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{M})$ given by

$$\bar{\mathcal{L}} \mapsto \hat{c}_1(\bar{\mathcal{L}}) = (\operatorname{div}(s), -\log\|s\|^2) \tag{2.7}$$

extends to the L_1^2-case, where s is a meromorphic section of $\mathcal{L}$. Let $\bar{\mathcal{L}}_0 = (\mathcal{L}, \|\cdot\|_0)$, then

$$\hat{c}_1(\bar{\mathcal{L}}) = \hat{c}_1(\bar{\mathcal{L}}_0) + (0, \phi\circ s), \tag{2.8}$$

and the corresponding first Chern form is

$$c_1(\bar{\mathcal{L}}) = c_1(\bar{\mathcal{L}}_0) + dd^c(\phi\circ s). \tag{2.9}$$

Via the map (2.7), we have then an intersection theory

$$\widehat{\mathrm{Pic}}(\mathcal{M}) \times \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \longrightarrow \mathbb{C}.$$

It can be computed directly as follows:

$$\langle \bar{\mathcal{L}}, (D, g)\rangle = h_{\bar{\mathcal{L}}}(D) + \frac{1}{2}\int_{\mathcal{M}(\mathbb{C})} g c_1(\bar{\mathcal{L}}), \tag{2.10}$$

where the height function $h_{\bar{\mathcal{L}}}(D)$ is defined as in [KRY1, pages 15-16]. The change from arithmetic surfaces to stacks is the same as in [KRY1, Section 4].

3. The Main Result

Let $\mathcal{M}_0$ be the moduli stack over $\mathbb{Z}$ of elliptic curves considered by Deligne and Rapoport [DR]. Over $\mathbb{C}$, it is the same as the orbifold

$$\mathcal{M}_0(\mathbb{C}) = [\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}],$$

where each elliptic curve $E \in \mathcal{M}_0(\mathbb{C})$ is counted with multiplicity $1/(\#\operatorname{Aut} E)$, i.e., with the order of its stabilizer in $\mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{M}$ be the Deligne–Rapoport compactification of $\mathcal{M}_0$, so $\mathcal{M}(\mathbb{C})$ is $\mathcal{M}_0(\mathbb{C})$ plus the cusp ∞.

For an integer $m \geq 1$ (with $m \equiv -1, 0 \mod 4$), let $\mathcal{Z}(m)$ be the moduli stack over $\mathbb{Z}$ of elliptic curves E such that there is $\mathcal{O}_m \hookrightarrow \mathrm{End}(E)$. Then $\mathcal{Z}(m)$ is a divisor on $\mathcal{M}$, and one has $\mathcal{Z}(m)(\mathbb{C}) = [Z(m)]$ with each elliptic curve counted $1/(\# \mathrm{Aut}\, E)$ times. This explains the weight in defining $\deg Z(m)$ and $h_{\mathrm{Fal}}(Z(m))$. We can view $\mathcal{Z}(m)$ as giving a class in $\mathrm{CH}^1(\mathcal{M})$.

For every integer m and a positive real number $v > 0$, Kudla constructed in [Ku1] a function $\Xi(m, v)$ on $\mathcal{M}_0(\mathbb{C})$. We will review this construction in Section 4 and sketch a proof of the following proposition.

Proposition 3.1. (1) *When $m > 0$, $\Xi(m, v)$ is also smooth at the cusp ∞ and is a Green's function for $\mathcal{Z}(m)$ on $\mathcal{M}(\mathbb{C})$. That is, there is a C^∞ $(1,1)$-form $\omega(m, v)$ on M such that*

$$dd^c g + \delta_{Z(m)} = [\omega(m, v)]$$

as currents.

(2) *When $-m > 0$ is not a square, $\Xi(m, v)$ is smooth everywhere, including at the unique cusp ∞.*

(3) *When $-m = n^2 > 0$ is square, $\Xi(m, v)$ is smooth in the upper half plane but is singular at the cusp ∞. As a current, it satisfies the Green's equation*

$$\partial\bar{\partial}\Xi(m, v) + g(n, v)\delta_\infty = [\omega(m, v)] \tag{3.1}$$

for a L^2_{-1}-form $\omega(m, v)$ on M of degree 2. Here $g(n, v)$ is given in (0.2).

(4) *When $m = 0$, $\Xi(0, v)$ is smooth in the upper half plane but is singular at the cusp ∞. As a current, it satisfies the Green's equation*

$$\partial\bar{\partial}\Xi(0, v) + \frac{1}{16\pi\sqrt{v}}\,\delta_\infty = [\omega(0, v)]$$

for a L^2_{-1}-form $\omega(0, v)$ on M of degree 2.

Because of this proposition, we can define the arithmetic Chow cycles with real coefficients $\hat{\mathcal{Z}}(m, v) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$ for $m \neq 0$ and $v > 0$ via

$$\hat{\mathcal{Z}}(m, v) = \begin{cases} (\mathcal{Z}(m), \Xi(m, v)) & \text{if } m > 0, \\ (0, \Xi(m, v)) & \text{if } -m > 0 \text{ is not a square}, \\ (g(n, v)\cdot\infty, \Xi(m, v)) & \text{if } -m = n^2 > 0. \end{cases} \tag{3.2}$$

To define $\hat{\mathcal{Z}}(0, v)$, we need the metrized Hodge bundle $\hat{\omega}$ on $\mathcal{M}$. Let $\mathcal{E}$ be the universal elliptic curve over $\mathcal{M}$ with zero section ε. Then the Hodge line bundle is $\omega = \varepsilon^*\Omega_{\mathcal{E}/\mathcal{M}}$. Notice that ω^2 is the relative differential bundle $\Omega_{\mathcal{M}/\mathbb{Z}}$, which is associated to modular forms of weight 2. So the Hodge bundle is associated to modular forms of weight one. The metric on $\omega_{\mathbb{C}}$ is defined as follows. For a section α of $\omega_{\mathbb{C}}$, α_z at $z \in \mathcal{M}(\mathbb{C})$ corresponds to a holomorphic 1-form on the associated elliptic curve $\mathcal{E}_z$, we define

$$\|\alpha_z\|^2 = \left|\frac{i}{2\pi} e^{-C} \int_{\mathcal{E}_z(\mathbb{C})} \alpha_z \wedge \bar{\alpha_z}\right| \tag{3.3}$$

where C is the constant as in (1.3). We remark that $\hat{\omega} = (\omega, \| \,\|) \in \widehat{\mathrm{Pic}}(\mathcal{M})$ has singularity at the cusp ∞. See [Bos2] and [Kun] for detailed discussion on this issue. In these papers, Bost and Kühn independently computed the self-intersection number of $\hat{\omega}$, and their result is

$$\langle \hat{\omega}, \hat{\omega} \rangle = \tfrac{1}{2}\zeta(-1) + \zeta'(-1) + \tfrac{1}{12}\, C, \tag{3.4}$$

where ζ is the usual Riemann zeta function and C is the constant in (1.3). In view of the map (2.7), we may view $\hat{\omega} \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M})$. Its first Chern form $c_1(\hat{\omega})$ is

$$\frac{1}{4\pi}\frac{dx \wedge dy}{y^2}.$$

Finally, we define

$$\hat{\mathcal{Z}}(0, v) = \Big(\frac{1}{16\pi\sqrt{v}}\infty, \Xi(0, v)\Big) - \hat{\omega} - (0, \log v) \in \widehat{\mathrm{CH}}^1_{\mathbb{R}}(\mathcal{M}) \tag{3.5}$$

and the generating function of arithmetic cycles

$$\hat{\phi}(\tau) = \sum_{m \in \mathbb{Z}} \hat{\mathcal{Z}}(m, v) q^m. \tag{3.6}$$

According to [Ku2], $\hat{\phi}(\tau)$ should be a modular form of weight 3/2 valued in the arithmetic Chow group. In particular, for any linear functional f on the arithmetic Chow group,

$$f(\hat{\phi}(\tau)) = \sum_{m \in \mathbb{Z}} f(\hat{\mathcal{Z}}(m, v)) q^m$$

should be a scalar modular form of weight 3/2. The main theorem below asserts that it is true when f is the degree map or the intersection map with $\hat{\omega}$. Here the degree map is given by

$$\deg \hat{D} = \int_{\mathcal{M}(\mathbb{C})} \omega_1(\hat{D}),$$

where $\omega_1(\hat{D}) = dd^c g + \delta_D$ if $\hat{D} = (D, g)$. To be more precise, we need to introduce the Eisenstein series. Following Hirzebruch and Zagier [HZ, pp. 91, 93] and using their notation, we let

$$E(\tau, s) = \sum_{\substack{m>0 \\ (m,2n)=1}} \frac{\left(\frac{n}{m}\right)\left(\frac{-1}{m}\right)^{1/2}}{(m\tau + n)^{3/2}|m\tau + n|^{2s}} \tag{3.7}$$

and

$$\mathcal{F}(\tau, s) = -\frac{1}{96}\Big((1 - i)E(s, \tau) - i\tau^{3/2}|\tau|^{-2s}E\Big(s, -\frac{1}{4\tau}\Big)\Big). \tag{3.8}$$

They are non-holomorphic modular forms of weight $\frac{3}{2}$. We renormalize it as

$$\mathcal{E}(\tau, s) = \frac{(\frac{1}{2}+s)\Lambda(1+2s)}{\Lambda(2)} \left(\frac{v}{2}\right)^{(1/2)(s-1/2)} \mathcal{F}(\tau, \tfrac{1}{2}(s-\tfrac{1}{2})) \tag{3.9}$$

We refer to [KRY1; KRY2] for adelic construction of such Eisenstein series, which seems more natural.

THEOREM 3.2. *Let the notation be as above.*

(1) $\mathcal{E}(\tau, s) = \mathcal{E}(\tau, -s)$.
(2) $\mathcal{E}(\tau, \frac{1}{2}) = E_{\text{Zagier}}(\tau) = 2\deg(\hat{\phi}(\tau))$.
(3) $\mathcal{E}'(\tau, \frac{1}{2}) = 4\phi_{\text{height}}(\tau)$, *where*

$$\phi_{\text{height}}(\tau) = \langle \hat{\phi}(\tau), \hat{\omega} \rangle = \sum_m \langle \hat{\mathcal{Z}}(m, v), \hat{\omega} \rangle q^m.$$

In particular, when $m > 0$ the unfolding of the height pairing gives

$$\langle \hat{\mathcal{Z}}(m, v), \hat{\omega} \rangle = h_{\text{Fal}}(Z(m)) + \frac{1}{8\pi} \int_{\mathcal{M}(\mathbb{C})} \Xi(m, v) \frac{dx\, dy}{y^2}. \tag{3.10}$$

So just like the degree, the generating function of the Faltings' height $\sum_{m>0} h_{\text{Fal}}(Z(m))q^m$ is part of a modular form of weight 3/2. Moreover, we also give some arithmetic interpretation of the negative terms of Zagier's Eisenstein series as well as its 'derivative'.

We would like to point out three interesting features of Theorem 3.2. Firstly $s = \frac{1}{2}$ is *not* the symmetric center although it is a critical point. Secondly, both the value and the derivative of the Eisenstein series have interesting arithmetic meanings. In particular, the derivative here is *not* the leading term but the second term. Finally, since Zagier's Eisenstein series has real numbers $g(n, v)$ as its Fourier coefficients, any arithmetic interpretation of the Eisenstein series as degree generating function would be forced to consider arithmetic Chow cycles with *real* coefficients as we did here and in [KRY2]. It is a little amusing to us that our Greens' function $\Xi(m, v)$, originally defined for the Heegner cycle $Z(m)$ for $m > 0$, gives such a cycle and matches up perfectly with the Fourier coefficients when $-m = n^2 > 0$.

COROLLARY 3.3. *Let* $\mathcal{E}(\tau, s)$ *be as in Theorem* 3.2. *Then, for* $m > 0$,

(1) $2\deg Z(m) = \mathcal{E}_m(\tau, \frac{1}{2})q^{-m}$;
(2) $4h_{\text{Fal}}(Z(m)) = \lim_{v\mapsto\infty} \mathcal{E}'_m(\tau, \frac{1}{2})q^{-m}$.

PROOF. (1) follows directly from Theorem 3.2. (2) follows from Theorem 3.2, (3.7) and the fact that the integral in (3.7) goes to zero when v goes to infinity. □

Now we describe the Fourier expansion of the Eisenstein series. Let

$$\Psi(a, b, t) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tr}(1+r)^{b-a-1} r^{a-1} dr \tag{3.11}$$

be the classical second confluent hypergeometric function for $a > 0$ and $t > 0$, and let

$$\Psi_n(s,t) = \Psi\big(\tfrac{1}{2}(1+n+s),\, 1+s,\, t\big). \tag{3.12}$$

This function satisfies the functional equation

$$\Psi_n(-s,t) = z^s \Psi_n(s,t). \tag{3.13}$$

Theorem 3.4. *The Eisenstein series $\mathcal{E}(\tau,s)$ in Theorem 3.2 has the following Fourier expansion*

$$\mathcal{E}(\tau,s) = \sum_{m \equiv 0,-1 \bmod 4} A_m(v,s) q^m,$$

where

(1) *for $m > 0$,*

$$A_m(v,s) = \Lambda(\tfrac{1}{2} - s, \chi_m)(4\pi m v)^{(1/2)(s-1/2)} \Psi_{-3/2}(s, 4\pi m v);$$

(2) *for $m < 0$,*

$$A_m(v,s) = \frac{(s^2 - \frac{1}{4})\Lambda(\frac{1}{2} - s, \chi_m)(4\pi |m| v)^{(1/2)(s-1/2)} \Psi_{\frac{3}{2}}(s, 4\pi |m| v)}{4\sqrt{\pi} e^{4\pi m v}};$$

(3) *the constant term is given by*

$$A_0(v,s) = -\frac{1}{2\pi}(G(s) + G(-s))$$

with

$$G(s) = (4v)^{(1/2)(1/2-s)}(s + \tfrac{1}{2})\Lambda(1+2s).$$

4. Construction of the Green's Function $\Xi(m,v)$

Although the construction is quite general [Ku1], we stick to the special case at hand.

Let

$$V = \{x = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \in M_2(\mathbb{Q}) : \operatorname{tr} x = 0\}, \tag{4.1}$$

with the quadratic form

$$Q(x) = \det x = ac - b^2.$$

It has signature $(1,2)$. Let D be the set of negative 2-planes in $V(\mathbb{R})$. Then D is in bijection with the upper half plane $\mathbb{H}$, given by

$$z = g(i) \in \mathbb{H} \;\longleftrightarrow\; z = \{g z_1 g^{-1}, g z_2 g^{-1}\} \in D$$

for any $g \in \mathrm{GL}_2(\mathbb{R})$, where

$$z_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here $\{v_1, v_2\}$ denotes the subspace of $V(\mathbb{R})$ spanned by v_1 and v_2 in $V(\mathbb{R})$. We will use z to stand for both the complex number in $\mathbb{H}$ and the associated negative two-plane by abuse of notation. Given $z \in \mathbb{H} = D$, one has the orthogonal decomposition

$$V(\mathbb{R}) = z \oplus z^{\perp}, \quad x = (\operatorname{pr}_z x, \operatorname{pr}_{z^\perp} x).$$

For $x \in V(\mathbb{R})$ as in (4.1) and $z \in \mathbb{H}$, define

$$R(x,z) = -(\operatorname{pr}_z x, \operatorname{pr}_z x) = \frac{1}{2}(az\bar{z} + b(z+\bar{z}) + c - \det x). \tag{4.2}$$

Then $R(x,z) \geq 0$ and it equals zero if and only if $z \perp x$. Notice that when $x > 0$ (meaning $Q(x) > 0$), $D_x = x^{\perp}$ is a negative 2-plane and thus a point in D. Notice that $R(x,z) = 0$ if and only if $z = D_x$ in this case.

Instead of the maximal integral lattice $M_2(\mathbb{Z}) \cap V$ used in [KRY2], we choose the lattice

$$L = \{x = \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix} : \, a, b, c \in \mathbb{Z}\} \subset V \tag{4.3}$$

(notice that $Q(x) = 4ac - b^2$), and for $m \neq 0$, let

$$L(m) = \{x \in L : \, \det x = m\}.$$

Then, for $m > 0$, one has the following identification

$$[L(m)/\operatorname{SL}_2(\mathbb{Z})] \leftrightarrow \mathcal{Z}(m)(\mathbb{C}), \quad x \mapsto D_x$$

where $x \in L(m)/\operatorname{SL}_2(\mathbb{Z})$ is counted with multiplicity $1/(\#\Gamma_x)$, and Γ_x is the stabilizer of x in $\operatorname{SL}_2(\mathbb{Z})$. For every integer m and real number $v > 0$, we define

$$\Xi(m,v)(z) = \frac{1}{2} \sum_{0 \neq x \in L(m)} \rho(x\sqrt{v}, z) \tag{4.4}$$

where

$$\rho(x,z) = \int_1^{\infty} e^{-2\pi R(x,z) r} \frac{dr}{r} = -\operatorname{Ei}(-2\pi R(x,z)). \tag{4.5}$$

Proposition 11.1 of [Ku1] asserts that as currents on D, one has

$$dd^c \rho(x,z) + \delta_{D_x} = [\phi_{KM}(x)] \tag{4.6}$$

where D_x is empty if $(x,x) \leq 0$ and

$$\phi_{KM}(x) = \left((x,z)^2 - \frac{1}{2\pi}\right) e^{-2\pi R(x,z)} \frac{i}{2} \frac{dz d\bar{z}}{(\operatorname{Im} z)^2} \tag{4.7}$$

is a smooth $(1,1)$-form in the upper half plane. Set

$$\omega(m,v) = \frac{1}{2} \sum_{0 \neq x \in L(m)} \phi_{KM}(x\sqrt{v}). \tag{4.8}$$

Note that, when $m = 0$, we sum over the set of nonzero null vectors $L(0) - \{0\}$. When $m > 0$, the series (4.8) is absolutely convergent and thus

$$dd^c \Xi(m, v) + \delta_{Z(m)} = [\omega(m, v)]. \tag{4.9}$$

This proves Proposition 3.1(1). The same is true when $-m > 0$ is not a square. However, when $-m \ge 0$ is a square, the series (4.8) is convergent not termwise integrable. It turns out that $\omega(m, v)$ has a logarithmic singularity at the cusp ∞ in this case and is in $L^1(\mathcal{M}(\mathbb{C}))$ according to Funke [Fu]. From this, one can check that $[\omega(m, v)]$ is a L^2_{-1}-current and $(\mathcal{Z}(m, v), \Xi(m, v))$ is an arithmetic divisor with real coefficient. Furthermore, Funke proved [Fu, Propositions 4.7, 4.8] that

$$\int_{\mathcal{M}(\mathbb{C})} \omega(m, v) = g(n, v) \tag{4.10}$$

in this case. This proves Proposition 3.1.

5. The proof of Theorem 3.2

Now we sketch a proof of Theorem 3.2 and refer to [KRY2] for details. The functional equation follows from Theorem 3.4 and functional equations (1.7) and (3.13).

Checking the identity $E_{\text{Zagier}}(\tau) = \deg \hat{\phi}(\tau)$ amounts to verifying that

$$\deg \hat{\mathcal{Z}}(m, v) = \int_{\mathcal{M}(\mathbb{C})} \omega(m, v) = \begin{cases} \frac{1}{2} H_0(m) & \text{if } m > 0, \\ 0 & \text{if } -m > 0 \text{ is not a square,} \\ g(n, v) & \text{if } -m = n^2 > 0, \\ 1/(16\pi\sqrt{v}) & \text{if } m = 0. \end{cases}$$

When $m < 0$, this is basically (4.10). When $m > 0$,

$$\deg \hat{\mathcal{Z}}(m, v) = \deg Z(m) = \tfrac{1}{2} H_0(m)$$

is just Theorem 1.1(1).

The identity $\mathcal{E}(\tau, \frac{1}{2}) = E_{\text{Zagier}}(\tau)$ follows from Theorem 3.4 directly. Indeed, when $m > 0$, the fact $\Psi_{-3/2}(\frac{1}{2}, t) = 1$ implies that

$$A_m(v, \tfrac{1}{2}) = \Lambda(0, \chi_m) q^m = E_{\text{Zagier}, m}(\tau).$$

When $-m > 0$, one has [KRY1, page 84]

$$\Psi_{\frac{3}{2}}(\frac{1}{2}, 4\pi|m|v) = \frac{e^{4\pi|m|v}}{\sqrt{4\pi|m|v}} \int_1^\infty e^{-4\pi|m|vr} r^{-3/2} dr.$$

So $A_m(v, \frac{1}{2}) = 0$ unless $\Lambda(s, \chi_m)$ has a pole at $s = 0$, i.e., $-m = n^2$ is a square. If $-m = n^2 > 0$, then

$$\Lambda(s, \chi_m) = \Lambda(s) n^s \prod_{p|n} b_p(n, s)$$

and

$$\lim_{s \longrightarrow 0} s\Lambda(-s, \chi_m) = n.$$

Therefore

$$A_m(v, \tfrac{1}{2}) = \frac{1}{8\pi\sqrt{v}} \int_1^\infty e^{-4\pi|m|vr} r^{-3/2} dr = E_{\text{Zagier},m}(\tau) q^{-m}.$$

The case $m = 0$ is similar.

Part (3) of Theorem 3.2 is new and is proved in a term-by-term manner. The case $m > 0$ is basically Theorem 1.1 together with a routine calculation of the integral

$$\int_{\mathcal{M}(\mathbb{C})} \Xi(m, v) \frac{dx\,dy}{y^2}. \tag{5.1}$$

The case where $-m > 0$ is not a square is also a routine calculation of the integral (5.1), and is done in [KRY1, Section 12]. The case $-m = 0$ involves the self-intersection of the Hodge bundle $\hat{\omega}$, which was given by (3.4).

Finally, when $-m = n^2 > 0$ is a square, the proof goes as follows. Notice [Bos2; Kun] that

$$\hat{c}_1(\hat{\omega}) = \tfrac{1}{12}(\infty, -\log|\Delta|^2) + (0, C) \tag{5.2}$$

where C is the constant defined in (1.3). So

$$\widehat{\mathcal{Z}}(m, v) = 12g(n, v)\hat{c}_1(\hat{\omega}) + (0, \beta(n, v)) \tag{5.3}$$

with

$$\beta(n, v) = \Xi(-n^2, v) - 12g(n, v)C - g(n, v)\log|\Delta(\tau)|^2. \tag{5.4}$$

Notice that $\beta(n, v)$ is smooth at ∞. Therefore

$$\langle \widehat{\mathcal{Z}}(m, v), \hat{\omega} \rangle = 12g(n, v)\langle \hat{\omega}, \hat{\omega} \rangle + \frac{1}{8\pi} \int_{\mathcal{M}(\mathbb{C})} \beta(n, v) \frac{dx\,dy}{y^2}. \tag{5.5}$$

Since the self-intersection $\langle \hat{\omega}, \hat{\omega} \rangle$ is computed by Bost and Kühn, the key is thus to compute the integral in (5.5) and compare it with $A'(-n^2, \frac{1}{2})$. The computation turns out to be quite involving and amusing to us.

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