

Finsler Geometry on Complex Vector Bundles

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1. Introduction

A Finsler metric of a manifold or vector bundle is defined as a smooth assignment for each base point a norm on each fibre space, and thus the class of Finsler metrics contains Riemannian metrics as a special sub-class. For this reason, Finsler geometry is usually treated as a generalization of Riemannian geometry. In fact, there are many contributions to Finsler geometry which contain Riemannian geometry as a special case (see e.g., [Bao et al. 2000], [Matsumoto 1986], and references therein).

On the other hand, we can treat Finsler geometry as a special case of Riemannian geometry in the sense that Finsler geometry may be developed as differential geometry of fibred manifolds (e.g., [Aikou 2002]). In fact, if a Finsler metric in the usual sense is given on a vector bundle, then it induces a Riemannian inner product on the vertical subbundle of the total space, and thus, Finsler geometry is translated to the geometry of this Riemannian vector bundle.

It is natural to question why we need Finsler geometry at all. To answer this question, we shall describe a few applications of complex Finsler geometry to some subjects which are impossible to study via Hermitian geometry.

The notion of complex Finsler metric is old and goes back at least to Carathéodory who introduced the so-called *Carathéodory metric*. The geometry of complex Finsler manifold, via tensor analysis, was started by [Rizza 1963], and afterwards, the connection theory on complex Finsler manifolds has been developed by [Rund 1972], [Icijiō 1994], [Fukui 1989], and [Cao and Wong 2003], etc..

Recently, from the viewpoint of the geometric theory of several complex variables, complex Finsler metric has become an interesting subject. In particular, an intrinsic metric on a complex manifold, namely the *Kobayashi metric*, is a holomorphic invariant metric on a complex manifold. The Kobayashi metric is, by its definition, a pseudo Finsler metric. However, by the fundamental work of [Lempert 1981], the Kobayashi metric on a smoothly bounded strictly convex domain in \mathbb{C}^n is a smooth pseudoconvex Finsler metric.

The interest in complex Finsler geometry also arises from the study of holomorphic vector bundles. The characterization of ample (or negative) vector bundles due to Kobayashi [Kobayashi 1975] shows the importance of Finsler geometry. In fact, he has proved that E is ample if and only if its dual E^* admits a “negatively curved” pseudoconvex Finsler metric (Theorem 3.2). The meaning of the term “negatively curved” is defined by using the curvature tensor of the Finsler connection on a Finsler bundle (E, F) .

Another example of interest in complex Finsler geometry arises from the geometry of *geometrically ruled surfaces* \mathcal{X} . A geometrically ruled surface \mathcal{X} is, by definition (see [Yang 1991]), an algebraic surface with a holomorphic projection $\phi : \mathcal{X} \rightarrow M$, M a compact Riemann surface, such that each fibre is isomorphic to the complex projective line \mathbb{P}^1 . Every geometrically ruled surface is isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle $\pi : E \rightarrow M$ of $\text{rank}(E) = 2$. Then, every geometrically ruled surface $\mathcal{X} = \mathbb{P}(E)$ is also a compact Kähler manifold by Lemma 6.37 in [Shiffmann and Sommese 1985], and any Kähler metric $g_{\mathcal{X}}$ on \mathcal{X} induces a Finsler metric F , which is *not* a Hermitian metric in general, on the bundle E . Thus the geometry of $(\mathcal{X}, g_{\mathcal{X}})$ is translated to the geometry of the Finsler bundle (E, F) .

In general, an algebraic curve (or polarized manifold) $\varphi : \mathcal{X} \rightarrow \mathbb{P}^N$ has a Kähler metric $\omega_{\mathcal{X}} = \varphi^* \omega_{FS}$ induced from the Fubini–Study metric ω_{FS} on \mathbb{P}^N . An interesting subject in complex geometry is to investigate how metrics of this kind are related to constant curvature metrics, and moreover, it is interesting to investigate how constant scalar curvature metrics should be related to algebro-geometric stability. LeBrun [LeBrun 1995] has investigated minimal ruled surfaces $\mathcal{X} = \mathbb{P}(E)$ over a compact Riemann surface M of genus $g(M) \geq 2$ with constant scalar curvature. He showed that, roughly speaking, \mathcal{X} admits such a Kähler metric $g_{\mathcal{X}}$ if and only if the bundle E is semi-stable in the sense of Mumford–Takemoto. By the statement above, an arbitrary Kähler metric on a minimal ruled surface \mathcal{X} determines a Finsler metric F on E by the identity $\omega_{\mathcal{X}} = \sqrt{-1} \partial \bar{\partial} \log F$ for the Kähler form $\omega_{\mathcal{X}}$. The geometry of such a minimal

ruled surface can also be investigated by the study of the Finsler bundle (E, F) (see [Aikou 2003b]).

In this article, we shall report on the geometry of complex vector bundles with Finsler metrics, i.e., *Finsler bundles*. Let F be a Finsler metric on a holomorphic vector bundle $\pi : E \rightarrow M$ over a complex manifold M . The geometry of a Finsler bundle (E, F) is the study of the vertical bundle $V_E = \ker \pi_*$ with a Hermitian metric G_{V_E} induced from the given Finsler metric.

The main tool of the investigation in Finsler geometry is the *Finsler connection*. The connection is a unique one on the Hermitian bundle (V_E, g_{V_E}) , satisfying some geometric condition (see definition below). Although it is natural to investigate (V_E, g_{V_E}) by using the Hermitian connection of (V_E, g_{V_E}) , it is convenient to use the Finsler connection for investigating some special Finsler metrics. For example, the flatness of the Hermitian connection of (V_E, g_{V_E}) implies that the Finsler metric F is reduced to a flat Hermitian metric. However, if the Finsler connection is flat, then the metric F belongs to an important class, the so-called *locally Minkowski metrics* (we simply call these special metrics *flat Finsler metrics*). If the Finsler connection is induced from a connection on E , then the metric F belongs to another important class, the so-called *Berwald metrics* (sometimes a Berwald metric is said to be *modeled on a Minkowski space*). In this sense, the big difference between Hermitian geometry and Finsler geometry is the connection used for the investigation of the bundle (V_E, g_{V_E}) .

2. Ampleness

2.1. Ample line bundles. Let L be a holomorphic line bundle over a compact complex manifold M . We denote by $\mathcal{O}(L)$ the sheaf of germs of holomorphic sections of L . Since M is compact, $\dim_{\mathbb{C}} H^0(M, \mathcal{O}(L))$ is finite. Let $\{f^0, \dots, f^N\}$ be a set of linearly independent sections of L , from the complex vector space of global sections. The vector space spanned by these sections is called a *linear system* on M . If the vector space consists of all global sections of L , it is called a *complete linear system* on M . Using these sections $\{f^0, \dots, f^N\}$, a rational map $\varphi_{|L|} : M \rightarrow \mathbb{P}^N$ is defined by

$$\varphi_{|L|}(z) = [f^0(z) : \dots : f^N(z)]. \tag{2.1}$$

This rational map is defined on the open set in M which is complementary to the common zero-set of the sections f^i ($0 \leq i \leq N$). It is verified that the rational map $\tilde{\varphi}_{|L|}$ obtained from another basis $\{\tilde{f}^0, \dots, \tilde{f}^N\}$ is transformed by an automorphism of \mathbb{P}^N .

DEFINITION 2.1. A line bundle L over M is said to be *very ample* if the rational map $\varphi_{|L|} : M \rightarrow \mathbb{P}^N$ determined by its complete linear system $|L|$ is an embedding. L is said to be *ample* if there exists some integer $m > 0$ such that $L^{\otimes m}$ is very ample.

Let L be a very ample line bundle over a compact complex manifold M , and $\{f^0, \dots, f^N\}$ a basis of $H^0(M, \mathcal{O}(L))$ which defines an embedding $\varphi_{|L|} : M \rightarrow \mathbb{P}^N$. Embedding M into \mathbb{P}^N , we may consider the line bundle as the hyperplane bundle over $M \subset \mathbb{P}^N$. We define an open covering $\{U_{(j)}\}$ of M by $U_{(j)} = \{z \in M : f^j(z) \neq 0\}$. With respect to this covering, the local trivialization φ_j over $U_{(j)} \times \mathbb{C}$ is given by $\varphi_j(f^i) = (z_{(j)}^\alpha, f_{(j)}^i)$. The transition cocycle $\{l_{jk} : U_{(j)} \cap U_{(k)} \rightarrow \mathbb{C}^*\}$ is given by

$$l_{jk}(z) = \frac{f_{(k)}^i}{f_{(j)}^i}. \quad (2.2)$$

Let $\{h_{jk}\}$ be the transition cocycle of the hyperplane bundle \mathbb{H} with respect to the standard covering $\{U_{(j)}\}$ of \mathbb{P}^N . Then, $\{h_{jk}\}$ is given by $h_{jk} = \xi^k / \xi^j$ in terms of the homogeneous coordinate system $[\xi^0 : \dots : \xi^N]$ of \mathbb{P}^N . Since (2.2) implies

$$h_{jk} \circ \varphi_{|L|} = \frac{f_{(k)}^i}{f_{(j)}^i} = l_{jk},$$

we obtain $L = \varphi_{|L|}^* \mathbb{H}$.

LEMMA 2.1. *Let L be a very ample line bundle over a complex manifold M . Then L is isomorphic to the pullback bundle $\varphi_{|L|}^* \mathbb{H}$ of the hyperplane bundle \mathbb{H} over the target space \mathbb{P}^N of $\varphi_{|L|}$.*

EXAMPLE 2.1. (1) The hyperplane bundle \mathbb{H} over \mathbb{P}^N is very ample.
 (2) Let E be a holomorphic line bundle and L an ample line bundle over a compact complex manifold M . For some sufficiently large integer k , the line bundle $E \oplus L^{\otimes k}$ is very ample (see [Griffith and Harris 1978, p. 192]).

As we can see from the above, it is an algebro-geometric issue to determine whether a holomorphic line bundle is ample or not. However, the Kodaira embedding theorem provides a differential geometric way to check ampleness; see Theorem 2.1 and Proposition 2.1 below. The key idea is to relate ampleness to the notion of positivity, defined as follows.

DEFINITION 2.2. A holomorphic line bundle L is said to be *positive* if its Chern class $c_1(L) \in H^2(M, \mathbb{R})$ is represented by a positive real $(1, 1)$ -form. A holomorphic line bundle L is said to be *negative* if its dual L^* is positive. Since $c_1(L^*) = -c_1(L)$, the holomorphic line bundle L is negative if $c_1(L)$ is represented by a negative real $(1, 1)$ -form.

A Hermitian metric g on L is given by the family $\{g_{(j)}\}$ of local positive functions $g_{(j)} : U_{(j)} \rightarrow \mathbb{R}$, satisfying $g_{(k)} = |l_{jk}|^2 g_{(j)}$ on $U_{(j)} \cap U_{(k)}$ for the transition cocycle $\{l_{jk}\}$ of L . Since l_{jk} are holomorphic, we have $\bar{\partial} \partial \log g_{(j)} = \bar{\partial} \partial \log g_{(k)}$, and thus $\{\bar{\partial} \partial \log g_{(j)}\}$ defines a global $(1, 1)$ -form on M , which will be denoted by $\bar{\partial} \partial \log g$, and is just the curvature form of (L, g) . The Chern form $c_1(L, g)$ defined by $c_1(L, g) = \sqrt{-1} \bar{\partial} \partial \log g$ is a representative of the Chern class $c_1(L)$.

By the definition above, a holomorphic line bundle L is positive if and only if L admits a Hermitian metric g whose Chern form $c_1(L, g)$ is positive-definite. A compact complex manifold M is called a *Hodge manifold* if there exists a positive line bundle L over M . If M is a Hodge manifold, then there exists a Hermitian line bundle (L, g) whose Chern form $c_1(L, g)$ is positive-definite, and thus $c_1(L, g)$ defines a Kähler metric on M . Consequently, every Hodge manifold is Kähler.

The hyperplane bundle \mathbb{H} over \mathbb{P}^N is positive. In fact, if we define a function $g_{(j)}$ on $V_{(j)} = \{\xi^0 : \dots : \xi^N \in \mathbb{P}^N : \xi^j \neq 0\}$ by

$$g_{(j)} = \frac{|\xi^j|^2}{\sum |\xi^k|^2}, \quad (2.3)$$

the family $\{g_{(j)}\}_{j=0, \dots, N}$ satisfies $g_{(k)} = |h_{jk}|^2 g_{(j)}$ on $V_{(j)} \cap V_{(k)}$, and thus it determines a Hermitian metric $g_{\mathbb{H}}$ on \mathbb{H} . Then we have

$$c_1(\mathbb{H}) = \left[\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log g_{\mathbb{H}} \right] > 0. \quad (2.4)$$

The closed real $(1, 1)$ -form representing $c_1(\mathbb{H})$ induces a Kähler metric on \mathbb{P}^N , which is called the *Fubini–Study metric* g_{FS} with the Kähler form

$$\omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \|\xi\|^2,$$

where we put $\|\xi\|^2 = \sum |\xi^i|^2$.

The following well-known theorem shows that every Hodge manifold M is algebraic, i.e., M is holomorphically embedded in a projective space \mathbb{P}^N .

THEOREM 2.1 (KODAIRA’S EMBEDDING THEOREM). *Let L be a holomorphic line bundle over a compact complex manifold M . If L is positive, then it is ample, i.e., there exists some integer $n_0 > 0$ such that for all $m \geq n_0$ the map $\varphi_{|L^{\otimes m}|} : M \rightarrow \mathbb{P}^N$ is a holomorphic embedding.*

Conversely, we suppose that L is ample. Then, by definition, there exists a basis $\{f^0, \dots, f^N\}$ of $H^0(M, \mathcal{O}(L^{\otimes m}))$ such that $\varphi_{|L^{\otimes m}|} : M \rightarrow \mathbb{P}^N$ defined by (2.1) is an embedding. By Lemma 2.1, the line bundle $L^{\otimes m}$ is identified with $\varphi_{|L^{\otimes m}|}^* \mathbb{H}$. Thus $L^{\otimes m}$ admits a Hermitian metric $g = \varphi_{|L^{\otimes m}|}^* g_{\mathbb{H}}$, and $c_1(L^{\otimes m})$ is given by

$$c_1(L^{\otimes m}) = m c_1(L) = \left[\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log g \right].$$

Since \mathbb{H} is positive, the $(1, 1)$ -form $\sqrt{-1} \bar{\partial} \partial \log g$ is positive, and thus

$$c_1(L) = \frac{1}{m} \left[\frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log g \right]$$

is positive. Consequently:

PROPOSITION 2.1. *A holomorphic line bundle L over a compact complex manifold M is ample if and only if L is positive.*

Let M be a compact Riemann surface. The integer $g(M)$ defined by

$$g(M) := \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M) = \dim_{\mathbb{C}} H^0(M, \mathcal{O}(K_M))$$

is called the *genus* of M , where $K_M = T_M^*$ is the canonical line bundle over M , and \mathcal{O}_M is the sheaf of germs of holomorphic functions on M . The *degree* of a line bundle L is defined by

$$\deg L = \int_M c_1(L) \in \mathbb{Z}.$$

Applying the *Riemann–Roch theorem*

$$\dim_{\mathbb{C}} H^0(M, \mathcal{O}(L)) - \dim_{\mathbb{C}} H^1(M, \mathcal{O}(L)) = \deg L + 1 - g(M)$$

to the case of $L = K_M$, we have

$$\dim_{\mathbb{C}} H^1(M, \mathcal{O}(K_M)) = \dim_{\mathbb{C}} H^0(M, \Omega^1(K_M^*)) = \dim_{\mathbb{C}} H^0(M, \mathcal{O}_M) = 1,$$

since M is compact. Consequently we have $\deg K_M = 2g(M) - 2$, and the Euler characteristic $\chi(M)$ is given by

$$\chi(M) = \int_M c_1(T_M) = -\deg K_M = 2 - 2g(M).$$

By the uniformisation theorem (e.g., Theorem 4.41 in [Jost 1997]), any compact Riemann surface M is determined completely by its genus $g(M)$. If $g(M) = 0$, then M is isomorphic to the Riemannian sphere $S^2 \cong \mathbb{P}^1$ and its holomorphic tangent bundle T_M is ample. In the case of $g(M) = 1$, then M is isomorphic to a torus $T = \mathbb{C}/\Lambda$, where Λ is a module over \mathbb{Z} of rank two, and T_M is trivial. In the last case of $g(M) \geq 2$, it is well-known that M is *hyperbolic*, i.e., M admits a Kähler metric of negative constant curvature, and T_M is negative since $c_1(T_M) < 0$.

In the case of $\dim_{\mathbb{C}} M \geq 2$, Hartshorne’s conjecture (“If the tangent bundle T_M is ample, then M is bi-holomorphic to the projective space \mathbb{P}^n ”) was solved affirmatively by an algebro-geometric method ([Mori 1979]). Then, it is natural to investigate complex manifolds with negative tangent bundles. We next discuss the *negativity* and *ampleness* of holomorphic vector bundles.

2.2. Ample vector bundles. Let $\pi : E \rightarrow M$ be a holomorphic vector bundle of $\text{rank}(E) = r + 1$ (≥ 2) over a compact complex manifold M , and $\phi : \mathbb{P}(E) = E^\times / \mathbb{C}^\times \rightarrow M$ the *projective bundle* associated with E . Here and in the sequel, we put $E^\times = E - \{0\}$ and $\mathbb{C}^\times = \mathbb{C} - \{0\}$. We also denote by $\mathbb{L}(E)$ the *tautological line bundle* over $\mathbb{P}(E)$, i.e.,

$$\mathbb{L}(E) = \{(V, v) \in \mathbb{P}(E) \times E \mid v \in V\}.$$

The dual line bundle $\mathbb{H}(E) = \mathbb{L}(E)^*$ is called the *hyperplane bundle* over $\mathbb{P}(E)$.

Since $\mathbb{L}(E)$ is obtained from E by blowing up the zero section of E to $\mathbb{P}(E)$, the manifold $\mathbb{L}(E)^\times$ is biholomorphic to E^\times . This biholomorphism is given by the holomorphic map

$$\tau : E^\times \ni v \rightarrow ([v], v) \in \mathbb{P}(E) \times E^\times. \quad (2.5)$$

Then, for a arbitrary Hermitian metric $g_{\mathbb{L}(E)}$ on $\mathbb{L}(E)$, we define the norm $\|v\|_E$ of $v \in E^\times$ by

$$\|v\|_E = \sqrt{g_{\mathbb{L}(E)}(\tau(v))}. \quad (2.6)$$

Extending this definition to the whole of E continuously, we obtain a function $F : E \rightarrow \mathbb{R}$ by

$$F(v) = \|v\|_E^2 \quad (2.7)$$

for every $v \in E$. This function satisfies the following conditions.

- (F.1) $F(v) \geq 0$, and $F(v) = 0$ if and only if $v = 0$,
- (F.2) $F(\lambda v) = |\lambda|^2 F(v)$ for any $\lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$,
- (F.3) $F(v)$ is smooth outside of the zero-section.

DEFINITION 2.3. Let $\pi : E \rightarrow M$ be a holomorphic vector bundle over a complex manifold M . A real valued function $F : E \rightarrow \mathbb{R}$ satisfying the conditions (F1) \sim (F3) is called a *Finsler metric* on E , and the pair (E, F) is called a *Finsler bundle*. If a Finsler metric F satisfies, in addition,

- (F.4) the real $(1, 1)$ -form $\sqrt{-1} \partial \bar{\partial} F$ is positive-definite on each fibre E_z ,

then F is said to be *pseudoconvex*. (Note: it's $\sqrt{-1} \partial \bar{\partial} F$, not $\sqrt{-1} \partial \bar{\partial} \log F$.)

This discussion shows that any Hermitian metric on $\mathbb{L}(E)$ defines a Finsler metric on E . Conversely, an arbitrary Finsler metric F on E determines a Hermitian metric $g_{\mathbb{L}(E)}$ on $\mathbb{L}(E)$, i.e., we obtain

PROPOSITION 2.2 [Kobayashi 1975]. *There exists a one-to-one correspondence between the set of Hermitian metrics on $\mathbb{L}(E)$ and the set of Finsler metrics on E .*

DEFINITION 2.4 [Kobayashi 1975]. A holomorphic vector bundle $\pi : E \rightarrow M$ over a compact complex manifold M is said to be *negative* if its tautological line bundle $\mathbb{L}(E) \rightarrow \mathbb{P}(E)$ is negative, and E is said to be *ample* if its dual E^* is negative.

The Chern class $c_1(\mathbb{L}(E))$ is represented by the closed real $(1, 1)$ -form

$$c_1(\mathbb{L}(E), F) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log F$$

for a Finsler metric F on E . Thus, E is negative if and only if E admits a Finsler metric F satisfying $c_1(\mathbb{L}(E), F) < 0$. Consequently:

PROPOSITION 2.3. *Let E be a holomorphic vector bundle over a compact complex manifold M . Then E is negative if and only if E admits a Finsler metric F satisfying $\sqrt{-1} \bar{\partial} \partial \log F < 0$.*

Given any negative holomorphic vector bundle E over M , we shall construct a pseudoconvex Finsler metric F on E , with $c_1(\mathbb{L}(E), F) < 0$ (see [Aikou 1999] and [Wong 1984]). By definition the line bundle $\mathbb{L}(E)$ is negative, and so $\mathbb{L}(E)^*$ is ample. Hence there exists a sufficiently large positive $m \in \mathbb{Z}$ such that $L := \mathbb{L}(E)^{* \otimes m}$ is very ample. By the definition of very ampleness, we can take $f^0, \dots, f^N \in H^0(\mathbb{P}(E), L)$ such that

$$\varphi_{|L|} : \mathbb{P}(E) \ni [v] \rightarrow [f^0([v]) : \dots : f^N([v])] \in \mathbb{P}^N$$

defines a holomorphic embedding $\varphi_{|L|} : \mathbb{P}(E) \rightarrow \mathbb{P}^N$. Then, by Lemma 2.1, we have $L \cong \varphi_{|L|}^* \mathbb{H}$ for the hyperplane bundle $\mathbb{H} \rightarrow \mathbb{P}^N$. Since $c_1(\mathbb{H})$ is given by (2.4), we have $c_1(L) = c_1(\varphi_{|L|}^* \mathbb{H}) = \varphi_{|L|}^* c_1(\mathbb{H}) > 0$, and the induced metric g_L is given by $g_L = \varphi_{|L|}^* g_{\mathbb{H}}$ for the metric $g_{\mathbb{H}}$ on \mathbb{H} defined by (2.3). Since $L = (\mathbb{L}(E)^*)^{\otimes m}$, we have $g_L = g_{\mathbb{L}(E)}^{-m}$, and thus the induced metric $g_{\mathbb{L}(E)}$ on $\mathbb{L}(E)$ is given by

$$g_{\mathbb{L}(E)} = \left[\varphi_{|L|}^* g_{\mathbb{H}} \right]^{-1/m} = \left[\frac{1}{\varphi_{|L|}^* g_{\mathbb{H}}} \right]^{1/m}.$$

Because of (2.3), the metric g_L is locally given by

$$g_{(j)} = \frac{|f^j|^2}{\sum |f^k|^2},$$

and the Finsler metric F on E corresponding to $g_{\mathbb{L}(E)}$ is given by

$$F(v) = \left[(\varphi_{|L|}^* g_{\mathbb{H}})(\tau(v)) \right]^{-1/m} = \left[\frac{\sum |f^k([v])|^2}{|f^j([v])|^2} \right]^{1/m} |v^j|^2 \quad (2.8)$$

for $v = (v^1, \dots, v^{r+1}) \in E_z$. The Finsler metric F obtained as above satisfies the condition $c_1(\mathbb{L}(E), F) < 0$. The pseudoconvexity of F will be shown by more local computations (see Theorem 3.2).

REMARK 2.1. Every pseudoconvex Finsler metric on a holomorphic vector bundle E is obtained from a pseudo-Kähler metric on $\mathbb{P}(E)$ (Propositions 4.1 and 4.2).

In a later section, we shall show a theorem of Kobayashi's (Theorem 3.2) which characterizes negative vector bundles in terms of the curvature of Finsler metrics. For this purpose, in the next section, we shall discuss the theory of Finsler connections on (E, F) .

3. Finsler Connections

Let $\pi : E \rightarrow M$ be a holomorphic vector bundle of $\text{rank}(E) = r + 1$ over a complex manifold M . We denote by T_M the holomorphic tangent bundle of M . We also denote by T_E the holomorphic tangent bundle of the total space E . Then we have an exact sequence of holomorphic vector bundles

$$0 \longrightarrow V_E \xrightarrow{i} T_E \xrightarrow{\pi_*} \pi^*T_M \longrightarrow 0, \quad (3.1)$$

where $V_E = \ker \pi_*$ is the *vertical subbundle* of T_E . A connection of the bundle $\pi : E \rightarrow M$ is a smooth splitting of this sequence.

DEFINITION 3.1. A *connection* of a fibre bundle $\pi : E \rightarrow M$ is a smooth V_E -valued $(1, 0)$ -form $\theta_E \in \Omega^{1,0}(V_E)$ satisfying

$$\theta_E(Z) = Z \quad (3.2)$$

for all $Z \in V_E$. A connection θ_E defines a smooth splitting

$$T_E = V_E \oplus H_E \quad (3.3)$$

of the sequence (3.1), where $H_E \subset T_E$ is the *horizontal subbundle* defined by $H_E = \ker \theta_E$.

The complex general linear group $GL(r + 1, \mathbb{C})$ acts on E in a natural way. A connection θ_E is called a *linear connection* if the horizontal subspace at each point is $GL(r + 1, \mathbb{C})$ -invariant. A Hermitian metric on E defines a unique linear connection θ_E .

On the other hand, the multiplier group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\} \cong \{c \cdot I \mid c \in \mathbb{C}^\times\} \subset GL(r + 1, \mathbb{C})$ also acts on the total space E by multiplication $L_\lambda : E \ni v \rightarrow \lambda v \in E$ on the fibres for all $v \in E$ and $\lambda \in \mathbb{C}^\times$. In this paper, we assume that a connection θ_E is \mathbb{C}^\times -invariant. We denote by $\mathcal{E} \in \mathcal{O}_E(V_E)$ the *tautological section* of V_E generated by the action of \mathbb{C}^\times , i.e., \mathcal{E} is defined by

$$\mathcal{E}(v) = (v, v)$$

for all $v \in E$. The invariance of θ_E under the action of \mathbb{C}^\times is equivalent to

$$\mathcal{L}_\mathcal{E}H_E \subset H_E, \quad (3.4)$$

where $\mathcal{L}_\mathcal{E}$ denotes the Lie derivative by \mathcal{E} . In this sense, a \mathbb{C}^\times -invariant connection θ_E is called a *non-linear connection*.

EXAMPLE 3.1. Let E be a holomorphic vector bundle with a Hermitian metric h . Let $\bar{\nabla}$ be the Hermitian connection of h , and $\nabla = \pi^*\bar{\nabla}$ the induced connection on V_E . If we define $\theta \in \Omega^{1,0}(V_E)$ by

$$\theta_E = \bar{\nabla}\mathcal{E}, \quad (3.5)$$

then θ_E defines a linear connection on the bundle $\pi : E \rightarrow M$.

If a pseudoconvex Finsler metric F is given on E , then it defines a canonical non-linear connection θ_E (see below).

If a connection θ_E is given on E , we put

$$X^V = \theta_E(X), \quad X^H = X - X^V,$$

and

$$(d^V f)(X) = df(X^V), \quad (d^H f)(X) = df(X^H)$$

for every $X \in T_E$ and $f \in C^\infty(E)$. These operators can be decomposed as $d^V = \partial^V + \bar{\partial}^V$ and $d^H = \partial^H + \bar{\partial}^H$. Furthermore the partial derivatives are also decomposed as $\partial = \partial^H + \partial^V$ and $\bar{\partial} = \bar{\partial}^H + \bar{\partial}^V$.

3.1. Finsler connection. We define a partial connection $\nabla^H : V_E \rightarrow \Omega^1(V_E)$ on V_E by

$$\nabla_Y^H Z = \theta_E([Y^H, Z]) \quad (3.6)$$

for all $Y \in T_E$ and $Z \in V_E$, where $[\cdot, \cdot]$ denotes the Lie bracket on T_E . This operator ∇^H is linear in X and satisfies the Leibnitz rule

$$\nabla^H(fZ) = (d^H f) \otimes Z + f \nabla^H Z$$

for all $f \in C^\infty(E)$.

On the other hand, since E is a fibre bundle over M , the projection map $\pi : E \rightarrow M$ can be used to pullback the said fibre bundle, generating a π^*E which sits over E . Note that $V_E \cong \pi^*E$. Thus V_E admits a canonical relatively flat connection $\nabla^V : V_E \rightarrow \Omega^1(V_E)$ defined by $\nabla_X^V(\pi^*s) = 0$ for every local holomorphic section s of E , i.e.,

$$\nabla^V = d^V. \quad (3.7)$$

Then a connection $\nabla : V_E \rightarrow \Omega^1(V_E)$ is defined by

$$\nabla Z = \nabla^H Z + d^V Z \quad (3.8)$$

for every $Y \in T_E$ and $Z \in V_E$. We note here that the connection ∇ is determined uniquely from the connection θ_E on the bundle $\pi : E \rightarrow M$.

PROPOSITION 3.1. *Let $\nabla : V_E \rightarrow \Omega^1(V_E)$ be the connection on V_E defined by (3.8), from a connection θ_E on the bundle $\pi : E \rightarrow M$. Then ∇ satisfies*

$$\nabla \mathcal{E} = \theta_E. \quad (3.9)$$

PROOF. Since θ_E is invariant by the action of \mathbb{C}^\times , we have $\nabla^H \mathcal{E} = 0$. In fact,

$$\nabla_X^H \mathcal{E} = \theta_E([X^H, \mathcal{E}]) = -\theta_E(\mathcal{L}_X \mathcal{E}) = 0.$$

Furthermore, since $(d^V \mathcal{E})(X) = X^V$, we obtain

$$\nabla_X \mathcal{E} = \nabla_X^H \mathcal{E} + (d^V \mathcal{E})(X) = X^V = \theta_E(X)$$

for every $X \in T_E$. Hence we have (3.9). \square

For the rest of this paper, we shall use the following local coordinate system on M and E . Let U be an open set in M with local coordinates (z^1, \dots, z^n) , and $s = (s_0, \dots, s_r)$ a holomorphic local frame field on U . By setting $v = \sum \xi^i s_i(z)$ on for each $v \in \pi^{-1}(U)$, we take $(z, \xi) = (z^1, \dots, z^n, \xi^0, \dots, \xi^r)$ as a local coordinate system on $\pi^{-1}(U) \subset E$. We use the notation

$$\partial_\alpha = \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad \partial_j = \frac{\partial}{\partial \xi^j}.$$

We also denote by $\partial_{\bar{\alpha}}$ and $\partial_{\bar{j}}$ their conjugates.

We suppose that a pseudoconvex Finsler metric F is given on E . Then, by definition in the previous section, the form $\omega_E = \sqrt{-1} \partial \bar{\partial} F$ is a closed real $(1, 1)$ -form on the total space E such that the restriction ω_z on each fibre $E_z = \pi^{-1}(z)$ defines a Kähler metric G_z on E_z , and ω_E defines a Hermitian metric G_{V_E} on the bundle V_E . Thus we shall investigate the geometry of the Hermitian bundle $\{V_E, G_{V_E}\}$.

A Finsler metric F on E is pseudoconvex if and only if the Hermitian matrix $(G_{i\bar{j}})$ defined by

$$G_{i\bar{j}}(z, \xi) = \partial_i \partial_{\bar{j}} F \tag{3.10}$$

is positive-definite. Each fibre E_z may be considered as a Kähler manifold with Kähler form $\omega_z = \sqrt{-1} \sum G_{i\bar{j}} d\xi^i \wedge d\bar{\xi}^j$. The family $\{E_z, \omega_z\}_{z \in M}$ is considered as a family of Kähler manifolds and the bundle is considered as the associated fibred manifold. The Hermitian metric G_{V_E} on V_E is defined by

$$G_{V_E}(s_i, s_j) = G_{i\bar{j}}, \tag{3.11}$$

where we consider $s = (s_0, \dots, s_r)$ as a local holomorphic frame field for $V_E \cong \pi^* E$. We denote by $\|\cdot\|_E$ the norm defined by the Hermitian metric G_{V_E} . Then, because of the homogeneity (F2), we have

$$\|\mathcal{E}\|_E^2 = G_{V_E}(\mathcal{E}, \mathcal{E}) = F(z, \xi) \tag{3.12}$$

and

$$\mathcal{L}_{\mathcal{E}} G_{V_E} = G_{V_E} \tag{3.13}$$

for the tautological section $\mathcal{E}(z, \xi) = \sum \xi^i s_i(z)$.

Let $\theta = (\theta^1, \dots, \theta^r)$ be the dual frame field for the dual bundle V_E^* , i.e., $\theta^i(s_j) = \delta_j^i$. A connection θ_E for the bundle $\pi : E \rightarrow M$ is written as

$$\theta_E = s \otimes \theta = \sum s_i \otimes \theta^i.$$

PROPOSITION 3.2. *Let F be a pseudoconvex Finsler metric on a holomorphic vector bundle E . Then there exists a unique connection θ_E such that (3.3) is the orthogonal splitting with respect to ω_E .*

PROOF. We can easily show that the required $(1, 0)$ -form θ^i is defined by $\theta^i = d\xi^i + \sum N_\alpha^i dz^\alpha$, where the local functions $N_\alpha^i(z, \xi)$ are given by

$$N_\alpha^i = \sum G^{i\bar{m}} \partial_\alpha \partial_{\bar{m}} F$$

for the inverse matrix $(G^{i\bar{m}})$ of $(G_{i\bar{m}})$. By the homogeneity (F2), these functions satisfy $N_\alpha^i(z, \lambda\xi) = \lambda N_\alpha^i(z, \xi)$ for every $\lambda \in \mathbb{C}^\times$, which implies that θ_E is \mathbb{C}^\times -invariant. \square

For the rest of this paper, we shall adopt the connection θ_E obtained in Proposition 3.2 on a pseudoconvex Finsler bundle (E, F) .

PROPOSITION 3.3 [Aikou 1998]. *The connection θ_E satisfies*

$$\partial^H \circ \partial^H \equiv 0. \quad (3.14)$$

Such a connection θ_E determines a unique connection ∇ on V_E .

DEFINITION 3.2. The connection $\nabla : V_E \rightarrow \Omega^1(V_E)$ on (V_E, G_{V_E}) defined by (3.8) is called the *Finsler connection* of (E, F) .

The connection ∇ defined by (3.8) is canonical in the following sense.

PROPOSITION 3.4 [Aikou 1998]. *Let ∇ be the Finsler connection on a pseudoconvex Finsler bundle (E, F) . Then $\nabla = \nabla^H + d^V$ satisfies the following metrical condition.*

$$d^H G_{V_E}(Y, Z) = G_{V_E}(\nabla^H Y, Z) + G_{V_E}(Y, \nabla^H Z) \quad (3.15)$$

for all $Y, Z \in V_E$.

The *connection form* $\omega = (\omega_j^i)$ of ∇ with respect to a local holomorphic frame field $s = (s_0, \dots, s_r)$ is defined by $\nabla s_j = \sum s_i \otimes \omega_j^i$. By the identity (3.15), ω is given by

$$\omega = G^{-1} \partial^H G. \quad (3.16)$$

3.2. Curvature. Let ∇ be the Finsler connection on (E, F) . We also denote by $\nabla : \Omega^k(V_E) \rightarrow \Omega^{k+1}(V_E)$ the covariant exterior derivative defined by ∇ .

DEFINITION 3.3. The section $R = \nabla \circ \nabla \in \Omega^2(\text{End}(V_E))$ is called the *curvature* of ∇ .

With respect to the local frame field $s = (s_0, \dots, s_r)$, we put

$$R(s_j) = \sum s_i \otimes \Omega_j^i.$$

In matrix notation, the *curvature form* $\Omega = (\Omega_j^i)$ of ∇ is given by

$$\Omega = d\omega + \omega \wedge \omega. \quad (3.17)$$

The curvature form Ω is decomposed as $\Omega = d^H \omega + \omega \wedge \omega + d^V \omega$, which can be simplified to

$$\Omega = \bar{\partial}^H \omega + d^V \omega. \quad (3.18)$$

This is made possible by

PROPOSITION 3.5. *The horizontal (2, 0)-part of Ω vanishes, i.e.,*

$$\partial^H \omega + \omega \wedge \omega \equiv 0. \quad (3.19)$$

PROOF. Since $\omega = G^{-1} \partial^H G$, (3.14) implies

$$\begin{aligned} \partial^H \omega + \omega \wedge \omega &= \partial^H (G^{-1} \partial^H G) + \omega \wedge \omega \\ &= -G^{-1} \partial^H G \wedge G^{-1} \partial^H G + G^{-1} \partial^H \circ \partial^H G + \omega \wedge \omega \\ &= G^{-1} \partial^H \circ \partial^H G = 0. \quad \square \end{aligned}$$

We give the definition of flat Finsler metrics.

DEFINITION 3.4. A Finsler metric F is said to be *flat* if there exists a holomorphic local frame field $s = (s_0, \dots, s_r)$ around every point of M such that $F = F(\xi)$, i.e., F is independent of the base point $z \in M$.

THEOREM 3.1 [Aikou 1999]. *A Finsler metric F is flat if and only if the curvature R vanishing identically.*

Let R^H be the curvature of the partial connection ∇^H , i.e., $R^H = \nabla^H \circ \nabla^H$. From (3.18), the curvature form Ω^H of R^H is given by $\Omega^H = \bar{\partial}^H \omega$. If we put

$$\bar{\partial}^H \omega_j^i = \sum R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta,$$

the curvature R^H is given by

$$R^H(s_j) = \sum s_i \otimes \left(R_{j\alpha\bar{\beta}}^i dz^\alpha \wedge d\bar{z}^\beta \right).$$

For the curvature form Ω^H of ∇^H , we define a horizontal (1, 1)-form Ψ by

$$\Psi(X, Y) = \frac{G_{V_E}(R_{XY}^H(\mathcal{E}), \mathcal{E})}{\|\mathcal{E}\|_E^2} = \frac{1}{F} \sum R_{i\bar{j}\alpha\bar{\beta}}(z, \xi) \xi^i \bar{\xi}^j X^\alpha \bar{Y}^\beta$$

for any horizontal vector fields X, Y at $(z, \xi) \in E$, where we put $R_{i\bar{j}\alpha\bar{\beta}} = \sum G_{m\bar{j}} R_{i\alpha\bar{\beta}}^m$. We set

$$\Psi_{\alpha\bar{\beta}}(z, \xi) = \frac{1}{F} \sum R_{i\bar{j}\alpha\bar{\beta}} \xi^i \bar{\xi}^j. \quad (3.20)$$

In [Kobayashi 1975], this (1, 1)-form $\Psi = \sum \Psi_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ is also called the *curvature* of F .

DEFINITION 3.5. If the curvature form Ψ satisfies the negativity condition, i.e., $\Psi(Y, Z) < 0$ for all $Y, Z \in H_E$, then we say that (E, F) is *negatively curved*.

Direct computation gives:

PROPOSITION 3.6 [Aikou 1998]. *For a pseudoconvex Finsler metric F on a holomorphic vector bundle E , the real (1, 1) form $\sqrt{-1} \partial \bar{\partial} \log F = -c_1(\mathbb{L}(E), F)$ is given by*

$$\sqrt{-1} \partial \bar{\partial} \log F = \sqrt{-1} \begin{pmatrix} -\Psi_{\alpha\bar{\beta}} & 0 \\ 0 & \partial_i \partial_{\bar{j}} \log F \end{pmatrix}$$

with respect to the orthogonal decomposition (3.3), i.e.,

$$\sqrt{-1} \partial \bar{\partial} \log F = \sqrt{-1} \left(- \sum \Psi_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta + \sum \partial_i \partial_{\bar{j}} (\log F) \theta^i \wedge \bar{\theta}^j \right). \quad (3.21)$$

Analyzing the negativity of the form $c_1(\mathbb{L}(E), F)$, we have the following theorem of Kobayashi.

THEOREM 3.2 [Kobayashi 1975]. *A holomorphic vector bundle $\pi : E \rightarrow M$ over a compact complex manifold M is negative if and only if E admits a negatively curved pseudoconvex Finsler metric.*

PROOF. By Proposition 2.3, E is negative if and only if there exists a Finsler metric F such that $\sqrt{-1} \partial \bar{\partial} \log F < 0$. Since ∂ and $\bar{\partial}$ anti-commute, this characterization is equivalent to $\sqrt{-1} \partial \bar{\partial} \log F > 0$. Thus, E is negative if and only if the right hand side of (3.21) is positive.

Denote by F_z the restriction of F to each fibre $E_z = \pi^{-1}(z)$. Then, we have

$$\sqrt{-1} \partial \bar{\partial} F_z = \sqrt{-1} F_z (\partial \bar{\partial} \log F_z + \partial \log F_z \wedge \bar{\partial} \log F_z).$$

If (3.21) has positive right hand side, then $\sqrt{-1} \Psi$ must be negative, and $\sqrt{-1} \partial \bar{\partial} \log F_z$ must be positive. The latter, in conjunction with the displayed formula, implies the positivity of $\sqrt{-1} \partial \bar{\partial} F_z$. Thus F is pseudoconvex and negatively curved.

Conversely, suppose F is pseudoconvex and negatively curved. That is, we have $\sqrt{-1} \partial \bar{\partial} F_z > 0$ and $\sqrt{-1} \Psi < 0$. Now, the pseudoconvexity of F implies that the second term on the right hand side of (3.21) is positive-definite (see section 4.1 for details). Thus the entire right hand side of (3.21) is positive. \square

REMARK 3.1. In this section, the horizontal $(1, 1)$ -part R^H of R plays an important role. In Finsler geometry, there are other important tensors. The V_E -valued 2-form T_E defined by

$$T_E = \nabla \theta_E \quad (3.22)$$

is called the *torsion form* of ∇ , which is expressed by

$$T_E = s \otimes (d\theta + \omega \wedge \theta) = \sum s_i \otimes \left(d\theta^i + \sum \omega_j^i \wedge \theta^j \right).$$

Because of (3.9), the torsion form T_E is also given by

$$T_E = R(\mathcal{E}).$$

The torsion form T_E vanishes if and only if the horizontal subbundle H_E defined by θ_E is holomorphic and integrable (see [Aikou 2003b]).

On the other hand, the mixed part R^{HV} of R defined by

$$R^{HV}(s_j) = \sum s_i \otimes d^V \omega_j^i$$

is also an important curvature form. The vanishing of R^{HV} shows that (E, F) is modeled on a complex Minkowski space, i.e., there exists a Hermitian metric h_F

on E such that the Finsler connection ∇ on (E, F) is obtained by $\nabla = \pi^* \bar{\nabla}$ for the Hermitian metric $\bar{\nabla}$ of (E, h_F) (see [Aikou 1995]). Hence a Finsler metric F is flat if and only if (E, F) is modeled on a complex Minkowski space and the associated Hermitian metric h_F is flat.

3.3. Holomorphic sectional curvature. We now study the holomorphic tangent bundle T_M of a complex manifold with a pseudoconvex Finsler metric $F : T_M \rightarrow \mathbb{R}$. The pair (M, F) is called a *complex Finsler manifold*. This is the special case of $E = T_M$ in Definition 2.3, and we have the natural identifications $V_E \cong H_E \cong \pi^* T_M$.

Let $\Delta(r) = \{\zeta \in \mathbb{C} : |\zeta| < r\}$ be the disk of radius r in \mathbb{C} with the *Poincaré metric*

$$g_r = \frac{4r^2}{(r^2 - |\zeta|^2)^2} d\zeta \otimes d\bar{\zeta}.$$

For every point $(z, \xi) \in T_M$, there exists a holomorphic map $\varphi : \Delta(r) \rightarrow M$ satisfying $\varphi(0) = z$ and

$$\varphi_*(0) := \varphi_* \left(\left(\frac{\partial}{\partial \zeta} \right)_{\zeta=0} \right) = \xi. \quad (3.23)$$

Then, the pullback $\varphi^* F$ defines a Hermitian metric on $\Delta(r)$ by

$$\varphi^* F = E(\zeta) d\zeta \otimes d\bar{\zeta},$$

where we put $E(\zeta) = F(\varphi(\zeta), \varphi_*(\zeta))$. The Gauss curvature $K_{\varphi^* F}(z, \xi)$ is defined by

$$K_{\varphi^* F}(z, \xi) = - \left(\frac{1}{E} \frac{\partial^2 \log E}{\partial \zeta \partial \bar{\zeta}} \right)_{\zeta=0}.$$

DEFINITION 3.6 [Royden 1986]. The *holomorphic sectional curvature* K_F of (M, F) at $(z, \xi) \in T_M$ is defined by

$$K_F(z, \xi) = \sup_{\varphi} \{K_{\varphi^* F}(z, \xi) : \varphi(0) = z, \varphi_*(0) = \xi\},$$

where φ ranges over all holomorphic maps from a small disk into M satisfying $\varphi(0) = z$ and (3.23).

Then K_F has a computable expression in terms of the curvature tensor of the Finsler connection ∇ .

PROPOSITION 3.7 [Aikou 1991]. *The holomorphic sectional curvature K_F of a complex Finsler manifold (M, F) is given by*

$$K_F(z, \xi) = \frac{\Psi(\mathcal{E}, \mathcal{E})}{\|\mathcal{E}\|_E^2} = \frac{1}{F^2} \sum R_{i\bar{j}k\bar{l}}(z, \xi) \xi^i \bar{\xi}^j \xi^k \bar{\xi}^l, \quad (3.24)$$

where $R_{i\bar{j}k\bar{l}} = \sum G_{m\bar{j}} R_{ik\bar{l}}^m$ is the curvature tensor of the Finsler connection ∇ on (T_M, F) .

Then we have a Schwarz-type lemma:

PROPOSITION 3.8 [Aikou 1991]. *Let F be a pseudoconvex Finsler metric on the holomorphic tangent bundle of a complex manifold M . Suppose that its holomorphic sectional curvature $K_F(z, \xi)$ at every point $(z, \xi) \in T_M$ is bounded above by a negative constant $-k$. Then, for every holomorphic map $\varphi : \Delta(r) \rightarrow M$ satisfying $\varphi(0) = z$ and (3.23), we have*

$$g_r \geq k\varphi^*F. \quad (3.25)$$

The Kobayashi metric F_M on a complex manifold M is a positive semidefinite pseudo metric defined by

$$F_M(z, \xi) = \inf_{\varphi} \left\{ \frac{1}{r} : \varphi(0) = z, \varphi_*(0) = \xi \right\}. \quad (3.26)$$

In general, F_M is not smooth. F_M is only upper semi-continuous, i.e., for every $X \in T_M$ and every $\varepsilon > 0$ there exists a neighborhood U of X such that $F_M(Y) < F_M(X) + \varepsilon$ for all $Y \in U$ (see [Kobayashi 1998], [Lang 1987]). Even though F_M is not a Finsler metric in our sense, the *decreasing principle* shows the importance of the Kobayashi metric, i.e., for every holomorphic map $\varphi : N \rightarrow M$, we have the inequality

$$F_N(X) \geq F_M(\varphi_*(X)). \quad (3.27)$$

This principle shows that F_M is holomorphically invariant, i.e., if $\varphi : N \rightarrow M$ is biholomorphic, then we have $F_N = \varphi^*F_M$. In this sense, F_M is an intrinsic metric on complex manifolds.

A typical example of Kobayashi metrics is the one on a domain in \mathbb{C}^n . It is well-known that, if M is a strongly convex domain with smooth boundary in \mathbb{C}^n , then F_M is a pseudoconvex Finsler metric in our sense (see [Lempert 1981]).

A complex manifold M is said to be *Kobayashi hyperbolic* if its Kobayashi metric F_M is a metric in the usual sense. If M admits a pseudoconvex Finsler metric F whose holomorphic sectional curvature K_F is bounded above by a negative constant $-k$, then (3.25) implies the inequality

$$F_M^2 \geq kF, \quad (3.28)$$

and thus M is Kobayashi hyperbolic.

THEOREM 3.3 [Kobayashi 1975]. *Let M be a compact complex manifold. If its holomorphic tangent bundle T_M is negative, then M is Kobayashi hyperbolic.*

PROOF. We suppose that T_M is negative. Then, Theorem 3.2 implies that there exists a pseudoconvex Finsler metric F on T_M with negative-definite Ψ . By the definition (3.20), the negativity of Ψ and (3.24) imply

$$K_F(z, \xi) = \frac{\Psi(\mathcal{E}, \mathcal{E})}{\|\mathcal{E}\|_E^2} < 0.$$

Since M is compact, $\mathbb{P}(E)$ is also compact. Moreover, since K_F is a function on $\mathbb{P}(E)$, the negativity of K_F shows that K_F is bounded by a negative constant $-k$. Hence we obtain (3.28), and M is Kobayashi hyperbolic. \square

REMARK 3.2. Recently Cao and Wong [Cao and Wong 2003] have introduced the notion of “mixed holomorphic bisectional curvature” for Finsler bundles (E, F) , which equals the usual holomorphic bisectional curvature in the case of $E = T_M$. They also succeeded in showing that a holomorphic vector bundle E of $\text{rank}(E) \geq 2$ over a compact complex manifold M is ample if and only if E admits a Finsler metric with positive mixed holomorphic bisectional curvature.

4. Ruled Manifolds

4.1. Projective bundle. Let $\phi : \mathbb{P}(E) \rightarrow M$ be the projective bundle associated with a holomorphic vector bundle E over M .

DEFINITION 4.1. A locally $\partial\bar{\partial}$ -exact real $(1, 1)$ -form $\omega_{\mathbb{P}(E)}$ on the total space $\mathbb{P}(E)$ is called a *pseudo-Kähler metric* on $\mathbb{P}(E)$ if its restriction to each fibre defines a Kähler metric on $\mathbb{P}(E_z) \cong \mathbb{P}^r$.

If a pseudo-Kähler metric $\omega_{\mathbb{P}(E)}$ is given on $\mathbb{P}(E)$, then its restriction to each fibre $\phi^{-1}(z) = \mathbb{P}(E_z)$ is a Kähler form on $\mathbb{P}(E_z)$. We shall show that a pseudoconvex Finsler metric on E defines a pseudo-Kähler metric on $\mathbb{P}(E)$. For this purpose, we use the so-called Euler sequence (e.g., [Zheng 2000]).

We denote by $\rho : E^\times \rightarrow \mathbb{P}(E)$ the natural projection. We also denote by $V_{\mathbb{P}(E)} := \ker \phi_*$ the vertical subbundle of $T_{\mathbb{P}(E)}$. Let $s = (s_0, \dots, s_r)$ be a holomorphic local frame field of E on an open set $U \subset M$, which is naturally considered as a holomorphic local frame field of V_E on $\pi^{-1}(U)$. Then, the vertical subbundle $V_{\mathbb{P}(E)} \subset T_{\mathbb{P}(E)}$ is locally spanned by $\{\rho_* s_0, \dots, \rho_* s_r\}$ with the relation

$$\rho_* \mathcal{E} = 0. \quad (4.1)$$

Then the Euler sequence

$$0 \longrightarrow \mathbb{L}(E) \xrightarrow{i} \phi^* E \longrightarrow \mathbb{L}(E) \otimes V_{\mathbb{P}(E)} \longrightarrow 0$$

implies the following exact sequence of vector bundles:

$$0 \longrightarrow 1_{\mathbb{P}(E)} \xrightarrow{i} \mathbb{H}(E) \otimes \phi^* E \xrightarrow{\mathcal{P}} V_{\mathbb{P}(E)} \longrightarrow 0, \quad (4.2)$$

where $\mathbb{H}(E) = \mathbb{L}(E)^*$ is the hyperplane bundle over $\mathbb{P}(E)$ and the bundle morphism $\mathcal{P} : \mathbb{H}(E) \otimes \phi^* E \rightarrow V_{\mathbb{P}(E)}$ is defined as follows. Since any section Z of $\mathbb{H}(E) \otimes \phi^* E$ can be naturally identified with a section $Z = \sum Z^j s_j$ of V_E satisfying the homogeneity $Z^j(\lambda v) = \lambda Z^j(v)$, the definition $\rho_* Z(v) = (\rho_* Z)([v])$ makes sense. Then \mathcal{P} is defined by

$$\mathcal{P}(Z) = \rho_* \left(\sum Z^j s_j \right). \quad (4.3)$$

Moreover, since $\rho_* \mathcal{E} = 0$, $1_{\mathbb{P}(E)} (= \ker \mathcal{P})$ is the trivial line bundle spanned by \mathcal{E} . Then, since $\ker \mathcal{P} = 1_{\mathbb{P}(E)}$ is spanned by \mathcal{E} , the morphism \mathcal{P} is expressed as

$$\mathcal{P}(Z) = Z - \frac{G_{V_E}(Z, \mathcal{E})}{\|\mathcal{E}\|_E^2} \mathcal{E}$$

for a Hermitian metric G_{V_E} on V_E . Since \mathcal{P} is surjective, for any sections \tilde{Z} and \tilde{W} of $V_{\mathbb{P}(E)}$, there exist sections Z and W of V_E satisfying $\mathcal{P}(Z) = \tilde{Z}$ and $\mathcal{P}(W) = \tilde{W}$. Then, a Hermitian metric $G_{V_{\mathbb{P}(E)}}$ on $V_{\mathbb{P}(E)}$ is defined by

$$G_{V_{\mathbb{P}(E)}}(\tilde{Z}, \tilde{W}) = \frac{\|\mathcal{E}\|_E^2 G_{V_E}(Z, W) - G_{V_E}(Z, \mathcal{E})G_{V_E}(\mathcal{E}, W)}{\|\mathcal{E}\|_E^4} \quad (4.4)$$

for the Hermitian metric G_{V_E} on V_E defined by (3.11), which induces the orthogonal decomposition

$$\mathbb{H}(E) \otimes \phi^* E = 1_{\mathbb{P}(E)} \oplus V_{\mathbb{P}(E)}.$$

Because of (4.1) and (4.4), the components of the metric $G_{V_{\mathbb{P}(E)}}$ with respect to the local frame $\{\rho_*(s_j)\}$ is given by

$$G_{V_{\mathbb{P}(E)}}(\rho_* s_i, \rho_* s_j) = \partial_i \partial_{\bar{j}} (\log F). \quad (4.5)$$

Consequently we have

PROPOSITION 4.1. *If F is a pseudoconvex Finsler metric on a holomorphic vector bundle E , then the real $(1, 1)$ -form $\sqrt{-1} \partial \bar{\partial} \log F$ defines a pseudo-Kähler metric on $\mathbb{P}(E)$.*

Conversely:

PROPOSITION 4.2. *If $\omega_{\mathbb{P}(E)}$ is a pseudo-Kähler metric on $\mathbb{P}(E)$, then $\omega_{\mathbb{P}(E)}$ defines a pseudoconvex Finsler metric F on E such that $\rho^* \omega_{\mathbb{P}(E)} = \sqrt{-1} \partial \bar{\partial} \log F$. Such a pseudoconvex Finsler metric F is unique up to a local positive function σ_U on $U \subset M$.*

PROOF. On each open set $U_{(j)} = \{[v] = (z, [\xi]) \in \phi^{-1}(U) : \xi^j \neq 0\}$ of $\mathbb{P}(E)$, we express the pseudo-Kähler metric $\omega_{\mathbb{P}(E)}$ by

$$\omega_{\mathbb{P}(E)}|_{U_{(j)}} = \sqrt{-1} \partial \bar{\partial} g_{(j)},$$

where $\{g_{(j)}\}$ is a family of local smooth functions $g_{(j)} : U_{(j)} \rightarrow \mathbb{R}$. Since the restriction of this form to each fibre $\mathbb{P}_z \subset U_{(j)}$ is a Kähler form ω_z on \mathbb{P}_z , we may put

$$\omega_z = \sqrt{-1} \partial \bar{\partial} g_{z,(j)},$$

where the local functions $g_{z,(j)} = g_{(j)}|_{\mathbb{P}_z}$ depend on $z \in U$ smoothly. Then we define a function $F_z : E_z^\times \rightarrow \mathbb{R}$ by

$$F_z(\xi) = |\xi^j|^2 \exp g_{z,(j)}.$$

Since F_z also depends on $z \in U$ smoothly, we extend this function to a smooth function $F : E^\times \rightarrow \mathbb{R}$ by $F(z, \xi) = F_z(\xi)$. It is easily verified that F defines a pseudoconvex Finsler metric on E .

We note that another Kähler potential $\{\tilde{g}_{(j)}\}$ for $\omega_{\mathbb{P}(E)}$ which induces the Kähler metric ω_z on each \mathbb{P}_z is given by

$$\tilde{g}_{(j)}(z, [\xi]) = \sigma_U(z) + g_{(j)}(z, [\xi]) \quad (4.6)$$

for some functions $\sigma_U(z)$ defined on U . Hence the Finsler metric \tilde{F} determined from the potential $\{\tilde{g}_j\}$ is connected to the function F by the relation $\tilde{F} = e^{\sigma_U(z)} F$ on each U . \square

Similar to Definition 3.4, we say a pseudo-Kähler metric $\omega_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$ is *flat* if there exists an open cover $\{U, s\}$ of E so that we can choose Kähler potentials $g_{(j)}$ for $\omega_{\mathbb{P}(E)}$ which are independent of the base point $z \in M$. Now we define the projective-flatness of Finsler metrics.

DEFINITION 4.2. A Finsler metric on E is said to be *projectively flat* if it is obtained from a flat pseudo-Kähler metric on $\mathbb{P}(E)$.

In a previous paper, we proved this result:

THEOREM 4.1 [Aikou 2003a]. *A pseudoconvex Finsler metric is projectively flat if and only if the trace-free part of the curvature form Ω vanishes identically, i.e.,*

$$\Omega = A(z) \otimes \text{Id} \quad (4.7)$$

for some $(1,1)$ -form A on M .

REMARK 4.1. A Finsler metric F is projectively flat if and only if there exists a local function $\sigma_U(z)$ on U such that F is of the form

$$F(z, \xi)|_U = \exp \sigma_U(z) \cdot |\xi^j|^2 \exp g_{(j)}([\xi]) \quad (4.8)$$

on each U . In other words, a Finsler metric F is projectively flat if and only if there exists a local function $\sigma_U(z)$ on U such that the local metric $e^{-\sigma_U(z)} F$ is a flat Finsler metric on U . In the previous paper [Aikou 1997], such a Finsler metric F is said to be *conformally flat*.

We suppose that a pseudoconvex Finsler metric F is projectively flat. Then, since the curvature form Ω is given by (3.18), we have $R^{HV} \equiv 0$, and thus (E, F) is modeled on a complex Minkowski space. We can easily show that the associated Hermitian metric h_F is also projectively flat. Hence:

THEOREM 4.2. *A holomorphic vector bundle E of $\text{rank}(E) = r + 1$ admits a projectively flat Finsler metric if and only if $\mathbb{P}(E)$ is flat, i.e.,*

$$\mathbb{P}(E) = \tilde{M} \times_{\rho} \mathbb{P}^r, \quad (4.9)$$

where \tilde{M} is the universal cover of M , and $\rho : \pi_1(M) \rightarrow PU(r)$ is a representation of the fundamental group $\pi_1(M)$ in the projective unitary group $PU(r)$.

4.2. Ruled manifolds. An algebraic surface \mathcal{X} is said to be *ruled* if it is birational to $M \times \mathbb{P}^1$, where M is a compact Riemann surface. An algebraic surface \mathcal{X} is said to be *geometrically ruled* if there exists a holomorphic projection $\phi : \mathcal{X} \rightarrow M$ such that every fibre $\phi^{-1}(z) = \mathcal{X}_z$ is holomorphically isomorphic to \mathbb{P}^1 . As is well known, a geometrically ruled surface is ruled (see [Beauville 1983]), and every geometrically ruled surface \mathcal{X} is holomorphically isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle $\pi : E \rightarrow M$ of $\text{rank}(E) = 2$ (see [Yang 1991], for example).

An algebraic manifold \mathcal{X} is said to be a *ruled manifold* if \mathcal{X} is a holomorphic \mathbb{P}^r -bundle with structure group $PGL(r+1, \mathbb{C}) = GL(r+1, \mathbb{C})/\mathbb{C}^\times$. Any holomorphic \mathbb{P}^r -bundle over M is classified by $H^1(M, PGL(r+1, \mathcal{O}_M))$, and any rank $r+1$ holomorphic vector bundle over M is classified by the elements of $H^1(M, GL(r+1, \mathcal{O}_M))$. The exact sequence

$$0 \rightarrow \mathcal{O}_M^* \rightarrow GL(r+1, \mathcal{O}_M) \rightarrow PGL(r+1, \mathcal{O}_M) \rightarrow 0$$

implies the sequence of cohomology groups:

$$\cdots \rightarrow H^1(M, GL(r+1, \mathcal{O}_M)) \rightarrow H^1(M, PGL(r+1, \mathcal{O}_M)) \rightarrow H^2(M, \mathcal{O}_M^*) \rightarrow \cdots$$

Since $H^2(M, \mathcal{O}_M^*) = 0$, the following is obtained.

PROPOSITION 4.3. *Every ruled manifold \mathcal{X} over a compact Riemann surface M is holomorphically isomorphic to $\mathbb{P}(E)$ for some holomorphic vector bundle $\pi : E \rightarrow M$ of $\text{rank}(E) = r+1$. Such a bundle E is uniquely determined up to tensor product with a holomorphic line bundle.*

If E is a holomorphic vector bundle over a compact Kähler manifold M , then $\mathbb{P}(E)$ is also a compact Kähler manifold. In fact, we can construct a Kähler metric $\omega_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$ of the form $\omega_{\mathbb{P}(E)} = \phi^* \omega_M + \varepsilon \eta$. Here, ω_M is a Kähler metric on M , ε is a small positive constant, and η is a closed $(1,1)$ -form on $\mathbb{P}(E)$ such that η is positive-definite on the fibres of ϕ (see Lemma (6.37) in [Shiffmann and Sommesse 1985]). Thus every ruled manifold \mathcal{X} over a compact Riemann surface M is a compact Kähler manifold, and $\phi : \mathcal{X} = \mathbb{P}(E) \rightarrow M$ is a holomorphic submersion from a compact Kähler manifold \mathcal{X} to M . Then we have

THEOREM 4.3. *Let \mathcal{X} be a ruled manifold over a compact Riemann surface M with a Kähler metric $\omega_{\mathcal{X}}$. Then there exists a negative vector bundle $\pi : E \rightarrow M$ such that $\mathcal{X} = \mathbb{P}(E)$, and a negatively curved pseudoconvex Finsler metric F on E satisfying $\rho_* \omega_{\mathcal{X}} = \sqrt{-1} \partial \bar{\partial} \log F$.*

PROOF. Let $\omega_{\mathcal{X}}$ be a Kähler metric on \mathcal{X} . Propositions 4.3 and 4.2 imply that there exists a holomorphic vector bundle E satisfying $\mathcal{X} = \mathbb{P}(E)$ with a pseudoconvex Finsler metric F . Then, since

$$\sqrt{-1} \partial \bar{\partial} \log F = \omega_{\mathcal{X}} > 0,$$

F is negatively curved, and hence Theorem 2.3 implies that E is negative. \square

LeBrun [LeBrun 1995] has investigated minimal ruled surfaces (i.e., geometrically ruled surface) over a compact Riemann surface of genus $g(M) \geq 2$ with constant negative scalar curvature. Roughly speaking, he proved that such a minimal ruled surface \mathcal{X} is obtained by a semi-stable vector bundle over M so that $\mathcal{X} = \mathbb{P}(E)$. Since the semi-stability of vector bundles over a compact Riemann surface is equivalent to the existence of a projectively flat Hermitian metric on E , such a surface is written in the form (4.9).

On the other hand, by Theorem 4.3, the geometry of a minimal ruled surface \mathcal{X} is naturally translated to the geometry of a negative vector bundle E with a negatively curved pseudoconvex Finsler metric F . From this viewpoint, we have also investigated minimal ruled surfaces, and we have concluded that each minimal ruled surface $\phi : \mathcal{X} \rightarrow M$ over a compact Riemann surface of genus $g(M) \geq 2$ with constant negative scalar curvature is a Kähler submersion with isometric fibres (see [Aikou 2003b]).

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