Riemann–Finsler Geometry
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Synopses

1. Part One

- **Álvarez–Thompson**: This paper is a concise introduction to the theory of volumes on normed and Finsler spaces. The definitions of Holmes–Thompson, Busemann, and Benson–Gromov are studied and their convexity (ellipticity) properties are discussed in detail. The authors show how the theory of volumes provides a unified context for a diverse range of geometric inequalities. The article is intended for students and researchers in differential, integral, and convex geometry.

- **Bellettini**: Crystalline motion driven by the mean curvature is an evolution process arising in material science and phase transitions. It is an anisotropic flow in an ambient space endowed with a piecewise linear norm. For three-dimensional crystals, the crystalline mean curvature of a facet is defined and identified with the initial velocity in the evolution process. Facets with constant crystalline mean curvature are important because they are expected not to break or curve under the evolution. The problem of characterizing such facets is discussed.

2. Part Two

- **Aikou**: This article highlights the essential role played by Finsler metrics in complex differential geometry. It describes a few situations for which techniques based solely on Hermitian metrics are hopelessly inadequate. These include Kobayashi's characterization of negative holomorphic vector bundles over compact complex manifolds, in terms of the existence of negatively curved pseudoconvex Finsler metrics.

- **Chandler–Wong**: The authors present the proof of the Kobayashi conjecture (1960) on the hyperbolicity of generic algebraic surfaces of degree $d \geqslant 5$ in $\mathbb{P}^3$. They also address the Green–Griffiths conjecture (1979) that every holomorphic map $f : \mathbb{C} \to X$ to a surface X of general type is algebraically degenerate. Their paper establishes the latter for the *special* case where X is minimal, $\mathrm{Pic}(X) \cong \mathbb{Z}$, and $p_g(X) > 0$. The main tool used is their generalization of the classical Schwarz lemma for complex curves, to varieties of every dimension. In this crucial step, algebraic geometric arguments are used to construct a Finsler metric of logarithmic type, thereby reducing the problem to one in which a certain estimate (the lemma of logarithmic derivatives) is applicable.

3. Part Three

- **Bao–Robles**: Many recent developments have advanced our understanding of the flag and Ricci curvatures of Finsler metrics. This paper is a uniform presentation of these results and their underlying techniques. Included is a geometric definition of Einstein–Finsler metrics. Einstein metrics of Randers type are studied via their representation as solutions to Zermelo navigation on Riemannian manifolds. This viewpoint leads to the classification of *all* constant flag curvature Randers metrics. It also yields a Schur lemma, and settles a question of rigidity in three dimensions, for Einstein–Randers metrics. The theory is illustrated with a diverse array of explicit examples.
- **Rademacher**: The author shows in detail how the classical Sphere Theorem in Riemannian geometry is extended to the case of nonreversible Finsler metrics. The proof hinges on a fruitful definition of the notion of reversibility, and how that can be used to effect some crucial estimates, such as the injectivity radius, the length of nonminimal geodesics between two fixed points, and the length of nonconstant geodesic loops. The proof also capitalizes on an idea of Klingenberg: that Morse theory of the energy functional allows us to circumvent Toponogov's comparison theorem for geodesic triangles. This idea renders irrelevant the "handicap" that, in Finsler geometry proper, there is no Toponogov's theorem.
- **Shen**: This paper is about the interaction between the generalized Riemann curvature and other non-Riemannian quantities in Finsler geometry. The latter include the S-curvature, whose vanishing is equivalent to having constant "distortion" along each geodesic. The S-curvature quantifies some aspect of the change in the Minkowski model ("infinitesimal color pattern") as one moves from one tangent space to another, along geodesics on a Finslerian landscape. The information it provides is complementary to that supplied by a certain contracted version of the Berwald curvature. These theoretical constructs are exemplified by Finsler metrics with a broad array of special curvature properties. The author also proves a number of local and global theorems using certain curvature equations along geodesics.