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Volumes on Normed and Finsler Spaces

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1. Introduction

The study of volumes and areas on normed and Finsler spaces is a relatively new field that comprises and unifies large domains of convexity, geometric tomography, and integral geometry. It opens many classical unsolved problems in these fields to powerful techniques in global differential geometry, and suggests new challenging problems that are delightfully geometric and simple to state.

Keywords: Minkowski geometry, Hausdorff measure, Holmes–Thompson volume, Finsler manifold, isoperimetric inequality.

The theory starts with a simple question: How does one measure volume on a finite-dimensional normed space? At first sight, this question may seem either unmotivated or trivial: normed spaces are metric spaces and we can measure volume using the Hausdorff measure, period. However, if one starts asking simple, naive questions one discovers the depth of the problem. Even if one is unwilling to consider that definitions of volume other than the Hausdorff measure are not only possible but may even be better, one is faced with questions such as these: What is the $(n-1)$-dimensional Hausdorff measure of the unit sphere of an n-dimensional normed space? Do flat regions minimize area? For what normed spaces are metric balls also the solutions of the isoperimetric problem? These questions, first posed in convex-geometric language by Busemann and Petty [1956], are still open, at least in their full generality. However, one should not assume too quickly that the subject is impossible. Some beautiful results and striking connections have been found. For example, the fact that the $(n-1)$-Hausdorff measure in a normed space determines the norm is equivalent to the fact that the areas of the central sections determine a convex body that is symmetric with respect to the origin. This, in turn, follows from the study of the spherical Radon transform. The fact that regions in hyperplanes are area-minimizing is equivalent to the fact that the intersection body of a convex body that is symmetric with respect to the origin is also convex.

But the true interest of the theory can only be appreciated if one is willing to challenge Busemann's dictum that the natural volume in a normed or Finsler space is the Hausdorff measure. Indeed, thinking of a normed or Finsler space as an anisotropic medium where the speed of a light ray depends on its direction, we are led to consider a completely different notion of volume, which has become known as the *Holmes–Thompson volume.* This notion of volume, introduced in [Holmes and Thompson 1979], uncovers striking connections between integral geometry, convexity, and Hamiltonian systems. For example, in a recent series of papers, [Schneider and Wieacker 1997], [Alvarez and Fernandes 1998], [Alvarez and Fernandes 1999], [Schneider 2001], and [Schneider 2002], it was shown that the classical integral geometric formulas in Euclidean spaces can be extended to normed and even to projective Finsler spaces (the solutions of Hilbert's fourth problem) if the areas of submanifolds are measured with the Holmes–Thompson definition. That extensions of this kind are not possible with the Busemann definition was shown by Schneider [Schneider 2001].

Using Finsler techniques, Burago and Ivanov [2001] have proved that a flat two-dimensional disc in a finite-dimensional normed space minimizes area among all other immersed discs with the same boundary. Ivanov [2001] has shown, among other things, that Pu's isosystolic inequality for Riemannian metrics in the projective plane extends to Finsler metrics, and the Finslerian extension of Berger's infinitesimal isosystolic inequality for Riemannian metrics on real projective spaces of arbitrary dimension has been proved by Álvarez [2002].

Despite these and other recent interdisciplinary results, we believe that the most surprising feature of the Holmes–Thompson definition is the way in which it organizes a large portion of convexity into a coherent theory. For example, the sharp upper bound for the volume of the unit ball of a normed space is given by the Blaschke–Santaló inequality; the conjectured sharp lower bound is Mahler's conjecture; and the reconstruction of the norm from the area functional is equivalent to the famous Minkowski's problem of reconstructing a convex body from the knowledge of its curvature as a function of its unit normals.

In this paper, we have attempted to provide students and researchers in Finsler and global differential geometry with a clear and concise introduction to the theory of volumes on normed and Finsler spaces. To do this, we have avoided the temptation to use auxiliary Euclidean structures to describe the various concepts and constructions. While these auxiliary structures may render some of the proofs simpler, they hinder the understanding of the subject and make the application of the ideas and techniques to Finsler spaces much more cumbersome. We also believe that by presenting the results and techniques in intrinsic terms we can make some of the beautiful results of convexity more accessible and enticing to differential geometers.

In the course of our writing we had to make some tough choices as to what material should be left out as either too advanced or too specialized. At the end we decided that we would concentrate on the basic questions and techniques of the theory, while doing our best to exhibit the general abstract framework that makes the theory of volumes on normed spaces into a sort of Grand Unified Theory for many problems in convexity and Finsler geometry. As a result there is little Finsler geometry *per se* in the pages that follow. However, just as tensors, forms, spinors, and Clifford algebras developed in invariant form have immediate use in Riemannian geometry, the more geometric constructions with norms, convex bodies, and k-volume densities that make up the heart of this paper have immediate applications to Finsler geometry.

2. A Short Review of the Geometry of Normed Spaces

This section is a quick review of the geometry of finite-dimensional normed spaces adapted to the needs and language of Finsler geometry. Unless stated otherwise, *all vector spaces in this article are real and finite-dimensional.* We suggest that the reader merely browse through this section and come back to it if and when it becomes necessary.

Definition 2.1. A *norm* on a vector space X is a function

$$\| \cdot \| : X \to [0, \infty)$$

satisfying the following properties of positivity, homogeneity, and convexity:

(1) If $\|\boldsymbol{x}\| = 0$, then $\boldsymbol{x} = \mathbf{0}$;

(2) If t is a real number, then $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$;
(3) For any two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in X, $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

If $(X, \|\cdot\|)$ is a finite-dimensional normed space, the set

$$B_X := \{\boldsymbol{x} \in X : \|\boldsymbol{x}\| \leq 1\}$$

is the *unit ball* of X and the boundary of B_X, S_X, is its *unit sphere.* Notice that B_X is a compact, convex set with nonempty interior. In short, it is a *convex body* in X. Moreover, it is symmetric with respect to the origin. Conversely, if $B \subset X$ is a *centered convex body* (*i.e.,* a convex body symmetric with respect to the origin), it is the unit ball of the norm

$$\|\boldsymbol{x}\| := \inf\{t \geq 0 : t\boldsymbol{y} = \boldsymbol{x} \text{ for some } \boldsymbol{y} \in B\}.$$

We shall now describe various classes of normed spaces that will appear repeatedly throughout the paper.

Euclidean spaces. A Euclidean structure on a finite-dimensional vector space X is a choice of a symmetric, positive-definite quadratic form $\mathcal{Q} : X \to \mathbb{R}$. The normed space $(X, \mathcal{Q}^{1/2})$ will be called a Euclidean space. Note that a normed space is Euclidean if and only if its unit sphere is an ellipsoid, which is if and only if the norm satisfies the parallelogram identity:

$$\|\boldsymbol{x}+\boldsymbol{y}\|^2 + \|\boldsymbol{x}-\boldsymbol{y}\|^2 = 2\|\boldsymbol{x}\|^2 + 2\|\boldsymbol{y}\|^2.$$

EXERCISE 2.2. Let $B \subset \mathbb{R}^n$ be a convex body symmetric with respect to the origin. Show that if the intersection of B with every 2-dimensional plane passing through the origin is an ellipse, then B is an ellipsoid.

The ℓ_p spaces. If $p \geq 1$ is a real number, the function

$$\|\boldsymbol{x}\|_p := \left(|x_1|^p + \cdots + |x_n|^p\right)^{1/p}$$

is a norm on $\mathbb{R}^n$. When p tends to infinity, $\|\boldsymbol{x}\|_p$ converges to

$$\|\boldsymbol{x}\|_\infty := \max\{|x_1|, \ldots, |x_n|\}.$$

The normed space $(\mathbb{R}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$, is denoted by ℓ_p^n.

The unit ball of ℓ_∞^n is the n-dimensional cube of side length two, while the unit ball of ℓ_1^n is the n-dimensional *cross-polytope.* In general, norms whose unit balls are polytopes are called *crystalline norms.*

Subspaces of $L_1([0,1], dx)$. The space of measurable functions $f : [0,1] \to \mathbb{R}$ satisfying

$$\|f\| := \int_0^1 |f(x)|\, dx < \infty$$

is a normed space denoted by $L_1([0,1], dx)$. The geometry of finite-dimensional subspaces of $L_1([0,1], dx)$ is closely related to problems of volume, area, and integral geometry on normed and Finsler spaces. In the next few paragraphs, we will summarize the properties of these subspaces that will be used in the rest of the paper. For proofs, references, and to learn more about hypermetric spaces, we recommend the landmark paper [Bolker 1969], as well as the surveys [Schneider and Weil 1983] and [Goodey and Weil 1993].

First we begin with a beautiful metric characterization of the subspaces of $L_1([0,1], dx)$.

Definition 2.3. A metric space (M, d) is said to be *hypermetric* if it satisfies the following stronger version of the triangle inequality: If $m_1, \dots, m_k$ are elements of M and $b_1, \dots, b_k$ are integers with $\sum_i b_i = 1$, then

$$\sum_{i,j=1}^{k} d(m_i, m_j) b_i b_j \leq 0.$$

Theorem 2.4. *A finite-dimensional normed space is hypermetric if and only if it is isometric to a subspace of $L_1([0,1], dx)$.*

An important analytic characterization of a hypermetric normed space can be given through the Fourier transform of its norm:

Theorem 2.5. *A norm on $\mathbb{R}^n$ is hypermetric if and only if its distributional Fourier transform is a nonnegative measure.*

The characterizations above, important as they are, are hard to grasp at first sight. A much more visual approach will be given after we review the duality of normed spaces.

Minkowski spaces. Minkowski spaces are normed spaces with strict smoothness and convexity properties. In precise terms, a norm $\|\cdot\|$ on a vector space X is said to be a *Minkowski norm* if it is smooth outside the origin and the Hessian of the function $\|\cdot\|^2$ at every nonzero point is a positive-definite quadratic form.

The unit sphere of a Minkowski space X is a smooth convex hypersurface S_X such that for any Euclidean structure on X the principal curvatures of S_X are positive.

2.1. Maps between normed spaces. An important feature of the geometry of normed spaces is that the space of linear maps between two normed spaces carries a natural norm.

Definition 2.6. If $T : X \to Y$ is a linear map between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we define the *norm of* T as the supremum of $\|T\boldsymbol{x}\|_Y$ taken over all vectors $\boldsymbol{x} \in X$ with $\|\boldsymbol{x}\|_X \leq 1$.

A linear map $T : X \to Y$ is said to be *short* if its norm is less than or equal to one. In other words, a short linear map does not increase distances. Two important types of short linear maps between normed spaces are isometric embeddings and isometric submersions:

Definition 2.7. An injective linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be an *isometric embedding* if $\|T\boldsymbol{x}\|_Y = \|\boldsymbol{x}\|_X$ for all vectors $\boldsymbol{x} \in X$.

Definition 2.8. A surjective linear map $T : X \to Y$ between normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ is said to be an *isometric submersion* if

$$\|T\boldsymbol{x}\|_Y = \inf\{\|\boldsymbol{v}\|_X : \boldsymbol{v} \in X \text{ and } T\boldsymbol{v} = T\boldsymbol{x}\}$$

for all vectors $x \in X$.

In terms of the unit balls, $T : X \to Y$ is an isometric embedding if and only if $T(B_X) = T(X) \cap B_Y$, and T is an isometric submersion if and only if $T(B_X) = B_Y$.

2.2. Dual spaces and polar bodies. From the previous paragraph, we see that if $(X, \|\cdot\|)$ is a normed space, then the set of all linear maps onto the one-dimensional normed space $(\mathbb{R}, |\cdot|)$ carries a natural norm. The resulting normed space is called the *dual* of $(X, \|\cdot\|)$ and is denoted by $(X^*, \|\cdot\|^*)$. It is easy to see that the double dual (*i.e.,* the dual of the dual) of a finite-dimensional normed space can be naturally identified with the space itself. The unit ball of $(X^*, \|\cdot\|^*)$ is said to be the *polar* of the unit ball of $(X, \|\cdot\|)$.

Example. Hölder's inequality implies that if $p > 1$, the dual of ℓ_p^n is ℓ_q^n, where $1/p + 1/q = 1$. Likewise, it is easy to see that the dual of ℓ_1^n is ℓ_∞^n.

If $T : X \mapsto Y$ is a linear map then the *dual map* $T^* : Y^* \mapsto X^*$ is defined by

$$(T^*\boldsymbol{\xi})(\boldsymbol{x}) = \boldsymbol{\xi}(T\boldsymbol{x}).$$

Note that $\|T^*\| = \|T\|$.

Exercise 2.9. Show that if $T : X \to Y$ is an isometric embedding between normed spaces X and Y, the dual map $T^* : Y^* \to X^*$ is an isometric submersion.

Many of the geometric constructions in convex geometry and the geometry of normed spaces are functorial. More precisely, if we denote by $\mathcal{N}$ the category whose objects are finite-dimensional normed spaces and whose morphisms are short linear maps, many classical constructions define functors from $\mathcal{N}$ to itself.

Exercise 2.10. Show that the assignment $(X, \|\cdot\|) \mapsto (X^*, \|\cdot\|^*)$ is a contravariant functor from $\mathcal{N}$ to $\mathcal{N}$.

Duals of hypermetric normed spaces. As advertised earlier in this section, the notion of duality allows us to give a more geometric characterization of hypermetric spaces.

Definition 2.11. A polytope in a vector space X is said to be a *zonotope* if all of its faces are symmetric. A convex body is said to be a *zonoid* if it is the limit (in the Hausdorff topology) of zonotopes.

Notice that an n-dimensional cube, as well as all its linear projections, are zonotopes. In fact, it can be shown that any zonotope is the linear projection of a cube (see, for example, Theorem 3.3 in [Bolker 1969]).

THEOREM 2.12. *Let X be a finite-dimensional normed space with unit ball B_X. The dual of X is hypermetric if and only if B_X is a zonoid.*

Notice that this immediately implies that the space ℓ_1^n, $n \geq 1$, is hypermetric.

Duality in Minkowski spaces. If $(X, \|\cdot\|)$ is a Minkowski space, the differential of the function $L := \|\cdot\|^2/2$,

$$dL(\boldsymbol{x})(\boldsymbol{y}) := \frac{1}{2}\frac{d}{dt}\|\boldsymbol{x} + t\boldsymbol{y}\|^2_{t=0},$$

is a continuous linear map from X to X^* that is smooth outside the origin and homogeneous of degree one. This map is usually called the *Legendre transform,* although that term is also used to describe some related concepts (see, for example, § 2.2 in [Hörmander 1994]). The following exercise describes the most important properties of the Legendre transform.

EXERCISE 2.13. Let $(X, \|\cdot\|)$ be a Minkowski space and let

$$\mathcal{L} : X \setminus \mathbf{0} \to X^* \setminus \mathbf{0}$$

be its Legendre transform.

(1) Show that if $\boldsymbol{x} \in X$ is a unit vector, then $\mathcal{L}(\boldsymbol{x})$ is the unique covector $\boldsymbol{\xi} \in X^*$ such that the equation $\boldsymbol{\xi} \cdot \boldsymbol{y} = 1$ describes the tangent plane to the unit sphere S_X at the point $\boldsymbol{x}$.
(2) Show that the Legendre transform defines a diffeomorphism between the unit sphere and its polar.
(3) Show that the inverse of the Legendre transform from $X \setminus \mathbf{0}$ to $X^* \setminus \mathbf{0}$ is just the Legendre transform from $X^* \setminus \mathbf{0}$ to $X \setminus \mathbf{0}$.
(4) Show that the Legendre transform is linear if and only if X is a Euclidean space.

EXERCISE 2.14. Show that a normed space is a Minkowski space if its unit sphere and the unit sphere of its dual are smooth.

2.3. Sociology of normed spaces. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on a finite-dimensional vector space X, it is easy to see that there are positive numbers m and M such that

$$m\|\cdot\|_2 \leq \|\cdot\|_1 \leq M\|\cdot\|_2.$$

If we take the numbers m and M such that the inequalities are sharp, then $\log(M/m)$ is a good measure of how far away one norm is from the other.

For example, the following well-known result states that we can always approximate a norm by one whose unit sphere is a polytope or by one such that its unit sphere and the unit sphere of its dual are smooth.

PROPOSITION 2.15. *Let $\|\cdot\|$ be a norm on the finite-dimensional vector space X. Given $\varepsilon > 0$, there exist a crystalline norm $\|\cdot\|_1$ and a Minkowski norm $\|\cdot\|_2$ on X such that*

$$\|\cdot\|_1 \leq \|\cdot\| \leq (1+\varepsilon)\|\cdot\|_1,$$
$$\|\cdot\|_2 \leq \|\cdot\| \leq (1+\varepsilon)\|\cdot\|_2.$$

For a short proof see Lemma 2.3.2 in [Hörmander 1994] .

In many circumstances, one wants to measure how far is one normed space from being isometric to another. The straightforward adaptation of the previous idea leads us to the following notion:

DEFINITION 2.16. The *Banach–Mazur distance* between n-dimensional normed spaces X and Y, is the infimum of the numbers $\log(\|T\|\|T^{-1}\|)$, where T ranges over all invertible linear maps from X to Y.

Notice that the Banach–Mazur distance is a distance on the set of isometry classes of n-dimensional normed spaces: two such spaces are at distance zero if and only if they are isometric.

An important question is to determine how far a general n-dimensional normed space is from being Euclidean. The answer rests on two results of independent interest:

THEOREM 2.17 (LOEWNER). *If B is a convex body in an n-dimensional vector space X, there exists a unique n-dimensional ellipsoid $E \subset B$ such that for any Lebesgue measure on X, the ratio* $\mathrm{vol}(B)/\mathrm{vol}(E)$ *is minimal.*

THEOREM 2.18 [John 1948]. *Let X be an n-dimensional normed space with unit ball B. If $E \subset B$ is the Loewner ellipsoid of B, then*

$$E \subset B \subset \sqrt{n}E.$$

EXERCISE 2.19. Show that the Banach–Mazur distance from an n-dimensional normed space to a Euclidean space is at most $\log(n)/2$.

The structure of the set of isometry classes of n-dimensional normed spaces is given by the following theorem (see [Thompson 1996, page 73] for references and some of the history on the subject):

THEOREM 2.20. *The set of isometry classes of n-dimensional normed spaces, $\mathcal{M}_n$, provided with the Banach–Mazur distance is a compact, connected metric space.*

The *Banach–Mazur compactum,* $\mathcal{M}_n$, enters naturally into Finsler geometry by the following construction: Let $\pi : \zeta \to M$ be a vector bundle with n-dimensional fibers such that every fiber $\zeta_m = \pi^{-1}(m)$ carries a norm that varies continuously with the base point (a *Finsler bundle*). The (continuous) map

$$\mathcal{I} : M \longrightarrow \mathcal{M}_n$$

that assigns to each point $m \in M$ the isometry class of ζ_m measures how the norms vary from point to point.

Currently, there are not many results that describe the map $\mathcal{I}$ under different geometric and/or topological hypotheses on the bundle. However the following exercise (and its extension in [Gromov 1967]) shows that such results are possible.

Exercise 2.21. Let $\pi : \zeta \to S^2$ be a Finsler bundle whose fibers are 2-dimensional. Show that if the bundle is nontrivial and the map $\mathcal{I}$ is constant, then the image of S^2 under $\mathcal{I}$ is the isometry class of 2-dimensional Euclidean spaces.

A corollary of this exercise is that if X is a three-dimensional normed space such that all its two-dimensional subspaces are isometric, then X is Euclidean. Another interesting corollary is that a Berwald (Finsler) metric on S^2 must be Riemannian.

3. Volumes on Normed Spaces

In defining the notion of volume on normed spaces, it is best to adopt an axiomatic approach. We shall impose some minimal set of conditions that are reasonable and then try to find out to what extent they can be satisfied, and to what point they determine our choices.

In a normed space, all translations are isometries. This suggests that we require the volume of a set to be invariant under translations. Since any finite-dimensional normed space is a locally compact, commutative group, we can apply the following theorem of Haar:

Theorem 3.1. *If μ is a translation-invariant measure on $\mathbb{R}^n$ for which all compact sets have finite measure and all open sets have positive measure*, *then μ is a constant multiple of the Lebesgue measure*.

Proofs of this theorem can be found in many places. A full account is given in [Cohn 1980] and an abbreviated version in [Thompson 1996].

In the light of Haar's theorem, in order to give a definition of volume in *every* normed space, we must assign to every normed space X a multiple of the Lebesgue measure. Since the Lebesgue measure is not intrinsically defined (it depends on a choice of basis for X), it is best to describe this assignment as a choice of a norm μ in the 1-dimensional vector space $\Lambda^n X$, where n is the dimension of X; if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in X$, we define $\mu(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)$ as the volume of the parallelotope formed by these vectors.

Another natural requirement is *monotonicity:* if X and Y are n-dimensional normed spaces and $T : X \to Y$ is a short linear map (*i.e.,* a linear map that does not increase distances), we require that T does not increase volumes. Notice that this implies that isometries between normed spaces are volume-preserving.

The monotonicity requirement makes a definition of volume on normed spaces into a functor from $\mathcal{N}$ to itself that takes the n-dimensional normed space $(X, \|\cdot\|)$ to the 1-dimensional normed space $(\Lambda^n X, \mu)$. While we shall often abandon this viewpoint, it is a guiding principle throughout the paper with which we would like to acquaint the reader early on.

Definition 3.2. A *definition of volume* on normed spaces assigns to every n-dimensional, $n \geq 1$, normed space X a normed space $(\Lambda^n X, \mu_X)$ with the following properties:

(1) If X and Y are n-dimensional normed spaces and $T : X \to Y$ is a short linear map , then the induced linear map $T_* : \Lambda^n X \to \Lambda^n Y$ is also short.
(2) The map $X \mapsto (\Lambda^n X, \mu_X)$ is continuous with respect to the topology induced by the Banach–Mazur distance.
(3) If X is Euclidean, then μ_X is the standard Euclidean volume on X.

Before presenting the principal definitions of volume in normed spaces, let us make the first link between these concepts and the affine geometry of convex bodies.

Exercise 3.3. Assume we have a definition of volume in normed spaces and use it to assign a number to any centrally symmetric convex body $B \subset \mathbb{R}^n$ by the following procedure: Consider $\mathbb{R}^n$ as the normed space X whose unit ball is B and compute

$$\mathcal{V}(B) := \mu_X(B) = \int_B \mu_X .$$

Show that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear map, then $\mathcal{V}(B) = \mathcal{V}(T(B))$, and write the monotonicity condition in terms of the affine invariant $\mathcal{V}$.

Notice that we can turn the tables and start by considering a suitable affine invariant $\mathcal{V}$ of centered convex bodies and give a definition of volume in normed spaces by prescribing that the volume of the unit ball B of a normed space X be given by $\mathcal{V}(B)$.

Exercise 3.4. Let μ be a definition of volume for 2-dimensional normed spaces. Use John's theorem to show that if B is the unit disc of a two-dimensional normed space X, then $\pi/2 \leq \mu_X(B) \leq 2\pi$.

3.1. Examples of definitions of volume in normed spaces. The study of the four definitions of volume we shall describe below makes up the most important part of the theory of volumes on normed and Finsler spaces.

The Busemann definition. The Busemann volume of an n-dimensional normed space is that multiple of the Lebesgue measure for which the volume of the unit ball equals the volume of the Euclidean unit ball in dimension n, ε_n, . In other words, we have chosen as our affine invariant the constant ε_n, where n is the dimension of the space.

Another way to define the Busemann volume of a normed space X is by setting

$$\mu^b(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n) = \frac{\varepsilon_n}{\mathrm{vol}(B; \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)},$$

where the notation $\mathrm{vol}(B; \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)$ indicates the volume of B in the Lebesgue measure determined by the basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$.

Using Brunn–Minkowski theory, Busemann showed in [1947] that the Busemann volume of an n-dimensional normed space equals its n-dimensional Hausdorff measure. Hence, from the viewpoint of metric geometry, this is a very natural definition.

Exercise 3.5. Show that the Busemann definition of volume satisfies the axioms in Definition 3.2.

The Holmes–Thompson definition. Let X be an n-dimensional normed space and let $B^* \subset X^*$ be the dual unit ball. If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is a basis of X and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ is its dual basis, define

$$\mu^{\mathrm{ht}}(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n) := \varepsilon_n^{-1}\ \mathrm{vol}(B^*; \boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_n).$$

Another way of defining the Holmes–Thompson volume is by considering the set $B \times B^*$ in the product space $X \times X^*$. Since $X \times X^*$ has a natural symplectic structure defined by

$$\omega((\boldsymbol{x}_1, \boldsymbol{\xi}_1), (\boldsymbol{x}_2, \boldsymbol{\xi}_2)) := \boldsymbol{\xi}_2(\boldsymbol{x}_1) - \boldsymbol{\xi}_1(\boldsymbol{x}_2),$$

it has a canonical volume (the *symplectic* or *Liouville* volume) defined by the n-th exterior power ω^n of ω, divided by $n!$. The Holmes–Thompson volume of the n-dimensional normed space X is the multiple of the Lebesgue measure for which the volume of the unit ball equals the Liouville volume of $B \times B^*$ divided by the volume of the Euclidean unit ball of dimension n. We mention in passing that in convex geometry it is usual to denote the Liouville volume of $B \times B^*$ as the *volume product* of B, $\mathrm{vp}(B)$.

The Holmes–Thompson definition — introduced in [Holmes and Thompson 1979] — was originally motivated by purely geometric considerations. However, from the physical point of view it is the natural definition of volume if we think of normed spaces as homogeneous, anisotropic media: media in which the speed of light varies with the direction of the light ray, but not with the point at which the propagation of light originates.

It is interesting to remark that the Busemann definition and the Holmes–Thompson definition are *dual functors:* to obtain the Holmes–Thompson volume of an n-dimensional normed space X we pass to the dual normed space X^*, we apply the "Busemann functor" to obtain $(\Lambda^n X^*, \mu^b_{X^*})$ and then pass to the dual of the normed space $(\Lambda^n X^*, \mu^b_{X^*})$.

Exercise 3.6. Consider a definition of volume $(X, \|\cdot\|) \mapsto (\Lambda^n X, \mu_X)$, where n is the dimension of X, and define its *dual definition* by the map $(X, \|\cdot\|) \mapsto$

$(\Lambda^n X, \mu^*_X) := (\Lambda^n X^*, \mu_{X^*})^*$. Show that μ^* also satisfies the axioms in Definition 3.2.

The notion of duality is somewhat mysterious and is closely related to the duality between intersections and projections proposed in [Lutwak 1988], and which led to the development of the dual Brunn–Minkowski theory. We shall have a little more to say about this duality after presenting a second dual pair of volume definitions.

Gromov's mass. If X is an n-dimensional normed space, define $\mu^m : \Lambda^n X \to [0, \infty)$ by the formula

$$\mu^m(a) := \inf\left\{ \prod_{i=1}^{n} \|\boldsymbol{x}_i\| : \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n = a \right\}.$$

Another way to define the mass of an n-dimensional normed space X is as the multiple of the Lebesgue measure for which the volume of the maximal cross-polytope inscribed to the unit ball is $2^n/n!$.

EXERCISE 3.7. Consider the 2-dimensional normed space whose unit disc D is a regular hexagon. What is $\mu^m(D)$?

The Benson definition or Gromov's mass∗. One way to make the Benson definition is as the dual of mass: given an n-dimensional normed space X together with a basis $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, we take the dual basis $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ in X^* and define

$$\mu^{m*}_X(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n) := \frac{1}{\mu^m_{X^*}(\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_n)}.$$

This is Gromov's definition [1983]. Benson [1962] originally defined the mass∗ of an n-dimensional normed space as the multiple of the Lebesgue measure for which the volume of a minimal parallelotope circumscribed to the unit ball is 2^n.

EXERCISE 3.8. Consider the 2-dimensional normed space whose unit disc D is a regular hexagon. What is $\mu^{m*}(D)$?

The following exercise gives a third characterization of mass∗.

EXERCISE 3.9. Let X be an n-dimensional normed space and let B be its unit ball. A basis $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ of X^* is said to be *short* if $|\boldsymbol{\xi}_i(\boldsymbol{x})| \le 1$ for all $\boldsymbol{x} \in B$ and all i, $1 \le i \le n$ (*i.e.,* if all the vectors in the basis are in the dual unit ball). Show that for any n-vector $a \in \Lambda^n X$

$$\mu^{m*}(a) = \sup\{|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_n\ (a)| : \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n \text{ is a short basis of } X^*\}$$

It is not hard to come up with other definitions of volume. For example, instead of considering inscribed cross-polytopes and circumscribed parallelotopes one might consider maximal inscribed or minimal circumscribed ellipsoids (as in Loewner's theorem cited above) and then specify the volume of either to be

ε_n. However, as we shall see in the next two sections, a good definition of volume must satisfy some additional conditions that are very hard to verify. The examples given above are important mainly because their study provides a common context to many problems in convex, integral, and differential geometry.

3.2. The volume of the unit ball. If we are given a definition of volume and a normed space, we would like to compute the volume of the unit ball. This is, of course, trivial if we work with the Busemann definition, but for the other definitions it is a challenging problem. Let us start with some simple experiments.

EXAMPLE 3.10. In the table below we use 7 different norms in $\mathbb{R}^3$ whose unit balls are, in order, the Euclidean unit ball; the cube with vertices at $(\pm 1, \pm 1, \pm 1)$; the octahedron with vertices at $\pm(1,0,0)$, $\pm(0,1,0)$, $\pm(0,0,1)$; the right cylinder over the unit circle in the xy-plane and with $-1 \le z \le 1$; its dual, the double cone which is the convex hull of the unit circle in the xy-plane and the points $\pm(0,0,1)$; the affine image of the cuboctahedron that has vertices at $\pm(1,0,0)$, $\pm(0,1,0)$, $\pm(0,0,1)$, $\pm(1,-1,0)$, $\pm(1,0,-1)$, $\pm(0,1,-1)$; and its dual, the affine image of the rhombic dodecahedron, that has vertices at $\pm(1,1,1)$, $\pm(0,1,1)$, $\pm(1,0,1)$, $\pm(1,1,0)$, $\pm(0,0,1)$, $\pm(0,1,0)$, $\pm(1,0,0)$. These are listed in the first column. In the subsequent columns are the volumes of each unit ball using the different definitions of volume.

The ball B	$\mu^b(B)$	$\mu^{\mathrm{ht}}(B)$	$\mu^{m*}(B)$	$\mu^m(B)$
ball	$4\pi/3$	$4\pi/3$	$4\pi/3$	$4\pi/3$
cube	$4\pi/3$	$8/\pi$	8	2
octahedron	$4\pi/3$	$8/\pi$	$16/3$	$4/3$
cylinder	$4\pi/3$	π	2π	π
double cone	$4\pi/3$	π	$4\pi/3$	$2\pi/3$
cuboctahedron	$4\pi/3$	$10/\pi$	$20/3$	$10/3$
rhombic dodecahedron	$4\pi/3$	$10/\pi$	4	2

EXERCISE 3.11. Verify these numbers.

Given a definition of volume, an interesting problem is to determine sharp upper and lower bounds for the volume of the unit balls of n-dimensional normed spaces. In the case of the Holmes–Thompson definition, this question has a classical reformulation: give sharp upper and lower bounds for the volume product of an n-dimensional centrally-symmetric convex body.

THEOREM 3.12 (BLASCHKE–SANTALÓ INEQUALITY). *The Holmes–Thompson volume of the unit ball of an n-dimensional normed space is less than or equal to the volume of the Euclidean unit ball of dimension n. Moreover, equality holds if and only if the space is Euclidean.*

The sharp lower bound for the Holmes–Thompson volume of unit balls is a reformulation of a long-standing conjecture of Mahler [1939]:

Conjecture. *The Holmes–Thompson volume of the unit ball of an n-dimensional normed space is greater than or equal to $4^n/\varepsilon_n n!$. Moreover, equality holds if and only if the unit ball is a parallelotope or a cross-polytope.*

This conjecture has been verified by Mahler [1939] in the two-dimensional case and by Reisner [1985, 1986] in the case when either the normed space or its dual is hypermetric.

For $\mu^{m*}(B)$ the upper bound of 2^n is attained for a parallelotope and for $\mu^m(B)$ the equivalent lower bound of $2^n/n!$ is attained by a cross-polytope. One also has $\mu^{m*}(B) \geq 2^n/n!$ and $\mu^m(B) \leq 2^n$ but these are far from optimal; better bounds will be obtained after studying the relationship between the different definitions of volume.

3.3. Relationship between the definitions of volume. There are several relationships between the various measures we are considering. For example, the Blaschke–Santaló inequality is clearly equivalent to the following theorem:

Theorem 3.13. *If X is an n-dimensional normed space, then $\mu_X^{\mathrm{ht}} \leq \mu_X^b$ with equality if and only if X is Euclidean.*

For mass and mass$*$ we have the following inequality:

Proposition 3.14. *If X is an n-dimensional normed space, then $\mu_X^m \leq \mu_X^{m*}$.*

Proof. Let P be a minimal circumscribed parallelotope to the unit ball B. Then (see for example [Thompson 1999], but there are many other possible references) the midpoint of each facet of P is a point of contact with B. The convex hull of these midpoints is a cross-polytope C inscribed to B. Also, if P is given the volume 2^n, then C has volume $2^n/n!$. Hence, in this situation, a *maximal* inscribed cross-polytope will have volume greater than or equal to $2^n/n!$. □

Theorem 3.15. *If X is an n-dimensional normed space, then $\mu_X^m \leq \mu_X^b$ with equality if and only if X is Euclidean.*

The proof depends on the following theorem of McKinney [1974]:

Theorem 3.16. *Let $K \subset X$ be a convex set symmetric about the origin and let S be a maximal simplex contained in K with one vertex at the origin, then for any Lebesgue measure λ*

$$\lambda(S)/\lambda(K) \geq 1/n!\varepsilon_n$$

with equality if and only if K is an ellipsoid.

Proof of Theorem 3.15. If B is the unit ball of X then $\mu^b(B) = \varepsilon_n$ and $\mu^m(B) = 2^n\lambda(B)/n!\lambda(C)$ where C is a maximal cross-polytope inscribed to B. Moreover, C is the convex hull of $S \cup -S$, where S is a maximal simplex inscribed to B with one vertex at the origin. It follows from the theorem that $\lambda(C)/\lambda(B) \geq 2^n/n!\varepsilon_n$ which, upon rearrangement, gives $\mu^b(B) \geq \mu^m(B)$. □

The relationship between mass∗ and the Holmes–Thompson volume follows from Theorem 3.15 and the following simple exercise:

EXERCISE 3.17. Let μ and ν be two definitions of volume, and let μ^* and ν^* be their dual definitions. Show that if for every normed space X

$$\mu_X \leq \nu_X, \quad \text{then} \quad \nu^*_X \leq \mu^*_X.$$

COROLLARY 3.18. *If X is an n-dimensional normed space, then $\mu^{\mathrm{ht}}_X \leq \mu^{m*}_X$ with equality if and only if X is Euclidean.*

The previous inequalities are summarized by the diagram

$$\begin{tikzcd}
\mu^b \arrow[d] \arrow[dr] & \mu^{m*} \arrow[dl] \arrow[d] \\
\mu^{\mathrm{ht}} & \mu^m
\end{tikzcd}$$

Notice that as a consequence of the Mahler–Reisner inequality we have the following lower bounds for the mass and mass∗ of unit balls in normed spaces and their duals.

COROLLARY 3.19. *For any unit ball B, we have $\mu^m(B) \leq \varepsilon_n$ and, if B is either a zonoid or the dual of a zonoid, $\mu^{m*}(B) \geq 4^n/n!\varepsilon_n$.*

PROBLEM. *Is the mass∗ of the unit ball of an n-dimensional normed space at least $4^n/n!\varepsilon_n$? This is a weaker version of Mahler's conjecture.*

3.4. Extension to Finsler manifolds

DEFINITION 3.20. A *volume density* on an$n$-dimensional manifold M is a continuous function

$$\Phi : \Lambda^n TM \longrightarrow \mathbb{R}$$

such that for every point $m \in M$ the restriction of Φ to $\Lambda^n T_m M$ is a norm. A volume density is said to be smooth if the function Φ is smooth outside the zero section.

If M is an oriented manifold and Φ is a volume density on M, then we can define a volume form Ω on M whose value at a basis $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ of $T_m M$ is $\Phi(m; \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)$ if the basis is positively oriented and $-\Phi(m; \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_n)$ if it is negatively oriented. For any positively oriented n-dimensional submanifold $U \subset M$ we have that

$$\int_U \Phi = \int_U \Omega.$$

However, the integral of a volume density does not depend on the orientation and volume densities can be defined in nonorientable manifolds like the projective plane where no volume form exists.

DEFINITION 3.21. A *continuous Finsler metric* F on a manifold M assigns a norm, $F(m,\cdot)$, to each tangent space T_mM in such a way that the norm varies continuously with the base point. A *continuous Finsler manifold* is a pair (M,F), where M is a manifold and F is a continuous Finsler metric on M.

An important class of examples of Finsler manifolds are finite-dimensional submanifolds of normed spaces. If M is a submanifold of a finite-dimensional normed space X, at each point $m \in M$ the tangent space T_mM can be thought of as a subspace of X and, as such, it inherits a norm.

If $\gamma : [a,b] \to M$ is a differentiable curve on a continuous Finsler manifold (M,F), we define

$$\mathrm{length}(\gamma) := \int_a^b F(\gamma(t), \dot{\gamma}(t))\, dt.$$

This definition can be extended in the obvious way to piecewise-differentiable curves. If x and y are two points in M, we define the *distance between x and y* as the infimum of the lengths of all piecewise-differentiable curves that join them. Thus, continuous Finsler manifolds are metric spaces and metric techniques can be used to study them.

Each definition of volume on normed spaces gives a definition of volume on continuous Finsler manifolds: if we are given a volume definition μ and an n-dimensional continuous Finsler manifold (M,F), then the map that assigns to every point m the norm μ_{T_mM} on $\Lambda^n T_mM$ is a volume density on M. Notice that, in particular, a definition of volume on normed spaces immediately yields a way to measure the volumes of submanifolds of a normed space X because, as remarked above, the tangent space T_mM of such a submanifold can be regarded as a subspace of X and so inherits both a norm and a volume. This will be studied from an extrinsic viewpoint and in much more detail in the next section.

EXERCISE 3.22. Show that if a definition of volume on normed spaces is used to define a volume on Finsler manifolds, it satisfies the following two properties:

(1) If the Finsler manifold is Riemannian, its volume is the standard Riemannian volume;
(2) If $\varphi : M \to N$ is a short map (*i.e.,* does not increase distances) between two Finsler manifolds of the same dimension, then φ does not increase volumes.

Extending our four volume definitions from normed spaces to continuous Finsler manifolds, we may speak of the Busemann, Holmes–Thompson, mass, and mass$*$ definition of volumes on continuous Finsler manifolds. To end the section we relate the Busemann and Holmes–Thompson definitions with well-known geometric constructions.

As was previously remarked, Finsler manifolds are metric spaces and, as such, we can define their volume as their Hausdorff measure: if (M,F) is an n-dimensional Finsler manifold and $r > 0$, we cover M by a family of metric balls of radius at most r, $B(m_1,r_1)$, $B(m_2,r_2)$, $\ldots$, and consider the quantity

$\varepsilon_n(r_1^n + r_2^n + \cdots)$. We now take the infimum of this quantity over all possible covering families and take the limit as r tends to zero. The resulting number is the *n-dimensional Hausdorff measure of* (M, F).

THEOREM 3.23 [Busemann 1947]. *The Busemann volume of a continuous Finsler manifold is equal to its Hausdorff measure.*

To explain the second construction, we need to recall some standard facts about the geometry of cotangent bundles.

If $\pi : T^*M \to M$ is the canonical projection, we define the *canonical 1-form* α on T^*M by the formula

$$\alpha(\boldsymbol{v}_p) := p(\pi_*(\boldsymbol{v}_p)).$$

In standard coordinates $(x_1, \ldots, x_n, p_1, \ldots, p_n)$, $\alpha := \sum_{i=1}^n p_i dq_i$. The *canonical symplectic form* on T^*M is defined as $\omega := -d\alpha$ and the *Liouville volume* is defined by $\Omega := \omega^n/n!$.

If (M, F) is a continuous Finsler manifold, each tangent space T_mM carries the norm $F(m, \cdot)$ and, hence, each cotangent space T_m^*M carries the dual norm $F^*(m, \cdot)$. Let us denote the unit ball in T_m^*M by B_m^* and define the *unit co-disc bundle* of M as the set

$$B^*(M) := \bigcup_{m \in M} B_m^* \subset T^*M.$$

PROPOSITION 3.24. *The Holmes–Thompson volume of an n-dimensional, continuous Finsler manifold is equal to the Liouville volume of its unit co-disc bundle divided by the volume of the n-dimensional Euclidean unit ball.*

PROOF. It suffices to verify the result on normed spaces where it easily follows from the definitions. □

THEOREM 3.25 [Duran 1998]. *If M is a Finsler manifold, then the Holmes–Thompson volume of M is less than or equal to its Hausdorff measure with equality if and only if M is Riemannian.*

PROOF. By the Blaschke–Santaló inequality, at each point $m \in M$ we have that $\mu^{\mathrm{ht}}_{T_mM} \leq \mu^{b}_{T_mM}$ with equality if and only if T_mM is Euclidean. The result now follows immediately from Theorems 3.23 and 3.24 . □

4. k-Volume Densities

The theory of volumes and areas on Euclidean and Riemannian spaces is based on the fact that a Euclidean structure on a vector space induces natural Euclidean structures on its exterior powers: if $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ is an orthonormal basis of a Euclidean space X, then the vectors

$$\boldsymbol{x}_{i_1} \wedge \boldsymbol{x}_{i_2} \wedge \cdots \wedge \boldsymbol{x}_{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n,$$

form an orthonormal basis of $\Lambda^k X$. If we want to know the area of the parallelogram formed by the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we need to compute the norm of $\boldsymbol{x} \wedge \boldsymbol{y}$ in $\Lambda^2 X$. In normed and Finsler spaces these simple algebraic constructions, which should be seen as functors from the category of Euclidean spaces onto itself, cannot be reproduced and we need to understand their geometry to see how they may be naturally extended to these spaces.

The first important remark is that in order to compute k-dimensional volumes (k-volumes from now on), we do not need to define a norm on all of $\Lambda^k X$. It suffices to define the magnitude of vectors of the form $\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_k$, where $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k$ are vectors in X. In this paper we shall refer to these k-vectors as *simple* and denote the set of all simple k-vectors in X by $\Lambda_s^k X$. Note that for $k = 1, n-1$ every k-vector is simple, which makes the study of $(n{-}1)$-volume in n-dimensional normed spaces a richer and more approachable subject than the study of volumes in higher codimension. Indeed, when $k \neq 1, n-1$, $\Lambda_s^k X$ is not a vector subspace of $\Lambda^k X$, but just an algebraic cone.

Exercise 4.1. Let $\mathcal{Q} : \Lambda^2\mathbb{R}^4 \to \Lambda^4\mathbb{R}^4$ be the quadratic form defined by $\mathcal{Q}(a) = a \wedge a$. Show that $\Lambda_s^2\mathbb{R}^4$ is the quadric $\mathcal{Q} = 0$ and use this to prove that the intersection of $\Lambda_s^2\mathbb{R}^4$ with the (Euclidean) unit sphere in $\Lambda^2\mathbb{R}^4$ is a product of two 2-dimensional spheres.

In general, the intersection of $\Lambda_s^k\mathbb{R}^n$ with the Euclidean unit sphere in $\Lambda^k\mathbb{R}^n$ is the Plücker embedding of the Grassmannian of oriented k-planes in $\mathbb{R}^n$, $G_k^+(\mathbb{R}^n)$. Let us recall that this Grassmannian is a smooth manifold of dimension $k(n{-}k)$.

Following our first remark, we see that in order to compute k-volumes in a normed space X, we need to define a "norm" on the cone of simple k-vectors of X. The fact that $\Lambda_s^k X$ is not a vector space complicates matters since it is not clear how to write the triangle inequality, and, even if an apparent analogue could be found, it would have to be justified in terms of its geometric significance. Nevertheless, the homogeneity and positivity of a norm are easy to generalize:

Definition 4.2. A *k-density* on a vector space X is a continuous function

$$\phi : \Lambda_s^k X \longrightarrow \mathbb{R}$$

that is homogeneous of degree one (*i.e.,* $\phi(\lambda a) = |\lambda|\,\phi(a)$). A k-density ϕ is said to be a *k-volume density* if $\phi(a) \geq 0$ with equality if and only if $a = 0$.

4.1. Examples of k-volume densities. In the previous chapter we studied four natural volume definitions on normed spaces. Each one of these definitions yields natural constructions of k-volume densities on the spaces of simple k-vectors of a normed space X.

Given a volume definition μ and a k-vector a in a k-dimensional normed space Y, we can compute $\mu(a)$. To be perfectly rigorous, we should include the normed space as a variable in μ, for example, by writing $\mu_Y(a)$. If a is a simple k-vector in an n-dimensional normed space X, then we may consider it as a k-vector on the k-dimensional normed space "spanned by a",

$$\langle a\rangle := \{\boldsymbol{x} \in X : a \wedge \boldsymbol{x} = 0\} \subset X,$$

(provided with the induced norm), and compute $\mu_{\langle a\rangle}(a)$. Thus, once we have chosen a way to define volume in *all* finite-dimensional normed spaces, we have a way to associate to each norm on an n-dimensional vector space X a family of k-volume densities, with $1 \leq k \leq n$.

The Busemann k-volume densities. Let X be a normed space of dimension n with unit ball B, and let k be a positive integer less than n. The Busemann k-volume density on X is defined by the formula

$$\mu^b(a) := \frac{\varepsilon_k}{\mathrm{vol}(B \cap \langle a\rangle; a)}.$$

The Holmes–Thompson k-volume densities. Let X be a normed space of dimension n, and let k be a positive integer less than n. If a is a simple k-vector spanning the k-dimensional subspace $\langle a\rangle$, we consider the inclusion of $\langle a\rangle$ into X and the dual projection $\pi : X^* \to \langle a\rangle^*$. Regarding a as a volume form on $\langle a\rangle^*$ we define

$$\mu^{\mathrm{ht}}(a) := \varepsilon_k^{-1} \int_{\pi(B_X^*)} |a|.$$

The mass k-volume densities. Let $(X, \|\cdot\|)$ be a normed space of dimension n, and let k be a positive integer less than n. The mass k-volume density on X is defined by the formula

$$\mu^m(a) := \inf\left\{ \prod_{i=1}^{k} \|\boldsymbol{x}_i\| : \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_k = a \right\}.$$

The mass$*$ k-volume densities. According to the characterization of mass$*$ given in Exercise 3.9, we may describe the mass$*$ k-volume densities as follows:

Let X be a normed space and let $W \subset X$ be a k-dimensional subspace. If a is a simple k-vector on X, we define $\mu^{m*}(a)$ as the supremum of the numbers $|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_k(a)|$, where $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ ranges over all short bases of $\langle a\rangle^*$.

However, there is a simpler description:

Exercise 4.3. Using the Hahn–Banach theorem and the notation above, show that $\mu^{m*}(a)$ is the supremum of the numbers $|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_k(a)|$, where $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k \in B_X^*$.

In the study of volumes and areas on Finsler manifolds, we shall also need to work with k-densities and smooth k-densities on manifolds. For this purpose we introduce the bundle of simple tangent k-vectors on a manifold M, $\Lambda_s^k TM$. This is a subbundle of algebraic cones of the vector bundle $\Lambda^k TM$, and if we omit the zero section it is a smooth manifold.

Definition 4.4. A k-density ϕ (*resp.* k-volume density) on a manifold M is a continuous function $\phi : \Lambda^k_s TM \to \mathbb{R}$ such that at each point m, the restriction of ϕ to $\Lambda^k_s T_m M$ is a k-density (*resp.* k-volume density). If the function ϕ is smooth outside the zero section, we shall say that the density is smooth.

Since every tangent space of a Finsler manifold (M, F) is a normed space, we may define the Busemann, Holmes–Thompson, mass, and mass$*$ k-volume densities on (M, F) by assigning to each tangent space its respective k-density. It is easy to show that if F is a smooth Finsler metric, then the Busemann and Holmes–Thompson k-volume densities are smooth. This is probably not the case with mass and mass$*$, but we have no explicit examples to illustrate this.

Just like differential k-forms, k-densities can be pulled back: if $f : N \to M$ is a smooth map and ϕ is a k-density on M, then

$$f^*\phi(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_k) = \phi\big(Df(\boldsymbol{v}_1) \wedge Df(\boldsymbol{v}_2) \wedge \cdots \wedge Df(\boldsymbol{v}_k)\big).$$

Remark that if $f : N \to M$ is an immersion and ϕ is a k-volume density on M, then $f^*\phi$ is a k-volume density on N.

Also like differential forms, k-densities can be integrated over k-dimensional submanifolds: if $N \subset M$ is a k-dimensional submanifold of M and $i : N \to M$ is the inclusion map, then $i^*\phi$ is a volume density on N, and its integral over N was defined in Section 3. This integral is independent of the parameterization and orientation of N. In the same way, we may define the integral of a k-density over a k-chain.

For the rest of the chapter, we associate to a given k-volume density ϕ on a vector space X the functional

$$N \longmapsto \int_N \phi,$$

and investigate the relationship between the behavior of the functional and certain convexity properties of ϕ. The easiest case is when ϕ is an $(n{-}1)$-volume density in an n-dimensional vector space.

4.2. Convexity of $(n{-}1)$-volume densities. This case is special because every $(n{-}1)$-vector in an n-dimensional vector space, X, is simple and we may impose the condition that an $(n{-}1)$-volume density be a norm in $\Lambda^{n-1}X$. This is, for example, satisfied by $(n{-}1)$-volume densities for the Busemann, Holmes–Thompson, and mass$*$ definitions of volume. Nevertheless, it remains to see why such a condition is desirable.

The next result is the first of four characterizations of norms on $\Lambda^{n-1}X$.

Theorem 4.5. *Let ϕ be an $(n{-}1)$-volume density on an n-dimensional vector space X. The following conditions on ϕ are equivalent*:

- *ϕ is a norm*;
- *If $P \subset X$ is a closed $(n-1)$-dimensional polyhedron in X, then the area of any one of its facets is less than or equal to the sum of the areas of the remaining facets.*

Before proving this theorem, we need to introduce a classical construction that associates to any k-dimensional polyhedral surface on X a set of simple k-vectors. This set will be called the *Gaussian image* of the polyhedron (see also [Burago and Ivanov 2002], where the almost identical notion of *Gaussian measure* is used).

If $P \subset X$ is a polyhedron with facets $F_1, \dots, F_m$ we associate to each facet F_i the simple k-vector a_i such that $\langle a_i \rangle$ is parallel to F_i and such that $\mathrm{vol}(F_i; a_i) = 1$. The Gaussian image of P is the set $\{a_1, \dots, a_m\} \subset \Lambda_s^k X$. If ϕ is a k-volume density in X, the k-volume of P (with respect to ϕ) is just $\phi(a_1) + \cdots + \phi(a_m)$.

EXERCISE 4.6. Show that if $\{a_1, \dots, a_m\} \subset \Lambda_s^k X$ is the Gaussian image of a closed polyhedron in X, then $a_1 + \cdots + a_m = 0$.

In general, the condition that the sum of a set of simple k-vectors be zero, does not imply that it is the Gaussian image of a closed k-dimensional polyhedron in X. However, in codimension one we have the following celebrated theorem of Minkowski.

THEOREM 4.7 (MINKOWSKI). *A set of $(n-1)$-vectors of an n-dimensional vector space X is the Gaussian image of a closed, convex polyhedron if and only if the $(n-1)$-vectors span $\Lambda^{n-1}X$ and their sum equals zero.*

To prove Theorem 4.5, we shall need an easy particular case of Minkowski's result:

EXERCISE 4.8. Let X be an n-dimensional vector space and let $a_1, \dots, a_n$ be a basis of $\Lambda^{n-1}X$. Show that there exists a simplex in X whose Gaussian image is the set $\{a_1, \dots, a_n, -(a_1 + \cdots + a_n)\}$.

PROOF OF THEOREM 4.5. Assume that ϕ is a norm, and let $P \subset X$ be an $(n-1)$-dimensional closed polyhedron with Gaussian image $\{a_0, a_1, \dots, a_m\}$. Since the sum of the a_i's is zero, we may use the triangle inequality to write

$$\phi(a_0) = \phi(a_1 + \cdots + a_m) \leq \phi(a_1) + \cdots + \phi(a_m).$$

In other words, the area of the facet corresponding to a_0 is less than or equal to the sum of the areas of the remaining facets.

To prove the converse, we take any two $(n-1)$-vectors a_1 and a_2, which we assume to be linearly independent, and use them as part of a basis $a_1, \dots, a_n$ of $\Lambda^{n-1}X$. By Exercise 4.8, the set $\{a_1, \dots, a_n, -(a_1 + \cdots + a_n)\}$ is the Gaussian image of a simplex in X. Then, by assumption,

$$\phi(a_1 + \cdots + a_n) \leq \phi(a_1) + \cdots + \phi(a_n).$$

By letting $a_3, \dots, a_n$ tend to zero in the above inequality we obtain the triangle inequality $\phi(a_1 + a_2) \leq \phi(a_1) + \phi(a_2)$, and, therefore, ϕ is a norm. $\square$

Exercise 4.9. Consider the tetrahedron in the normed space ℓ^3_∞ with vertices $(0,0,0)$, $(-1,1,1)$, $(1,-1,1)$, $(1,1,-1)$, and show that the mass of the facet opposite the origin is greater than the sum of the masses of the three other facets. *Hint.* Use the definition of the mass 2-volume density in terms of minimal circumscribed parallelograms.

By Theorem 4.5, the previous exercise shows that the mass $(n-1)$-volume density of a normed space X is not necessarily a norm in $\Lambda^{n-1}X$. As we shall see in the rest of this chapter, this is a good reason to disqualify mass as a satisfactory definition of volume on normed spaces.

Our second characterization of norms in $\Lambda^{n-1}X$ is another variation on the theme of *flats minimize.*

Theorem 4.10. *Let ϕ be an $(n-1)$-volume density on an n-dimensional vector space X. The following conditions on ϕ are equivalent*:

- *ϕ is a norm*;
- *Whenever C and C' are $(n-1)$-chains with real coefficients such that $\partial C = \partial C'$ and the image of C is contained in a hyperplane*, *then the area of C is less than or equal to the area of C'.*

In order to prove this theorem we need to introduce the concept of *calibration* formalized by Harvey and Lawson [1982].

Definition 4.11. A closed k-form ω is said to *calibrate* a k-density ϕ if for all simple k-vectors a in TM we have that $\omega(a) \leq \phi(a)$ and equality is attained on a nonempty subset of $\Lambda^k_s TM$.

The homogeneity of ω and ϕ allows us to consider the set where they coincide as a subset E of the bundle of oriented k-dimensional subspaces of TM, $G^+_k(TM)$.

Proposition 4.12 [Harvey and Lawson 1982]. *Let ϕ be a k-volume density on a manifold M, let ω be a closed k-form on M that calibrates ϕ and let $E \subset G^+_k(TM)$ be the set where ϕ and ω coincide. If $N \subset M$ is a k-dimensional oriented submanifold all of whose tangent planes belong to E, and N' is another submanifold of M homologous to N, then*

$$\int_N \phi \leq \int_{N'} \phi.$$

Proof. Using that $\phi = \omega$ on the tangent spaces of N and Stokes' formula, we have

$$\int_N \phi = \int_N \omega = \int_{N'} \omega \leq \int_{N'} \phi. \qquad \square$$

Proof of Theorem 4.10. Assume that ϕ is a norm, let C and C' be as in the statement of the theorem, and let a be an $(n-1)$-vector on X such that $\phi(a) = 1$ and the subspace $\langle a \rangle$ is parallel to the hyperplane containing the image of C. Since the unit sphere in $(\Lambda^{n-1}X, \phi)$ is a convex hypersurface, it has a supporting hyperplane that touches it at a. This hyperplane can be given as the set $\omega = 1$, where ω is a constant-coefficient $(n-1)$-form on X. Since the unit sphere lies in the half-space $\omega \leq 1$, we have $\omega \leq \phi$ and, thus, ω calibrates ϕ.

Using that $\omega = \phi$ on C, that $d\omega = 0$, and that $C + (-C')$ is a closed chain homologous to zero, we have

$$\int_C \phi = \int_C \omega = \int_{C'} \omega \leq \int_{C'} \phi.$$

To prove the converse, note that the second condition in the theorem immediately implies that the $(n-1)$-volume of the facet of a closed polyhedron is less than or equal to the sum of the $(n-1)$-volumes of the remaining facets. By Theorem 4.5, this implies that ϕ is a norm. $\square$

In Euclidean geometry, the orthogonal projection onto a k-dimensional subspace is area-decreasing. This can be generalized as follows:

Theorem 4.13. *Let ϕ be an $(n-1)$-volume density on an n-dimensional vector space X. The following conditions on ϕ are equivalent*:

- *ϕ is a norm*;
- *For every $(n-1)$-dimensional subspace $W \subset X$ there is a ϕ-decreasing linear projection $P_W : X \to W$.*

The proof of this theorem rests on a simple lemma in multi-linear algebra.

Lemma 4.14. *Let X be an n-dimensional vector space and let $W \subset X$ be a k-dimensional subspace. If $\boldsymbol{w}_1, \dots, \boldsymbol{w}_k$ is a basis of W and $\omega \in \Lambda^k X^*$ is such that $\omega(\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k) = 1$, then the linear map*

$$P\boldsymbol{x} := \sum_{i=1}^{k} (-1)^i \omega(\boldsymbol{x} \wedge \boldsymbol{w}_1 \wedge \cdots \wedge \hat{\boldsymbol{w}}_i \wedge \cdots \wedge \boldsymbol{w}_k)\boldsymbol{w}_i$$

is a projector with range W. Moreover, ω is simple if and only if for any vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in X$,

$$P\boldsymbol{x}_1 \wedge P\boldsymbol{x}_2 \wedge \cdots \wedge P\boldsymbol{x}_k = \omega(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_k)\ \boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k.$$

The proof of the lemma is left as an exercise to the reader.

Proof of Theorem 4.13. Assume that ϕ is a norm in $\Lambda^{n-1}X$ and let $W \subset X$ be an $(n-1)$-dimensional subspace. Choose a basis of W, $\boldsymbol{w}_1, \dots, \boldsymbol{w}_{n-1}$, such that $\phi(\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_{n-1}) = 1$ and consider the support hyperplane of the unit sphere of $(\Lambda^{n-1}X, \phi)$ at the point $\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_{n-1}$. This hyperplane is given by an equation of the form $\omega = 1$, where ω is an $(n-1)$-form with constant coefficients. In other words, $\omega \in \Lambda^{n-1}X^*$.

We claim that the linear projection

$$P\boldsymbol{x} := \sum_{i=1}^{n-1} (-1)^i \omega(\boldsymbol{x} \wedge \boldsymbol{w}_1 \wedge \cdots \wedge \hat{\boldsymbol{w}}_i \wedge \cdots \wedge \boldsymbol{w}_{n-1})\boldsymbol{w}_i$$

is ϕ-decreasing. Indeed, since ω is an $(n-1)$-form on an n-dimensional space, it is simple. Using the second part of Lemma 4.14, we have, for any $(n-1)$-vector $a := \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}$,

$$\phi(P\boldsymbol{x}_1 \wedge P\boldsymbol{x}_2 \wedge \cdots \wedge P\boldsymbol{x}_{n-1}) = |\omega(a)|\,\phi(\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_{n-1}) = |\omega(a)| \leq \phi(a).$$

To prove the converse, we note that the existence of a ϕ-decreasing linear projection onto any given hyperplane implies that the $(n-1)$-volume of the facet of any closed $(n-1)$-dimensional polyhedron is less than or equal to the sum of the areas of the remaining facets. The argument is quite simple: if the closed polyhedron has facets $F_0, F_1, \ldots, F_m$, we use the ϕ-decreasing projection P to project the whole polyhedron onto the hyperplane containing F_0. Note that $P(F_1) \cup \cdots \cup P(F_m)$ contains $P(F_0) = F_0$ and, therefore, the sum of the $(n-1)$-volumes of the $P(F_i)$, $1 \leq i \leq m$, is greater than or equal to the $(n-1)$-volume of F_0. Since P is ϕ-decreasing, this gives us that the sum of the $(n-1)$-volumes of the F_i, $1 \leq i \leq m$, is greater than or equal to the $(n-1)$-volume of F_0. $\square$

We now state the fourth and last of our characterizations of norms on the space of $(n-1)$-vectors in an n-dimensional normed space.

THEOREM 4.15. *Let ϕ be an $(n-1)$-volume density on an n-dimensional vector space X. The following conditions on ϕ are equivalent*:

- *ϕ is a norm*;
- *If $K \subset K'$ are two nested convex bodies in X, then the area of ∂K is less than or equal to the area of $\partial K'$ with equality if and only if K equals K'.*

The proof of this theorem is a simple consequence of the relation between norms in the space of $(n-1)$-vectors and the theory of mixed volumes that will be developed in Section 6.

At the beginning of this section we mentioned that for any n-dimensional normed space X the Busemann, Holmes–Thompson, and mass$*$ $(n-1)$-volume densities on X are norms in $\Lambda^{n-1}X$. For the Busemann definition this is a celebrated theorem of Busemann [1949a]. For the mass$*$ definition this result is due to Benson [1962]. We shall follow [Gromov 1983] and give a proof of a much stronger result later in this section. For the Holmes–Thompson definition, the result — under a different formulation — goes back to Minkowski.

THEOREM 4.16 (MINKOWSKI). *The Holmes–Thompson $(n-1)$-volume density of an n-dimensional normed space X is itself a norm in $\Lambda^{n-1}X$.*

In order to prove the convexity of the Holmes–Thompson $(n-1)$-volume density, we shall first give an integral representation for it. This representation depends, in turn, on two classical constructions: the *Gauss map* and the *surface-area measure.* Our approach follows [Fernandes 2002].

Let X be an n-dimensional vector space and let ϕ be an $(n-1)$-volume density on X. If $N \subset X$ is an oriented hypersurface and $n \in N$, we define $\mathcal{G}_\phi(n)$ as the unique $(n-1)$-vector in $\Lambda^{n-1}T_nN \subset \Lambda^{n-1}X$ that is positively oriented and satisfies $\phi(\mathcal{G}_\phi(n)) = 1$. Notice that when N is a strictly convex hypersurface, the Gauss map

$$\mathcal{G}_\phi : N \longrightarrow \Sigma := \{a \in \Lambda^{n-1}X : \phi(a) = 1\}$$

is a diffeomorphism. In this case, we define the *surface-area measure* of N as the $(n-1)$-volume density $dS_N := \mathcal{G}_\phi^{-1*}\phi$ on Σ.

LEMMA 4.17. *Let $\pi : X \to Y$ be a surjective linear map between an n-dimensional vector space X and an $(n-1)$-dimensional vector space Y, and let ϕ be an $(n-1)$-volume density on X with unit sphere $\Sigma \subset \Lambda^{n-1}X$. If $N \subset X$ is a smooth, strictly convex hypersurface and ω is a volume form on Y, then*

$$\int_{\pi(N)} |\omega| = \frac{1}{2}\int_{a\in\Sigma} |\pi^*\omega(a)|\, dS_N.$$

PROOF. By the definition of the Gauss map, if $n \in N$ and $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1} \in \Lambda^{n-1}T_nN$,

$$\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1} = \phi(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1})\mathcal{G}_\phi(n).$$

Therefore, $\pi^*|\omega|(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}) = |\pi^*\omega(\mathcal{G}_\phi(n))|\phi(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1})$. Then

$$\begin{aligned}\int_{\pi(N)} |\omega| &= \frac{1}{2}\int_N \pi^*|\omega| = \frac{1}{2}\int_{n\in N} |\pi^*\omega(\mathcal{G}_\phi(n))|\,\phi \\ &= \frac{1}{2}\int_\Sigma \mathcal{G}_\phi^{-1*}\,|\pi^*\omega(\mathcal{G}_\phi(n))|\,\phi = \frac{1}{2}\int_{a\in\Sigma} |\pi^*\omega(a)|\,\mathcal{G}_\phi^{-1*}\phi \\ &= \frac{1}{2}\int_{a\in\Sigma} |\pi^*\omega(a)|\, dS_N. \end{aligned}$$ □

PROOF OF THEOREM 4.16. By a standard approximation argument, it suffices to consider the case where the dual unit sphere $\partial B^* \subset X^*$ is smooth and strictly convex. Applying the previous lemma to the surface $N = \partial B^*$ and to an arbitrary $(n-1)$-volume density on X^*, we have

$$\mu^{\mathrm{ht}}(a) = \varepsilon_{n-1}^{-1}\int_{\pi(B^*)} |a| = \varepsilon_{n-1}^{-1}\int_{\xi\in\Sigma} |a(\xi)|\, dS_{\partial B^*}.$$

Since the surface-area measure $dS_{\partial B^*}$ is nonnegative,

$$\begin{aligned}\mu^{\mathrm{ht}}(a+b) &= \varepsilon_{n-1}^{-1}\int_{\xi\in\Sigma}|a(\xi)+b(\xi)|\,dS_{\partial B^*}\\ &\le \varepsilon_{n-1}^{-1}\int_{\xi\in\Sigma}|a(\xi)|\,dS_{\partial B^*}+\varepsilon_{n-1}^{-1}\int_{\xi\in\Sigma}|b(\xi)|\,dS_{\partial B^*}\\ &= \mu^{\mathrm{ht}}(a)+\mu^{\mathrm{ht}}(b). \qquad \square\end{aligned}$$

EXERCISE 4.18. Let X be an n-dimensional vector space and let ϕ be an $(n-1)$-volume density on X. Show that if ϕ is a norm, then compact hypersurfaces cannot by minimal.

4.3. Convexity properties of k-volume densities. We now pass to the more delicate subject that Busemann, Ewald, and Shephard studied extensively under the heading of *convexity on Grassmannians.* Most of what follows can be found in their papers, "Convex bodies and convexity on Grassmannian cones" I–XI, but we have tried to make the language and proofs more accessible.

We shall see that there are several notions and degrees of convexity for k-volume densities. These are closely related to the concept of *ellipticity* in geometric measure theory and, historically, to the generalization of the Legendre condition for variational problems.

Weakly convex k-densities. Let X be an n-dimensional vector space and let $\Lambda_s^k X$, $1 \le k \le n-1$, be the cone of simple k-vectors on X. If Y is a $(k+1)$-dimensional subspace of X, then the subspace $\Lambda^k Y \subset \Lambda^k X$ lies inside $\Lambda_s^k X$. This motivates a definition:

DEFINITION 4.19. A k-volume density ϕ on an n-dimensional vector space X, $n > k$, is said to be *weakly convex* if for any linear subspace Y of dimension $k+1$, the restriction of ϕ to the linear space $\Lambda^k Y$ is a norm.

From the previous section, we know that the k-volume densities of any normed space for the Busemann, Holmes–Thompson, or mass$*$ definitions of volume are weakly convex.

EXERCISE 4.20. Show that a k-volume density in a vector space X is weakly convex if for every $(k+1)$-dimensional simplex in X the area of any one facet is less than or equal to the sum of the areas of the remaining facets.

Extendibly convex k-volume densities

DEFINITION 4.21. A k-volume density ϕ on an n-dimensional vector space X, $n > k$, is said to be *extendibly convex* if it is the restriction of a norm on $\Lambda^k X$ to the cone of simple k-vectors in X.

Equivalently, ϕ is extendibly convex if and only if there is a support hyperplane for the unit sphere

$$\mathcal{S} := \{a \in \Lambda_s^k X : \phi(a) = 1\}$$

passing through any of its points.

THEOREM 4.22. *If ϕ is an extendibly convex k-volume density on a vector space X, then any k-chain with real coefficients whose image is contained in a k-dimensional flat is ϕ-minimizing.*

The proof — by the method of calibrations — is nearly identical to the proof of Theorem 4.10 and is left as an exercise for the reader. Notice that a corollary to Theorem 4.22 is that if $P \subset X$ is a closed k-dimensional polyhedron, the area of any of its facets is less than or equal to the sum of the areas of the remaining facets.

The problem of determining whether the Busemann k-volume densities are extendibly convex was posed by Busemann in several of his papers as a major problem in convexity. So far, there are no results in this direction.

PROBLEM. *Is the Busemann 2-volume density of a 4-dimensional normed space extendibly convex?*

In the case of the Holmes–Thompson definition, Busemann, Ewald, and Shephard have given explicit examples of norms for which the k-volume densities, $1 < k < n-1$, are not extendibly convex (see [Busemann et al. 1963]). A simpler example has been given recently by Burago and Ivanov:

THEOREM 4.23 [Burago and Ivanov 2002]. *Consider the norm $\|\cdot\|$ on $\mathbb{R}^4$ whose dual unit ball in $\mathbb{R}^{4*}$ is the convex hull of the curve*

$$\gamma(t) := (\sin t,\, \cos t,\, \sin 3t,\, \cos 3t), \quad 0 \le t \le 2\pi.$$

The Holmes–Thompson 2-volume density for $(\mathbb{R}^4, \|\cdot\|)$ is not extendibly convex.

Despite these examples, in many important cases the Holmes–Thompson k-volume densities are extendibly convex.

THEOREM 4.24. *The Holmes–Thompson k-volume densities of a hypermetric normed space are extendibly convex.*

In order to prove this result, we shall derive a formula for the Holmes–Thompson k-volume densities of a Minkowski space in terms of the Fourier transform of its norm. In a somewhat different guise, this formula was first obtained by W. Weil [1979]. In the present form it was rediscovered by Álvarez and Fernandes in [1999], where it was shown to follow from the Crofton formula for Minkowski spaces.

Let ϕ be a smooth, even, homogeneous function of degree one on an n-dimensional vector space X, let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ be a basis of X, and let $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n$ be the dual basis in X^*. Using the basis $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ and its dual to identify both X and X^* with $\mathbb{R}^n$, we can compute the (distributional) Fourier transform of ϕ,

$$\hat{\phi}(\boldsymbol{\xi}) := \int_{\mathbb{R}^n} e^{i\boldsymbol{\xi}\cdot\boldsymbol{x}} \phi(\boldsymbol{x})\, dx.$$

The form $\hat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n$ does not depend on the choice of basis in X. Up to a constant factor, we define the form $\check{\phi}$ as the contraction of this n-form with the Euler vector field, $X_E(\boldsymbol{\xi}) = \boldsymbol{\xi}$, in X^*:

$$\check{\phi} := \frac{-1}{4(2\pi)^{n-1}} \hat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n \rfloor X_E.$$

It is known (see [Hörmander 1983, pages 167–168]) that $\hat{\phi}$ is smooth on $X^* \setminus \mathbf{0}$ and homogeneous of degree $-n-1$; therefore $\check{\phi}$ is a smooth differential form on $X^* \setminus \mathbf{0}$ that is homogeneous of degree -1.

Denoting by $\check{\phi}^k$ the product form in the product space $(X^* \setminus \mathbf{0})^k$, we have the following result:

Theorem 4.25. *Let (X, ϕ) be an n-dimensional Minkowski space. For any simple k-vector a on X, $1 \le k < n$, we have*

$$\mu^{\mathrm{ht}}(a) = \frac{1}{\varepsilon_k} \int_{(\boldsymbol{\xi}_1, \dots \boldsymbol{\xi}_k) \in S^{*k}} |\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_k \cdot a|\, \check{\phi}^k,$$

where S^* is any closed hypersurface in $X^* \setminus \mathbf{0}$ that is star-shaped with respect to the origin.

Notice that this formula allows us to extend the definition of the Holmes–Thompson k-volume density of any Minkowski space to all of $\Lambda^k X$. It remains to see when this extension is a norm.

Proof of Theorem 4.24. It is enough to prove convexity in the case the hypermetric space (X, ϕ) is also a Minkowski space. This allows us to use the integral representation given above. Since X is hypermetric, Theorem 2.5 tells us that the form $\hat{\phi}\, d\xi_1 \wedge \cdots \wedge d\xi_n$ is a volume form, and, therefore, the restriction of $\check{\phi}^k$ to the manifold S^{*k} defines a nonnegative measure. Then for any two k-vectors a and b we have

$$\begin{aligned} \mu^{\mathrm{ht}}(a+b) &= \int_{S^{*k}} |\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_k \cdot (a+b)|\, \check{\phi}^k \\ &\le \int_{S^{*k}} |\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_k \cdot a|\, \check{\phi}^k + \int_{S^{*k}} |\boldsymbol{\xi}_1 \wedge \cdots \wedge \boldsymbol{\xi}_k \cdot b|\, \check{\phi}^k \\ &= \mu^{\mathrm{ht}}(a) + \mu^{\mathrm{ht}}(b). \end{aligned}$$

□

Totally convex k-densities

Definition 4.26. A k-density ϕ on an n-dimensional vector space X, $n > k$, is said to be *totally convex* if through every point of the unit sphere of $\Lambda^k X$ there passes a supporting hyperplane of the form $\xi = 1$ with ξ a simple k-vector in $\Lambda^k X^*$.

Total convexity implies extendible convexity and, in turn, weak convexity. The following result, stated in [Busemann 1961] gives an important characterization of totally convex k-densities in terms of what Gromov [1983] calls the *compressing property.*

Theorem 4.27. *A k-density ϕ on an n-dimensional vector space X is totally convex if and only if for every k-dimensional linear subspace there exists a ϕ-decreasing linear projection onto that subspace.*

The proof, using Lemma 4.14, is nearly identical to the proof of Theorem 4.13.

Of all the four volume definitions we have studied, mass∗ has by far the strongest convexity property:

Theorem 4.28. *The mass∗ k-volume densities of an n-dimensional normed space X, $1 \leq k \leq n-1$, are totally convex.*

Proof. By Theorem 4.27, it is enough to show that given any k-dimensional subspace W, there exists a linear projection $P : X \to W$ that is mass∗-decreasing.

Choose a basis $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$ of W^* which satisfies two properties:

(1) It is short (*i.e.,* $|\boldsymbol{\xi}_i(\boldsymbol{x})| \leq 1$ for all $\boldsymbol{x} \in B \cap W$);
(2) The integral of the volume density $|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_k|$ over $B \cap W$ is maximal among all short bases.

Notice that for any basis $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ of W, we have that

$$\mu^{m*}(\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k) = |\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_k \ (\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k)|,$$

and that if $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ is dual to $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_k$, then $\mu^{m*}(\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k) = 1$.

By the Hahn–Banach theorem, there exist covectors $\hat{\boldsymbol{\xi}}_1, \ldots, \hat{\boldsymbol{\xi}}_k \in X^*$ such that

(1) $|\hat{\boldsymbol{\xi}}_i(\boldsymbol{x})| \leq 1$ for all $\boldsymbol{x} \in B$ and for all i, $1 \leq i \leq k$;
(2) the restriction of $\hat{\boldsymbol{\xi}}_i$ to W equals $\boldsymbol{\xi}_i$ for all i, $1 \leq i \leq k$.

We may now define the projection $P : X \to W$ by the formula

$$P(\boldsymbol{x}) := \sum_{i=1}^{k} \hat{\boldsymbol{\xi}}_i(\boldsymbol{x})\boldsymbol{w}_i,$$

and show that it is μ^{m*}-decreasing. Indeed, if $a = \boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_k$ is a simple k-vector in X,

$$P(\boldsymbol{v}_1) \wedge P(\boldsymbol{v}_2) \wedge \cdots \wedge P(\boldsymbol{v}_k) = \hat{\boldsymbol{\xi}}_1 \wedge \hat{\boldsymbol{\xi}}_2 \wedge \cdots \wedge \hat{\boldsymbol{\xi}}_k \ (a)\boldsymbol{w}_1 \wedge \boldsymbol{w}_2 \wedge \cdots \wedge \boldsymbol{w}_k,$$

and therefore

$$\mu^{m*}(P(\boldsymbol{v}_1) \wedge P(\boldsymbol{v}_2) \wedge \cdots \wedge P(\boldsymbol{v}_k)) = |\hat{\boldsymbol{\xi}}_1 \wedge \hat{\boldsymbol{\xi}}_2 \wedge \cdots \wedge \hat{\boldsymbol{\xi}}_k \ (a)|.$$

Since the restriction of $\hat{\boldsymbol{\xi}}_i$, $1 \leq i \leq k$, to $\langle a \rangle$ form a short basis of $\langle a \rangle^*$, we have

$$\mu^{m*}(P(\boldsymbol{v}_1) \wedge P(\boldsymbol{v}_2) \wedge \cdots \wedge P(\boldsymbol{v}_k)) = |\hat{\boldsymbol{\xi}}_1 \wedge \hat{\boldsymbol{\xi}}_2 \wedge \cdots \wedge \hat{\boldsymbol{\xi}}_k \ (a)| \leq \mu^{m*}(a). \qquad \square$$

We end the section with an exercise and an open problem:

Exercise 4.29. Show that the sum of two totally convex 2-volume densities in $\mathbb{R}^4$ is not necessarily totally convex. On the other had, show that the maximum of two totally convex k-volume densities is a totally convex k-volume density.

PROBLEM. *For what (hypermetric) normed spaces are the Holmes–Thompson k-volume densities totally convex?*

5. Length and Area in Two-Dimensional Normed Spaces

Before going into the rich and beautiful theory of (hypersurface) area on finite-dimensional normed spaces, we shall sharpen our intuition by carefully considering the case of two-dimensional normed spaces. This case is fundamentally simpler because the notion of hypersurface area coincides with that of length and is thus independent of our volume definition. Nevertheless, we shall see that the theory of length and area on two-dimensional normed spaces is far from trivial and provides a platform from which to jump to higher dimensions.

We start with two theorems that involve solely the notion of length:

THEOREM 5.1 [Gołąb 1932]. *The perimeter of the unit circle of a two-dimensional normed space is between six and eight. Moreover, the length is equal to six if and only if the unit ball is an affine regular hexagon and is equal to eight if and only if it is a parallelogram.*

Full proofs can be found in [Schäffer 1967] and [Thompson 1996]. We stress that the length of the unit circle is measured with the definition of length given by the norm:

If $\gamma : [a, b] \to X$ is a continuous curve on the normed space $(X, \|\cdot\|)$, the length of γ is defined as the supremum of the quantities

$$\sum_{i=0}^{n-1} \|\gamma(t_{i+1}) - \gamma(t_i)\|$$

taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. Notice that if γ is differentiable, we can also compute its length by the integral

$$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt.$$

It is convenient to denote the length of a curve γ on the normed space X with unit disc B by $\ell_B(\gamma)$. Note that $\ell_B(\partial B)$ is an affine invariant of the convex body B.

THEOREM 5.2 [Schäffer 1973]. *If B and D are unit balls of two norms in a two-dimensional space X and if B^* and D^* are the dual balls in X^* then*

$$\ell_D(\partial B) = \ell_{B^*}(\partial D^*).$$

In particular,

$$\ell_B(\partial B) = \ell_{B^*}(\partial B^*).$$

A complete proof is available in [Thompson 1996].

For those who like simply stated open problems, we pass on the following question of Schäffer (private communication):

Problem. *Given an arbitrary convex body* $B \subset \mathbb{R}^3$ *that is symmetric with respect to the origin, does there always exist a plane* Π *passing through the origin and for which* $\ell_{\Pi\cap B}(\partial(\Pi \cap B))$ *is less than or equal to* 2π?

Next we discuss the relation between length and area on two-dimensional normed spaces. The first important question that arises is the isoperimetric problem: Of all convex bodies in a two-dimensional normed space X with a given perimeter find those that enclose the largest area.

The solution of this problem passes through the representation of the length as a mixed volume (in this case a mixed area). This permits the use of Brunn–Minkowski theory to solve the isoperimetric problem and to also give further properties of the length functional. The reader is referred to [Schneider 1993] for a complete discussion of the theory, but our needs can be met in just a few paragraphs.

Let X be an n-dimensional vector space and let λ be a Lebesgue measure on X. If K and L are two subsets of X, the *Minkowski sum* of K and L is the set

$$K + L := \{\boldsymbol{x} + \boldsymbol{y} \in X : \boldsymbol{x} \in K, \boldsymbol{y} \in L\}.$$

If L is the unit ball of a norm in X, we may think of $K + L$ as the set of all points in X whose distance from K is less than or equal to one. In other words, the tube of radius one about the set K.

The *mixed volume* $V(K[n-1], L)$ of two closed, bounded convex sets K and L in X is defined as a “directional derivative” of the Lebesgue measure:

$$V(K[n-1], L) = \frac{1}{n} \lim_{t\to+0} \frac{\lambda(K + tL) - \lambda(K)}{t}.$$

In the two-dimensional, case $V(K, L) := V(K[1], L)$ is linear and monotonic in each variable. The key result in the solution of the isoperimetric problem in normed spaces is the *Minkowski mixed volume inequality:*

$$V(K[n-1], L) \geq \lambda(K)^{n-1}\lambda(L).$$

Moreover, if K and L are convex bodies, then equality holds if and only if K is obtained from L by translation and dilation.

Back to the two-dimensional case, if we're given a centered convex body B, we may define the magnitude of a vector $\boldsymbol{x}$ in two different ways:

(1) Take B to be the unit ball of a norm $\|\cdot\|$ on X and set the magnitude of $\boldsymbol{x}$ to be $\|\boldsymbol{x}\|$.
(2) Let $[\boldsymbol{x}] \subset X$ denote the line segment from the origin to $\boldsymbol{x}$ and define the magnitude of $\boldsymbol{x}$ as $V([\boldsymbol{x}], B)$.

Exercise 5.3. Show that for any convex body B that is symmetric with respect to the origin, the map $\boldsymbol{x} \mapsto V([\boldsymbol{x}], B)$ is a norm, but that in general its unit disc is *different* from B.

The first step in solving the isoperimetric problem in a normed space X is to find a centrally symmetric convex body I such that

$$\|\boldsymbol{x}\| = V([\boldsymbol{x}], I), \text{ for all } \boldsymbol{x} \in X.$$

Of course, I will also depend on the choice of Lebesgue measure λ used to define the mixed volume. However, given a volume definition the body I will be defined intrinsically in terms of the norm.

The construction of I is extremely simple: Let B be the unit ball of X and let Ω be the volume form on X that satisfies $\Omega(\boldsymbol{x} \wedge \boldsymbol{y}) = \lambda(\boldsymbol{x} \wedge \boldsymbol{y})$ for all positive bases $\boldsymbol{x}, \boldsymbol{y}$ of X (we are forced to take an orientation of X at this point, but the result will not depend on the choice). If

$$i_\Omega : X \longrightarrow X^*$$

is defined by $i_\Omega(\boldsymbol{v})(\boldsymbol{w}) := \Omega(\boldsymbol{v} \wedge \boldsymbol{w})$, the set I is given by $(i_\Omega B)^*$.

Summarizing:

Proposition 5.4. *Let X be a two-dimensional normed space with unit ball B and volume form Ω. If I denotes the body $(i_\Omega B)^*$, then*

$$\|\boldsymbol{x}\| = V([\boldsymbol{x}], I)$$

for all vectors $\boldsymbol{x} \in X$.

The proof will be postponed to the next section where we will treat the n-dimensional version of the proposition.

Notice that if the orientation of X is changed, the form Ω changes sign, but the symmetry of the unit disc B implies that the body I stays the same.

Exercise 5.5. Show that if K is a convex body in X, its perimeter equals $2V(K, I)$ and that, in particular, the perimeter of I is twice its area. *Hint:* Try first with bodies whose boundaries are polygons and use the previous proposition.

The representation of length as a mixed volume gives an easy proof of the following monotonicity property of length in two-dimensional normed spaces.

Proposition 5.6. *If $K_1 \subset K_2$ are nested convex bodies in a two-dimensional normed space, then $\ell(\partial K_1) \leq \ell(\partial K_2)$.*

The proof is left as a simple exercise to the reader. The following related exercise is, perhaps, somewhat harder.

Exercise 5.7. Show that a Finsler metric on the plane satisfies the monotonicity property in the previous proposition if and only if its geodesics are straight lines.

Theorem 5.8. *Let X be a two-dimensional normed space with unit disc B and area form $\Omega \in \Lambda^2 X^*$. Of all convex bodies in X with a given perimeter the one that encloses the largest area is, up to translations, a dilate of $I := (i_\Omega B)^*$.*

PROOF. Let $K \subset X$ be a convex body and let

$$\ell_B(\partial K) = 2V(K, I)$$

be its perimeter. By *Minkowski's mixed volume inequality,* we have

$$\frac{\ell_B(\partial K)^2}{4} = V(K, I)^2 \geq \lambda(K)\lambda(I)$$

with equality if an only if K and I are homothetic. Thus, the area enclosed by K is maximal for a given perimeter if and only if K is a dilate of I. $\square$

DEFINITION 5.9. Let $Y \mapsto (\Lambda^2 Y, \mu_Y)$ be a volume definition for two-dimensional normed spaces. If X is a two-dimensional normed space with unit ball B, the *isoperimetrix* of X corresponding to the volume definition μ is the body $\mathbb{I}_X := \big(i_{\Omega_X}(B)\big)^*$, where Ω_X is a 2-form on X satisfying $|\Omega_X| = \mu_X$.

We shall denote the isoperimetrices of a two-dimensional normed space X with respect to the Busemann, Holmes–Thompson, mass, and mass$*$ volume definitions by $\mathbb{I}^b_X$, $\mathbb{I}^{\mathrm{ht}}_X$, $\mathbb{I}^m_X$, and $\mathbb{I}^{m*}_X$.

If $T : X \to X$ is an invertible linear transformation, the isoperimetrix, with respect to any volume definition, of the norm with unit ball $T(B_X)$ is $T(\mathbb{I}_X)$.

EXERCISE 5.10. If X is a two-dimensional normed space with unit ball B and if μ is a particular choice of volume definition, then

$$\ell_B(\partial \mathbb{I}_X) = 2\mu_X(\mathbb{I}_X) \text{ and } \mu_X(\mathbb{I}_X) = \mu^*_X(B^*).$$

Using this exercise, we can give sharp estimates on the area and perimeter of the isoperimetrix of a two-dimensional normed space for Busemann, Holmes–Thompson, and mass$*$ volume definitions.

Indeed, it follows trivially from the exercise that $\mu^b_X(\mathbb{I}^b_X) = \mathrm{vp}(B_X)/\pi$ and that $\mu^{\mathrm{ht}}_X(\mathbb{I}^{\mathrm{ht}}_X) = \pi$. Using the Mahler and Blaschke–Santaló inequalities, we have

$$\frac{8}{\pi} \leq \mu^b_X(\mathbb{I}^b_X) \leq \pi.$$

The fact that $\mu^{m*}_X(\mathbb{I}^{m*}_X) \leq \pi$ with equality if and only if X is Euclidean is equivalent to the inequality $\mu^{m*} \geq \mu^{\mathrm{ht}}$ for two-dimensional normed spaces.

EXERCISE 5.11. Find the sharp lower bound for $\mu^{m*}_X(\mathbb{I}^{m*}_X)$.

It is interesting to note that the Blaschke–Santaló inequality implies that

$$\mu^b_X(B_X) \geq \mu^b_X(\mathbb{I}^b_X) \quad \text{and} \quad \mu^{\mathrm{ht}}_X(B_X) \leq \mu^{\mathrm{ht}}_X(\mathbb{I}^{\mathrm{ht}}_X),$$

with equality in both cases if and only if B is an ellipse. Of course this implies that for both the Busemann and Holmes–Thompson definitions $B_X = \mathbb{I}_X$ if and only if X is Euclidean. Notice that whether a unit disc is equal to its isoperimetrix depends on the volume definition we are using. However, whether the unit disc is a dilate of its isoperimetrix does not depend on such a choice.

Definition 5.12. Let X be a two-dimensional normed space. If B_X is a dilate of $\mathbb{I}_X$ for one (and, therefore, any) volume definition, the unit circle, ∂B_X, is said to be a *Radon curve.*

For comparison with the higher-dimensional case we summarize the properties of the map $\mathbb{I}$ that sends a unit disc B_X to $\mathbb{I}_X$. This maps sends convex bodies to convex bodies; it is a bijection; it commutes with linear maps in the sense that $T(B_X)$ is sent to $T(\mathbb{I}_X)$ for all invertible linear maps T; it maps polygons to polygons, smooth bodies to strictly convex bodies and strictly convex bodies to smooth bodies; and the only fixed points for the μ^b and μ^{ht} normalizations are ellipses.

A good, very elementary account of the construction of the isoperimetrix from first principles and its relationship to physics and symplectic geometry (the ball is used for measuring position and the isoperimetrix for measuring velocity) is given by Wallen [1995].

Finally, we explore the relationship between the perimeter and area of the unit ball. The motivation is that $\ell(\partial\mathbb{I}) = 2\mu(\mathbb{I}_B)$ and that in the Euclidean case this holds for the ball.

Theorem 5.13. *If X is a two-dimensional normed space with unit ball B then*

$$2\mu^m(B) \le \ell(\partial B) \le 2\mu^{m*}(B)$$

with equality on the left if and only if ∂B is a Radon curve and on the right if and only if ∂B is an equiframed curve.

For the definition of equiframed curves and a proof of the theorem we refer the reader to [Martini et al. 2001] where the history of this result is also discussed.

Exercise 5.14. Use this result and properties of $\mathbb{I}_X$ to show that $B_X = \mathbb{I}_X^m$ if and only if ∂B_X is a Radon curve; and that $B_X = \mathbb{I}_X^{m*}$ if and only if ∂B_X is equiframed.

There is a further recent result in this direction.

Theorem 5.15 [Moustafaev]. *If X is a two-dimensional normed space, then*

$$2\mu_X^{\mathrm{ht}}(B_X) \le \ell(\partial B_X),$$

with equality if and only if X is Euclidean.

Proof. By definition of the isoperimetrix and Minkowski's mixed volume inequality, we have

$$\ell(B_X)^2 = 4V(B_X, \mathbb{I}_X^{\mathrm{ht}}) \ge 4\mu_X^{\mathrm{ht}}(B_X)\mu_X^{\mathrm{ht}}(\mathbb{I}_X^{\mathrm{ht}}).$$

Using that $\mu_X^{\mathrm{ht}}(\mathbb{I}_X^{\mathrm{ht}}) = \pi$ and that $\mu_X^{\mathrm{ht}}(B_X)/\pi \le 1$, we have

$$\ell(B_X)^2 \ge 4\pi\mu_X^{\mathrm{ht}}(B_X) \ge 4\pi\mu_X^{\mathrm{ht}}(B_X)\frac{\mu_X^{\mathrm{ht}}(B_X)}{\pi} = 4\mu_X^{\mathrm{ht}}(B_X)^2. \qquad \square$$

EXERCISE 5.16. If X is a two-dimensional normed space, show that

$$2 \le \mu_X^m(B_X) \le \pi,$$
$$3 \le \mu_X^{m*}(B_X) \le 4,$$

and (using inequalities from Section 3)

$$8/\pi \le \mu_X^{\mathrm{ht}}(B_X) \le \pi.$$

Give the equality cases.

6. Area on Finite-Dimensional Normed Spaces

In Section 4, we saw that the Busemann, Holmes–Thompson, and mass$*$ volume definitions induce k-volume densities that are weakly convex. In the special case where the dimension of the normed space X is $n = k + 1$, then the $(n-1)$-volume densities are norms on the space $\Lambda^{n-1}X$.

It follows from the properties of the volume definitions that, in all three cases, the map that assigns to the normed space X the normed space $\Lambda^{n-1}X$ has the following properties:

(1) If $T : X \to Y$ is a short linear map between normed spaces X and Y, then the induced map $T_* : \Lambda^{n-1}X \to \Lambda^{n-1}Y$ is also short.
(2) The map $X \mapsto \Lambda^{n-1}X$ is continuous with respect to the topology induced by the Banach–Mazur distance.
(3) If X is a Euclidean space, then the $(n-1)$-volume density is the standard Euclidean area on X.
(4) If the dimension of X is two, the map $X \mapsto \Lambda^1 X$ is the identity.

Notice that property (1) states that for the Busemann, Holmes–Thompson, and mass$*$ definitions, the map that takes the normed space X to the normed space $\Lambda^{n-1}X$ is a covariant functor in the category $\mathcal{N}$ of finite-dimensional normed spaces.

DEFINITION 6.1. A *definition of area* on normed spaces assigns to every n-dimensional, $n \ge 2$, normed space X a normed space $(\Lambda^{n-1}X, \sigma_X)$ in such a way that properties (1)–(4) above are satisfied.

For simplicity, we shall speak of the Busemann, Holmes–Thompson, and mass$*$ definitions of area to refer to the definitions of area induced, respectively, by the Busemann, Holmes–Thompson, and mass$*$ volume definitions.

Definitions of area in normed spaces are related to important constructions in convex geometry such as intersection bodies, projection bodies, and Wulff shapes. However, let us start by posing a few natural questions that arise whenever we have a definition of area. The answer to some of these questions, once specialized to the Busemann and Holmes–Thompson definitions, are deep results in the theory of convex bodies. Other questions are long-standing open problems, and yet others seem to be new.

Given a definition of area $X \mapsto (\Lambda^{n-1}X, \sigma_X)$ on normed spaces, we may ask: Is the map $X \mapsto (\Lambda^{n-1}X, \sigma_X)$ injective? What is its range? Does it send crystalline norms to crystalline norms? Does it send Minkowski spaces to Minkowski spaces? In what numeric range is the area of the unit sphere of an n-dimensional normed space?

Other problems arise when we consider the relationship between length, area, and volume, but, for now, let us concentrate on the questions we have just posed.

6.1. Injectivity and range of the area definition Let us start the study of the injectivity and range of the Busemann definition of area by describing the unit ball of the $(n-1)$-volume density in terms of a well-known construction in convex geometry.

Busemann area and intersection bodies. Consider $\mathbb{R}^n$ with its Euclidean structure and its unit sphere S^{n-1}. If $K \subset \mathbb{R}^n$ is a star-shaped body containing the origin, the *intersection body* of K, IK, is defined by the following simple construction: if $\boldsymbol{x} \in \mathbb{R}^n$ is a unit vector, let $A(K \cap \boldsymbol{x}^\perp)$ denote the area of the intersection of K with the hyperplane perpendicular to $\boldsymbol{x}$, and let IK be the star-shaped body enclosed by the surface

$$\{\boldsymbol{x}/A(K \cap \boldsymbol{x}^\perp) \in \mathbb{R}^n : \boldsymbol{x} \in S^{n-1}\}.$$

A celebrated theorem of Busemann, which is equivalent to the weak convexity of the Busemann volume definition, states that if K is a centered convex body, then IK is also a centered convex body.

Let X be an n-dimensional normed space. Choose a basis of X and use it to identify X with $\mathbb{R}^n$. Take the Euclidean structure in $\mathbb{R}^n$ for which the basis is orthonormal and use the resulting Euclidean structure to identify the spaces X^* and $\Lambda^{n-1}X$, as well as to define the unit sphere S^{n-1} in $\mathbb{R}^n$.

Exercise 6.2. Show that with all these identifications, the convex body $\{\boldsymbol{x} \in \mathbb{R}^n : \sigma_X^b(\boldsymbol{x}) \leq 1\}$ is ε_{n-1} times the intersection body of B_X; (here σ_X^b is the norm induced on X^* by the norm on $\Lambda^{n-1}X$).

Notice that we can now write the question of whether the Busemann definition of area is injective in the following classical form: Is a centered convex body determined uniquely by the area of its intersections with hyperplanes passing through the origin? The answer is affirmative (see [Lutwak 1988] and [Gardner 1995]), and so we have the following result:

Theorem 6.3. *The Busemann area definition is injective.*

Determining the range of the Busemann area definition is somewhat trickier. Thanks to the efforts of R. Gardner, G. Zhang, and others in the solution of the first of the Busemann–Petty problems, it is known (see [Gardner 1994], [Gardner et al. 1999], and [Zhang 1999] and the references therein) that in dimensions two, three, and four every convex body symmetric with respect to the origin is the intersection body *of some star-shaped body.* It is not clear at this point whether those bodies that are intersection bodies of centered convex bodies can be characterized effectively. For dimensions greater than four, not every centered convex body is an intersection body ([Gardner et al. 1999]). For further information about intersection bodies see, for example, [Gardner 1995] and [Lutwak 1988].

Examples in [Thompson 1996] show that the Busemann area definition does not take crystalline norms to crystalline norms. We don't know whether it takes Minkowski norms to Minkowski norms.

Let us now pass to the Holmes–Thompson definition.

Holmes–Thompson area and projection bodies. Consider $\mathbb{R}^n$ with its Euclidean structure and its unit sphere S^{n-1}. If $K \subset \mathbb{R}^n$ is a convex body, the *projection body* of K, ΠK, is given by the following simple construction: if $\boldsymbol{x} \in \mathbb{R}^n$ is a unit vector, let $A(K|\boldsymbol{x}^\perp)$ denote the area of the orthogonal projection of K onto the hyperplane perpendicular to $\boldsymbol{x}$, and let the *polar* of ΠK be the body enclosed by the surface

$$\{A(K|\boldsymbol{x}^\perp)\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \in S^{n-1}\}.$$

As in the case of the Busemann definition of area, identifying a normed space X with $\mathbb{R}^n$ allows us to write the unit ball for the $(n-1)$-volume density in terms of this nonintrinsic construction.

Exercise 6.4. Show that by identifying a normed space X with $\mathbb{R}^n$ as in the previous exercise, the convex body $\{\boldsymbol{x} \in \mathbb{R}^n : \sigma_X^{\mathrm{ht}}(\boldsymbol{x}) \le 1\}$ is $1/\varepsilon_{n-1}$ times the polar of the projection body of B_X^*.

The question of the injectivity of the Holmes–Thompson definition of area can now be formulated in classical terms: Is a centered convex body determined uniquely by the area of its orthogonal projections onto hyperplanes? The answer, in the affirmative, follows from a celebrated result of Alexandrov [1933] (see also [Gardner 1995]). We then have the following result:

Theorem 6.5. *The Holmes–Thompson area definition is injective.*

It is known, basically from the time of Minkowski, that a centered convex body B is the projection body of another if and only if it is a zonoid (see [Gardner 1995]). By Theorem 2.12, this means that for any n-dimensional normed space X the normed space $(\Lambda^{n-1}X, \sigma_X^{\mathrm{ht}})$ is hypermetric.

Moreover, because of the integral formula for the Holmes–Thompson $(n-1)$-volume density in terms of the surface area measure of the dual sphere given in the proof of Theorem 4.16, the problem of reconstructing the norm from the Holmes–Thompson $(n-1)$-volume density is precisely the famous *Minkowski problem:* Reconstruct a convex body from the knowledge of its Gauss curvature as a function of its unit normals. The next two theorems follow directly from the work of Minkowski, Pogorelov, and Nirenberg (see [Pogorelov 1978] for a detailed presentation).

THEOREM 6.6. *The range of the Holmes–Thompson area definition is the set of hypermetric normed spaces.*

THEOREM 6.7. *Let X be an n-dimensional vector space and let $\sigma : \Lambda^{n-1}X \to [0,\infty)$ be a Minkowski norm. If $(\Lambda^{n-1}X, \sigma)$ is hypermetric, then there exists a unique Minkowski norm $\|\cdot\|$ on X such that σ is the Holmes–Thompson $(n-1)$-volume density of the normed space $(X, \|\cdot\|)$.*

Another important feature of the Holmes–Thompson area is the following (for a proof see [Thompson 1996]):

THEOREM 6.8. *The Holmes–Thompson area definition takes Minkowski spaces to Minkowski spaces and crystalline norms to crystalline norms.*

Mass area and wedge bodies.* Let B be a centered convex body in an n-dimensional vector space X, and let B^{n-1} be the $(n-1)$-fold product of B in the $n(n-1)$-dimensional space X^{n-1}. If $\mathrm{Alt} : X^{n-1} \to \Lambda^{n-1}X$ denotes the (nonlinear) map

$$(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}) \longmapsto \boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1},$$

we define the *wedge body* of B, denoted by $\mathcal{W}B$, as the convex hull of $\mathrm{Alt}(B^{n-1})$ in $\Lambda^{n-1}X$.

We remark that even if $B^{n-1} \subset X^{n-1}$ is a centered convex body, $\mathrm{Alt}(B^{n-1})$ is not necessarily convex.

THEOREM 6.9. *The unit ball in $\Lambda^{n-1}X$ for the mass$*$ $(n-1)$-volume density of a normed space X is the body $(\mathcal{W}B_X^*)^*$.*

PROOF. By Exercise 4.3, we have

$$\sigma_X^{m*}(a) = \sup\{|\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_{n-1} \cdot a| : \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1} \in B_X^*\}.$$

But this is just the supremum of $|\eta \cdot a|$, where $\eta \in \mathrm{Alt}\big((B_X^*)^{n-1}\big)$. Therefore σ_X^{m*} is the dual to the norm in $\Lambda^{n-1}X^*$ whose unit ball is $\mathcal{W}B_X^*$. $\square$

It is quite easy to do calculations for $\mathcal{W}B^*$ in the case when the centered convex body B is a simple object. The following statements are based on such calculations, the details of which are left as exercises (see also [Thompson 1999]).

PROPOSITION 6.10. *The mass$*$ area definition is not injective.*

SKETCH OF THE PROOF.. All we must do is find two centered convex bodies B and K such that $\mathcal{W}B^* = \mathcal{W}K^*$, but $B \neq K$.

Let B be the cube with vertices at $(\pm1, \pm1, \pm1)$. In this case, B^* is the octahedron with vertices $(\pm1, 0, 0)$, $(0, \pm1, 0)$, $(0, 0, \pm1)$ and $\mathcal{W}B^* = B^*$.

Let K be the cuboctahedron with vertices $(\pm 1, \pm 1, 0)$, $(\pm 1, 0, \pm 1)$, $(0, \pm 1, \pm 1)$. The dual ball K^* is the rhombic dodecahedron with vertices $\pm(1,0,0)$, $\pm(0,1,0)$, $\pm(0,0,1)$ and $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. A simple calculation shows that $\mathcal{W}B^* = WK^*$.

In fact, if L is any centered convex body that lies between the cube and the cube-octahedron then $\mathcal{W}L^* = \mathcal{W}B^*$. □

While it seems unlikely that the wedge body of the unit ball in a Minkowski space is the ball of a Minkowski space, it is not hard to show that the wedge body of a polytope is a polytope. Then:

Proposition 6.11 [Thompson 1999]. *The mass$*$ area definition takes crystalline norms to crystalline norms.*

The question of determining the range for the mass$*$ area definition is completely open. Is it possible that any centered convex body is a wedge body?

6.2. Area of the unit sphere. In this section we give the higher-dimensional analogues (as far as we know them) of the theorems of Schäffer and Gołąb discussed in Section 5.

The Holmes–Thompson definition was designed originally to yield a generalization of Schäffer's result and we have the following theorem.

Theorem 6.12 [Holmes and Thompson 1979]. *If B and K are the unit balls of two norms $\|\cdot\|_B$ and $\|\cdot\|_K$ in the vector space X, the Holmes–Thompson area of ∂K in the normed space $(X, \|\cdot\|_B)$ equals the Holmes–Thompson area of ∂B^* in the normed space $(X^*, \|\cdot\|_K^*)$.*

Notice that in particular, the Holmes–Thompson area of the unit sphere of a normed space equals the Holmes–Thompson area of the unit sphere of its dual. Simple calculations show that neither the Busemann, the mass$*$, nor the mass definition have this property. In fact, Daniel Hug (private communication) has shown that Theorem 6.12 characterizes the Holmes–Thompson definition. However, the following question remains open.

Problem [Thompson 1996]. *Is the Holmes–Thompson definition of volume characterized by the fact that the area of the unit sphere of a normed space equals the area of the unit sphere of its dual*?

The first result extending Gołąb's theorem to higher dimension is the following sharp upper bound for the Busemann area of a unit sphere.

Theorem 6.13 [Busemann and Petty 1956]. *The Busemann area of the unit sphere of an n-dimensional normed space is at most $2n\varepsilon_{n-1}$ with equality if and only if B is a parallelotope.*

For $n \geq 3$ no sharp lower bound for the Busemann area of the unit sphere of an n-dimensional normed space has been proved. It is conjectured that the minimum is $n\varepsilon_n$ attained by the Euclidean ball. However, when $n = 3$ it is also attained by the rhombic dodecahedron.

Since $\mu^b \geq \mu^{\mathrm{ht}}$, an upper bound for the Busemann area is also an upper bound for the Holmes–Thompson area.

Corollary 6.14. *The Holmes–Thompson area of the unit sphere of an n-dimensional normed space is less than $2n\varepsilon_{n-1}$.*

While the sharp upper bound for the Holmes–Thompson area of the unit sphere in any dimension greater than two is not known, the sharp lower bound in dimension three is given by the following unpublished result of Álvarez, Ivanov, and Thompson:

Theorem 6.15. *The Holmes–Thompson area of the unit sphere of a three-dimensional normed space is at least $36/\pi$. Moreover, equality holds if the unit ball is a cuboctahedron or a rhombic dodecahedron.*

Since $\mu^b \geq \mu^h t$ and $\mu^{m*} \geq \mu^{\mathrm{ht}}$, we have the following lower bound for the Busemann and mass$*$ areas of the unit sphere of a three-dimensional normed space.

Corollary 6.16. *The Busemann and mass$*$ areas of unit sphere of a three-dimensional normed space is greater than $36/\pi$.*

Although these bounds are not sharp, they are the best bounds known so far.

It is possible to use a variety of inequalities including the Petty projection inequality (in the case of σ^{ht}) and the Busemann intersection inequality (in the case of σ^b) to give nonsharp lower bounds. The reader is referred to [Thompson 1996] for examples of what one can get.

6.3. Mixed volumes and the isoperimetrix. We now pass to questions concerning the relationship between areas and volumes, and, in particular, to the solution of the isoperimetric problem in finite-dimensional normed spaces. The subject is classical and has been studied from different viewpoints by convex geometers, geometric measure theorists, and crystallographers (see, for example, [Busemann 1949b], [Taylor 1978], and [Ambrosio and Kirchheim 2000]). Nevertheless, being interested in a particular intrinsic viewpoint and relations to area on normed and Finsler spaces that are not treated elsewhere, we shall give a short account of the subject.

Let X be an n-dimensional vector space and let λ be a Lebesgue measure on X. If $I \subset X$ is a centered convex body, we can define an $(n-1)$-volume density on X by the following construction: given $n-1$ linearly independent vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1} \in X$, we denote the parallelotope they define by $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}]$ and set

$$\sigma_I(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}) := \frac{1}{n} \lim_{t \to +0} \frac{\lambda([\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}] + tI) - \lambda([\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}])}{t}.$$

It is easy to see that σ_I is well defined and that by changing λ for another Lebesgue measure on X we simply multiply σ_I by a constant. Note also that although the measure of $[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{n-1}]$ is zero, we have included it in the formula to stress its relationship with the n-dimensional mixed volume of two bodies,

$$V(K[n-1], L) := \frac{1}{n} \lim_{t \to +0} \frac{\lambda(K + tL) - \lambda(K)}{t}.$$

With this definition, if K is a convex body in X,

$$\int_{\partial K} \sigma_I = nV(K[n-1], I).$$

EXERCISE 6.17. Show that the $(n-1)$-volume density σ_I constructed above is a norm on $\Lambda^{n-1}(X)$, and that

$$\int_{\partial I} \sigma_I = n\lambda(I).$$

We would also like to reverse this construction: Starting from a norm $\sigma : \Lambda^{n-1}X \to [0, \infty)$ and a Lebesgue measure λ on X construct a convex body $I \subset X$ such that $\sigma = \sigma_I$. The construction is quite simple: Let Ω be a volume form on X such that $|\Omega| = \lambda$ and consider the linear isomorphism

$$i_\Omega : \Lambda^{n-1}X \longrightarrow X^*$$

defined by $i_\Omega(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1})(\boldsymbol{x}) = \Omega(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1} \wedge \boldsymbol{x})$. The body I is given by $(i_\Omega \mathcal{B})^*$, where $\mathcal{B} \subset \Lambda^{n-1}X$ is the unit ball of σ.

In terms of mixed volumes, we have the following result:

PROPOSITION 6.18. *Let X be an n-dimensional vector space, let σ be a norm on $\Lambda^{n-1}X$ with unit ball $\mathcal{B}$ and let λ be a Lebesgue measure on X. Using the notation above, if $I := (i_\Omega \mathcal{B})^*$, we have*

$$\int_{\partial K} \sigma = nV(K[n-1], I)$$

for all convex bodies $K \subset X$.

To prove the proposition, let us give a simpler, more visual relationship between σ and $I := (i_\Omega \mathcal{B})^*$ that is of independent interest. Given a nonzero $(n-1)$-vector $a \in \Lambda^{n-1}X$, we shall say that a vector $\boldsymbol{v} \in X$ is *normal to a with respect to I* if $\boldsymbol{v} \in \partial I$, the hyperplane parallel to $\langle a \rangle$ and passing through $\boldsymbol{v}$ supports I, and $\Omega(a \wedge \boldsymbol{v}) > 0$. When I is smooth and strictly convex the normal is unique, but this is of no importance to what follows. Notice, and this is important, that $\boldsymbol{v}$ is constructed in such a way that

$$\Omega(a \wedge \boldsymbol{v}) = \sup\{|\Omega(a \wedge \boldsymbol{x})| : \boldsymbol{x} \in I\}.$$

LEMMA 6.19. *Let X be an n-dimensional vector space, let σ be a norm on $\Lambda^{n-1}X$ with unit ball $\mathcal{B}$ and let $\Omega \in \Lambda^n X^*$ be a volume form on X. If a is a nonzero $(n-1)$-vector on X and $\boldsymbol{v} \in X$ is normal to a with respect to $I := (i_\Omega \mathcal{B})^*$, then*

$$\sigma(a) = \Omega(a \wedge \boldsymbol{v}).$$

Proof. Let $\|\cdot\|^*$ denote the norm in X^* whose unit ball is $I^* = i_\Omega \mathcal{B}$. Trivially, we have $\sigma(a) = \|i_\Omega(a)\|^*$ for any $a \in \Lambda^{n-1}X$. Therefore,

$$\sigma(a) = \sup\{|\Omega(a \wedge \boldsymbol{x})| : \boldsymbol{x} \in I\} = \Omega(a \wedge \boldsymbol{v}). \qquad \square$$

In other terms, if $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}$ are linearly independent vectors in X and $\boldsymbol{v}$ is normal to $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}$ with respect to I, then the volume of the n-dimensional parallelotope $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{v}]$ is the area of the $(n-1)$-dimensional parallelotope $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}]$.

Proof of Proposition 6.18. Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}$ be linearly independent vectors in X and let $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}]$ denote the parallelotope spanned by them. Notice that if $\boldsymbol{v}$ is normal to $\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}$ with respect to I, then for any $t > 0$, the union of the n-dimensional parallelotopes

$$[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, t\boldsymbol{v}] \text{ and } [\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, -t\boldsymbol{v}],$$

which we denote by $P(t)$, is contained in the set $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}] + tI$. Moreover, since up to terms of order 2 and higher in t the volumes of $P(t)$ and $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}] + tI$ are the same, we have

$$\frac{1}{n} \lim_{t \to +0} \frac{\lambda([\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}] + tI)}{t} = \frac{1}{n} \lim_{t \to +0} \frac{\lambda(P(t))}{t} = \sigma(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \cdots \wedge \boldsymbol{x}_{n-1}),$$

and this concludes the proof. $\square$

We are now ready to solve the isoperimetric problem for convex bodies:

Theorem 6.20. *Let X be an n-dimensional vector space, let σ be a norm on $\Lambda^{n-1}X$ with unit ball $\mathcal{B}$, and let $\Omega \in \Lambda^n X^*$ be a volume form on X. Of all convex bodies in X with a given surface area the one that encloses the largest volume is, up to translations, a dilate of $I := (i_\Omega \mathcal{B})^*$.*

Proof. Let $K \subset X$ be a convex body and let

$$\int_{\partial K} \sigma = nV(K[n-1], I)$$

be its surface area. By Minkowski's mixed volume inequality, we have

$$\left(\int_{\partial K} \sigma \right)^n = n^n V(K[n-1], I)^n \geq n^n \lambda(K)^{n-1} \lambda(I)$$

with equality if an only if K and I are homothetic. Thus, the volume enclosed by K is maximal for a given surface area if and only if K is a dilate of I. $\square$

We shall denote the isoperimetrices of a normed space X with respect to the Busemann, Holmes–Thompson, and mass$*$ definitions by $\mathbb{I}_X^b$, $\mathbb{I}_X^{\mathrm{ht}}$, and $\mathbb{I}_X^{m*}$, respectively. In the case of the Busemann and Holmes–Thompson definitions, the isoperimetrices can be given, nonintrinsically, in terms of intersection and projection bodies.

Exercise 6.21. Using Exercises 6.2 and 6.4, and the construction of the isoperimetrix, show that

$$\mathbb{I}^b_X = \frac{\varepsilon_{n-1}}{\varepsilon_n}\lambda(B_X)(IB_X)^* \text{ and } \mathbb{I}^{\mathrm{ht}}_X = \frac{\varepsilon_n}{\varepsilon_{n-1}}\lambda^*(B^*_X)^{-1}\Pi B^*_X,$$

where λ and λ^* are, respectively, the Euclidean volumes on X and X^* given by their identification with $\mathbb{R}^n$.

Exercise 6.22. Describe the isoperimetrix $\mathbb{I}^{m*}_X B$ in terms of wedge bodies.

6.4. Geometry of the isoperimetrix. We now turn our attention to problems relating the unit ball of a normed space and its isoperimetrix with respect to some volume definition. Let us start with the deceptively simple problem of estimating the volume of the isoperimetrix.

Identifying the normed space X with $\mathbb{R}^n$ as in Exercise 6.21, we see that the Holmes–Thompson volume of $\mathbb{I}^{\mathrm{ht}}_X$ is

$$\mu^{\mathrm{ht}}_X(\mathbb{I}^{\mathrm{ht}}_X) = \varepsilon_n^{-1}\left(\frac{\varepsilon_n}{\varepsilon_{n-1}}\right)^n \lambda^*(B^*_X)^{-n+1}\lambda(\Pi B^*_X).$$

The statement that this quantity is greater than or equal to ε_n with equality if and only if X is Euclidean is known as *Petty's conjectured projection inequality,* and is one of the major open problems in the theory of affine geometric inequalities.

Sharp lower bounds for $\mu^b_X(\mathbb{I}^b_X)$ and $\mu^{m*}_X(\mathbb{I}^{m*}_X)$ are also unknown, although as observed in [Thompson 1996] the inequality $\mu^b_X(\mathbb{I}^b_X) \geq \varepsilon_n$ for $n \geq 3$ would easily yield (exercise!) that the Busemann area of unit sphere of a normed space of dimension n is at least $n\varepsilon_n$.

Another interesting affine invariant involving the isoperimetrix is the symplectic volume of $B_X \times \mathbb{I}^*_X$ in $X \times X^*$. In the two-dimensional case this simply yields the square of the area of the unit disc, but in higher dimension it is a much more interesting invariant:

Exercise 6.23. Pick up either [Gardner 1995] or [Thompson 1996] and, using Exercise 6.21, prove that the inequality

$$\mathrm{svol}(B_X \times \mathbb{I}^*_X) \leq \varepsilon_n \mu_X(B_X)$$

is true for the Busemann (resp. Holmes–Thompson) definition by showing that it is equivalent to Busemann's intersection inequality (resp. Petty's projection inequality).

It would be interesting to complete the picture by having a sharp upper bound for $\mathrm{svol}(B_X \times (\mathbb{I}^{m*}_X)^*)$ in terms of ${\mu^{m*}}_X(B_X)$.

We finish the paper by considering some questions relating length, area, and volume. In terms of the isoperimetrix they have very simple statements: Given a volume definition, when is the isoperimetrix equal to the unit ball, when is it a multiple of the ball, and when is it inside the ball?

These simple questions are really about the existence of a coarea formula or inequality for the different definitions of volume on normed and Finsler spaces. Many Riemannian and Euclidean results depend, or seem to depend, on the simple fact that volume = base × height. To what extent is this true in normed and Finsler spaces?

In order to relate the coarea formula and inequality with the geometry of the isoperimetrix, let us first define the height of a parallelotope $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{x}_n]$ in a vector space X with respect to a centered convex body $B \subset X$ by the following construction: let $\boldsymbol{\xi} \in X^*$ be a covector in ∂B^* such that $\boldsymbol{\xi}(\boldsymbol{x}_i) = 0$ for all i between 1 and $n-1$. The quantity $|\boldsymbol{\xi}(\boldsymbol{x}_n)|$, which is independent of the choice of $\boldsymbol{\xi}$, will be called the *height* of $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{x}_n]$ with respect to B.

By the construction of the isoperimetrix, we know that the volume of the parallelotope $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{x}_n]$ equals the area of its base, $[\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}]$, times its height with respect to the isoperimetrix. Therefore, if the volume of every parallelotope in a normed space equals the area of its base times its height with respect to the unit ball, the ball equals the isoperimetrix. If the volume is greater than the area of the base times the height with respect to the unit ball, then the isoperimetrix is contained in the ball, and so on.

The first clear sign that the relationship between length, area and volume may not go smoothly on normed and Finsler spaces is the following result of Thompson:

PROPOSITION 6.24 [Thompson 1996]. *The isoperimetric of a normed space X for the Holmes–Thompson definition is contained in the unit ball if and only if the space is Euclidean. In which case, the ball and the isoperimetric are equal.*

In other words, the coarea equality or inequality "volume ≥ base × height" for the Holmes–Thompson definition is true only for Euclidean spaces.

PROOF. If $\mathbb{I}_X^{\mathrm{ht}} \subset B_X$, then $B_X^* \subset (\mathbb{I}_X^{\mathrm{ht}})^*$ and, therefore,

$$\operatorname{svol}(B_X \times (\mathbb{I}_X^{\mathrm{ht}})^*) \geq \operatorname{svol}(B_X \times B_X^*) = \varepsilon_n \mu_X^{\mathrm{ht}}(B_X).$$

By Exercise 6.23, the only way this can happen is if X is Euclidean. □

However, for the mass∗ definition the coarea inequality is always true:

THEOREM 6.25 [Gromov 1983]. *If X is a finite-dimensional normed space, then $\mathbb{I}_X^{m*} \subset B_X$.*

PROOF. We must show that if $[\boldsymbol{v}_1, \dots, \boldsymbol{v}_n]$ is a parallelotope,

$$\mu^{m*}(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_n) \geq \sigma^{m*}(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_{n-1}) |\boldsymbol{\xi}(\boldsymbol{v}_n)|,$$

where $\boldsymbol{\xi} \in \partial B_X^*$ and $\boldsymbol{\xi}(\boldsymbol{v}_i) = 0$, $1 \leq i \leq n-1$.

Without loss of generality we may suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots \boldsymbol{v}_{n-1}$ is an extremal basis in the subspace $V \subset X$ they span, *i.e.* each vector $\boldsymbol{v}_i$ is a point of contact between $B_X \cap V$ and a minimal circumscribing parallelotope for $B \cap V$. Let $\boldsymbol{u}$

be such that $\|\boldsymbol{u}\| = \boldsymbol{\xi}(\boldsymbol{u}) = 1$ and set $\boldsymbol{v}_n = \alpha\boldsymbol{u} + \boldsymbol{x}$ where $\boldsymbol{x} \in V$. The right hand side of the above inequality is $|\alpha|$.

Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{n-1}$ be the dual basis to the $\boldsymbol{v}_i$'s in V and extend these to the whole of X by setting $\boldsymbol{\xi}_i(\boldsymbol{u}) = 0$. Then $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots, \boldsymbol{\xi}_{n-1}, \boldsymbol{\xi}$ are all of norm 1 and form the dual basis to $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{u}$. Now

$$\begin{aligned}\mu_X^{m*}(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_n) &= |\alpha| \mu_X^{m*}(\boldsymbol{v}_1 \wedge \boldsymbol{v}_2 \wedge \cdots \wedge \boldsymbol{v}_{n-1} \wedge \boldsymbol{u}) \\ &= |\alpha| (\mu_{X^*}^m(\boldsymbol{\xi}_1 \wedge \boldsymbol{\xi}_2 \wedge \cdots \wedge \boldsymbol{\xi}_{n-1} \wedge \boldsymbol{\xi}))^{-1} \\ &\geq |\alpha| (\|\boldsymbol{\xi}\| \prod \|\boldsymbol{\xi}_i\|)^{-1} = |\alpha|.\end{aligned}$$

The inequality comes from the definition of mass. □

As we have said, the problem of determining for what normed spaces metric balls are solutions to the isoperimetric problem, *i.e.* when is the isoperimetrix a multiple of the unit ball, is completely open for all three definitions of volume in dimensions greater than two.

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