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Ricci and Flag Curvatures in Finsler Geometry

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Contents

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Introduction

It is our goal in this article to present a current and uniform treatment of flag and Ricci curvatures in Finsler geometry, highlighting recent developments. (The flag curvature is a natural extension of the Riemannian sectional curvature to Finsler manifolds.) Of particular interest are the Einstein metrics, constant Ricci curvature metrics and, as a special case, constant flag curvature metrics. Our understanding of Einstein spaces is inchoate. Much insight may be gained by considering the examples that have recently proliferated in the literature. This motivates us to discuss many of these metrics.

Happily, the theory is developing as well. The Einstein and constant flag curvature metrics of spaces of Randers type, a fecund class of Finsler spaces, are now properly understood. Enlightenment comes from being able to identify the class as solutions to Zermelo's problem of navigation, a perspective that allows a very apt characterisation of the Einstein spaces. When specialised to flag curvature, the navigation description yields a complete classification of the constant flag curvature Randers metrics.

We hope to bring out the rich variety of behaviour displayed by these metrics. For example, Finsler metrics of constant flag curvature exhibit qualities not found in their constant sectional curvature Riemannian counterparts.

- Beltrami's theorem guarantees that a Riemannian metric is projectively flat if and only if it has constant sectional curvature. On the other hand, there are many Finsler metrics of constant flag curvature which are *not* projectively flat. See Section 3.2.3 and [Shen 2004].
- Every Riemannian metric of constant sectional curvature K is locally isometric to a round sphere, Euclidean space, or a hyperbolic space, depending on K. Hence, for each K, there is only one Riemannian standard model, up to isometry. By contrast, on S^n, $\mathbb{R}^n$, and the unit ball B^n, there are numerous nonisometric Randers metrics of constant flag curvature K. In fact, isometry classes of Randers type standard models make up a moduli space $\mathcal{M}_K$ whose dimension is linear in n. See the table on page 242 and [Bryant 2002].

A roadmap. This article is written with a variety of readers in mind, ranging from the geometric neophyte to the Finsler aficionado. We anticipate that these users will approach the manuscript with distinct aims. The outline below is intended to help readers navigate the article efficiently.

Section 1 introduces Finsler metrics and their curvatures, as well as tools and constructions that are endemic to non-Riemannian Finsler geometry. The reader conversant with Finsler metrics might only skim this section to glean our notation and conventions.

In Section 2 we develop a useful characterisation of the Einstein spaces among a ubiquitous class of Finsler metrics. This description generalises a characterisation of constant flag curvature Randers metrics ([Bao–Robles 2003] and

[Matsumoto–Shimada 2002]) to the Einstein realm. The resulting conditions form a tensorial, coupled system of nonlinear second order partial differential equations, whose unknowns consist of Riemannian metrics a and 1-forms b. These equations provide a substantial step forward in computational efficiency over the defining Einstein criterion (which stipulates that the 'average flag curvature' is to be a function of position only). Indeed, study of these equations has led to the construction of the Finslerian Poincaré metric ([Okada 1983] and [Bao et al. 2000]), as well as the S^3 metric [Bao–Shen 2002]. However, while the characterisation improves the computational accessibility of Einstein metrics, it does little to advance our understanding of their geometry. It is in the following section that we pursue this geometric insight.

The *sine qua non* here is Shen's observation that Randers metrics may be identified with solutions to Zermelo's problem of navigation on Riemannian manifolds. This navigation structure establishes a bijection between Randers spaces $(M, F = \alpha+\beta)$ and pairs (h, W) of Riemannian metrics h and vector fields W on the manifold M. From this perspective, the characterisation of Section 2 is parlayed into a breviloquent geometric description of Einstein metrics. Explicitly, the Randers metric F with navigation data (h, W) is Einstein if and only if h is Einstein and W is an infinitesimal homothety of h. (In particular, these h and W *solve* the system of partial differential equations in Section 2.) The transparent nature of the navigation description immediately yields a Schur lemma for the Ricci scalar, together with a certain rigidity in three dimensions.

The variety of examples in the article may be categorised as follows.

- Metrics in their defining form: Sections 1.1.1, 1.2.1, 2.1.1, 2.3.2, 3.1.2
- Solutions to Zermelo navigation: Sections 1.1.1, 3.1.1, 3.1.2, 3.2.3
- Of constant flag curvature: Sections 1.2.1, 2.3.2, 3.1.2, 3.2.3, 4.1.1
- Einstein but not of const. flag curvature: Sections 4.1.1, 4.1.2, 4.2.3, 4.3.3
- Ricci-flat Berwald: Sections 3.1.1 (locally Minkowski), 4.3.3 (not loc. Mink.)

In Section 4, the emphasis is on Einstein metrics of nonconstant flag curvature, especially those on compact boundaryless manifolds. The spaces studied include Finsler surfaces with Ricci scalar a function on M alone (the scalar is *a priori* a function on the tangent bundle), as well as non-Riemannian Ricci-constant solutions of Zermelo navigation on Cartesian products and Kähler–Einstein manifolds. Section 5 discusses open problems.

1. Flag and Ricci Curvatures

1.1. Finsler metrics

1.1.1. Definition and examples. A Finsler metric is a continuous function

$$F : TM \to [0, \infty)$$

with the following properties:

(i) *Regularity*: F is smooth on uniformized in favor of : $TM \smallsetminus 0 := \{(x,y) \in TM : y \neq 0\}$.
(ii) *Positive homogeneity*: $F(x,cy) = cF(x,y)$ for all $c > 0$.
(iii) *Strong convexity*: the fundamental tensor

$$g_{ij}(x,y) := \left(\tfrac{1}{2}F^2\right)_{y^i y^j}$$

is positive definite for all $(x,y) \in TM \smallsetminus 0$. Here the subscript $_{y^i}$ denotes partial differentiation by y^i.

Strong convexity implies that $\{y \in T_xM : F(x,y) \leqslant 1\}$ is a strictly convex set, but not vice versa; see [Bao et al. 2000].

The function F for a Riemannian metric a is $F(x,y) := \sqrt{a_{ij}(x)y^iy^j}$. In this case, one finds that $g_{ij} := (\frac{1}{2}F^2)_{y^iy^j}$ is simply a_{ij}. Thus the fundamental tensor for general Finsler metrics may be thought of as a direction-dependent Riemannian metric. This viewpoint is treated more carefully in Section 1.1.2.

Many calculations in Finsler geometry are simplified, or magically facilitated, by *Euler's theorem* for homogeneous functions:

Let ϕ be a real valued function on $\mathbb{R}^n$, differentiable at all $y \neq 0$. The following two statements are equivalent.

- $\phi(cy) = c^r\phi(y)$ *for all* $c > 0$ *(positive homogeneity of degree r).*
- $y^i\phi_{y^i} = r\phi$; *that is, the radial derivative of ϕ is r times ϕ.*

(See, for example, [Bao et al. 2000] for a proof.) This theorem, for instance, lets us invert the defining relation of the fundamental tensor given above to get

$$F^2(x,y) = g_{ij}(x,y)y^iy^j.$$

Consequently, strong convexity implies that F must be positive at all $y \neq 0$. The converse, however, is false; positivity does not in general imply strong convexity. This is because while $g_{ij}(x,y)y^iy^j = F^2(x,y)$ may be positive for $y \neq 0$, the quadratic $g_{ij}(x,y)\tilde{y}^i\tilde{y}^j$ could still be $\leqslant 0$ for some nonzero $\tilde{y}$.

Here are some 2-dimensional examples. Being in two dimensions, we revert to the common notation of denoting position coordinates by x, y rather than x^1, x^2, and components of tangent vectors by u, v rather than y^1, y^2.

EXAMPLE (QUARTIC METRIC). Let

$$F(x,y;u,v) := (u^4 + v^4)^{1/4}.$$

Positivity is manifest. However, $\det(g_{ij}) = 3u^2v^2/(u^4+v^4)$ along the tangent vector $u\partial_x + v\partial_y$ based at the point (x,y). Thus (g_{ij}) fails to be a positive definite matrix when u or v vanishes. For instance, if $u = 0$ but $v \neq 0$, we have $g_{ij}(x,y;0,v) = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, which is not positive definite. It is shown in [Bao et al. 2000] that F can be regularised to restore strong convexity. ♢

We shall see in Section 2.1.2 that, surprisingly, positivity of F does imply strong convexity for Randers metrics.

Next, consider a surface S given by the graph of a smooth function $f(x, y)$. Parametrise S via $(x, y) \mapsto (x, y, f)$. By a slight abuse of notation, set $\partial_x := (1, 0, f_x)$ and $\partial_y := (0, 1, f_y)$, and denote the natural dual of this basis by dx, dy. The Euclidean metric of $\mathbb{R}^3$ induces a Riemannian metric on S:

$$h := \left(1 + f_x^2\right)\, dx \otimes dx + f_x f_y \,(dx \otimes dy + dy \otimes dx) + \left(1 + f_y^2\right)\, dy \otimes dy.$$

If $Y := u\partial_x + v\partial_y$ is an arbitrary tangent vector on S, we have

$$|Y|^2 := h(Y, Y) = u^2 + v^2 + (uf_x + vf_y)^2.$$

We note for later use that the contravariant description of $df = f_x\, dx + f_y\, dy$ is the vector field

$$(df)^\sharp = \frac{1}{1 + f_x^2 + f_y^2}\,(f_x\,\partial_x + f_y\,\partial_y), \quad \text{with} \quad |(df)^\sharp|^2 = \frac{f_x^2 + f_y^2}{1 + f_x^2 + f_y^2}.$$

Example (Metric from Zermelo navigation). If we assume that gravity acts perpendicular to S, a person's weight does not affect his motion along the surface. Now introduce a wind $W = W^x\partial_x + W^y\partial_y$ blowing tangentially to S. The norm function F that measures travel time on S can be derived using a procedure (Section 3.1) due to Zermelo and generalised by Shen. With

$$|W|^2 = (W^x)^2 + (W^y)^2 + (W^x f_x + W^y f_y)^2,$$
$$h(W, Y) = uW^x + vW^y + (uf_x + vf_y)(W^x f_x + W^y f_y),$$

and $\lambda := 1 - |W|^2$, the formula for F reads

$$F(x, y; u, v) = \frac{\sqrt{h(W, Y)^2 + |Y|^2\lambda}}{\lambda} - \frac{h(W, Y)}{\lambda}.$$

This Zermelo navigation metric F is strongly convex if and only if $|W| < 1$. The unit circle of h in each tangent plane represents the destinations reachable in one unit of time when there is no wind. It will be explained (Section 3.1) that the effect of the wind is to take this unit circle and translate it rigidly by the amount W. The resulting figure is off-centered and represents the locus of unit time destinations under windy conditions, namely the indicatrix of F. Since the latter lacks central symmetry, F could not possibly be Riemannian; the above formula makes explicit this fact. $\diamondsuit$

Example (Matsumoto's slope-of-a-mountain metric). Take the same surface S, but without the wind. View S as the slope of a mountain resting on level ground, with gravity pointing down instead of perpendicular to S. A person who can walk with speed c on level ground navigates this hillside S along a path that makes an angle θ with the steepest downhill direction. The acceleration of gravity (of magnitude g), being perpendicular to level ground, has a component of magnitude $g_{\|} = g\sqrt{(f_x^2+f_y^2)/(1+f_x^2+f_y^2)}$ along the steepest downhill direction. The hiker then experiences an acceleration $g_{\|}\cos\theta$ along her path, and compensates against the $g_{\|}\sin\theta$ which tries to drag her off-course. Under suitable assumptions about frictional forces, the acceleration $g_{\|}\cos\theta$ *rapidly* effects a terminal addition $\frac{1}{2}g_{\|}\cos\theta$ to the pace c generated by her leg muscles. In other words, her speed is effectively of the form $c + a\cos\theta$, where a is independent of θ. Thus the locus of unit time destinations is a limaçon. The unit circle of h, instead of undergoing a rigid translation as in Zermelo navigation, has now experienced a direction-dependent deformation. The norm function F with this limaçon as indicatrix measures travel time on S. It was worked out by Matsumoto, after being inspired by a letter from P. Finsler, and reads

$$\frac{|Y|^2}{c|Y| - (g/2)\,(uf_x + vf_y)};$$

see [Matsumoto 1989] and [Antonelli et al. 1993].

For simplicity, specialise to the case $c = g/2$. Multiplication by c then converts this norm function to

$$F(x,y;u,v) := \frac{|Y|^2}{|Y| - (uf_x + vf_y)} = |Y|\,\varphi\Big(\frac{(df)(Y)}{|Y|}\Big),$$

with $\varphi(s) := 1/(1-s)$. We see from [Shen 2004] in this volume that metrics of the type $\alpha\varphi(\beta/\alpha)$ are strongly convex whenever the function $\varphi(s)$ satisfies $\varphi(s) > 0$, $\varphi(s) - s\varphi'(s) > 0$, and $\varphi''(s) \geqslant 0$. For the φ at hand, this is equivalent to $(df)(Y) < \frac{1}{2}|Y|$, which is in turn equivalent to $|(df)^\sharp| < \frac{1}{2}$. (In one direction, set $Y = (df)^\sharp$; the converse follows from a Cauchy–Schwarz inequality.) Using the formula for $|(df)^\sharp|^2$ presented earlier, this criterion produces

$$f_x^2 + f_y^2 < \tfrac{1}{3}.$$

Whenever this holds, F defines a Finsler metric. Such is the case for $f(x,y) := \frac{1}{2}x$ but not for $f(x,y) := x$, even though the surface S is an inclined plane in both instances. As for the elliptic paraboloid given by the graph of $f(x,y) := 100 - x^2 - y^2$, we have strong convexity only in a circular vicinity of the hilltop. $\diamondsuit$

The functions F in these two examples are not absolutely homogeneous (and therefore non-Riemannian) because at any given juncture, the speed with which one could move forward typically depends on the direction of travel. Our discussion also raises a tantalising question: if the wind were blowing on the slope of a mountain, would the indicatrix of the resulting F be a rigid translate of the limaçon?

1.1.2. The pulled-back bundle and the fundamental tensor. The matrix g_{ij} involved in the definition of strong convexity is known as the fundamental tensor, and has a geometric meaning, which is most transparent through the use of the pulled-back bundle introduced by Chern.

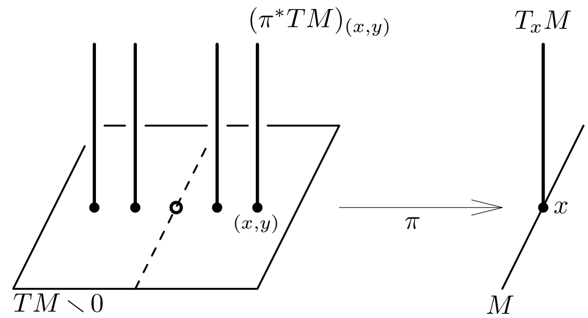


Figure 1. The pulled-back tangent bundle π^*TM is a vector bundle over the "parameter space" $TM \smallsetminus 0$. The fiber over any point (x, y) is a copy of T_xM. The dotted part is the deleted zero section.

The g_{ij} depend on both x and $y \in T_xM$. Over each fixed point $(x, y) \in TM \smallsetminus 0$, the bundle π^*TM provides a copy of T_xM. Endow this T_xM with the symmetric bilinear form

$$g := g_{ij}(x, y)\, dx^i \otimes dx^j .$$

Since the Finsler metric is strongly convex, this bilinear form is positive definite, which renders it an inner product. Thus, every Finsler metric endows the fibres of the pulled-back bundle with a Riemannian metric. Two facts stand out:

(1) The fundamental tensor $g_{ij}(x, y)$ is invariant under $y \mapsto \lambda y$ for all $\lambda > 0$. This invariance directly follows from the hypothesis that F is positively homogeneous of degree 1 in y. Thus, over the points $\{(x, \lambda y) : \lambda > 0\}$ in the parameter space $TM \smallsetminus 0$, not only the fibres are identical, but the inner products are too.
(2) That g_{ij} arises as the y-Hessian of $\frac{1}{2}F^2$ imposes a stringent symmetry condition. Namely, $(g_{ij})_{y^k}$ must be totally symmetric in its three indices i, j, k. For this reason, not every Riemannian metric on the fibres of the pulled-back bundle comes from a Finsler metric.

Property (1) is a redundancy that will be given a geometrical interpretation below. We will also explain why the symmetry criterion (2) is an integrability condition in disguise.

Let's switch our perspective from the pulled-back bundle π^*TM to the tangent bundle TM itself. Recall that a Riemannian metric on M is a smooth assignment of inner products, one for each tangent space T_xM. By contrast, a Finsler metric F gives rise to a family of inner products $g_{ij}(x, y)\, dx^i \otimes dx^j$ on each tangent space T_xM. Item (1) above means that there is exactly one inner product for each *direction*. This sphere's worth of inner products on each tangent space has to satisfy the symmetry condition described in (2).

The converse is also true. Suppose we are given a family of inner products $g_{ij}(x, y)$ on each tangent space T_xM, smoothly dependent on x and nonzero

$y \in T_xM$, invariant under positive rescaling of y (that is, maps $y \mapsto \lambda y$, for $\lambda > 0$) and such that $(g_{ij})_{y^k}$ is totally symmetric in i, j, and k. We construct a Finsler function as follows:

$$F(x,y) := \sqrt{g_{pq}(x,y)\, y^p y^q}.$$

This F is smooth on the entire slit tangent bundle $TM \smallsetminus 0$. It is positively homogeneous of degree 1 in y because $g_{pq}(x,y)$ is invariant under positive rescaling in y. Also, with the total symmetry of $(g_{ij})_{y^k}$, we have

$$\begin{aligned}(F^2)_{y^i y^j} &= (g_{pq}\, y^p y^q)_{y^i y^j} = \big((g_{pq})_{y^i}\, y^p y^q + g_{iq}\, y^q + g_{pi}\, y^p\big)_{y^j} \\ &= (g_{iq})_{y^j}\, y^q + g_{ij} + (g_{pi})_{y^j}\, y^p + g_{ji} = 2\, g_{ij}.\end{aligned}$$

Thus F is strongly convex because, for each nonzero y, the matrix $g_{ij}(x,y)$ is positive definite.

In the calculation above, the quantity $(g_{pq})_{y^i}\, y^p$ on the first line and the terms $(g_{iq})_{y^j}\, y^q$ and $(g_{pi})_{y^j}\, y^p$ on the second line are all zero because of the hypothesis that $(g_{rs})_{y^t}$ is totally symmetric in r, s, t. For instance,

$$(g_{pq})_{y^i}\, y^p = (g_{iq})_{y^p}\, y^p = 0,$$

where the last equality follows from Euler's theorem, and the assumption that $g_{iq}(x,y)$ is positive homogeneous of degree 0 in y. The symmetry hypothesis therefore plays the role of an integrability condition.

We conclude that *the concept of a Finsler metric on M is equivalent to an assignment of a sphere's worth of inner products $g_{ij}(x,y)$ on each tangent space T_xM, such that $(g_{ij})_{y^k}$ is totally symmetric in i, j, k and appropriate smoothness holds*. *Similarly*, *a Finsler metric on M is also equivalent to a smooth Riemannian metric on the fibres of the pulled-back bundle π^*TM, satisfying the above redundancy* (1) *and integrability condition* (2).

The pulled-back bundle π^*TM and its natural dual π^*T^*M each contains an important *global* section. They are

$$\ell := \frac{y^i}{F(x,y)}\, \partial_{x^i},$$

the *distinguished section*, and

$$\omega := F_{y^i}\, dx^i,$$

the *Hilbert form*. As another application of the integrability condition, we differentiate the statement $F^2 = g_{ij}\, y^i y^j$ to find

$$F_{y^i} = \frac{{}^g y_i}{F} = g_{ij}\, \ell^j =: \ell_i, \quad \text{with} \quad {}^g y_i := g_{ij}\, y^j.$$

This readily gives

$$\ell^i\, \ell_i = 1 \quad \text{and} \quad \ell_{i;j} = g_{ij} - \ell_i \ell_j,$$

where the semicolon abbreviates the differential operator $F\partial_{y^i}$.

1.1.3. Geodesic spray coefficients and the Chern connection. We have seen in Section 1.1.1 that if F is the Finsler function of a Riemannian metric, its fundamental tensor has no y-dependence. The converse follows from the fact that $g_{ij}\, y^i y^j$ reconstructs F^2 for us, thanks to Euler's theorem. Thus, the y derivative of g_{ij} measures the extent to which F fails to be Riemannian. More formally, we define the *Cartan tensor*

$$A_{ijk} := \tfrac{1}{2} F\, (g_{ij})_{y^k} = \tfrac{1}{4} F\, (F^2)_{y^i y^j y^k}\,,$$

which is totally symmetric in all its indices. As an illustration of Euler's theorem, note that since g is homogeneous of degree zero in y, we have $y^i A_{ijk} = 0$.

Besides the Cartan tensor, we can also associate to g its formal Christoffel symbols of the second kind ,

$$\gamma^i{}_{jk} := \tfrac{1}{2} g^{is} (g_{sj,x^k} - g_{jk,x^s} + g_{ks,x^j}),$$

and the *geodesic spray coefficients*

$$G^i := \tfrac{1}{2} \gamma^i{}_{jk}\, y^j y^k .$$

The latter are so named because the (constant speed) geodesics of F are the solutions of the differential equation $\ddot{x}^i + \dot{x}^j \dot{x}^k \gamma^i{}_{jk}(x, \dot{x}) = 0$, which may be abbreviated as $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$.

Caution: the G^i defined here is equal to *half* the G^i in [Bao et al. 2000].

Covariant differentiation of local sections of the pulled-back bundle $\pi^* TM$ requires a connection, for which there are many name-brand ones. All of them have their genesis in the *nonlinear connection*

$$N^i{}_j := (G^i)_{y^j} .$$

This $N^i{}_j$ is a connection in the Ehresmann sense, because it specifies a distribution of *horizontal* vectors on the manifold $TM \smallsetminus 0$, with basis

$$\frac{\delta}{\delta x^j} := \partial_{x^j} - N^i{}_j\, \partial_{y^i} .$$

As another application of Euler's theorem, note that $N^i{}_j\, y^j = 2G^i$, since G^i is homogeneous of degree 2 in y. We also digress to observe that the Finsler function F is constant along such horizontal vector fields:

$$F_{|j} := \frac{\delta}{\delta x^j} F = 0.$$

The key lies in the following sketch of a computation:

$$N^i{}_j\, \ell_i = (G^i\, \ell_i)_{y^j} - \frac{1}{F} G^i\, \ell_{i;j} = F_{x^j}\,,$$

in which establishing $(\ell^i\, G_i)_{y^j} = \tfrac{1}{2}(F_{x^j} + y^k\, F_{x^k y^j})$ and its companion statement $-(1/F)\, G^i\, \ell_{i;j} = \tfrac{1}{2}(F_{x^j} - y^k\, F_{x^k y^j})$ takes up the bulk of the work.

Using A, γ, and N, we can now state the formula of the *Chern connection* in natural coordinates [Bao et al. 2000]:

$$\Gamma^i{}_{jk} = \gamma^i{}_{jk} - \frac{1}{F} g^{is}(A_{sjt} N^t{}_k - A_{jkt} N^t{}_s + A_{kst} N^t{}_j),$$

with associated connection 1-forms $\omega_j{}^i := \Gamma^i{}_{jk}\, dx^k$. These $\omega_j{}^i$ represent the unique, torsion-free ($\Gamma^i{}_{kj} = \Gamma^i{}_{jk}$) connection which is almost g-compatible:

$$dg_{ij} - g_{kj}\,\omega_i{}^k - g_{ik}\,\omega_j{}^k = \frac{2}{F} A_{ijk}\,(dy^k + N^k{}_l dx^l).$$

As yet another application of Euler's Theorem, we shall show that the nonlinear connection N can be recovered from the Chern connection Γ as follows:

$$\Gamma^i{}_{jk}\, y^k = N^i{}_j.$$

A crucial ingredient in the derivation is $(\gamma^i{}_{st})_{y^j}\, y^s y^t = 2(g^{ip})_{y^j}\, G_p$. Indeed,

$$\begin{aligned}(\gamma^i{}_{st})_{y^j}\, y^s y^t &= (g^{ip}\gamma_{pst})_{y^j}\, y^s y^t \\ &= (g^{ip})_{y^j}\,\gamma_{pst}\, y^s y^t + g^{ip}\left(\left(\frac{1}{F}A_{psj}\right)_{x^t} - \left(\frac{1}{F}A_{stj}\right)_{x^p} + \left(\frac{1}{F}A_{tpj}\right)_{x^s}\right) y^s y^t \\ &= 2(g^{ip})_{y^j}\, G_p + 0.\end{aligned}$$

Whence, with the help of $A_{ijk}\, y^k = 0$ and $N^t{}_k\, y^k = 2G^t$, we have

$$\begin{aligned}\Gamma^i{}_{jk}\, y^k &= \gamma^i{}_{jk}\, y^k - \frac{2}{F} g^{is} A_{stj}\, G^t = \gamma^i{}_{jk}\, y^k - g^{is}(g_{st})_{y^j}\, G^t \\ &= \gamma^i{}_{jk}\, y^k + (g^{is})_{y^j}\, g_{st}\, G^t \\ &= \gamma^i{}_{jk}\, y^k + \tfrac{1}{2}(\gamma^i{}_{st})_{y^j}\, y^s y^t \\ &= (\tfrac{1}{2}\gamma^i{}_{st}\, y^s y^t)_{y^j} = (G^i)_{y^j} = N^i{}_j,\end{aligned}$$

as claimed.

It is now possible to covariantly differentiate sections of π^*TM (and its tensor products) along the horizontal vector fields $\delta/\delta x^k$ of the manifold $TM \smallsetminus 0$. For example,

$$T^i{}_{j|k} := \frac{\delta}{\delta x^k} T^i{}_j + T^s{}_j\,\Gamma^i{}_{sk} - T^i{}_s\,\Gamma^s{}_{jk}.$$

If F arises from a Riemannian metric a, then $A = \frac{1}{2}F(a_{ij})_{y^k} = 0$ because a has no y-dependence. In that case, Γ is given by the Christoffel symbols of a. If the tensor T also has no y-dependence, $T^i{}_{j|k}$ reduces to the familiar covariant derivative in Riemannian geometry.

Let's return to the Finsler setting. Using the symmetry $\Gamma^i{}_{sk} = \Gamma^i{}_{ks}$ and the recovery property $y^s\,\Gamma^i{}_{ks} = N^i{}_k$, we have

$$y^i{}_{|k} = \frac{\delta}{\delta x^k} y^i + y^s\,\Gamma^i{}_{ks} = 0, \quad \text{hence} \quad \ell^i{}_{|k} = 0.$$

Also, the covariant derivative of the Cartan tensor A along the special horizontal vector field $\ell^s\,\delta/\delta x^s$ gives the *Landsberg tensor*, which makes frequent appearances in Finsler geometry:

$$\dot{A}_{ijk} := A_{ijk|s}\,\ell^s.$$

Note that $\dot{A}$ is totally symmetric, and its contraction with y vanishes.

The Chern connection gives rise to two curvature tensors:

$$R_j{}^i{}_{kl} = \frac{\delta}{\delta x^k}\Gamma^i{}_{jl} - \frac{\delta}{\delta x^l}\Gamma^i{}_{jk} + \Gamma^i{}_{sk}\Gamma^s{}_{jl} - \Gamma^i{}_{sl}\Gamma^s{}_{jk},$$

$$P_j{}^i{}_{kl} = -F\frac{\partial}{\partial y^l}\Gamma^i{}_{jk} \qquad \text{(note the symmetry: } P_j{}^i{}_{kl} = P_k{}^i{}_{jl}\text{)},$$

both invariant under positive rescaling in y. In the case of Riemannian metrics, Γ reduces to the standard Levi-Civita (Christoffel) connection, which is independent of y; hence $\frac{\delta}{\delta x}\Gamma$ becomes $\frac{\partial}{\partial x}\Gamma$. The curvature R is then the usual Riemann tensor, and P is zero.

In Finsler geometry, there are many Bianchi identities. A leisurely account of their derivation can be found in [Bao et al. 2000].

1.2. Flag curvature. This is a generalisation of the sectional curvature of Riemannian geometry. Alternatively, flag curvatures can be treated as Jacobi endomorphisms [Foulon 2002]. The flag curvature has also led to a pinching (sphere) theorem for Finsler metrics; see [Rademacher 2004] in this volume.

1.2.1. The flag curvature versus the sectional curvature. Installing a flag on a Finsler manifold (M, F) implies choosing

- a basepoint $x \in M$ at which the flag will be planted,
- a flagpole given by a nonzero $y \in T_xM$, and
- an edge $V \in T_xM$ transverse to the flagpole.

See Figure 2. Note that the flagpole $y \neq 0$ singles out an inner product

$$g_y := g_{ij}(x,y)\,dx^i \otimes dx^j$$

from among the sphere's worth of inner products described in Section 1.1.2. This g_y allows us to measure the angle between V and y. It also enables us to calculate the area of the parallelogram formed by V and $\ell := y/F(x,y)$.

The flag curvature is defined as

$$K(x,y,V) := \frac{V^i\,(y^j\,R_{jikl}\,y^l)\,V^k}{g_y(y,y)\,g_y(V,V) - g_y(y,V)^2}\,,$$

where the index i on $R_j{}^i{}_{kl}$ has been lowered by g_y. When the Finsler function F comes from a Riemannian metric, g_y is simply the Riemannian metric, R_{jikl} is the usual Riemann tensor, and $K(x,y,V)$ reduces to the familiar sectional curvature of the 2-plane spanned by $\{y, V\}$.

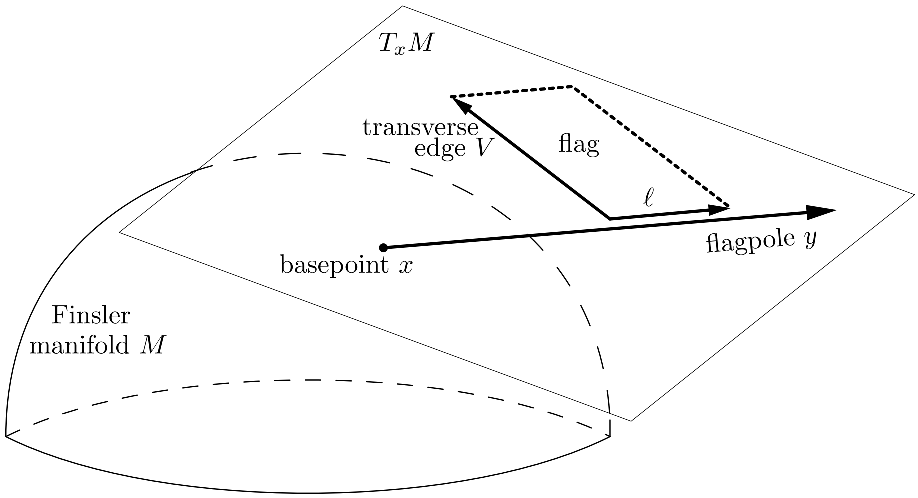


Figure 2. A typical flag, based at the point x on a Finsler manifold M. The flagpole is y, and the "cloth" part of the flag is $\ell \wedge V$. The entire flag lies in the tangent space T_xM.

Since $g_{_y}(y,y) = F^2(x,y)$, the flag curvature can be reexpressed as

$$K(x,y,V) = \frac{V^i\,(\ell^j\,R_{jikl}\,\ell^l)\,V^k}{g_{_y}(V,V) - g_{_y}(\ell,V)^2}\,.$$

The denominator here is the area-squared of the parallelogram formed by V and the $g_{_y}$ unit vector $\ell = y/F$.

The tensor $R_{ik} := \ell^j\,R_{jikl}\,\ell^l$ is called the *predecessor* of the flag curvature. It is proved in [Bao et al. 2000] that $R_{ki} = R_{ik}$.

We also note that $\ell^i\,R_{ik} = 0 = R_{ik}\,\ell^k$. The second equality is immediate because $R_j{}^i{}_{kl}$ is manifestly skew-symmetric (Section 1.1.3) in k and l. The first equality follows from the symmetry of R_{ik}.

One can use $y^i\,R_{ik} = 0 = R_{ik}\,y^k$ to show that if $\{y, V_1\}$ and $\{y, V_2\}$ have the same span, then $K(x,y,V_1) = K(x,y,V_2)$. In other words, K depends on x, y, and $\operatorname{span}\{y,V\}$.

A Finsler metric is *of scalar curvature* if $K(x,y,V)$ does not depend on V, that is, if $R_{ik}\,V^iV^k = K(x,y)\,(g_{ik} - \ell_i\ell_k)\,V^iV^k$. This says that two symmetric bilinear forms generate the same quadratic form. A polarisation identity then tells us that the bilinear forms in question must be equal. So, Finsler metrics of scalar curvature are described by

$$R_{ik} = K(x,y)\,(g_{ik} - \ell_i\ell_k),$$

where $\ell_i = F_{y^i} = {}^g y_i/F$ (see Section 1.1.2).

Example (Numata metrics). Numata [1978] has shown that the Finsler metrics

$$F(x,y) = \sqrt{q_{ij}(y)\, y^i y^j} + b_i(x) y^i,$$

where q is positive definite and b is *closed*, are of scalar curvature. Note that the first term of F is a locally *Minkowski* norm, Riemannian only when the q_{ij} are constant.

Consider the case that $q_{ij} = \delta_{ij}$ and $b = df$, where f is a smooth function on $\mathbb{R}^n$. If necessary, scale f so that the open set $M := \left\{ x \in \mathbb{R}^n : \sqrt{\delta^{ij} f_{x^i} f_{x^j}} < 1 \right\}$ is nonempty. Then a straightforward calculation reveals that F is of scalar curvature on M with

$$K(x,y) = \frac{3}{4}\frac{1}{F^4}(f_{x^i x^j} y^i y^j)^2 - \frac{1}{2}\frac{1}{F^3}(f_{x^i x^j x^k} y^i y^j y^k).$$

The computation in [Bao et al. 2000] utilises the spray curvature and Berwald's formula, to be discussed in Section 1.2.3. The Numata metrics are projectively flat. For Finsler metrics of (nonconstant) scalar curvature but not projectively flat, see [Shen 2004] in this volume. ♢

The flag curvature is an important geometric invariant because its sign governs the growth of Jacobi fields, which in turn gives qualitative information about short geodesic rays with close initial data. See [Bao et al. 2000]. To bring out the essential difference between the Finslerian and Riemannian settings, we consider the case of surfaces. There, once the basepoint x and the flagpole y are chosen, span$\{y, V\}$ is equal to the tangent plane $T_x M$ for all transverse edges V. Thus every Finsler surface is of scalar curvature $K(x,y)$.

Remark. It is evident from the Numata metrics that the sign of $K(x,y)$ can depend on the direction of the flagpole $y \in T_x M$. By contrast, when the surface is Riemannian, this $K(x,y)$ reduces to the usual Gaussian curvature $K(x)$, which does not depend on y. The implication of this difference is profound when we survey the immediate vicinity of any fixed x. If the landscape is Riemannian, the sign of $K(x)$ creates only one type of geometry near x: hyperbolic, flat, or spherical. If the landscape is Finslerian, the sign of $K(x,y)$ can depend on the direction y of our line of sight, making it possible to encounter all three geometries during the survey!

If K is a constant (namely, it depends neither on V, nor y, nor x), the Finsler metric F is said to be *of constant flag curvature*. The tensorial criterion is $R_{ik} = K(g_{ik} - \ell_i \ell_k)$, with K constant. For later purposes, we rewrite it as

$$F^2 \, R^i{}_k = K(F^2 \, \delta^i{}_k - y^i \, {}^g y_k),$$

where ${}^g y_k := g_{ks} \, y^s$ (see Section 1.1.2).

Example (Bryant's metrics). Bryant discovered an interesting 2-parameter family of projectively flat Finsler metrics on the sphere S^2, with constant flag curvature $K = 1$. Here we single out one metric from this family for presentation; see [Bryant 1997] for the geometry behind the construction.

Parametrise the hemispheres of S^2 via the map $(x, y) \mapsto \big(x,\ y,\ s\sqrt{1-x^2-y^2}\big)$, with $s = \pm 1$. Denote tangent vectors by $(u, v) = u\partial_x + v\partial_y \in T_{(x,y)}S^2$, and introduce the following abbreviations:

$$\begin{aligned} r^2 &:= x^2 + y^2, \quad P^2 := 1 - r^2, \quad B := 2r^2 - 1; \\ R^2 &:= u^2 + v^2, \quad C := xu + yv; \\ a &:= (1 + B^2)\big((P^2 R^2 + C^2) + B(P^2 R^2 - C^2)\big) + 8(1 + B)C^2 P^4; \\ b &:= (1 + B^2)\big((P^2 R^2 - C^2) - B(P^2 R^2 + C^2)\big) - 8(0 + B)C^2 P^2. \end{aligned}$$

We emphasise that in b, the very last term contains $C^2 P^2$ and not $C^2 P^4$. The formula for the Finsler function is then

$$F(x, y; u, v) = \frac{1}{1 + B^2}\left(\frac{1}{P}\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + 2C\right).$$

Note that a is a quadratic and $a^2 + b^2$ is a *quartic*. All the geodesics of F are arcs of great circles with Finslerian length 2π. As a comparison, the corresponding Finsler function for the standard Riemannian metric on S^2 is simply $(1/P)\sqrt{P^2 R^2 + C^2}$. For additional discussions, see [Bao et al. 2000; Sabau 2003; Shen 2004]. $\diamondsuit$

In dimension greater than two, a Schur lemma says that if K does not depend on V and y, it must be constant. This was proved in [del Riego 1973], then in [Matsumoto 1986]; see also [Berwald 1947].

In two dimensions, K can be a function of position x only without being constant. All Riemannian surfaces with nonconstant Gaussian curvature belong to this category; for non-Riemannian examples, see Section 4.1.1.

1.2.2. Rapscák's identity. We now prepare to relate the flag curvature to one of Berwald's spray curvatures. Suppose F and $\mathcal{F}$ are two arbitrary Finsler metrics, with geodesic spray coefficients G^i and $\mathcal{G}^i$. We think of $\mathcal{F}$ as a "background" metric, and use a colon to denote horizontal covariant differentiation with respect to the Chern connection of $\mathcal{F}$. (Note that $\mathcal{F}_{:j}$ vanishes by Section 1.1.3, but not $F_{:j}$.) Finally, let g^{ij} denote the inverse of the fundamental tensor of F, and let the subscript 0 abbreviate contraction with y: for instance, $F_{:0} = F_{:j}\, y^j$.

Then Rapcsák's identity [Rapcsák 1961] reads

$$G^i = \mathcal{G}^i + y^i\left(\frac{1}{2F}F_{:0}\right) + \tfrac{1}{2}F g^{ij}\big((F_{:0})_{y^j} - 2F_{:j}\big).$$

Since Finsler metrics generally involve square roots, another form of Rapcsák's identity is more user-friendly:

$$G^i = \mathcal{G}^i + \tfrac{1}{4}g^{ij}\big((F^2{}_{:0})_{y^j} - 2F^2{}_{:j}\big).$$

The derivation of the identity involves three key steps, in which the basic fact $F F_{y^i} = g_{ij}\, y^j$ will be used repeatedly without mention.

We first verify that $2G_i = (F F_{x^k})_{y^i}\, y^k - F F_{x^i}$.

$$\begin{aligned} 2G_i = \gamma_{ijk}\, y^j y^k &= \tfrac{1}{2}(g_{ij,x^k} - g_{jk,x^i} + g_{ki,x^j})\, y^j y^k \\ &= (g_{ij}\, y^j)_{x^k}\, y^k - \tfrac{1}{2}(g_{jk}\, y^j y^k)_{x^i} = (\tfrac{1}{2}F^2)_{y^i x^k}\, y^k - (\tfrac{1}{2}F^2)_{x^i}, \end{aligned} \qquad (\dagger)$$

where $g_{rs} := (\frac{1}{2}F^2)_{y^r y^s}$, and the last displayed equality follows from two applications of Euler's theorem.

Next, observe that $F F_{x^r} = F F_{:r} + \mathcal{N}^j{}_r\, {}^g y_j$, where ${}^g y_j := g_{js}\, y^s$ and $\mathcal{N}$ is the nonlinear connection of $\mathcal{F}$. Indeed, horizontal differentiation with respect to $\mathcal{F}$ is given by $(\cdots)_{:r} = (\cdots)_{x^r} - \mathcal{N}^j{}_r\, (\cdots)_{y^j}$. Thus

$$F F_{x^r} = F(F_{:r} + \mathcal{N}^j{}_r\, F_{y^j}) = F F_{:r} + \mathcal{N}^j{}_r\, (g_{js}\, y^s). \qquad (\ddagger)$$

Here we have chosen to reexpress F_{x^r} using the nonlinear connection of $\mathcal{F}$ rather than that of F, thereby opening the door for $\mathcal{G}$ to enter the picture.

Finally, we substitute equality (‡) into the purpose of (†), getting

$$2G_i = F_{y^i}\, F_{:k}\, y^k + F(F_{:k})_{y^i}\, y^k - F F_{:i} + (\mathcal{N}^j{}_k)_{y^i}\, {}^g y_j\, y^k + \mathcal{N}^j{}_k ({}^g y_j)_{y^i}\, y^k - \mathcal{N}^j{}_i\, {}^g y_j.$$

Note that $(F_{:k})_{y^i}\, y^k = (F_{:0})_{y^i} - F_{:i}$. Also, since $\mathcal{N}^j{}_k = (\mathcal{G}^j)_{y^k}$, Euler's theorem gives

$$\begin{aligned} &(\mathcal{N}^j{}_k)_{y^i}\, {}^g y_j\, y^k = (\mathcal{G}^j)_{y^i}\, {}^g y_j = \mathcal{N}^j{}_i\, {}^g y_j, \\ &\mathcal{N}^j{}_k ({}^g y_j)_{y^i}\, y^k = (2\mathcal{G}^j)\big((\tfrac{1}{2}F^2)_{y^j y^s y^i}\, y^s + g_{js}\, \delta^s{}_i\big) = (2\mathcal{G}^j)(0 + g_{ji}). \end{aligned}$$

These two statements constitute the heart of the entire derivation. After using them to simplify the above expression for $2G_i$, and relabelling i as r, we have

$$2G_r = 2\mathcal{G}^j\, g_{jr} + \frac{1}{F}\, {}^g y_r\, F_{:0} + F\big((F_{:0})_{y^r} - 2F_{:r}\big).$$

Contracting with $\frac{1}{2}\, g^{ir}$ yields Rapcsák's identity in its original form. The user-friendly version follows without much trouble.

1.2.3. Spray curvatures and Berwald's formulae. The definition of the flag curvature through a connection (for example, Chern's) has theoretical appeal, but is not practical if one wants to compute it. Even for relatively simple Finsler metrics, the machine computation of any name-brand connection is already a daunting and often insurmountable task, let alone the curvature. This is where Berwald's spray curvatures come to the rescue. They originate from his study of systems of coupled second order differential equations, and are defined entirely in terms of the geodesic spray coefficients [Berwald 1929].

$$\begin{aligned} K^i{}_k &:= 2(G^i)_{x^k} - (G^i)_{y^j}\,(G^j)_{y^k} - y^j\,(G^i)_{x^j y^k} + 2G^j\,(G^i)_{y^j y^k}, \\ G_j{}^i{}_{kl} &:= (G^i)_{y^j y^k y^l}. \end{aligned}$$

These spray curvatures are related to the predecessor of the flag curvature, and to the P curvature of Chern's, in the following manner:

$$F^2 R^i{}_k = y^j R_j{}^i{}_{kl} y^l = K^i{}_k,$$
$$\frac{1}{F} P_j{}^i{}_{kl} = -G_j{}^i{}_{kl} + (\dot{A}^i{}_{jk})_{y^l}.$$

The statement about P follows from the fact that the Berwald connection $(G)_{yy}$ can be obtained from the Chern connection Γ by adding $\dot{A}$. This, together with a companion formula, is discussed in the reference [Bao et al. 2000] (whose G^i is *twice* the G^i in the present article). Explicitly,

$$\dot{A}^i{}_{jk} = (G^i)_{y^j y^k} - \Gamma^i{}_{jk} \quad \text{and} \quad \dot{A}_{ijk} = -\tfrac{1}{2}\, {}^g y_s (G^s)_{y^i y^j y^k},$$

where ${}^g y_s := g_{st}\, y^t$. The key that helps establish the first claim is the realisation that $G^i = \frac{1}{2}\Gamma^i{}_{pq} y^p y^q$, which holds because contracting A with y gives zero. When calculating the y-Hessian of G^i, we need the latter part of Section 1.1.3 and the Bianchi identity $\ell^j P_j{}^i{}_{kl} = -\dot{A}^i{}_{kl}$. As for the second claim, here is a sketch of the derivation:

- Start with $\dot{A}^s{}_{jk} = (G^s)_{y^j y^k} - \Gamma^s{}_{jk}$, apply ∂_{y^i}, contract with ${}^g y_s$, and use $P = -F\,\partial_y \Gamma$ (Section 1.1.3). We get ${}^g y_s (\dot{A}^s{}_{jk})_{y^i} = {}^g y_s (G^s)_{y^i y^j y^k} + \ell^s P_{jski}$.
- With the two Bianchi identities (3.4.8) and (3.4.9) of [Bao et al. 2000], it can be shown that $\ell^s P_{jski} = -\ell^s P_{sjki} = \dot{A}_{jki}$.
- The term ${}^g y_s (\dot{A}^s{}_{jk})_{y^i}$ on the left is equal to $\ell_s (\dot{A}^s{}_{jk})_{;i}$, which in turn $=$ $(\ell_s \dot{A}^s{}_{jk})_{;i} - \ell_{s;i} \dot{A}^s{}_{jk} = 0 - (g_{si} - \ell_i \ell_s)\dot{A}^s{}_{jk} = -\dot{A}_{ijk}$.
- These manoeuvres produce $\dot{A}_{ijk} = -\frac{1}{2}\, {}^g y_s (G^s)_{y^i y^j y^k}$ as stated.

We are now in a position to express the Chern curvatures in terms of the Berwald spray curvatures. Note that applying ∂_{y^l} to $\dot{A}^i{}_{jk} = (G^i)_{y^j y^k} - \Gamma^i{}_{jk}$ immediately yields the statement involving P. The derivation of the formula $F^2 R^i{}_k = K^i{}_k$ makes frequent use of the relationship

$$y^k \Gamma^i{}_{jk} = N^i{}_j, \tag{$*$}$$

from Section 1.1.3. We first demonstrate that

$$y^r \frac{\delta T}{\delta x^s} = \frac{\delta}{\delta x^s}(y^r T) + N^r{}_s T. \tag{$**$}$$

This arises from $0 = y^r{}_{|s} = \frac{\delta}{\delta x^s} y^r + y^k \Gamma^r{}_{ks}$; see Section 1.1.3. Property $(*)$ gives $y^k \Gamma^r{}_{ks} = y^k \Gamma^r{}_{sk} = N^r{}_s$. Thus $\frac{\delta}{\delta x^s} y^r = -N^r{}_s$, and $(**)$ follows from the product rule.

Using $(**)$ and $(*)$, we ascertain that

$$F^2 R^i{}_k = y^j \left(\frac{\delta}{\delta x^k} N^i{}_j - \frac{\delta}{\delta x^j} N^i{}_k \right). \tag{$***$}$$

Indeed, $F^2 R^i{}_k$ is obtained by contracting with $y^j y^l$ the explicit formula for $R_j{}^i{}_{kl}$ (Section 1.1.3). After several uses of $(*)$, we get the intermediate expression

$y^j(y^l\,\delta_{x^k}\Gamma^i{}_{jl}) - y^l(y^j\,\delta_{x^l}\Gamma^i{}_{jk}) + y^j\,\Gamma^i{}_{hk}N^h{}_j - N^i{}_h N^h{}_k$. Now apply $(**)$ to the first two sets of parentheses, and use $(*)$ again. The result simplifies to $(***)$.

Finally, recall from Section 1.1.3 that $N^i{}_j := (G^i)_{y^j}$, so Euler's theorem gives $y^j N^i{}_j = 2G^i$. Consequently,

$$\begin{aligned} y^j\,\frac{\delta}{\delta x^k}N^i{}_j &= \frac{\delta}{\delta x^k}(2G^i) + N^i{}_j N^j{}_k \qquad \text{by } (**) \\ &= (2G^i)_{x^k} - N^j{}_k(2G^i)_{y^j} + N^i{}_j N^j{}_k = (2G^i)_{x^k} - (G^i)_{y^j}(G^j)_{y^k}, \\ -y^j\,\frac{\delta}{\delta x^j}N^i{}_k &= -y^j(N^i{}_k)_{x^j} + (y^j N^h{}_j)(N^i{}_k)_{y^h} = -y^j(G^i)_{y^k x^j} + 2G^j(G^i)_{y^k y^j}. \end{aligned}$$

Summing these two conclusions gives Berwald's formula for $K^i{}_k$. (For ease of exposition, we shall refer to $K^i{}_k$ simply as the *spray curvature*.)

We now return to the setting of two arbitrary Finsler metrics F and $\mathcal{F}$, as discussed in the previous subsection. Denote their respective spray curvatures by $K^i{}_k$ and $\mathcal{K}^i{}_k$. According to Rapcsák's identity, the geodesic coefficients of F and $\mathcal{F}$ are related by $G^i = \mathcal{G}^i + \zeta^i$. Inspired by Shibata–Kitayama, we now show that substituting this decomposition into Berwald's formula for $K^i{}_k$ allows us to rewrite the latter in a split and covariantised form

$$K^i{}_k = \mathcal{K}^i{}_k + 2(\zeta^i)_{:k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} - y^j(\zeta^i{}_{:j})_{y^k} + 2\zeta^j(\zeta^i)_{y^j y^k} + 3\zeta^j\dot{\mathcal{A}}^i{}_{jk},$$

where the colon refers to horizontal covariant differentiation with respect to the Chern connection of $\mathcal{F}$, and $\dot{\mathcal{A}}$ is associated to $\mathcal{F}$ as well.

Remarks. 1. Had we used the Berwald connection (which equals the Chern connection plus $\dot{\mathcal{A}}$), that $3\zeta\dot{\mathcal{A}}$ term in the above formula would have been absorbed away.

2. If the background metric were Riemannian, $\mathcal{F}^2 = a_{ij}(x)y^i y^j$, then $\mathcal{A}_{ijk} = \frac{1}{2}\mathcal{F}(a_{ij})_{y^k}$ would be zero because a is independent of y. Hence $\dot{\mathcal{A}} = 0$ as well. Since $\mathcal{A} = 0$, the Chern connection is given by the usual Christoffel symbols of a. Also, the Chern and Berwald connections coincide for Riemannian metrics because they differ merely by $\dot{\mathcal{A}}$.

Here is a sketch of the derivation of the split and covariantised formula. First we replace G by $\mathcal{G} + \zeta$ in Berwald's original formula, obtaining

$$\begin{aligned} K^i{}_k = \mathcal{K}^i{}_k + 2(\zeta^i)_{x^k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} - y^j(\zeta^i)_{x^j y^k} + 2\zeta^j(\zeta^i)_{y^j y^k} \\ - \mathcal{N}^i{}_j(\zeta^j)_{y^k} - (\zeta^i)_{y^j}\mathcal{N}^j{}_k + 2\mathcal{G}^j(\zeta^i)_{y^j y^k} + 2\zeta^j(\mathcal{G}^i)_{y^j y^k}. \quad (\dagger) \end{aligned}$$

Next, let $\tilde{\Gamma}$ denote the Chern connection of $\mathcal{F}$. Then horizontal covariant differentiation of ζ is

$$\zeta^i{}_{:j} = \big((\zeta^i)_{x^j} - \mathcal{N}^l{}_j(\zeta^i)_{y^l}\big) + \zeta^l\tilde{\Gamma}^i{}_{lj},$$

from which we solve for $(\zeta^i)_{x^j}$ in terms of $\zeta^i{}_{:j}$. The result is used to replace ("covariantise") all the x-derivatives in (†). The outcome reads

$$\begin{aligned} K^i{}_k = \mathcal{K}^i{}_k + 2(\zeta^i)_{:k} - (\zeta^i)_{y^j}(\zeta^j)_{y^k} - y^j(\zeta^i{}_{:j})_{y^k} + 2\zeta^j(\zeta^i)_{y^j y^k} \\ - 2\zeta^l\tilde{\Gamma}^i{}_{lk} + 2(\zeta^i)_{y^l}\mathcal{N}^l{}_k + y^j(\zeta^l)_{y^k}\tilde{\Gamma}^i{}_{lj} + y^j\zeta^l(\tilde{\Gamma}^i{}_{lj})_{y^k} \\ - y^j(\mathcal{N}^l{}_j)_{y^k}(\zeta^i)_{y^l} - y^j\mathcal{N}^l{}_j(\zeta^i)_{y^l y^k} - \mathcal{N}^i{}_j(\zeta^j)_{y^k} - (\zeta^i)_{y^j}\mathcal{N}^j{}_k \\ + 2\mathcal{G}^j(\zeta^i)_{y^j y^k} + 2\zeta^j(\mathcal{G}^i)_{y^j y^k}. \end{aligned}$$

It remains to check that the last ten terms on the above right-hand side actually coalesce into the single term $3\zeta^j\dot{\mathcal{A}}^i{}_{jk}$. To that end, homogeneity and Euler's theorem enable us to make the substitutions

$$\begin{aligned} y^j(\mathcal{N}^l{}_j)_{y^k} &= (\mathcal{N}^l{}_j y^j)_{y^k} - \mathcal{N}^l{}_k = 2(\mathcal{G}^l)_{y^k} - \mathcal{N}^l{}_k = \mathcal{N}^l{}_k, \\ y^j(\tilde{\Gamma}^i{}_{lj})_{y^k} &= (\tilde{\Gamma}^i{}_{lj}y^j)_{y^k} - \tilde{\Gamma}^i{}_{lk} = (\mathcal{N}^i{}_l)_{y^k} - \tilde{\Gamma}^i{}_{lk}, \\ y^j\tilde{\Gamma}^i{}_{lj} &= y^j\tilde{\Gamma}^i{}_{jl} = \mathcal{N}^i{}_j, \\ y^j\mathcal{N}^l{}_j &= 2\mathcal{G}^l. \end{aligned}$$

After some cancellations, those final ten terms consolidate into

$$3\zeta^j y^l(\tilde{\Gamma}^i{}_{jl})_{y^k} = 3\zeta^j\big((\mathcal{G}^i)_{y^j y^k} - \tilde{\Gamma}^i{}_{jk}\big) = 3\zeta^j\dot{\mathcal{A}}^i{}_{jk}.$$

1.3. Ricci curvature. The importance of Ricci curvatures (defined below, in Section 1.3.1) can be seen from the following Bonnet–Myers theorem:

Let (M,F) be a forward-complete connected Finsler manifold of dimension n. Suppose its Ricci curvature has the uniform positive lower bound

$$\mathcal{R}ic \geqslant (n-1)\lambda > 0;$$

equivalently, $y^i y^j \operatorname{Ric}_{ij}(x,y) \geqslant (n-1)\lambda F^2(x,y)$, *with* $\lambda > 0$. *Then:*

(i) *Every geodesic of length at least $\pi/\sqrt{\lambda}$ contains conjugate points.*
(ii) *The diameter of M is at most $\pi/\sqrt{\lambda}$.*
(iii) *M is in fact compact.*
(iv) *The fundamental group $\pi(M,x)$ is finite.*

The Riemannian version of this result is one of the most useful comparison theorems in differential geometry; see [Cheeger–Ebin 1975]. It was first extended to Finsler manifolds in [Auslander 1955]. See [Bao et al. 2000] for a leisurely exposition and references.

1.3.1. Ricci scalar and Ricci tensor. Our geometric definition of the Ricci curvature begins with $K(x,y,V) = V^i R_{ik} V^k / \big(g_y(V,V) - g_y(\ell,V)^2\big)$, a formula for the flag curvature (Section 1.2.1). If, with respect to g_y, the transverse edge V has unit length and is orthogonal to the flagpole y, that formula simplifies to

$$K(x,y,V) = V^i R_{ik} V^k.$$

Using g_y to measure angles and length, we take any collection of $n-1$ orthonormal transverse edges $\{e_\nu : \nu = 1, \dots, n-1\}$ perpendicular to the flagpole. They give rise to $n-1$ flags whose flag curvatures are $K(x, y, e_\nu) = (e_\nu)^i R_{ik} (e_\nu)^k$. The inclusion of $e_n := \ell = y/F$ completes our collection into a g_y orthonormal basis $\mathcal{B}$ for T_xM. Note that $K(x, y, e_\nu)$ is simply $R_{\nu\nu}$ (no sum), the (ν, ν) component of the tensor R_{ik} with respect to $\mathcal{B}$. Also, as mentioned in Section 1.2.1, $\ell^i R_{ik} \ell^k = 0$. Thus $R_{nn} = 0$ with respect to the orthonormal basis $\mathcal{B}$.

Define, geometrically, the *Ricci scalar* $\mathcal{R}ic(x, y)$ as the sum of those $n-1$ flag curvatures $K(x, y, e_\nu)$. Then

$$\mathcal{R}ic(x,y) := \sum_{\nu=1}^{n-1} R_{\nu\nu} = \sum_{a=1}^{n} R_{aa} = R^a{}_a = R^i{}_i = \frac{1}{F^2}\,(y^j\, R_j{}^i{}_{il}\, y^l) = \frac{1}{F^2}\, K^i{}_i,$$

where the last equality follows from Section 1.2.3.

Remarks. 1. The indices on R are to be manipulated by the fundamental tensor, and the latter is the Kronecker delta in the g_y orthonormal basis $\mathcal{B}$. Thus each component R_{aa} has the same numerical value as $R^a{}_a$ (no sum).

2. The fact that R_{ik} is a tensor ensures that its trace is independent of the basis used to carry that out. Hence $R^a{}_a = R^i{}_i$. Consequently, the definition of the Ricci scalar is independent of the choice of those $n-1$ orthonormal transverse edges.

3. The invariance of $R_j{}^i{}_{kl}$ under positive rescaling in y makes clear that $\mathcal{R}ic(x, y)$ has the same property. It is therefore meant to be a function on the projectivised sphere bundle of (M, F), but could just as well live on the slit tangent bundle $TM \smallsetminus 0$. In any case, being a function justifies the name *scalar*.

We obtain the *Ricci tensor* from the Ricci scalar as follows:

$$\mathrm{Ric}_{ij} := (\tfrac{1}{2} F^2\, \mathcal{R}ic)_{y^i y^j} = \tfrac{1}{2} (y^k\, R_k{}^s{}_{sl}\, y^l)_{y^i y^j}.$$

This definition, due to Akbar-Zadeh, is motivated by the fact that, when F arises from any Riemannian metric a, the curvature tensor depends on x alone and the y-Hessian in question reduces to the familiar expression ${}^aR_i{}^s{}_{sj}$, which is ${}^a\mathrm{Ric}_{ij}$.

The Ricci tensor has the same geometrical content as the Ricci scalar. It can be shown that

$$\mathcal{R}ic = \ell^i \ell^k\, \mathrm{Ric}_{ik},$$

$$\mathrm{Ric}_{ik} = g_{ik}\, \mathcal{R}ic + \tfrac{3}{4} (\ell_i\, \mathcal{R}ic_{\,;k} + \ell_k\, \mathcal{R}ic_{\,;i}) + \tfrac{1}{4} (\mathcal{R}ic_{\,;i;k} + \mathcal{R}ic_{\,;k;i}),$$

where the semicolon means $F\partial_y$. See [Bao et al. 2000].

1.3.2. Einstein metrics. We defined the Ricci scalar $\mathcal{R}ic$ as the sum of $n-1$ appropriately chosen flag curvatures. We showed in the last section that this sum depends only on the position x and the flagpole y, not on the specific $n-1$ flags with transverse edges orthogonal to y. Thus it is legitimate to think of $\mathcal{R}ic$ as $n-1$ times the average flag curvature at x in the direction y. In Riemannian geometry this is the average sectional curvature among sections spanned by y and a vector orthogonal to y; so generically the result again depends on both x and y.

Using the above perspective, it would seem quite remarkable if the said average does not depend on the flagpole y. Finsler metrics F with such a property, namely $\mathcal{R}ic = (n-1)K(x)$ for some function K on M, are called *Einstein metrics*. This nomenclature is due to Akbar-Zadeh. The reciprocal relationship (Section 1.3.1) between the Ricci scalar and the Ricci tensor tells us that

$$\mathcal{R}ic = (n-1)K(x) \quad \Longleftrightarrow \quad \mathrm{Ric}_{ij} = (n-1)K(x)\, g_{ij}.$$

Going one step further, if that average does not depend on the location x either, F is said to be *Ricci-constant*; in this case, the function K is constant.

REMARKS. 1. Every Riemannian surface is Einstein (because $\mathcal{R}ic$ equals the familiar Gaussian curvature $K(x)$), but not necessarily Ricci-constant. On the other hand, Finsler surfaces are typically not Einstein, with counterexamples provided by the Numata metrics in Section 1.2.1.

2. In dimension at least 3, a Schur type lemma ensures that every Riemannian Einstein metric is necessarily Ricci-constant. The proof uses the second Bianchi identity for the Riemann curvature tensor.

3. It is not known at the moment whether such a Schur lemma holds for Finsler Einstein metrics in general. However, if we restrict our Finsler metrics to those of Randers type, then there is indeed a Schur lemma for $\dim M \geqslant 3$; see [Robles 2003] and Section 3.3.1.

A good number of non-Riemannian Einstein metrics and Ricci-constant metrics are presented in Section 4.

It follows immediately from our geometric definition of $\mathcal{R}ic$ that every Finsler metric of constant flag curvature K must be Einstein with constant Ricci scalar $(n-1)K$. As a consistency check, we derive the same fact from the constant flag curvature criterion in Section 1.2.1, $R^i{}_k = K(\delta^i{}_k - y^i{}^g y_k/F^2)$. Indeed, tracing on i and k, and noting that $y^i{}^g y_i = y^i y^j g_{ij} = F^2$, we get $\mathcal{R}ic = R^i{}_i = (n-1)K$.

1.3.3. Known rigidity results and topological obstructions. We summarise a few basic results on Riemannian Einstein manifolds. References for this material include [Besse 1987; LeBrun–Wang 1999].

In two dimensions: Every 2-dimensional manifold M admits a complete Riemannian metric of constant (Gaussian) curvature, which is therefore Einstein. The construction involves gluing together manifolds with boundary, and is due to Thurston; see [Besse 1987] for a summary. If M is compact, there is a variational approach. In [Berger 1971], existence was established by starting with any Riemannian metric on M and deforming it conformally to one with constant curvature. For an exposition of the hard analysis behind this so-called (Melvyn) Berger problem, see [Aubin 1998].

In three dimensions: For 3-dimensional Riemannian manifolds (M, h), the Weyl conformal curvature tensor is automatically zero. This has the immediate consequence that h is Einstein if and only if it is of constant sectional curvature. Though such rigidity extends to Finsler metrics of Randers type (Section 3.3.2), it is not known whether the same holds for arbitrary Finsler metrics. The said rigidity in the Riemannian setting precludes some topological manifolds from admitting Einstein metrics. Take, for example, $M = S^2 \times S^1$. Were M to admit an Einstein metric h, the latter would perforce be of constant sectional curvature. Now M is compact, so h is complete and Hopf's classification of Riemannian space forms implies that the universal cover of M is either compact or contractible. This is a contradiction because the universal cover of M is $S^2 \times \mathbb{R}$.

In four dimensions: The basic result is the Hitchin–Thorpe Inequality [Hitchin 1974; Thorpe 1969; LeBrun 1999]: *If a smooth compact oriented 4-dimensional manifold M admits an Einstein metric, then*

$$\chi(M) \geqslant \tfrac{3}{2} |\tau(M)|.$$

Here $\chi(M)$ is the Euler characteristic, $\tau(M) = \frac{1}{3} p_1(M)$ is the signature, and $p_1(M)$ is the first Pontryagin number. The key lies in the following formulae peculiar to four dimensions [Besse 1987]:

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) \sqrt{h}\, dx,$$

$$\chi(M) = \frac{1}{8\pi^2} \int_M (|S|^2 + |W^+|^2 + |W^-|^2) \sqrt{h}\, dx,$$

where S and W are respectively the scalar curvature part and the Weyl part of the Riemann curvature tensor of h. In the second formula, the fact that h is Einstein has already been used to zero out an otherwise negative term from the integrand. The complex projective space $\mathbb{C}P^2$ has the Fubini–Study metric, which is Einstein; there, $\tau = 1$ and $\chi = 3$. On the other hand, the connected sum of 4 or more copies of $\mathbb{C}P^2$ fails the inequality and hence cannot admit any Einstein metric. Finally, the said inequality is not sufficient. See [LeBrun 1999] for compact simply connected 4-manifolds which satisfy $\chi > \frac{3}{2}|\tau|$, but which do not admit Einstein metrics.

In three dimensions, rigidity equips us with all the tools available to space forms. In particular, there are well-understood universal models. Without this structure in dimensions at least 4, the analysis of Einstein metrics becomes considerably more difficult. The saving grace in four dimensions may be attributed to the fact [Singer–Thorpe 1969] that a 4-manifold is Einstein if and only if its curvature operator (as a self-adjoint linear operator on 2-forms) commutes with the Hodge star operator. This is the property which leads us to the Hitchin–Thorpe Inequality. The tools above do not apply in dimensions greater than four, where there are no known topological obstructions.

2. Randers Metrics in Their Defining Form

2.1. Basics

2.1.1. Definition and examples. Randers metrics were introduced by Randers [1941] in the context of general relativity, and later named by Ingarden [1957]. In the positive definite category, they are Finsler spaces built from

- a Riemannian metric $a := a_{ij}\, dx^i \otimes dx^j$, and
- a 1-form $b := b_i\, dx^i$, with equivalent description $b^\sharp := b^i\, \partial_{x^i}$,

both living globally on the smooth n-dimensional manifold M. The Finsler function of a Randers metric has the simple form $F = \alpha + \beta$, where

$$\alpha(x,y) := \sqrt{a_{ij}(x)\, y^i y^j}\,, \quad \beta(x,y) := b_i(x)\, y^i.$$

Generic Randers metrics are only positively homogeneous. No Randers metric can satisfy absolute homogeneity $F(x, cy) = |c|\, F(x,y)$ unless $b = 0$, in which case it is Riemannian.

Examples. The Zermelo navigation metric (page 201) is a Randers metric with defining data

$$a_{ij} = \frac{\lambda h_{ij} + W_i W_j}{\lambda^2} \quad \text{and} \quad b_i = \frac{-W_i}{\lambda}, \quad \text{where } \lambda := 1 - h(W,W).$$

Matsumoto's slope-of-a-mountain metric (page 201) is *not* of Randers type. This is because it has the form $F = \alpha^2/(\alpha - \beta)$, where α comes from the Riemannian metric on the graph of a certain function f, and $\beta = df$.

Of the examples in Section 1.2.1, a subclass of the Numata metrics — those with constant q_{ij} (which serves as our a_{ij}) and closed 1-forms b — is Randers, while Bryant's metrics are manifestly not Randers. ♢

2.1.2. Criterion for strong convexity. In order that $F = \alpha + \beta : TM \to \mathbb{R}$ be a Finsler function, it must be nonnegative, regular, positively homogeneous, and strongly convex (Section 1.1.1). Regularity and positive homogeneity can be established by inspection. Strong convexity concerns the positive definiteness of the fundamental tensor, which for Randers metrics is

$$g_{ij} = \frac{F}{\alpha}\left(a_{ij} - \frac{{}^a y_i}{\alpha}\frac{{}^a y_j}{\alpha}\right) + \left(\frac{{}^a y_i}{\alpha} + b_i\right)\left(\frac{{}^a y_j}{\alpha} + b_j\right), \quad \text{where } {}^a y_i := a_{ij}\, y^j.$$

It turns out that the following three criteria are equivalent:

(1) The a-norm $\|b\|$ of b is strictly less than 1 on M.
(2) $F(x,y)$ is positive for all $y \neq 0$.
(3) The fundamental tensor $g_{ij}(x,y)$ is positive definite at all $y \neq 0$.

Proof: (1) $\implies$ (2). Suppose $\|b\| < 1$. A Cauchy–Schwarz type argument gives

$$\pm\beta \leqslant |\beta| = |b_i\, y^i| \leqslant \|b\|\, \|y\| < 1 \cdot \sqrt{a_{ij}\, y^i y^j} = \alpha.$$

In particular, $F = \alpha + \beta$ is positive.

(2) $\implies$ (3). (As stressed in Section 1.1.1, this is false for general Finsler metrics.) Suppose $F = \alpha + \beta$ is positive. Then so is $F_t := \alpha + t\beta$, where $|t| \leqslant 1$. Let g_t denote the fundamental tensor of F_t. Starting with the cited formula for the fundamental tensor of Randers metrics, a standard matrix identity gives

$$\det(g_t) = \left(\frac{F_t}{\alpha}\right)^{n+1} \det(a).$$

For a leisurely treatment, see [Bao et al. 2000]. This tells us that g_t has positive determinant, hence none of its eigenvalues can vanish. These eigenvalues depend continuously on t. At $t = 0$, they are all positive because $g_0 = a$ is Riemannian. If any eigenvalue were to become nonpositive, it would have to go through zero at some t, in which case $\det(g_t)$ could not possibly remain positive. Thus all eigenvalues stay positive; in particular, $g = g_1$ is positive definite.

(3) $\implies$ (2). As in Section 1.1.1, this follows from $F^2(x,y) = g_{ij}(x,y)\, y^i y^j$.

(2) $\implies$ (1). Suppose F is positive. Then $F(x, -b(x)) = \|b\|\,(1 - \|b\|)$ forces $\|b\| < 1$ wherever $b(x) \neq 0$. At points where $b(x)$ vanishes, the said inequality certainly holds.

2.1.3. Explicit formula of the spray curvature. Let ${}^aK^i{}_k$ denote the spray curvature tensor of the Riemannian metric a. Then the spray curvature tensor $K^i{}_k$ of the Randers metric $F(x,y) := \alpha + \beta = \sqrt{a_{ij}(x) y^i y^j} + b_i(x) y^i$ can be expressed in terms of ${}^aK^i{}_j$ and the quantities

$$\begin{aligned} &\mathrm{lie}_{ij} := b_{i|j} + b_{j|i}, \quad &&\mathrm{curl}_{ij} := b_{i|j} - b_{j|i}, \quad &&\theta_j := b^i \,\mathrm{curl}_{ij}, \\ &{}^a y_i := a_{ij}\, y^j, && \xi := \tfrac12 \,\mathrm{lie}_{00} - \alpha\,\theta_0, && \xi_{|0} := \tfrac12 \,\mathrm{lie}_{00|0} - \alpha\,\theta_{0|0}, \end{aligned}$$

through the use of Berwald's formula (Section 1.2.3) in a split and covariantised form. When applying Section 1.2.3, set the background metric $\mathcal{F}$ to be the Riemannian a, with Christoffel symbols ${}^a\gamma^i{}_{jk}$, and let $_{|}$ instead of $_{:}$ denote the corresponding covariant differentiation. We also need the fact, derived in [Bao et al. 2000], that $G^i = {}^aG^i + \zeta^i$ with $2\zeta^i = (y^i/F)\,\xi + \alpha\, \mathrm{curl}^i{}_0$. The resulting formula for the spray curvature (also independently obtained by Shen) reads

$$\begin{aligned} K^i{}_k = {}^aK^i{}_k &+ \text{yy-Coeff}\; y^i\, {}^a y_k + \text{yb-Coeff}\; y^i\, b_k + \delta\text{-Coeff}\; \delta^i{}_k \\ &+ \tfrac14\, \mathrm{curl}^i{}_j\, \mathrm{curl}^j{}_0\, {}^a y_k - \tfrac14\, \alpha^2\, \mathrm{curl}^i{}_j\, \mathrm{curl}^j{}_k + \tfrac34\, \mathrm{curl}^i{}_0\, \mathrm{curl}_{k0} \\ &+ \tfrac14\,(\alpha^2/F)\, y^i\, \theta_j\, \mathrm{curl}^j{}_k - \tfrac34\,(1/F)\, y^i\, \theta_0\, \mathrm{curl}_{k0} \\ &+ \tfrac12\,(\alpha/F)\, y^i\, \mathrm{curl}^j{}_0\, \mathrm{lie}_{jk} - \tfrac14\,(\alpha/F)\, y^i\, \mathrm{lie}_{j0}\, \mathrm{curl}^j{}_k \\ &+ \alpha\, \mathrm{curl}^i{}_{0|k} - \tfrac12\, \alpha\, \mathrm{curl}^i{}_{k|0} - \tfrac12\,(1/\alpha)\, \mathrm{curl}^i{}_{0|0}\, {}^a y_k \\ &+ \tfrac12\,(\alpha/F)\, y^i\, \theta_{k|0} - (\alpha/F)\, y^i\, \theta_{0|k} + \tfrac12\,(1/F)\, y^i\, \mathrm{lie}_{00|k} - \tfrac12\,(1/F)\, y^i\, \mathrm{lie}_{k0|0}. \end{aligned}$$

The three suppressed coefficients are

$$\text{yy-Coeff} := \big(\alpha/(2F^2) - 1/(4F)\big)\,\text{curl}^j{}_0\,\theta_j - \big(1/(2F^2) + 1/(4F\alpha)\big)\,\text{curl}^j{}_0\,\text{lie}_{j0} + \tfrac{1}{2}\,\theta_{0|0}/(F\alpha) - \tfrac{3}{4}\,\xi^2/(F^3\alpha) + \tfrac{1}{2}\,\xi_{|0}/(F^2\alpha),$$

$$\text{yb-Coeff} := \tfrac{1}{2}(\alpha^2/F^2)\,\text{curl}^j{}_0\theta_j - \tfrac{1}{2}(\alpha/F^2)\,\text{curl}^j{}_0\,\text{lie}_{j0} - \tfrac{3}{4}(1/F^3)\xi^2 + \tfrac{1}{2}(1/F^2)\xi_{|0},$$

$$\delta\text{-Coeff} := -\tfrac{1}{2}(\alpha^2/F)\,\text{curl}^j{}_0\theta_j + \tfrac{1}{2}(\alpha/F)\,\text{curl}^j{}_0\,\text{lie}_{j0} + \tfrac{3}{4}(1/F^2)\xi^2 - \tfrac{1}{2}(1/F)\xi_{|0}.$$

Remark. Covariant differentiation with respect to the Riemannian metric a, indicated by our vertical slash, can be lifted horizontally to $TM \smallsetminus 0$, using the nonlinear connection and the Christoffel symbols of a. The section y of π^*TM then satisfies $y^i{}_{|k} = 0$; see Section 1.1.3. So, in the above expressions, we can interpret the subscript 0 as contraction with y either before or after the vertical slash has been carried out, with no difference in the outcome.

2.2. Characterising Einstein–Randers metrics. In this section we derive necessary and sufficient conditions on a and b for the Randers metric to be Einstein. Recall that F is Einstein with Ricci scalar $\mathcal{R}ic(x)$ if and only if $K^i{}_i = \mathcal{R}ic(x)F^2$ (Section 1.3.2). We begin by assuming that this equality holds, and deduce the necessary conditions for the metric to be Einstein. Then we show that these necessary conditions are also sufficient.

Compute $K^i{}_i$ by tracing the expression for $K^i{}_k$ in Section 2.1.3 to arrive at

$$\begin{aligned} 0 &= K^i{}_i - F^2\,\mathcal{R}ic(x) \\ &= {}^a\text{Ric}_{00} + \alpha\,\text{curl}^i{}_{0|i} + \tfrac{1}{2}(n-1)\frac{\alpha}{F}\,\theta_{0|0} - \tfrac{1}{4}(n-1)\frac{1}{F}\,\text{lie}_{00|0} \\ &\qquad + \tfrac{1}{2}(n-1)\frac{\alpha}{F}\,\text{curl}^i{}_0\,\text{lie}_{i0} - \tfrac{1}{2}(n-1)\frac{\alpha^2}{F}\,\theta^i\,\text{curl}_{i0} + \tfrac{1}{2}\,\text{curl}^i{}_0\,\text{curl}_{i0} \\ &\qquad + \tfrac{1}{4}\alpha^2\,\text{curl}^{ij}\,\text{curl}_{ij} + \tfrac{3}{16}(n-1)\frac{1}{F^2}\,(\text{lie}_{00})^2 - \tfrac{3}{4}(n-1)\frac{\alpha}{F^2}\,\text{lie}_{00}\,\theta_0 \\ &\qquad + \tfrac{3}{4}(n-1)\frac{\alpha^2}{F^2}\,(\theta_0)^2 - F^2\,\mathcal{R}ic(x). \end{aligned}$$

Here, we have used the fact that ${}^aK^i{}_k$, the spray curvature of the Riemannian metric a, is related to the latter's Riemann tensor via ${}^aK^i{}_k = y^j\,{}^aR_j{}^i{}_{kl}\,y^l$, as shown in Section 1.2.3. Hence ${}^aK^i{}_i = y^j\,{}^aR_j{}^i{}_{il}\,y^l = y^j\,{}^a\text{Ric}_{jl}\,y^l = {}^a\text{Ric}_{00}$.

Multiplying this displayed equation by F^2 removes y from the denominators. The criterion for a Randers metric to be Einstein then takes the form

$$\text{Rat} + \alpha\text{Irrat} = 0, \quad \text{where} \quad \alpha := \sqrt{a_{ij}(x)y^iy^j}.$$

Here Rat and Irrat are homogeneous polynomials in y, of degree 4 and 3 respectively, whose coefficients are functions of x. Their formulae are given below.

As observed by Crampin, the displayed equation becomes $\text{Rat} - \alpha\,\text{Irrat} = 0$ if we replace y by $-y$. The two equations then effect $\text{Rat} = 0$ and $\alpha\,\text{Irrat} = 0$.

Being homogeneous in y, Irrat certainly vanishes at $y = 0$. At nonzero y, we have $\alpha > 0$ because a_{ij} is positive-definite. Hence Irrat $= 0$.

LEMMA 1. *Let* $F(x,y) := \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$ *be a Randers metric with positive definite* (*i.e., Riemannian*) a_{ij}. *Then* F *is Einstein if and only if* Rat $= 0$ *and* Irrat $= 0$.

The formulae for Rat and Irrat are

$$\begin{aligned}\mathrm{Rat} = {} & (\alpha^2+\beta^2)\,{}^a\mathrm{Ric}_{00} + 2\alpha^2\beta\,\mathrm{curl}^i{}_{0|i} + \tfrac{1}{2}(\alpha^2+\beta^2)\,\mathrm{curl}^i{}_0\,\mathrm{curl}_{i0} \\ & + \tfrac{1}{4}\alpha^2(\alpha^2+\beta^2)\,\mathrm{curl}^{ij}\,\mathrm{curl}_{ij} - (\alpha^4+6\alpha^2\beta^2+\beta^4)\,\mathcal{R}ic(x) \\ & + \tfrac{1}{2}(n-1)\big(\alpha^2\theta_{0|0} - \tfrac{1}{2}\beta\,\mathrm{lie}_{00|0} + \alpha^2\,\mathrm{curl}^i{}_0\,\mathrm{lie}_{i0} \\ & \qquad - \alpha^2\beta\theta^i\,\mathrm{curl}_{i0} + \tfrac{3}{8}(\mathrm{lie}_{00})^2 + \tfrac{3}{2}\alpha^2(\theta_0)^2\big),\end{aligned}$$

$$\begin{aligned}\mathrm{Irrat} = {} & 2\beta\,{}^a\mathrm{Ric}_{00} + (\alpha^2+\beta^2)\,\mathrm{curl}^i{}_{0|i} + \beta\,\mathrm{curl}^i{}_0\,\mathrm{curl}_{i0} \\ & + \tfrac{1}{2}\alpha^2\beta\,\mathrm{curl}^{ij}\,\mathrm{curl}_{ij} - 4\beta(\alpha^2+\beta^2)\,\mathcal{R}ic(x) \\ & + \tfrac{1}{2}(n-1)\big(\beta\theta_{0|0} - \tfrac{1}{2}\,\mathrm{lie}_{00|0} + \beta\,\mathrm{curl}^i{}_0\,\mathrm{lie}_{i0} - \alpha^2\theta^i\,\mathrm{curl}_{i0} - \tfrac{3}{2}\,\mathrm{lie}_{00}\,\theta_0\big).\end{aligned}$$

From these two expressions we will derive the preliminary form of three necessary and sufficient conditions for a Randers metric to be Einstein.

2.2.1. Preliminary form of the characterisation. Assume F is Einstein, so that Rat $= 0$ and Irrat $= 0$. For convenience abbreviate $\mathcal{R}ic(x)$ by $\mathcal{R}ic$. Then the weaker statement Rat $- \beta$ Irrat $= 0$ certainly holds, and reads

$$\begin{aligned}0 = (\alpha^2-\beta^2)\big({}^a\mathrm{Ric}_{00} + \beta\,\mathrm{curl}^i{}_{0|i} + \tfrac{1}{2}\,\mathrm{curl}^i{}_0\,\mathrm{curl}_{i0} + \tfrac{1}{4}\alpha^2\,\mathrm{curl}^{ij}\,\mathrm{curl}_{ij} - (\alpha^2+3\beta^2)\,\mathcal{R}ic \\ + \tfrac{1}{2}(n-1)(\mathrm{curl}^i{}_0\,\mathrm{lie}_{i0} + \tfrac{3}{2}(\theta_0)^2 + \theta_{0|0})\big) \\ + \tfrac{3}{16}(n-1)\big(\mathrm{lie}_{00} + 2\beta\theta_0\big)^2.\end{aligned}$$

Fix x. Considering the right-hand side as a polynomial in y, we see that $\alpha^2-\beta^2$ divides $(\mathrm{lie}_{00}+2\beta\theta_0)^2$. The polynomial $\alpha^2-\beta^2$ is irreducible, because if it were to factor — necessarily into two linear terms — its zero set would contain a hyperplane, contradicting the strong convexity condition ($\|b\| < 1$), which requires that it be positive at all $y \neq 0$ (Section 2.1.2).

Being irreducible, $\alpha^2 - \beta^2$ must divide not just the square but $\mathrm{lie}_{00} + 2\beta\theta_0$ itself. Thus there exists a scalar function $\sigma(x)$ on M such that

$$\mathrm{lie}_{00} + 2\beta\theta_0 = \sigma(x)(\alpha^2-\beta^2).$$

This is our *Basic Equation*, the first necessary condition for a Randers metric to be Einstein. Differentiating with respect to y^i and y^k gives an equivalent version:

$$\mathrm{lie}_{ik} + b_i\theta_k + b_k\theta_i = \sigma(x)(a_{ik} - b_ib_k).$$

To recover the original version, just contract this with y^iy^k. The Basic Equation is equivalent to the statement that the S-curvature of the Randers metric F is given by $S = \frac{1}{4}(n+1)\sigma(x)F$; see [Chen–Shen 2003].

Now return to the expression for $0 = \mathrm{Rat} - \beta\,\mathrm{Irrat}$. We use the Basic Equation to replace $\mathrm{lie}_{00} + 2\beta\theta_0$ with $\sigma(x)(\alpha^2 - \beta^2)$, and then divide off by a uniform factor of $\alpha^2 - \beta^2$. The result reads

$$
\begin{aligned}
{}^a\mathrm{Ric}_{00} = (\alpha^2 + 3\beta^2)\,\mathcal{R}ic - \beta\,\mathrm{curl}^j{}_{0|j} - \tfrac{1}{4}\alpha^2\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac{1}{2}\,\mathrm{curl}^j{}_0\,\mathrm{curl}_{j0} \\
- \tfrac{1}{2}(n-1)\big(\tfrac{3}{8}\sigma^2(x)(\alpha^2 - \beta^2) + \mathrm{curl}^j{}_0\,\mathrm{lie}_{j0} + \tfrac{3}{2}(\theta_0)^2 + \theta_{0|0}\big).
\end{aligned}
$$

This is the *Ricci Curvature Equation*, so named because it describes the Ricci tensor of a. We obtain the indexed version by differentiating with respect to y^i and y^k, and making use of the symmetry ${}^a\mathrm{Ric}_{ik} = {}^a\mathrm{Ric}_{ki}$.

$$
\begin{aligned}
{}^a\mathrm{Ric}_{ik} = {} & (a_{ik} + 3b_ib_k)\,\mathcal{R}ic - \tfrac{1}{2}(b_i\,\mathrm{curl}^j{}_{k|j} + b_k\,\mathrm{curl}^j{}_{i|j}) \\
& - \tfrac{1}{4}\,a_{ik}\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac{1}{2}\,\mathrm{curl}^j{}_i\,\mathrm{curl}_{jk} \\
& - \tfrac{1}{2}(n-1)\big(\tfrac{3}{8}\sigma^2(x)(a_{ik} - b_ib_k) + \tfrac{1}{2}(\mathrm{curl}^j{}_i\,\mathrm{lie}_{jk} + \mathrm{curl}^j{}_k\,\mathrm{lie}_{ji}) \\
& \qquad + \tfrac{3}{2}\,\theta_i\theta_k + \tfrac{1}{2}(\theta_{i|k} + \theta_{k|i})\big).
\end{aligned}
$$

From the Basic and Ricci Curvature Equations we derive the final characterising condition, which we call the E_{23} *Equation* (the number 23 being of some chronological significance in our research notes). Two pieces of information from the Basic Equation are required. To reduce clutter, abbreviate $\sigma(x)$ as σ. First, differentiate to obtain

$$
\mathrm{lie}_{00|0} = \sigma_{|0}(\alpha^2 - \beta^2) - \mathrm{lie}_{00}(\sigma\beta + \theta_0) - 2\beta\theta_{0|0}.
$$

Next, contract the indexed form of the Basic Equation with $y^i\,\mathrm{curl}^k{}_0$ to get

$$
\mathrm{curl}^j{}_0\,\mathrm{lie}_{j0} = -\beta\theta^j\,\mathrm{curl}_{j0} - (\theta_0)^2 - \sigma\beta\theta_0.
$$

Return to the equation $0 = \mathrm{Irrat}$. Replace the term ${}^a\mathrm{Ric}_{00}$ by the right-hand side of the Ricci Curvature Equation. Then, wherever possible, insert the expressions for lie_{00}, $\mathrm{lie}_{00|0}$ and $\mathrm{curl}^j{}_0\,\mathrm{lie}_{j0}$ given by the Basic Equation. After dividing off a factor of $\alpha^2 - \beta^2$, we obtain the E_{23} Equation:

$$
\mathrm{curl}^j{}_{0|j} = 2\,\mathcal{R}ic\,\beta + (n-1)\big(\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\theta_0 + \tfrac{1}{2}\theta^j\,\mathrm{curl}_{j0} + \tfrac{1}{4}\sigma_{|0}\big). \qquad (\mathrm{E}_{23})
$$

Again, differentiating by y^i produces the indexed version

$$
\mathrm{curl}^j{}_{i|j} = 2\,\mathcal{R}ic\,b_i + (n-1)\big(\tfrac{1}{8}\sigma^2 b_i + \tfrac{1}{2}\sigma\theta_i + \tfrac{1}{2}\theta^j\,\mathrm{curl}_{ji} + \tfrac{1}{4}\sigma_{|i}\big).
$$

The Basic, Ricci Curvature and E_{23} Equations are all necessary conditions for the Randers metric F to be Einstein. Together, they are also sufficient. In view of Lemma 1 (page 221), we can demonstrate this by showing that they imply $\mathrm{Rat} = 0 = \mathrm{Irrat}$.

Recall that we deduced the E_{23} Equation from $\mathrm{Irrat} = 0$ by

- expressing ${}^a\mathrm{Ric}_{00}$ via the Ricci Curvature Equation,
- computing lie_{00}, $\mathrm{lie}_{00|0}$ and $\mathrm{curl}^j{}_0\,\mathrm{lie}_{j0}$ with the Basic Equation, and
- dividing by a uniform factor of $\alpha^2 - \beta^2$.

Reversing these three algebraic steps allows us to recover Irrat $= 0$ from the E_{23} Equation.

Likewise, the Ricci Curvature Equation came from $\text{Rat} - \beta\, \text{Irrat} = 0$ by

- ◦ using the Basic Equation to replace $\text{lie}_{00} + 2\beta\theta_0$ with $\sigma(\alpha^2 - \beta^2)$, and
- ◦ dividing by $\alpha^2 - \beta^2$.

Again, reversing the two steps above will give us $\text{Rat} - \beta\, \text{Irrat} = 0$, whence $\text{Rat} = 0$ because $\text{Irrat} = 0$.

To summarise, *the Basic Equation, the Ricci Curvature Equation and the E_{23} Equation characterise strongly convex Einstein Randers metrics.*

In the next section we will refine the three characterising equations by showing that σ must be constant.

2.2.2. Constancy of the S-curvature. In the previous section we commented that the S-curvature of any Randers metric F satisfying the Basic Equation is given by $S = \frac{1}{4}(n+1)\sigma(x)F$ [Chen–Shen 2003]. The S-curvature is positively homogeneous of degree 1 in y. In Section 1 we demonstrated a strong preference for working with objects that are positively homogeneous of degree zero in y. That is because such objects naturally live on the projectivised sphere bundle SM as well as the larger slit tangent bundle $TM \setminus 0$. The compact parameter space provided by the sphere bundle is generally better suited for global and analytic considerations. So the object we are really interested in is not S, but S/F, which is homogeneous of degree zero in y. In this context, when we say that the S-curvature of any Randers metric satisfying the Basic Equation is *isotropic*, we mean that the quotient $\frac{S}{F}$ is a function of x alone. Similarly, when $\sigma(x)$ is constant, F is said to be a metric of *constant S-curvature*.

The following lemma plays a crucial role in establishing the constancy of the S-curvature for Einstein Randers metrics.

LEMMA 2. *The covariant derivative of the tensor* curl *associated to any Randers metric is given by*

$$\text{curl}_{ij|k} = -2b^{s}\,{}^{a}R_{ksij} + \text{lie}_{ik|j} - \text{lie}_{kj|i}.$$

PROOF. Using Ricci identities and the definition of lie_{ij} we have

$$\begin{aligned} b_{i|j|k} - b_{i|k|j} &= b^{s}\,{}^{a}R_{isjk}, \\ b_{i|k|j} + b_{k|i|j} &= \text{lie}_{ik|j}, \\ -b_{k|i|j} + b_{k|j|i} &= -b^{s}\,{}^{a}R_{ksij}, \\ -b_{k|j|i} - b_{j|k|i} &= -\text{lie}_{kj|i}, \\ b_{j|k|i} - b_{j|i|k} &= b^{s}\,{}^{a}R_{jski}. \end{aligned}$$

Summing these five equalities and applying the first Bianchi identity produces the desired formula. □

Proposition 3. *Let F be a strongly convex Randers metric on a connected manifold, satisfying the Basic Equation (with σ a function of x) and the Ricci Curvature Equation. Then F is of constant S-curvature (i.e., σ is constant) if and only if the E_{23} Equation holds.*

In practice, the E_{23} Equation has proved to be remarkably useful. Proposition 3 shows us that such efficacy is attributable to the constancy of the S-curvature. Also, since strongly convex Einstein Randers metrics satisfy the Basic, Ricci Curvature and E_{23} Equations, the following corollary is immediate.

Corollary 4. *Any strongly convex Einstein Randers metric on a connected manifold is necessarily of constant S-curvature.*

Proof of Proposition. The key is to compute a formula for the tensor $\mathrm{curl}^i{}_{0|i}$. Lemma 2 plays a pivotal role. We first contract that lemma with $a^{ik}y^j$ to obtain

$$\mathrm{curl}^i{}_{0|i} = 2b^{i\,a}\mathrm{Ric}_{i0} + \mathrm{lie}^i{}_{i|0} - \mathrm{lie}^i{}_{0|i}, \tag{$*$}$$

a preliminary formula around which all further analysis is centered. Another contracted version of Lemma 2 will be needed when we calculate a certain term in $2b^{i\,a}\mathrm{Ric}_{i0}$ and $-\,\mathrm{lie}^i{}_{0|i}$. Before plunging into details, here is an outline:

If σ is constant, we use the Basic and Ricci Curvature Equations to finish calculating the right-hand side of $(*)$. A third contracted version of Lemma 2 will come into play. The outcome is none other than the E_{23} Equation.

Conversely, if the E_{23} Equation is presumed to hold, we immediately get one formula for $\mathrm{curl}^i{}_{0|i}$. We use the Basic, Ricci Curvature, and E_{23} Equations to finish calculating the right-hand side of $(*)$, thereby deducing a second formula for $\mathrm{curl}^i{}_{0|i}$. A comparison of the two then tells us that σ is constant.

Now for the calculations. We first reexpress the terms in the right-hand side of $(*)$. The last two terms are handled using the Basic Equation:

$$\begin{aligned}\mathrm{lie}^i{}_{i|0} &= (n-\|b\|^2)\,\sigma_{|0} - \sigma\,(1-\|b\|^2)\,(\sigma\beta+\theta_0),\\ \mathrm{lie}^i{}_{0|i} &= \sigma_{|0} - \beta\, b^i\sigma_{|i} - \tfrac{1}{2}\sigma^2(n-2\|b\|^2+1)\,\beta + \tfrac{1}{2}\sigma(2\|b\|^2-n)\theta_0\\ &\qquad + \tfrac{1}{2}\beta\theta_i\theta^i + \tfrac{1}{2}\theta^i\,\mathrm{curl}_{i0} - \beta\theta^i{}_{|i} - b^i\theta_{0|i}.\end{aligned}$$

The remaining term, $2b^{i\,a}\mathrm{Ric}_{i0}$, is handled by the Ricci Curvature Equation:

$$\begin{aligned}2b^{i\,a}\mathrm{Ric}_{i0} &= \theta^i\,\mathrm{curl}_{i0} - (n{-}1)\big(\tfrac{1}{4}\,\|b\|^2\,\sigma_{|0} + \tfrac{1}{2}\,b^i(\theta_{i|0}+\theta_{0|i})\big)\\ &\qquad + \beta\Big(2\,(1+\|b\|^2)\,\mathcal{R}ic - \tfrac{1}{2}\,\mathrm{curl}^{ij}\,\mathrm{curl}_{ij} - (n{-}1)\big(\tfrac{1}{8}\,\sigma^2(3-\|b\|^2) + \tfrac{1}{4}\,b^i\,\sigma_{|i}\big)\Big).\end{aligned}$$

Next we compute the quantities $b^i\theta_{i|0}$, $b^i\theta_{0|i}$, and $\theta^i{}_{|i}$ that occur in these formulae. We will use without explicit mention the equalities

$$b_{i|j} = \tfrac{1}{2}(\mathrm{lie}_{ij} + \mathrm{curl}_{ij}) \quad\text{and}\quad b_{i|j}\,\mathrm{curl}^{ij} = \tfrac{1}{2}\,\mathrm{curl}_{ij}\,\mathrm{curl}^{ij}.$$

For $b^i\theta_{i|0}$, notice that $b^i\theta_i = b^ib^j\,\mathrm{curl}_{ij} = 0$, because curl_{ij} is skew-symmetric. Differentiating $b^i\theta_i = 0$ and using the Basic Equation gives

$$b^i\theta_{i|0} = \tfrac{1}{2}\theta_i\theta^i\beta - \tfrac{1}{2}\sigma\theta_0 - \tfrac{1}{2}\theta^i\,\mathrm{curl}_{i0}. \tag{$**$}$$

To compute $b^i\theta_{0|i}$, expand it as $\frac{1}{2}b^i(\mathrm{lie}_{ik} - \mathrm{curl}_{ik})\,\mathrm{curl}^k{}_0 + b^ib^k\,\mathrm{curl}_{i0|k}$. By Lemma 2, $b^ib^k\,\mathrm{curl}_{i0|k} = b^ib^k(-2b^j\,{}^aR_{ijk0} + \mathrm{lie}_{ik|0} + \mathrm{lie}_{i0|k})$. However, $b^ib^j\,{}^aR_{ijk0}$ vanishes because ${}^aR$ is skew-symmetric in the first two indices. Hence

$$b^i\theta_{0|i} = \tfrac{1}{2}b^i(\mathrm{lie}_{ik} - \mathrm{curl}_{ik})\,\mathrm{curl}^k{}_0 + b^ib^k(\mathrm{lie}_{ik|0} + \mathrm{lie}_{i0|k}).$$

Our calculation of $b^i\theta_{0|i}$ can now be completed in three steps as follows.

- Use the Basic Equation to remove all occurrences of the tensor lie and its covariant derivatives.
- Replace the $b^i\theta_{i|0}$ term, which resurfaces twice, by the right-hand side of $(**)$.
- Simplification leads to an expression of the form $(1-\|b\|^2)(\cdots)$ for the quantity $(1-\|b\|^2)b^i\theta_{0|i}$. Strong convexity (Section 2.1.2) allows us to divide both sides by $1-\|b\|^2$ to obtain

$$b^i\theta_{0|i} = \tfrac{1}{2}\theta_i\theta^i\beta + \tfrac{1}{2}\sigma\theta_0 - \tfrac{1}{2}\theta^i\,\mathrm{curl}_{i0} + \|b\|^2\sigma_{|0} - b^i\sigma_{|i}\beta.$$

The last term of interest, $\theta^i{}_{|i}$, is computed separately for each direction of the proof. First, if we assume that F is of constant S-curvature (σ is constant), the Ricci Curvature Equation simplifies. Carrying out an appropriate trace on Lemma 2, followed by contracting with b and rearranging, we find that

$$\theta^i{}_{|i} = \tfrac{1}{2}\,\mathrm{curl}_{ij}\,\mathrm{curl}^{ij} - b^i(2b^j\,{}^a\mathrm{Ric}_{ij} + \mathrm{lie}^j{}_{j|i} - \mathrm{lie}^j{}_{i|j}).$$

Now replace the Ricci tensor by the right-hand side of the Ricci Curvature Equation, and use the Basic Equation (with σ constant) to remove all occurrences of lie and its covariant derivatives. After simplifying we find that each term contains a factor of $1+\|b\|^2$ (not the $1-\|b\|^2$ occurring previously). Dividing out by that factor gives

$$\theta^i{}_{|i} = \tfrac{1}{2}\,\mathrm{curl}_{ij}\,\mathrm{curl}^{ij} - \big(2\,\mathcal{R}ic + \tfrac{1}{8}(n-1)\sigma^2\big)\,\|b\|^2 + \tfrac{1}{2}(n-1)\,\theta_i\theta^i. \qquad (\dagger)$$

Conversely, assume that the E_{23} Equation holds. Without the constancy of σ it is not possible to compute $\theta^i{}_{|i}$ via Lemma 2. Happily, the hypothesised E_{23} Equation saves the day:

$$\begin{aligned}\theta^i{}_{|i} &= (b_i\,\mathrm{curl}^{ij})_{|j} = \tfrac{1}{2}\,\mathrm{curl}^{ij}\,\mathrm{curl}_{ij} - b^i\,\mathrm{curl}^j{}_{i|j}\\ &= \tfrac{1}{2}\,\mathrm{curl}_{ij}\,\mathrm{curl}^{ij} - \big(2\,\mathcal{R}ic + \tfrac{1}{8}(n-1)\sigma^2\big)\,\|b\|^2 + \tfrac{1}{2}(n-1)\,\theta_i\theta^i - \tfrac{1}{4}(n-1)\,b^i\sigma_{|i}. \qquad (\ddagger)\end{aligned}$$

Remark. This is the only place where the E_{23} Equation gets used in the proof.

We are now ready to complete the proof. Consider the expressions for $2b^i\,{}^a\mathrm{Ric}_{i0}$, $\mathrm{lie}^i{}_{i|0}$, and $-\,\mathrm{lie}^i{}_{0|i}$ found on the previous page. By $(*)$, the sum of the three is $\mathrm{curl}^i{}_{0|i}$. Now substitute into this sum the formulae for $b^i\theta_{i|0}$, $b^i\theta_{0|i}$, and $\theta^i{}_{|i}$ just found, and simplify.

Under the hypothesis that F is of constant S-curvature, using $(\dagger)$ for the value of $\theta^i{}_{|i}$, we get

$$\mathrm{curl}^i{}_{0|i} = 2\,\mathcal{R}ic\,\beta + (n-1)\big(\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\theta_0 + \tfrac{1}{2}\theta^i\,\mathrm{curl}_{i0}\big).$$

This is the E_{23} Equation (with σ constant, i.e. $\sigma_{|i} = 0$).

If instead we assume that F satisfies the E_{23} Equation, and use (‡) as the value of $\theta^i{}_{|i}$, we obtain

$$\mathrm{curl}^i{}_{0|i} = 2\,\mathcal{R}ic\,\beta + (n-1)\big(\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\theta_0 + \tfrac{1}{2}\theta^i\,\mathrm{curl}_{i0} + \sigma_{|0} - \tfrac{3}{4}\,\|b\|^2\,\sigma_{|0}\big).$$

Comparing this formula for $\mathrm{curl}^i{}_{0|i}$ with the one given by the E_{23} Equation indicates that $\frac{3}{4}(1-\|b\|^2)\sigma_{|0} = 0$. Since $\|b\| < 1$, we must have $\sigma_{|0} = 0$; equivalently, all covariant derivatives $\sigma_{|i}$ vanish. But σ is a function of x, so all its partial derivatives are zero. Therefore σ is constant on the connected M. □

2.2.3. Final characterisation of Einstein–Randers metrics. In Section 2.2.1 we showed that strongly convex Einstein Randers metrics are characterised by the preliminary form of the Basic, Ricci Curvature and E_{23} Equations. The constancy of σ, established in Corollary 4, can now be used to refine these conditions to their final form.

Let's begin with the final form of the *Basic Equation*. Since the equation involves no derivatives of σ, it undergoes little cosmetic alteration. We simply write σ instead of $\sigma(x)$ to emphasise the constancy of the function:

$$\mathrm{lie}_{00} + 2\beta\,\theta_0 = \sigma\,(\alpha^2 - \beta^2).$$

Equivalently, the indexed form reads

$$\mathrm{lie}_{ik} + b_i\,\theta_k + b_k\,\theta_i = \sigma\,(a_{ik} - b_i b_k).$$

The final form of the *Ricci Curvature Equation* is derived in two steps. First, use the Basic Equation above to remove the tensor lie and its covariant derivatives from the preliminary expression in Section 2.2.1. Then replace the $\mathrm{curl}^j{}_{0|j}$ term with the formula given by the E_{23} Equation in Section 2.2.1. Keep in mind that the covariant derivatives $\sigma_{|i}$ vanish because σ is constant. After simplifying, the result is

$$\begin{aligned}{}^a\mathrm{Ric}_{00} = (\alpha^2+\beta^2)\,\mathcal{R}ic(x) - \tfrac{1}{4}\alpha^2\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac{1}{2}\,\mathrm{curl}^j{}_0\,\mathrm{curl}_{j0}& \\ -(n-1)\big(\tfrac{1}{16}\sigma^2(3\alpha^2-\beta^2) + \tfrac{1}{4}(\theta_0)^2 + \tfrac{1}{2}\theta_{0|0}\big)&.\end{aligned}$$

Differentiating by y^i and y^k and applying the symmetry of ${}^a\mathrm{Ric}_{ik}$ produces the indexed version

$$\begin{aligned}{}^a\mathrm{Ric}_{ik} = (a_{ik}+b_ib_k)\,\mathcal{R}ic(x) - \tfrac{1}{4}a_{ik}\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac{1}{2}\,\mathrm{curl}^j{}_i\,\mathrm{curl}_{jk}& \\ -(n-1)\big(\tfrac{1}{16}\sigma^2(3a_{ik}-b_ib_k) + \tfrac{1}{4}\theta_i\theta_k + \tfrac{1}{4}(\theta_{i|k}+\theta_{k|i})\big)&.\end{aligned}$$

The constancy of σ updates the *E_{23} Equation* to

$$\mathrm{curl}^j{}_{0|j} = 2\,\mathcal{R}ic(x)\beta + (n-1)\big(\tfrac{1}{8}\sigma^2\beta + \tfrac{1}{2}\sigma\theta_0 + \tfrac{1}{2}\theta^j\,\mathrm{curl}_{j0}\big),$$

or

$$\mathrm{curl}^j{}_{i|j} = 2\,\mathcal{R}ic(x)\,b_i + (n-1)\big(\tfrac{1}{8}\sigma^2 b_i + \tfrac{1}{2}\sigma\theta_i + \tfrac{1}{2}\theta^j\,\mathrm{curl}_{ji}\big).$$

Remark. The final forms of the Basic, E_{23} and Ricci Curvature Equations are equivalent to the preliminary forms. In the case of the Basic and E_{23} Equations, this follows immediately from the constancy of σ. As for the Ricci Curvature Equation, its final form was deduced from the preliminary form by replacing the terms lie_{00}, $\mathrm{lie}_{00|0}$, $\mathrm{curl}^j{}_0\,\mathrm{lie}_{j0}$ and $\mathrm{curl}^j{}_{0|j}$ with the expressions given by the Basic and E_{23} Equations. Reversing this algebraic substitution resurrects the preliminary form of the Ricci Curvature Equation.

We saw in Section 2.2.1 that the preliminary forms characterise strongly convex Einstein metrics. Therefore the final forms of the Basic, Ricci Curvature and E_{23} Equations are necessary and sufficient conditions for the metric to be Einstein. Moreover, Proposition 3 assures us that, with σ constant, the Basic and Ricci Curvature Equations alone do the trick.

Theorem 5 (Einstein Characterisation). *Let $F = \alpha + \beta$ be a strongly convex Randers metric on a smooth manifold M of dimension $n \geqslant 2$, with $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$. Then (M, F) is Einstein with Ricci scalar $\mathcal{R}ic(x)$ if and only if the Basic Equation*

$$\mathrm{lie}_{ik} + b_i\,\theta_k + b_k\,\theta_i = \sigma(a_{ik} - b_ib_k)$$

and the Ricci Curvature Equation

$$\begin{aligned}{}^a\mathrm{Ric}_{ik} = (a_{ik} + b_ib_k)\,\mathcal{R}ic(x) - \tfrac{1}{4}\,a_{ik}\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac{1}{2}\,\mathrm{curl}^j{}_i\,\mathrm{curl}_{jk}\\ -(n-1)\big(\tfrac{1}{16}\,\sigma^2(3a_{ik} - b_ib_k) + \tfrac{1}{4}\,\theta_i\theta_k + \tfrac{1}{4}(\theta_{i|k} + \theta_{k|i})\big)\end{aligned}$$

are satisfied for some constant σ.

Tracing the Basic Equation tells us that σ, besides being related to the S-curvature (Section 2.2.2), also has the geometrically significant value

$$\sigma = \frac{2\,\mathrm{div}\,b^\sharp}{n - \|b\|^2},$$

where $\mathrm{div}\,b^\sharp := b^i{}_{|i}$ is the divergence of the vector field $b^\sharp := b^i\partial_{x^i}$.

Remark. The Basic Equation, Ricci Curvature Equation, and E_{23} Equation are tensorial equations, and highly nonlinear due to the presence of ${}^a\mathrm{Ric}_{ik}$. They constitute a coupled system of second order partial differential equations.

Their redeeming feature is being polynomial in the tangent space coordinates y^i, whereas the original Einstein criterion is not (unless $b = 0$). This greatly reduces computational complexity. While testing Randers metrics to see whether they satisfy the Einstein criterion $K^i{}_i = F^2\,\mathcal{R}ic(x)$, we encountered cases in which the software Maple was unable to complete computations of $K^i{}_i$. So, for those examples we could not directly verify the Einstein criterion. Maple was able, however, to work efficiently with the three *indirect* characterising equations.

2.3. Characterising constant flag curvature Randers metrics

2.3.1. The result. Recall from Section 1.3.1 that the Ricci scalar $\mathcal{R}ic$ is the sum of $n-1$ appropriately chosen flag curvatures. Thus, Finsler metrics of constant flag curvature K necessarily have constant Ricci scalar $(n-1)K$, and are therefore Einstein. By Corollary 4, they must have constant S-curvature.

Computationally, the equation $K^i{}_k = KF^2(\delta^i{}_k - \ell^i \ell_k)$ characterizing constant flag curvature is even more challenging than the Einstein equation, which already gives Maple trouble (see end of previous section). The need for machine-friendly characterisation equations is acute.

Partly motivated by this, Randers metrics of constant flag curvature have been characterised in [Bao–Robles 2003]. The same conclusion was simultaneously obtained in [Matsumoto–Shimada 2002], albeit by a different method. The result is similar to that described in Theorem 5.

Theorem 6 (Constant flag curvature characterisation). *Let $F = \alpha+\beta$ be a strongly convex Randers metric on a smooth manifold M of dimension $n \geqslant 2$, with $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. Then (M,F) is of constant flag curvature K if and only if there exists a constant σ such that the Basic Equation*

$$\mathrm{lie}_{ik} + b_i\theta_k + b_k\theta_i = \sigma\,(a_{ik} - b_i b_k)$$

holds and the Riemann tensor of a satisfies the Curvature Equation

$$\begin{aligned} {}^a\!R_{hijk} = \xi\,(a_{ij}\,a_{hk} - a_{ik}\,a_{hj}) &- \tfrac14\,a_{ij}\,\mathrm{curl}^t{}_h\,\mathrm{curl}_{tk} + \tfrac14\,a_{ik}\,\mathrm{curl}^t{}_h\,\mathrm{curl}_{tj} \\ &+ \tfrac14\,a_{hj}\,\mathrm{curl}^t{}_i\,\mathrm{curl}_{tk} - \tfrac14\,a_{hk}\,\mathrm{curl}^t{}_i\,\mathrm{curl}_{tj} \\ &- \tfrac14\,\mathrm{curl}_{ij}\,\mathrm{curl}_{hk} + \tfrac14\,\mathrm{curl}_{ik}\,\mathrm{curl}_{hj} + \tfrac12\,\mathrm{curl}_{hi}\,\mathrm{curl}_{jk}, \end{aligned}$$

with $\xi := (K - \frac{3}{16}\sigma^2) + (K + \frac{1}{16}\sigma^2)\,\|b\|^2 - \frac14\,\theta^i\theta_i$.

2.3.2. Utility. Theorem 6 provides an *indirect* but efficient way for checking whether a given Randers metric is of constant flag curvature.

As we mentioned in Section 2.2.3, tracing the Basic Equation reveals that the constant σ, whenever it exists, is equal to $2\,\mathrm{div}\,b^\sharp/(n-\|b\|^2)$. Hence, this quotient is one of the first items we must compute. If the answer is not a constant, it is pointless to proceed any further.

If the computed value for σ is constant, surviving the Basic Equation constitutes the next checkpoint. After that, we solve the Curvature Equation for K, and see whether it is constant.

Example (Finslerian Poincaré disc). This metric is implicit in [Okada 1983], and is extensively discussed in [Bao et al. 2000]. Let r and θ denote polar coordinates on the open disc of radius 2 in $\mathbb{R}^2$. The Randers metric in question is defined by the following Riemannian metric a and 1-form b:

$$a = \frac{dr\otimes dr + r^2 d\theta\otimes d\theta}{(1-\frac14 r^2)^2}\,, \qquad b = \frac{r\,dr}{(1+\frac14 r^2)(1-\frac14 r^2)}\,.$$

Note that a is the Riemannian Poincaré model of constant sectional curvature -1, and $b = d\log\big((4{+}r^2)/(4{-}r^2)\big)$ is exact.

This metric is interesting because its geodesic trajectories agree with those of the Riemannian Poincaré model a. However, as shown in [Bao et al. 2000], the travel time from the boundary to the center is finite ($\log 2$ seconds), while the return trip takes infinite time! We summarise below three key steps in ascertaining that our Randers metric has constant flag curvature $K = -\frac{1}{4}$.

- The value of the S-curvature σ is computed to be 2.
- Since b is exact, it is closed and curl $= 0$. In particular, $\theta = 0$ and $\mathrm{lie}_{ik} = 2b_{i|k}$. The Basic Equation is shown to hold with $\sigma = 2$.
- Since curl $= 0$ and a has constant curvature -1, the Curvature Equation reduces to $(K + \frac{1}{4})(1 + \|b\|^2) = 0$, which gives $K = -\frac{1}{4}$. ♢

Another byproduct of Theorem 6 is the *corrected* Yasuda–Shimada theorem, proved in [Bao–Robles 2003; Matsumoto–Shimada 2002], and discussed near the end of [Bao et al. 2003]. That theorem characterises, within the family of Randers metrics satisfying $\theta = 0$, those that have constant flag curvature K. (Those with $K > 0$ and $\theta = 0$ were *classified* in [Bejancu–Farran 2002; 2003].)

Lastly, Theorem 6 provides an important link in the complete classification of constant flag curvature Randers metrics; see [Bao et al. 2003].

2.3.3. Comparing with the Einstein case. In Sections 1.3.2 and 2.3.1, we pointed out that Finsler metrics of constant flag curvature are necessarily Einstein. In particular, Randers metrics characterised by Theorem 6 should satisfy the criteria stipulated in Theorem 5. This is indeed the case. The two Basic Equations are identical; and tracing the Curvature Equation of Theorem 6 produces the Ricci Curvature Equation of Theorem 5.

In [Bao–Robles 2003], the characterisation result includes a third condition, called the CC(23) Equation, which gives a formula for the covariant derivative $\mathrm{curl}_{ij|k}$ of curl. We excluded this equation from our statement of Theorem 6 because, like the E_{23} Equation, it is automatically satisfied whenever the Basic and Curvature Equations hold with constant σ.

Proposition 7. *Suppose F is a strongly convex Randers metric on a connected manifold, satisfying the preliminary form of the Basic and Curvature Equations described in* [Bao–Robles 2003]. *Then F is of constant S-curvature* (σ *is constant*) *if and only if the CC*(23) *Equation of* [Bao–Robles 2003] *holds.*

We omit the proof because it is structurally similar to the one we gave for Proposition 3. For *constant* σ, the *CC*(23) *Equation* reads:

$$\mathrm{curl}_{ij|k} = a_{ik}\left((2K + \tfrac{1}{8}\sigma^2)\,b_j + \tfrac{1}{2}\,\mathrm{curl}^h{}_j\,\theta_h + \tfrac{1}{2}\,\sigma\,\theta_j\right) - a_{jk}\left((2K + \tfrac{1}{8}\sigma^2)\,b_i + \tfrac{1}{2}\,\mathrm{curl}^h{}_i\,\theta_h + \tfrac{1}{2}\,\sigma\,\theta_i\right).$$

Tracing the CC(23) Equation on its first and third indices gives the E_{23} Equation (with σ constant) in Section 2.2.1.

3. Randers Metrics Through Zermelo Navigation

3.1. Zermelo navigation. Zermelo [1931] posed and answered the following question (see also [Carathéodory 1999]): Consider a ship sailing on the open sea in calm waters. Suppose a mild breeze comes up. How must the ship be steered in order to reach a given destination in the shortest time?

Zermelo assumed that the open sea was $\mathbb{R}^2$ with the flat/Euclidean metric. Recently, Shen generalised the problem to the setting where the sea is an arbitrary Riemannian manifold (M, h). Shen [2002] finds that, when the wind is time-independent, the paths of shortest time are the geodesics of a Randers metric. This will be established in Section 3.1.1. For the remainder of this section, we develop some intuition by considering the problem on the infinitesimal scale.

Given any Riemannian metric h on a differentiable manifold M, denote the corresponding norm-squared of tangent vectors $y \in T_xM$ by

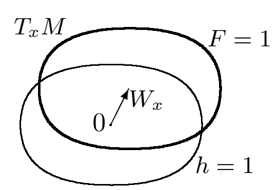
$$|y|^2 := h_{ij}\, y^i y^j = h(y, y).$$

Think of $|y|$ as measuring the *time* it takes, using an engine with a fixed power output, to travel from the base-point of the vector y to its tip. Note the symmetry property $|-y| = |y|$.

The unit tangent sphere in each T_xM consists of all those tangent vectors u such that $|u| = 1$. Now introduce a vector field W such that $|W| < 1$, the spatial velocity vector of our mild wind on the Riemannian landscape (M, h). Before W sets in, a journey from the base to the tip of any u would take 1 unit of time, say, 1 second. The effect of the wind is to cause the journey to veer off course (or merely off target if u is collinear with W). Within the same 1 second, we traverse not u but the resultant $v = u + W$ instead.

As an example, suppose $|W| = \frac{1}{2}$. If u points along W (that is, $u = 2W$), then $v = \frac{3}{2}u$. Alternatively, if u points opposite to W ($u = -2W$), then $v = \frac{1}{2}u$. In these two scenarios, $|v|$ equals $\frac{3}{2}$ and $\frac{1}{2}$ instead of 1. So, with the wind present, our Riemannian metric h no longer gives the travel time along vectors. This prompts the introduction of a function F on the tangent bundle TM, to keep track of the travel time needed to traverse tangent vectors y under windy conditions. For all those resultants $v = u + W$ mentioned above, we have $F(v) = 1$. In other words, within each tangent space T_xM, the unit sphere of F is simply the W-translate of the unit sphere of h. Since this W-translate is no longer centrally symmetric about the origin 0 of T_xM, the Finsler function F cannot be Riemannian.

Given any Finsler manifold (M, F), the *indicatrix* in each tangent space is $S_x(F) := \{y \in T_xM : F(x, y) = 1\}$. The indicatrices of h and the Randers metric F with navigation data (h, W) are related by a rigid translation: $S_x(F) = S_x(h) + W_x$. In particular, the Randers indicatrix is simply an ellipse centered at the tip of W_x.



In the following section we will algebraically derive an expression for F, showing that it is a Randers metric. Then we demonstrate that the paths of shortest time are indeed the geodesics of this F.

3.1.1. Algebraic and calculus-of-variations aspects. We return to the earlier discussion and consider those $u \in T_xM$ with $|u| = 1$; equivalently, $h(u,u) = 1$. Into this, we substitute $u = v - W$ and then $h(v,W) = |v|\,|W|\cos\theta$. Introducing the abbreviation $\lambda := 1 - |W|^2$, we have

$$|v|^2 - (2\,|W|\cos\theta)\,|v| - \lambda = 0.$$

Since $|W| < 1$, the resultant v is never zero, hence $|v| > 0$. This leads to $|v| = |W|\cos\theta + \sqrt{|W|^2\cos^2\theta + \lambda}$, which we abbreviate as $p + q$. Since $F(v) = 1$, we see that

$$F(v) = 1 = |v|\frac{1}{q+p} = |v|\frac{q-p}{q^2-p^2} = \frac{\sqrt{[h(W,v)]^2 + |v|^2\lambda}}{\lambda} - \frac{h(W,v)}{\lambda}\,.$$

It remains to deduce $F(y)$ for an arbitrary $y \in TM$. Note that every nonzero y is expressible as a positive multiple c of some v with $F(v) = 1$. For $c > 0$, traversing $y = cv$ under the windy conditions should take c seconds. Consequently, F is positively homogeneous: $F(y) = cF(v)$. Using this homogeneity and the formula derived for $F(v)$, we find that

$$F(y) = \frac{\sqrt{[h(W,y)]^2 + |y|^2\lambda}}{\lambda} - \frac{h(W,y)}{\lambda}\,.$$

Here, $F(y)$ abbreviates $F(x,y)$; the basepoint x has been suppressed temporarily.

As promised, F is a Randers metric. Namely, it has the form $F(x,y) = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$, where a is a Riemannian metric and b a differential 1-form. Explicitly,

$$a_{ij} = \frac{h_{ij}}{\lambda} + \frac{W_i}{\lambda}\frac{W_j}{\lambda}\,, \qquad b_i = \frac{-W_i}{\lambda}\,.$$

Here $W_i := h_{ij}W^j$ and $\lambda = 1 - W^iW_i$. In particular, there is a canonical Randers metric associated to each Zermelo navigation problem with data (h, W). Incidentally, the inverse of a is given by

$$a^{ij} = \lambda(h^{ij} - W^iW^j), \quad \text{and} \quad b^i := a^{ij}b_j = -\lambda W^i.$$

Under the influence of W, the most efficient navigational paths are no longer the geodesics of the Riemannian metric h; instead, they are the geodesics of the Finsler metric F. To see this, let $x(t)$, for $t \in [0,\tau]$, be a curve in M from point p to point q. Return to our imaginary ship sailing about M, with velocity vector u but not necessarily with constant speed. If the ship is to travel along the curve $x(t)$ while the wind blows, the captain must continually adjust the ship's direction $u/|u|$ so that the resultant $u + W$ is tangent to $x(t)$. The travel time along any infinitesimal segment $\dot{x}\,dt$ of the curve is $F(x, \dot{x}\,dt) = F(x,\dot{x})\,dt$, because as explained above it is the positively homogeneous F (not h) that keeps track of travel times. The captain's task is to select a path $x(t)$ from p to q that minimises the total travel time

$$\int_0^\tau F(x, \dot{x})\, dt.$$

This quantity is independent of orientation-preserving parametrisations due to the positive homogeneity of F and the change-of-variables theorem. Such an efficient path is precisely a geodesic of the Finsler metric F, which is said to have solved Zermelo's problem of navigation under the external influence W.

Let's look at some 2-dimensional examples. Being in two dimensions, we revert to the common notation of denoting position coordinates by x, y rather than x^1, x^2, and components of tangent vectors by u, v rather than y^1, y^2.

EXAMPLE (MINKOWSKI SPACE). Consider $\mathbb{R}^2$ equipped with the Euclidean metric $h_{ij} = \delta_{ij}$. Let $W = (p, q)$ be a constant vector field. The resulting Randers metric is of Minkowski type, and is given by

$$F(x, y; u, v) = \frac{\sqrt{(pu+qv)^2 + (u^2+v^2)\big(1-(p^2+q^2)\big)}}{1-(p^2+q^2)} + \frac{-(pu+qv)}{1-(p^2+q^2)}.$$

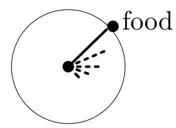
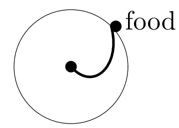
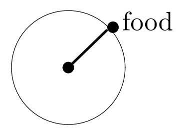
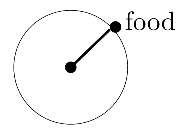
The condition $|W|^2 = p^2 + q^2 < 1$ ensures that F is strongly convex. The geodesics of both h and F are straight lines. Thus, when navigating on flat-land under the influence of a constant wind, the correct strategy is to steer the ship so that it travels along a straight line. This means the captain should aim the ship, not straight toward the desired destination, but slightly off course with a velocity V, selected so that $V + W$ points at the destination. $\diamondsuit$

EXAMPLE (SHEN'S FISH POND). Fish are kept in a pond with a rotational current. The pond occupies the unit disc in $\mathbb{R}^2$, and h_{ij} is again the Euclidean metric δ_{ij}. The current's velocity field is $W := -y\partial_x + x\partial_y$, which describes a counterclockwise circulation of angular speed 1. These navigation data give rise to the Randers metric [Shen 2002]

$$F(x, y; u, v) = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2} + \frac{-yu + xv}{1 - x^2 - y^2},$$

with $|W|^2 = x^2 + y^2 < 1$. A feeding station is placed at a fixed location along the perimeter of the pond. The fish, eager to get to the food, swim along geodesics of F. As observed by a stationary viewer at the pond's edge, these geodesics are spirals because it is the fish's instinct to approach the perimeter quickly, where it can obtain the most help from the current's linear velocity (which equals angular speed times radial distance). An experienced fish will aim itself with a velocity V, selected so that $V + W$ is tangent to the spiral. $\diamondsuit$

For the pond described above, Shen has found numerically that the geodesics of F are spirals as expected. Now suppose that instead of fish we have an excited octopus which secretes ink as it swims toward the feeding station. The resulting ink trail comoves with the swirling water. Its exact shape can be deduced by reexpressing the above spiral with respect to a frame which is comoving with the water. Shen agrees with this trend of thought and finds, somewhat surprisingly, that the trail in question is a straight ray.

fish pond	ink trail	geodesic in space
counterclockwise current		
no current		

3.1.2. Inverse problem. A question naturally arises: Can every strongly convex Randers metric be realised through the perturbation of some Riemannian metric h by some vector field W satisfying $|W| < 1$?

The answer is yes. Indeed, let us be given an arbitrary Randers metric F with Riemannian metric a and differential 1-form b, satisfying $\|b\|^2 := a^{ij}\, b_i b_j < 1$. Set $b^i := a^{ij}\, b_j$ and $\varepsilon := 1 - \|b\|^2$. Construct h and W as follows:

$$h_{ij} := \varepsilon\,(a_{ij} - b_i b_j), \quad W^i := -b^i/\varepsilon.$$

Note that F is Riemannian if and only if $W = 0$, in which case $h = a$. Also,

$$W_i := h_{ij}\, W^j = -\varepsilon\, b_i.$$

Using this, it can be directly checked that perturbing h by the stipulated W gives back the Randers metric we started with. Furthermore,

$$|W|^2 := h_{ij}\, W^i W^j = a^{ij}\, b_i b_j =: \|b\|^2 < 1.$$

Incidentally, the inverse of h_{ij} is

$$h^{ij} = \varepsilon^{-1}\, a^{ij} + \varepsilon^{-2}\, b^i b^j.$$

This h^{ij}, together with W^i, defines a Cartan metric F^* of Randers type on the cotangent bundle T^*M. A comparison with [Hrimiuc–Shimada 1996] shows that F^* is the Legendre dual of the Finsler–Randers metric F on TM. It is remarkable that the Zermelo navigation data of any strongly convex Randers metric F is so simply related to its Legendre dual. See also [Ziller 1982; Shen 2002; 2004].

We summarise:

Proposition 8. *A strongly convex Finsler metric F is of Randers type if and only if it solves the Zermelo navigation problem on some Riemannian manifold (M,h), under the influence of a wind W with $h(W,W) < 1$. Also, F is Riemannian if and only if $W = 0$.*

Example (Finslerian Poincaré disc). As an illustration of the inverse procedure, we apply it to the Randers metric F that describes the Finslerian Poincaré disc (Section 2.3.2). With r and θ denoting polar coordinates on the open disc of radius 2 in $\mathbb{R}^2$, the Randers metric in question is defined by

$$a = \frac{dr\otimes dr + r^2 d\theta\otimes d\theta}{(1-\frac{1}{4}r^2)^2}, \quad b = \frac{r\,dr}{(1+\frac{1}{4}r^2)(1-\frac{1}{4}r^2)}.$$

The underlying Zermelo navigation data is

$$h = \frac{(1-\frac{1}{4}r^2)^2\,dr\otimes dr + r^2(1+\frac{1}{4}r^2)^2\,d\theta\otimes d\theta}{(1+\frac{1}{4}r^2)^4}, \quad W = \frac{-r(1+\frac{1}{4}r^2)}{1-\frac{1}{4}r^2}\partial_r.$$

It turns out that this Riemannian landscape h on which the navigation takes place is flat (!), and the associated 1-form

$$W^\flat = -r\,\frac{1-\frac{1}{4}r^2}{1+\frac{1}{4}r^2}\,dr = -8\,d\frac{r^2}{(4+r^2)^2}$$

is exact. The wind here is blowing radially toward the center of the disc, and its strength decreases to zero there.

The geodesics of h are straight lines. Those of F have been analysed in detail in [Bao et al. 2000]. Their *trajectories* coincide with those of the Riemannian Poincaré model: straight rays to and from the origin, and circular arcs intersecting the rim of the disc at Euclidean right angles. ♢

3.1.3. General relation between two covariant derivatives. Our goal in Section 3.2 will be to reexpress the Einstein characterisation of Theorem 5 in terms of the navigation data (h,W). To that end, it is helpful to first relate the covariant derivative $b_{i|j}$ of b (with respect to a) to the covariant derivative $W_{i:j}$ of W (with respect to h). Let ${}^a\gamma^i{}_{ij}$ and ${}^h\gamma^i{}_{jk}$ denote the respective Christoffel symbols of the Riemannian metrics a and h. We have

$$b_{i|j} = b_{i,x^j} - b_s\,{}^a\gamma^s{}_{ij} \quad \text{and} \quad W_{i:j} = W_{i,x^j} - W_s\,{}^h\gamma^s{}_{ij}.$$

Since

$${}^a\gamma^i{}_{ij} = ({}^aG^i)_{y^j y^k}, \qquad {}^h\gamma^i{}_{ij} = ({}^hG^i)_{y^j y^k},$$

it suffices to compare the geodesic spray coefficients

$${}^aG^i := \tfrac{1}{2}\,{}^a\gamma^i{}_{jk}y^jy^k =: \tfrac{1}{2}\,{}^a\gamma^i{}_{00} \quad \text{and} \quad {}^hG^i := \tfrac{1}{2}\,{}^h\gamma^i{}_{jk}y^jy^k =: \tfrac{1}{2}\,{}^h\gamma^i{}_{00}.$$

The tool that effects this comparison is Rapcsák's identity (Section 1.2.2).

The symbol α denotes the Finsler norm associated to the Riemannian metric a: $\alpha^2(x,y) := a_{ij}(x)y^iy^j = a(y,y)$. According to Section 1.2.2, the user-friendly form of Rapcsák's identity gives

$$^aG^i = {}^hG^i + \tfrac{1}{4}\, a^{ij}\big((\alpha^2{}_{:0})_{y^j} - 2\alpha^2{}_{:j}\big).$$

Using the formula $a_{ij} = \frac{1}{\lambda} h_{ij} + \frac{1}{\lambda^2} W_i W_j$ derived in Section 1.3.1, we find that

$$a^{ij} = \lambda(h^{ij} - W^iW^j) \quad \text{and} \quad \alpha^2 = \frac{1}{\lambda} h_{00} + \frac{1}{\lambda^2} W_0{}^2.$$

Here, $W_i := h_{ij}W^j$, $\lambda := 1 - W^iW_i$, and the formula for the inverse a^{ij} is ascertained by inspection. A straightforward but tedious computation of the right-hand side of Rapcsák's identity yields

$$^aG^i = {}^hG^i + \zeta^i,$$

where (using $W^{s:i}$ to abbreviate $W^s{}_{:r}h^{ri}$)

$$\zeta^i := \frac{1}{\lambda} y^i W^s W_{s:0} + \tfrac{1}{2} W^i W_{0:0} + \Big(\frac{1}{2\lambda} h_{00} + \frac{1}{\lambda^2} W_0{}^2\Big)(W^iW^sW^tW_{s:t} - W_sW^{s:i}) + \frac{1}{2\lambda} W_0\big(W^iW^s(W_{s:0} + W_{0:s}) + (W^i{}_{:0} - W_0{}^{:i})\big).$$

Differentiating with respect to y^j, y^k gives $^a\gamma^i{}_{jk} = {}^h\gamma^i{}_{jk} + (\zeta^i)_{y^jy^k}$, whence $b_{j|k} = b_{j:k} - b_i(\zeta^i)_{y^jy^k}$. Into the right-hand side we substitute $b_s = -(1/\lambda)W_s$ (Section 1.3.1), resulting in a formula for $b_{j|k}$ in terms of the covariant derivatives of W with respect to h. For later purposes, it is best to split the answer into its symmetric and anti-symmetric parts,

$$b_{j|k} = \tfrac{1}{2}(b_{j|k} + b_{k|j}) + \tfrac{1}{2}(b_{j|k} - b_{k|j}) =: \tfrac{1}{2}\,\mathrm{lie}_{jk} + \tfrac{1}{2}\,\mathrm{curl}_{jk},$$

and to introduce the abbreviations

$$\mathcal{L}_{jk} := W_{j:k} + W_{k:j}, \quad \mathcal{C}_{jk} := W_{j:k} - W_{k:j}.$$

Then the said computation gives $b_{j|k} = \frac{1}{2}\,\mathrm{lie}_{jk} + \frac{1}{2}\,\mathrm{curl}_{jk}$, with

$$\mathrm{lie}_{jk} = -\mathcal{L}_{jk} - \Big(\frac{1}{\lambda} h_{jk} + \frac{2}{\lambda^2} W_jW_k\Big) W^sW^t\mathcal{L}_{st} + \frac{1+|W|^2}{\lambda^2} W^i(W_{i:j}W_k + W_{i:k}W_j) - \frac{1}{\lambda}(W_jW_{k:i} + W_kW_{j:i})W^i,$$

$$\mathrm{curl}_{jk} = -\frac{1}{\lambda}\mathcal{C}_{jk} + \frac{2}{\lambda^2} W^i(W_{i:j}W_k - W_{i:k}W_j).$$

Observe that since $\mathrm{curl}_{jk} = \partial_{x^k} b_j - \partial_{x^j} b_k$ and $\mathrm{lie}_{jk} = b^i\,\partial_{x^i} a_{jk} + a_{ik}\,\partial_{x^j} b^i + a_{ji}\,\partial_{x^k} b^i$ (where $b^i = -\lambda W^i$), the last two conclusions could have been obtained without relying on the explicit formula of $^a\gamma$. In any case, the relation above between $b_{j|k}$ and the covariant derivatives of W is valid *without* any assumption on b.

3.2. Navigation description of curvature conditions. Theorem 5 (p. 227) characterises Einstein Randers metrics via the defining Riemannian metric a and 1-form b. It says that a Randers metric F is Einstein if and only if both the Basic and Ricci Curvature Equations hold with constant σ (or S-curvature). Though this is a substantial improvement over the Einstein criterion $K^i{}_i = \mathcal{R}ic(x)F^2$, most notably in the realm of computation, the characterisation does little to describe the *geometry* of these metrics. Surprisingly, the breakthrough lies in a change of *dependent* variables. We find that replacing the defining data (a, b) by the navigation data (h, W) (discussed in Section 3.1.2) yields a breviloquent geometric description of Einstein Randers metrics. Explicitly, this change of variables reveals that the Riemannian metric h must be an Einstein metric itself, and the vector field W an infinitesimal homothety of h. The next two subsections are devoted to developing this "navigation description".

3.2.1. Consequences of the Basic Equation. Our first step is to derive the navigation version of the Basic Equation $\text{lie}_{ik} + b_i\theta_k + b_k\theta_i = \sigma(a_{ik} - b_ib_k)$ of page 226. For that we replace a_{ik} by $(1/\lambda)h_{ik} + (1/\lambda^2)W_iW_k$, b_i by $-(1/\lambda)W_i$, and lie_{ik} by the expression derived on the previous page. We also use the formula for curl_{jk} there to compute $\theta_k := b^j \,\text{curl}_{jk}$. Since $b^j := a^{js}b_s = -\lambda W^j$, we get

$$\theta_k = \mathcal{T}_k - \frac{1}{\lambda}(W^iW^j\mathcal{L}_{ij})W_k + \frac{2}{\lambda}|W|^2W^iW_{i:k}.$$

Here, an abbreviation has been introduced for the ubiquitous quantity

$$\mathcal{T}_k := W^j(W_{j:k} - W_{k:j}) = W^j\mathcal{C}_{jk}.$$

These manoeuvres, followed by some rearranging, convert the Basic Equation to $\lambda\mathcal{L}_{ik} + \mathcal{L}(W, W)h_{ik} = -\sigma h_{ik}$, where $\mathcal{L}(W, W)$ stands for $\mathcal{L}_{st}W^sW^t$. Contracting with W^i, W^k and using $\lambda := 1 - |W|^2$ shows that $\mathcal{L}(W, W) = -\sigma|W|^2$. Consequently, the navigation version of the Basic Equation is

$$\mathcal{L}_{ik} := W_{i:k} + W_{k:i} = -\sigma h_{ik}, \quad \text{that is,} \quad \mathcal{L}_W h = -\sigma h.$$

We name this the $\mathcal{L}_W$ *Equation.* It says that W is an *infinitesimal homothety* of h; see [Kobayashi–Nomizu 1996, p. 309]. In this equation,

σ must be zero whenever h is not flat.

(In particular, σ must vanish whenever h is not Ricci-flat.) Indeed, let φ_t denote the time t flow of the vector field W. The $\mathcal{L}_W$ Equation tells us that $\varphi_t^*h = e^{-\sigma t}h$. Since φ_t is a diffeomorphism, $e^{-\sigma t}h$ and h must be isometric; therefore they have the same sectional curvatures. If h is not flat, this condition on sectional curvatures mandates that $e^{-\sigma t} = 1$, hence $\sigma = 0$. The argument we presented was pointed out to us by Bryant.

Incidentally, the use of $h = \varepsilon(a - b \otimes b)$ and $W = -b^\sharp/\varepsilon$ (Section 3.1.2) allows us to recover the Basic Equation from the $\mathcal{L}_W$ Equation. Thus the two equations are equivalent. This remains so even if σ were to be a function of x, because neither equation contains any derivative of σ.

We now turn to the derivation of the navigation versions of the Ricci Curvature Equation (Theorem 5, page 227) and the Curvature Equation (Theorem 6, page 228). The $\mathcal{L}_W$ Equation affords simplified expressions for many quantities that enter into the curvature equations of Theorems 5 and 6. The key in all such simplifications can invariably be traced back to the statement

$$W_{i:j} = \tfrac{1}{2}(\mathcal{L}_{ij} + \mathcal{C}_{ij}) = -\tfrac{1}{2}\sigma h_{ij} + \tfrac{1}{2}\mathcal{C}_{ij}.$$

We first address all but one of the terms on the right-hand sides of the curvature equations. Keep in mind that indices on curl, θ, and ${}^a y_i := a_{ij}\, y^j$ are manipulated by the Riemannian metric a, while those on $\mathcal{C}$, $\mathcal{T}$, and ${}^h y_i := h_{ij}\, y^j$ are manipulated by the Riemannian metric h. The relevant formulae are

$$\begin{aligned}
\mathrm{curl}_{ij} &= -\frac{1}{\lambda}\mathcal{C}_{ij} + \frac{1}{\lambda^2}(\mathcal{T}_i W_j - \mathcal{T}_j W_i), & {}^a y_i &= \frac{1}{\lambda}\, {}^h y_i + \frac{1}{\lambda^2} W_i W_0,\\
\mathrm{curl}^i{}_j &= -\mathcal{C}^i{}_j + \frac{1}{\lambda}\mathcal{T}^i W_j, & \theta_i &= \frac{1}{\lambda}\mathcal{T}_i,\\
\mathrm{curl}^{ij} &= -\lambda \mathcal{C}^{ij}; & \theta^i &= \mathcal{T}^i.
\end{aligned}$$

This does not address the term $\theta_{0|0}$ (equivalently $\theta_{i|k} + \theta_{k|i}$), which appears on the right-hand side of the Ricci Curvature Equation (Section 2.2.3). To tackle this, as well as the left-hand sides of those curvature equations, we shall need the relation between the geodesic spray coefficients ${}^a G^i$ and ${}^h G^i$. That relation, first derived in Section 3.1.3, undergoes a dramatic simplification in the presence of the $\mathcal{L}_W$ Equation. The result is

$${}^a G^i = {}^h G^i + \zeta^i,$$

where

$$\zeta^i = y^i \frac{1}{2\lambda}(\mathcal{T}_0 - \sigma W_0) - \mathcal{T}^i \left(\frac{1}{4\lambda} h_{00} + \frac{1}{2\lambda^2} {W_0}^2 \right) + \frac{1}{2\lambda}\mathcal{C}^i{}_0 W_0.$$

Hence

$$\theta_{0|0} = \theta_{0:0} - 2\theta_i \zeta^i = \frac{1}{\lambda}\mathcal{T}_{0:0} - \frac{1}{\lambda^2} W_0 \mathcal{T}^i \mathcal{C}_{i0} + \left(\frac{1}{2\lambda^2} h_{00} + \frac{1}{\lambda^3} {W_0}^2 \right) \mathcal{T}^i \mathcal{T}_i.$$

Differentiating these two statements with respect to y^j, y^k gives an explicit relation between the Christoffel symbols ${}^a\gamma^i{}_{jk}$ and ${}^h\gamma^i{}_{jk}$, as well as a formula for $\theta_{j|k} + \theta_{k|j}$ in terms of the navigation data (h, W). However, we refrain from doing so. In the following two subsections, we shall determine the navigation version of the Ricci Curvature Equation in Theorem 5 and of the Curvature Equation in Theorem 6. It is found, in retrospect, that the computational tedium is significantly lessened by working with ${}^a\mathrm{Ric}_{00}$, ${}^a K^i{}_k$ rather than ${}^a\mathrm{Ric}_{ij}$, ${}^a R_{hijk}$. Consequently, the relation ${}^a G^i = {}^h G^i + \zeta^i$ and the formula for $\theta_{0|0}$ should suffice.

3.2.2. Einstein–Randers metrics. The contracted form of the Ricci Curvature Equation (Section 2.2.3) reads

$$ {}^{a}\mathrm{Ric}_{00} = (\alpha^2+\beta^2)\,\mathcal{R}ic(x) - \tfrac14\alpha^2\,\mathrm{curl}^{hj}\,\mathrm{curl}_{hj} - \tfrac12\,\mathrm{curl}^{j}{}_{0}\,\mathrm{curl}_{j0} \\ -(n-1)\big(\tfrac1{16}\sigma^2(3\alpha^2-\beta^2)+\tfrac14(\theta_0)^2+\tfrac12\theta_{0|0}\big). $$

With $\alpha^2 = a_{00} = (1/\lambda)\,h_{00} + (1/\lambda^2)W_0{}^2$ and $\beta = b_0 = -(1/\lambda)\,W_0$, the simplified formulae in Section 3.2.1, and $\mathcal{R}ic(x) =: (n-1)K(x)$, all the terms on the right-hand side are accounted for.

For the left-hand side, note first that ${}^{a}\mathrm{Ric}_{00} = {}^{a}K^{i}{}_{i}$. Specialising the split and covariantised form of Berwald's formula (Section 1.2.3) to $F = a$ and $\mathcal{F} = h$, and taking the natural trace, we have

$$ {}^{a}\mathrm{Ric}_{00} = {}^{h}\mathrm{Ric}_{00} + (2\zeta^i)_{:i} - (\zeta^i)_{y^j}(\zeta^j)_{y^i} - y^j(\zeta^i{}_{:j})_{y^i} + 2\zeta^j(\zeta^i)_{y^j y^i}. $$

Though ζ^i has a simplified formula (Section 3.2.1), computing the four terms dependent on ζ is still tedious. That task is helped by the $\mathcal{L}_W$ Equation and the navigation description [Robles 2003] of the E_{23} Equation:

$$ W^{i}{}_{:0:i} = (n-1)\big(K(x) + \tfrac1{16}\sigma^2\big)\,W_0. $$

The result is unexpectedly elegant:

$$ {}^{h}\mathrm{Ric}_{00} = (n-1)\big(K(x) + \tfrac1{16}\sigma^2\big)\,h_{00}. $$

Differentiating away the contracted y^i, y^k gives ${}^{h}\mathrm{Ric}_{ik}$.

Conversely, it has been checked that, via $h = \varepsilon(a - b\otimes b)$ and $W = -b^\sharp/\varepsilon$ (Section 3.1.2), the above navigation description reproduces the characterisation in Theorem 5. Thus the characterisation (in terms of a, b) is equivalent to the navigation description (in terms of h, W). In particular, Theorem 5 implies:

THEOREM 9 (EINSTEIN NAVIGATION DESCRIPTION). *Suppose the Randers metric F solves Zermelo's problem of navigation on the Riemannian manifold (M, h) under the external influence W, $|W| < 1$. Then (M, F) is Einstein with Ricci scalar $\mathcal{R}ic(x) =: (n-1)K(x)$ if and only if there exists a constant σ such that*

(i) *h is Einstein with Ricci scalar $(n-1)\big(K(x)+\tfrac1{16}\sigma^2\big)$, that is,*

$$ {}^{h}\mathrm{Ric}_{ik} = (n-1)\big(K(x) + \tfrac1{16}\sigma^2\big)\,h_{ik}, $$

and

(ii) *W is an infinitesimal homothety of h, namely,*

$$ (\mathcal{L}_W h)_{ik} = W_{i:k} + W_{k:i} = -\sigma h_{ik}. $$

Furthermore, σ must vanish whenever h is not Ricci-flat.

We call this a 'description' rather than a 'characterisation' because, in contrast with Theorem 5, it makes explicit the underlying geometry of Einstein–Randers metrics. Section 4 will illustrate Theorem 9 with a plethora of examples.

3.2.3. Constant flag curvature Randers metrics. For Randers metrics of constant flag curvature, the equation destined to be recast into navigational form is given in Theorem 6. Contracting it with y^h, y^k, raising the index i with a, and relabelling j as k, we obtain the following expression for the spray curvature ${}^a\!K^i{}_k$ $(= y^j\, {}^a\!R_j{}^i{}_{kl}\, y^l$; Section 1.2.3):

$$\begin{aligned} {}^a\!K^i{}_k &= \big((K - \tfrac{3}{16}\sigma^2) + (K + \tfrac{1}{16}\sigma^2)\, b^s b_s - \tfrac{1}{4}\theta^s\theta_s\big)\,(\delta^i{}_k\,\alpha^2 - y^i\, {}^a\!y_k) \\ &\quad + \tfrac{1}{4}\,\mathrm{curl}^s{}_0 (\mathrm{curl}_s{}^i\, {}^a\!y_k + y^i\,\mathrm{curl}_{sk} - \mathrm{curl}_{s0}\,\delta^i{}_k) - \tfrac{1}{4}\,\alpha^2\,\mathrm{curl}^{si}\,\mathrm{curl}_{sk} - \tfrac{3}{4}\,\mathrm{curl}^i{}_0\,\mathrm{curl}_{k0}. \end{aligned}$$

Here ${}^a\!y_k := a_{ik} y^i$. This is equivalent to the Curvature Equation because

$${}^a\!R_{hijk} = \tfrac{1}{3}\left(({}^a\!K_{ij})_{y^k y^h} - ({}^a\!K_{ik})_{y^j y^h}\right). \qquad (*)$$

We now recast the equality just given for ${}^a\!K^i{}_k$. All the terms on the right-hand side are routinely computed, using $\alpha^2 = a_{00} = (1/\lambda)\, h_{00} + (1/\lambda^2) W_0{}^2$, $b_s = -(1/\lambda)\, W_s$, $b^s = -\lambda W^s$, and the simplified formulae in Section 3.2.1.

For the left-hand side, we first specialise the split and covariantised form of Berwald's formula (Section 1.2.3) to $F = a$ and $\mathcal{F} = h$, getting

$${}^a\!K^i{}_k = {}^h\!K^i{}_k + (2\zeta^i)_{:k} - (\zeta^i)_{y^s}(\zeta^s)_{y^k} - y^s(\zeta^i{}_{:s})_{y^k} + 2\zeta^s(\zeta^i)_{y^s y^k}.$$

Into this we substitute the simplified formula for ζ (Section 3.2.1). The ensuing computation is assisted by the prodigious use of the $\mathcal{L}_W$ Equation and the navigation description of the CC(23) Equation:

$$W_{i:j:k} = \big(K + \tfrac{1}{16}\sigma^2\big)\,(h_{ik} W_j - h_{jk} W_i).$$

The result is as elegant as the Einstein case:

$${}^h\!K^i{}_k = \big(K + \tfrac{1}{16}\sigma^2\big)\,(\delta^i{}_k\, h_{00} - y^i\, {}^h\!y_k),$$

where ${}^h\!y_k := h_{ik}\, y^i$. Lowering the index i with the Riemannian metric h and differentiating in the same fashion as formula $(*)$ gives ${}^h\!R_{hijk}$.

We have verified that the use of $h = \varepsilon(a - b \otimes b)$ and $W = -b^\sharp/\varepsilon$ (Section 3.1.2) converts the above navigation description in terms of h, W back to the characterisation in terms of a, b presented by Theorem 6. So the two pictures are indeed equivalent, and Theorem 6 implies:

THEOREM 10 (CONSTANT FLAG CURVATURE NAVIGATION DESCRIPTION). *Suppose the Randers metric F solves Zermelo's problem of navigation on the Riemannian manifold (M, h) under the external influence W, $|W| < 1$. Then (M, F) is of constant flag curvature K if and only if there exists a constant σ such that*

(i) *h is of constant sectional curvature $(K + \frac{1}{16}\sigma^2)$, that is,*

$${}^h\!R_{hijk} = \big(K + \tfrac{1}{16}\sigma^2\big)\,(h_{ij} h_{hk} - h_{ik} h_{hj}),$$

and

(ii) *W is an infinitesimal homothety of h, namely,*

$$(\mathcal{L}_W h)_{ik} = W_{i:k} + W_{k:i} = -\sigma h_{ik}.$$

Furthermore, σ must vanish whenever h is not flat.

EXAMPLES. We present three favorite examples to illustrate the use of Theorem 10. In the examples below: position coordinates are denoted x, y, z rather than x^1, x^2, x^3; components of tangent vectors are u, v, w instead of y^1, y^2, y^3.

FUNK DISC. The Finslerian Poincaré example was discussed in Section 2.3.2 and Section 3.1.2. Here we revisit it a third time, for the sake of those who prefer to work with simple navigation data.

Fix the angle θ and contract the radius via $r \mapsto r/\big(1+\frac{1}{4}r^2\big)$. This map is an isometry of the Finslerian Poincaré model onto the Funk metric of the unit disc [Funk 1929; Okada 1983; Shen 2001]. The navigation data of the Funk metric is simple: h is the Euclidean metric and the radial $W = -r\partial_r$ is an infinitesimal homothety with $\sigma = 2$. Writing tangent vectors at (r,θ) as $u\partial_r + v\partial_\theta$, we have

$$F = \frac{\sqrt{u^2 + r^2(1-r^2)v^2}}{1-r^2} + \frac{ru}{1-r^2}, \quad \text{with } r^2 = x^2 + y^2.$$

By Theorem 10, $K + \frac{1}{16}\sigma^2 = 0$. Hence the Funk metric on the unit disc has constant flag curvature $K = -\frac{1}{4}$.

The isometry above is a global change of coordinates which transforms the navigation data in the Section 3.1.2 example into a more computationally friendly format. ♢

A 3-SPHERE THAT IS NOT PROJECTIVELY FLAT. We start with the unit sphere S^3 in $\mathbb{R}^4$, parametrised by its tangent spaces at the poles, as in [Bao–Shen 2002]. For each constant $K > 1$, let h be $1/K$ times the standard Riemannian metric induced on S^3. The rescaled metric has sectional curvature K. Perturb h by the Killing vector field

$$W = \sqrt{K-1}\big(-s(1+x^2), z - sxy, -y - sxz\big),$$

with $s = \pm 1$ depending on the hemisphere. Then $|W| = \sqrt{(K-1)/K}$ and W is tangent to the S^1 fibers in the Hopf fibration of S^3. By Theorem 10, the resulting Randers metric F has constant flag curvature K. Explicitly, $F = \alpha + \beta$, where

$$\alpha = \frac{\sqrt{K(su - zv + yw)^2 + (zu + sv - xw)^2 + (-yu + xv + sw)^2}}{1 + x^2 + y^2 + z^2},$$

$$\beta = \frac{\sqrt{K-1}(su - zv + yw)}{1 + x^2 + y^2 + z^2}.$$

This Randers metric is not projectively flat [Bao–Shen 2002]. This is in stark contrast with the Riemannian case because, according to Beltrami's theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. ♢

Shen's fish tank. This example, first presented in [Shen 2002], is a three-dimensional variant of Shen's fish pond (Section 3.1.1). Consider a cylindrical fish tank $x^2+y^2<1$ in $\mathbb{R}^3$, equipped with the standard Euclidean metric h. Suppose the tank has a rotational current with velocity vector $W = y\partial_x - x\partial_y + 0\partial_z$, and a big inquisitive mosquito hovers just above the water. Wishing to reach the bug as soon as possible, the hungry fish swim along a path of shortest time — that is, along a geodesic of the Randers metric $F=\alpha+\beta$ with Zermelo navigation data h and the infinitesimal rotation W. Explicitly,

$$\alpha = \frac{\sqrt{(-yu+xv)^2+(u^2+v^2+w^2)(1-x^2-y^2)}}{1-x^2-y^2},$$
$$\beta = \frac{-yu+xv}{1-x^2-y^2}, \qquad \text{with} \qquad |W|^2 = x^2+y^2.$$

Since W is a Killing field of Euclidean space we have $\sigma=0$, and Theorem 10 tells us that F is of constant flag curvature $K=0$. The same conclusion holds for the fish pond. ♢

Theorem 10 implies that every constant flag curvature Randers metric is locally isometric to a "standard model" with navigation data (h,W), where h is a standard Riemannian space form (sphere, Euclidean space, or hyperbolic space), and W is one of its infinitesimal homotheties. It remains to sort these standard models into Finslerian isometry classes.

Let (M_1,F_1), (M_2,F_2) be any two Randers spaces, with navigation data (h_1,W_1) and (h_2,W_2). It is a fact that F_1, F_2 are isometric as Finsler metrics if and only if there exists a Riemannian isometry $\varphi:(M_1,h_1)\to(M_2,h_2)$ such that $\varphi_*W_1=W_2$. For each standard Riemannian space form (M,h), the isometry group G of h leaves h invariant, but acts on its infinitesimal homotheties W via push-forward. By the cited fact, all (h,W) which lie on the same G-orbit generate mutually isometric standard models. This redundancy can be suppressed by collapsing each G-orbit to a point. For any fixed K, the resulting collection of such "points" constitutes the *moduli space* $\mathcal{M}_K$ for strongly convex Randers metrics of constant flag curvature K. Lie theory effects (a parametrisation and hence) a dimension count of $\mathcal{M}_K$; see [Bao et al. 2003] for details. The table on the next page includes, for comparison, similar information about the Riemannian setting and the case $\theta := \mathrm{curl}(b^\sharp,\cdot)=0$ (Section 2.3.2).

3.3. Issues resolved by the navigation description

3.3.1. Schur lemma for the Ricci scalar. In essence, this lemma constrains the geometry of Einstein metrics in dimension $\geqslant 3$ by forcing the Ricci scalar to be constant. Historically, this is the second Schur lemma in (non-Riemannian) Finsler geometry. The first Finslerian Schur lemma concerns the flag curvature; see [del Riego 1973; Matsumoto 1986; Berwald 1947]. An exposition can be found in [Bao et al. 2000].

<table>
<tr><td colspan="2"></td><td colspan="4">Dimension of moduli space</td></tr>
<tr><td rowspan="2">CFC metrics</td><td rowspan="2">dim M</td><td rowspan="2">$K > 0$</td><td rowspan="2">$K = 0$</td><td colspan="2">$K < 0$</td></tr>
<tr><td>$\sigma = 0$</td><td>$\sigma \neq 0$</td></tr>
<tr><td>Riemannian
b equiv. $W = 0$</td><td>$n \geqslant 2$</td><td colspan="3">0</td><td>empty</td></tr>
<tr><td rowspan="2">Yasuda–Shimada
$\theta = 0$</td><td>even n</td><td>0^*</td><td rowspan="2">1</td><td rowspan="2">0^*</td><td rowspan="2">$0^\dagger$</td></tr>
<tr><td>odd n</td><td>1</td></tr>
<tr><td rowspan="2">Unrestricted
Randers</td><td>even n</td><td colspan="4">$n/2$</td></tr>
<tr><td>odd n</td><td colspan="3">$(n+1)/2$</td><td>$(n-1)/2$</td></tr>
<tr><td colspan="6">The moduli spaces of dimension 0 consist of a single point.
* The single isometry class is Riemannian.
† The single isometry class is non-Riemannian, of Funk type.</td></tr>
</table>

Table 1. The dimension of the moduli space for several families of constant flag curvature (CFC) Randers metrics.

In two dimensions, the Ricci scalar of a Riemannian metric is the Gaussian curvature of the surface. Hence all Riemannian surfaces are Einstein in the sense of Section 1.3.2. Since the Gaussian curvature is not constant in general, the Schur lemma fails for Riemannian (and therefore Randers) metrics in two dimensions. It is natural to ask whether the Schur lemma also fails for non-Riemannian ($W \neq 0$) Randers *surfaces*. The answer is yes. Section 4.1 develops a class of non-Riemannian Randers surfaces whose Ricci scalars are nonconstant functions of x alone. In particular, these non-Riemannian surfaces are Einstein, but fail the Schur lemma.

In dimension $n \geqslant 3$, every Riemannian Einstein metric h is Ricci-constant. This follows readily from tracing the second Bianchi identity and realising that, for such metrics, ${}^h\mathrm{Ric}_{ij} = (S/n)\, h_{ij}$, where S denotes the scalar curvature of h.

Lemma 11 (Schur lemma). *The Ricci scalar of any Einstein Randers metric in dimension greater than two is necessarily constant*.

Proof. Suppose F is an Einstein metric of Randers type, with navigation data (h, W) and Ricci scalar $\mathcal{R}ic(x) = (n-1)K(x)$. Theorem 9 says that h must be Einstein with Ricci scalar $(n-1)\big(K + \frac{1}{16}\sigma^2\big)$, for some constant σ. Since $n > 2$ here, the Riemannian Schur lemma forces $K + \frac{1}{16}\sigma^2$ to be constant. The same must then hold for K and $\mathcal{R}ic = (n-1)K$. □

Another proof of the Schur lemma, based on the Einstein characterisation of Section 2.2.3 (Theorem 5 on page 227), is given in [Robles 2003].

3.3.2. Three dimensional Einstein–Randers metrics. For Riemannian metrics in three dimensions, being Einstein and having constant sectional curvature are equivalent conditions because the conformal Weyl curvature tensor automatically vanishes. It is not known whether this rigidity holds for Einstein–Finsler metrics in general. However, the said rigidity does hold for Randers metrics. The proof rests on a comparison between the navigation descriptions for Einstein (Section 3.2.2) and constant flag curvature (Section 3.2.3) Randers metrics.

PROPOSITION 12 (THREE-DIMENSIONAL RIGIDITY). *Let F be a Randers metric in three dimensions. Then F is Einstein if and only if it has constant flag curvature.*

PROOF. Metrics of constant flag curvature are always Einstein. As for the converse, let F be an Einstein Randers metric with navigation data (h, W). The Ricci scalar of F is $(n-1)K = 2K$; in view of the Schur lemma above, K has to be constant. According to Theorem 9, h is Einstein with Ricci scalar $2\big(K + \frac{1}{16}\sigma^2\big)$, for some constant σ. By Riemannian three-dimensional rigidity, h must have constant sectional curvature $K + \frac{1}{16}\sigma^2$. The navigation description in Theorem 10 then forces F to be of constant flag curvature K. □

Interestingly, the two navigation descriptions also tell us that *any Einstein Randers metric that arises as a solution to Zermelo's problem of navigation on a Riemannian space form must be of constant flag curvature.*

3.3.3. The Matsumoto identity. This identity first came to light in a letter from Matsumoto to the first author. It says that any Randers metric of *constant* flag curvature K satisfies

$$\sigma\left(K + \tfrac{1}{16}\sigma^2\right) = 0.$$

Since metrics of constant flag curvature are Einstein, it is natural to wonder whether this identity can be extended to Einstein Randers metrics. The answer is yes, by the following result:

PROPOSITION 13. *Let F be an Einstein Randers metric whose Ricci scalar $\mathcal{R}ic(x)$ we reexpress as $(n-1)K(x)$. Then*

$$\sigma\big(K + \tfrac{1}{16}\sigma^2\big) = \begin{cases} W^i K_{:i} & \textit{when } n = 2, \\ 0 & \textit{when } n > 2. \end{cases}$$

Here, σ is the constant supplied by the navigation data (h, W) of F. According to Theorem 9, h must be Einstein with Ricci scalar $(n-1)(K + \frac{1}{16}\sigma^2)$, and W satisfies the $\mathcal{L}_W$ Equation $W_{i:j} + W_{j:i} = -\sigma\, h_{ij}$.

PROOF. We begin with the Ricci identity for the tensor $\mathcal{C}_{ij} := W_{i:j} - W_{j:i}$, namely $\mathcal{C}_{ij:k:h} - \mathcal{C}_{ij:h:k} = \mathcal{C}_{sj}\, {}^h R_i{}^s{}_{kh} + \mathcal{C}_{is}\, {}^h R_j{}^s{}_{kh}$, where ${}^h R_h{}^i{}_{jk}$ is the curvature tensor of h. Trace this identity on (i, k) and (h, j) to obtain

$$\big(W^{i:j}{}_{:i} - W^{j:i}{}_{:i}\big)_{:j} = \big(W^{i:j} - W^{j:i}\big)\, {}^h\mathrm{Ric}_{ij} = 0,$$

where the second equality follows because ${}^h\mathrm{Ric}_{ij}$ is symmetric.

Next, we compute $W^{i:j}{}_{:i} - W^{j:i}{}_{:i}$. To that end, differentiating the $\mathcal{L}_W$ Equation gives $W_{p:q:r} + W_{q:p:r} = 0$. This and the Ricci identity for W imply that

$$\begin{aligned} W_{i:j:k} - W_{j:i:k} &= (W_{i:j:k} - W_{i:k:j}) - (W_{k:i:j} - W_{k:j:i}) + (W_{j:k:i} - W_{j:i:k}) \\ &= W^s\, {}^hR_{isjk} - W^s\, {}^hR_{ksij} + W^s\, {}^hR_{jski}. \end{aligned}$$

Using h to trace on (i,k) and raise j, we get $W^{i:j}{}_{:i} - W^{j:i}{}_{:i} = 2W_s\, {}^h\mathrm{Ric}^{sj}$. Since ${}^h\mathrm{Ric} = (n-1)(K + \frac{1}{16}\sigma^2)\, h$, we are led to

$$W^{i:j}{}_{:i} - W^{j:i}{}_{:i} = 2\,(n-1)\left(K + \tfrac{1}{16}\sigma^2\right) W^j. \tag{$*$}$$

Finally, tracing the $\mathcal{L}_W$ Equation gives $2W^j{}_{:j} = -n\sigma$, whence

$$0 = (W^{i:j}{}_{:i} - W^{j:i}{}_{:i})_{:j} = 2(n-1)\big(W^j K_{:j} - \tfrac{1}{2} n\sigma(K + \tfrac{1}{16}\sigma^2)\big).$$

The identity now follows from the Schur lemma (Section 3.3.1). □

Remark. If we assume that the $\mathcal{L}_W$ Equation (or the Basic Equation, which amounts to the same) holds, the E_{23} Equation (Section 2.2.3) can be reexpressed as $(*)$. Thus $(*)$ is the *navigation version* of the E_{23} Equation. It can be further refined, using $W_{i:j:k} + W_{j:i:k} = 0$, to read

$$W^i{}_{:j:i} = (n-1)\left(K + \tfrac{1}{16}\sigma^2\right) W_j.$$

A second derivation of the Matsumoto Identity, based on Theorem 5 (page 227), may be found in [Robles 2003].

4. Einstein Metrics of Nonconstant Flag Curvature

We now present a variety of non-Riemannian Randers metrics that are either Einstein or Ricci-constant. Apart from the 2-sphere, which is included merely because of its simplicity, all examples have nonconstant flag curvatures. Section 3.1.1 will be used without mention.

4.1. Examples with Riemannian–Einstein navigation data

4.1.1. Surfaces of revolution. Our first class of examples comprises surfaces of rotation in $\mathbb{R}^3$. We shall see that solutions to Zermelo's problem of navigation under infinitesimal rotations are Einstein, with Ricci scalar $\mathcal{R}ic(x)$ equal to the Gaussian curvature of the original Riemannian surface. Among the examples below, two (the elliptic paraboloid and the torus) have nonconstant Ricci scalar. These solutions of Zermelo navigation are non-Riemannian counterexamples to Schur's lemma in dimension 2.

To begin, take any surface of revolution M, obtained by revolving a profile curve

$$\varphi \mapsto \big(0,\, f(\varphi),\, g(\varphi)\big)$$

in the right half of the yz-plane around the z axis. The ambient Euclidean space induces a Riemannian metric h on M. Parametrise M by

$$(\theta,\varphi) \mapsto \big(f(\varphi)\cos\theta,\ f(\varphi)\sin\theta,\ g(\varphi)\big), \quad 0 \leqslant \theta \leqslant 2\pi.$$

Now consider the infinitesimal isometry $W := \varepsilon\partial_\theta$, where ε is a constant. By limiting the size of our profile curve if necessary, there is no loss of generality in assuming that f is bounded. Choose ε so that $\varepsilon|f| < 1$ for all φ. Expressing h in the given parametrisation, we find that the solution to Zermelo's problem is the Randers metric $F = \alpha + \beta$ on M, with

$$\alpha = \frac{\sqrt{u^2 f^2 + v^2(1-\varepsilon^2 f^2)(\dot f^2 + \dot g^2)}}{1-\varepsilon^2 f^2}, \quad \beta = \frac{-\varepsilon u f^2}{1-\varepsilon^2 f^2},$$

and $\|b\|^2 = \varepsilon^2 f^2 = |W|^2$. Here, $u\partial_\theta + v\partial_\varphi$ represents an arbitrary tangent vector on M, and $\dot f$, $\dot g$ are the derivatives of f, g with respect to φ. Note that the hypothesis $\varepsilon|f| < 1$ ensures strong convexity.

Because W is a Killing vector field, σ vanishes. The Einstein navigation description (Theorem 9 on page 238) then says that the Ricci scalar $\mathcal{R}ic$ of F is identical to that of the Riemannian metric h. The latter is none other than the Gaussian curvature K of h. Hence

$$\mathcal{R}ic(x) = K(x) = \frac{\dot g(\dot f \ddot g - \ddot f \dot g)}{f(\dot f^2 + \dot g^2)^2},$$

where the dots indicate derivatives with respect to φ.

We examine three special cases of surfaces of revolution:

Sphere. The unit sphere is given as a surface of revolution by $f(\varphi) = \cos\varphi$ and $g(\varphi) = \sin\varphi$. We will consider the infinitesimal rotation $W = \varepsilon\partial_\theta$, with $\varepsilon < 1$ to effect the necessary $\varepsilon|f| < 1$. The Randers metric solving Zermelo's problem of navigation on the sphere under the influence of W is of constant flag curvature $K = 1$, with

$$\alpha = \frac{\sqrt{u^2\cos^2\varphi + v^2(1-\varepsilon^2\cos^2\varphi)}}{1-\varepsilon^2\cos^2\varphi}, \quad \beta = \frac{-\varepsilon u\cos^2\varphi}{1-\varepsilon^2\cos^2\varphi}. \qquad \diamondsuit$$

Elliptic paraboloid. This is the surface $z = x^2 + y^2$ in $\mathbb{R}^3$. Set the multiple ε in W to be 1. The resulting Randers metric lives on the $x^2 + y^2 < 1$ portion of the elliptic paraboloid, and has Ricci scalar $4/(1+4x^2+4y^2)^2$. It reads

$$\alpha = \frac{\sqrt{(-yu+xv)^2 + \big((1+4x^2)u^2 + 8xyuv + (1+4y^2)v^2\big)(1-x^2-y^2)}}{1-x^2-y^2},$$

$$\beta = \frac{yu-xv}{1-x^2-y^2}, \qquad \text{with } \|b\|^2 = x^2+y^2. \qquad \diamondsuit$$

Torus. Specialize to a torus of revolution with parametrisation

$$(\theta,\varphi) \mapsto \big((2+\cos\varphi)\cos\theta,\ (2+\cos\varphi)\sin\theta,\ \sin\varphi\big).$$

Set the multiple ε in W to be $\frac{1}{4}$. The resulting Randers metric on the torus has Ricci scalar $\cos\varphi/(2+\cos\varphi)$. It is given by

$$\alpha = \frac{4\sqrt{16\,(2+\cos\varphi)^2\,u^2 + \big(16-(2+\cos\varphi)^2\big)\,v^2}}{16-(2+\cos\varphi)^2},$$

$$\beta = \frac{-4\,(2+\cos\varphi)^2\,u}{16-(2+\cos\varphi)^2}, \quad \text{with } \|b\|^2 = \tfrac{1}{16}\,(2+\cos\varphi)^2. \qquad \diamondsuit$$

4.1.2. Certain Cartesian products. Recall from Section 1.3.2 the geometrical definition of the Ricci scalar and of Einstein metrics. When specialised to Riemannian n-manifolds, it says that the Ricci scalar $\mathcal{R}ic$ is obtained by summing the sectional curvatures of $n-1$ appropriately chosen sections that share a common flagpole. A Ricci-constant metric is remarkable because this sum is a constant. A moment's thought convinces us of the following:

The Cartesian product of two Riemannian Einstein metrics with the same constant Ricci scalar ρ is again Ricci-constant, and has $\mathcal{R}ic = \rho$.

As we will illustrate, this allows us to construct a wealth of Ricci-constant Randers metrics with nonconstant flag curvature.

Fix ρ. For $i = 1, 2$, let M_i be an n_i-dimensional Riemannian manifold with constant sectional curvature $\rho/(n_i - 1)$. Therefore M_i is Einstein with Ricci scalar ρ. Let W_i be a Killing field on M_i. Let h denote the product Riemannian metric on the Cartesian product $M = M_1 \times M_2$. Then h has constant Ricci scalar ρ, and admits $W = (W_1, W_2)$ as a Killing field.

By Theorem 9 (page 238), the Randers metric F generated by the navigation data (h, W) on M is Einstein, with constant Ricci scalar ρ. When ρ is nonzero, the Einstein metric h on M is not of constant sectional curvature. Hence Theorem 10 (page 239) assures us that F will not be of constant flag curvature. Proposition 8 (page 234) then says that the Randers metric F is non-Riemannian if and only if the wind W is nonzero. So it suffices to select a nonzero W_1.

To that end, let $\widetilde{M}_1$ be the n_1-dimensional, complete, simply connected standard model of constant sectional curvature $\rho/(n_1-1)$. The space of globally defined Killing fields on $\widetilde{M}_1$ is a Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}n_1(n_1+1)$. Select a nonzero $\tilde{W}_1$ from $\mathfrak{g}$. The isometry group G of $\widetilde{M}$ acts on $\mathfrak{g}$ via push-forwards. Let H be any finite subgroup of the isotropy group of $\tilde{W}_1$. Then we have a natural projection $\pi : \widetilde{M}_1 \to \widetilde{M}_1/H$. The quotient space $M_1 := \widetilde{M}_1/H$ is of constant sectional curvature $\rho/(n_1-1)$, and has a nonzero Killing field $W_1 := \pi_* \tilde{W}_1$.

As a concrete illustration, we specialise the discussion to spheres. For simplicity, we specify the finite subgroup H to be trivial.

Example. Let M_i $(i = 1, 2)$ be the n_i-sphere of radius $\sqrt{n_i - 1}$, $n_i \geqslant 2$. Then M_i has constant sectional curvature $1/(n_i - 1)$, and is therefore Einstein with Ricci scalar 1. The Cartesian product $M = M_1 \times M_2$, equipped with the product metric h, is an (n_1+n_2)-dimensional Riemannian Einstein manifold with Ricci scalar 1, and it is *not* of constant sectional curvature.

The Lie algebra of Killing fields on the n-sphere $S^n(r)$ with radius r is isomorphic to $\mathfrak{so}(n+1)$, regardless of the size of r. The following description accounts for all such vector fields. View points $p \in S^n(r)$ as row vectors in $\mathbb{R}^{n+1}$. For each $\Omega \in \mathfrak{so}(n+1)$, the assignment $p \mapsto p\Omega \in T_p(S^n(r))$ is a globally defined Killing vector field on $S^n(r)$.

Now, for $i = 1, 2$, take $\Omega_i \in \mathfrak{so}(n_i+1)$. Denote points of M_1, M_2 by p and q, respectively. The map $(p, q) \mapsto (p\Omega_1, q\Omega_2) \in T_{(p,q)}M$ defines a Killing field W of h. This W is nonzero as long as Ω_1, Ω_2 are not both zero. Zermelo navigation on (M, h) under the influence of W generates a non-Riemannian Randers metric with constant Ricci scalar 1, and which is not of constant flag curvature. ♢

4.2. Examples with Kähler–Einstein navigation data. In this section, we construct Einstein metrics of Randers type, with navigation data h, which is a Kähler–Einstein metric, and W, which is a Killing field of h. We choose h from among Kähler metrics of constant *holomorphic* sectional curvature, because the formula can be explicitly written down. There exist much more general Kähler–Einstein metrics, for instance those with positive sectional curvature; see [Tian 1997].

4.2.1. Kähler manifolds of constant holomorphic sectional curvature Suppose (M, h) is a Kähler manifold of complex dimension m (real dimension $n = 2m$) with complex structure J. Let $(z^1, \dots, z^m)$, where $z^\alpha := x^\alpha + ix^{\bar\alpha}$, denote local complex coordinates. Here, $x^1, \dots, x^m; x^{\bar 1}, \dots, x^{\bar m}$ are the $2m$ real coordinates. In our notation, lowercase Greek indices run from 1 to m. The complex coordinate vector fields are

$$Z_\alpha := \partial_{z^\alpha} = \tfrac{1}{2}(\partial_{x^\alpha} - i\partial_{x^{\bar\alpha}}),$$
$$Z_{\bar\alpha} := \partial_{\bar z^\alpha} = \tfrac{1}{2}(\partial_{x^\alpha} + i\partial_{x^{\bar\alpha}}),$$

respectively eigenvectors of J with eigenvalues $+i$ and $-i$. In what follows, uppercase Latin indices run through $1, \dots, m, \bar 1, \dots, \bar m$. Also, $\bar z^\alpha := x^\alpha - ix^{\bar\alpha}$ abbreviates the complex conjugate of z^α.

The Kähler metric h is a Riemannian (real) metric with the J-invariance property $h(JX, JY) = h(X, Y)$, and such that the 2-form $(X, Y) \mapsto h(X, JY)$, known as the *Kähler form*, is closed.

Let $h_{AB} := h(Z_A, Z_B)$. The J-invariance of h implies $h_{\alpha\beta} = 0 = h_{\bar\alpha\bar\beta}$. Expanding in terms of the complex basis gives

$$h = h_{\alpha\bar\beta}(dz^\alpha \otimes d\bar z^\beta + d\bar z^\beta \otimes dz^\alpha),$$

where $(h_{\alpha\bar\beta})$ is an $m \times m$ complex Hermitian matrix. By contrast, using the real basis and setting $H_{AB} := h(\partial_{x^A}, \partial_{x^B})$, we have $h = H_{AB}\, dx^A \otimes dx^B$, with $H_{\bar\alpha\bar\beta} = H_{\alpha\beta}$ and $H_{\bar\alpha\beta} = -H_{\alpha\bar\beta}$: the two diagonal blocks are symmetric and identical, while the off-diagonal blocks are skew-symmetric and negatives of each other.

The Kähler form being closed is equivalent to either $Z_\gamma(h_{\alpha\bar\beta}) = Z_\alpha(h_{\gamma\bar\beta})$ or $Z_{\bar\gamma}(h_{\alpha\bar\beta}) = Z_{\bar\beta}(h_{\alpha\bar\gamma})$, which puts severe restrictions on the usual Riemannian connection of h. Consequently, if we expand the curvature operator ${}^hR(Z_C, Z_D)Z_B$ as ${}^hR_B{}^A{}_{CD} Z_A$ in the complex basis, it is not surprising to find considerable economy among the coefficients:

$$ {}^hR_{\bar\beta}{}^\alpha{}_{CD} = {}^hR_\beta{}^{\bar\alpha}{}_{CD} = 0 = {}^hR_B{}^A{}_{\gamma\delta} = {}^hR_B{}^A{}_{\bar\gamma\bar\delta}. $$

For each Y in the real tangent space T_xM, the 2-plane spanned by $\{Y, JY\}$ is known as a *holomorphic section* because it is invariant under J, and the corresponding sectional curvature (in the usual Riemannian sense) is called a *holomorphic sectional curvature*. If all such curvatures are equal to the same constant $\mathfrak{c}$, the Kähler metric is said to be of constant holomorphic sectional curvature $\mathfrak{c}$. Such metrics are characterised by their curvature tensor having the following form in the complex basis $\{Z_A\}$:

$$ {}^hR_{\beta\bar\alpha\gamma\bar\delta} = \frac{\mathfrak{c}}{2}(h_{\beta\bar\alpha} h_{\gamma\bar\delta} + h_{\gamma\bar\alpha} h_{\beta\bar\delta}). $$

Equivalently,

$$ {}^hR_\beta{}^\alpha{}_{\gamma\bar\delta} = \frac{\mathfrak{c}}{2}(\delta_\beta{}^\alpha h_{\gamma\bar\delta} + \delta^\alpha{}_\gamma h_{\beta\bar\delta}). $$

Discussion of all this may be found in [Kobayashi–Nomizu 1996]; but one must adjust for the fact that their K_{ABCD} is our R_{BACD}, and that their definition of the curvature operator agrees with ours.

Remarks. 1. Suppose h is a Kähler metric of constant *holomorphic* sectional curvature $\mathfrak{c}$. Return to the expression for ${}^hR_\beta{}^\alpha{}_{\gamma\bar\delta}$ above. Tracing on the indices α and γ, we see that ${}^h\mathrm{Ric} = (m+1)(\mathfrak{c}/2)\,h$. That is, h must be an Einstein metric with constant Ricci scalar $(m+1)\mathfrak{c}/2$.

2. If a Kähler metric h were to satisfy the stronger condition of constant sectional curvature $\mathfrak{c}$, then it would necessarily be Einstein with Ricci scalar $(n-1)\mathfrak{c} = (2m-1)\mathfrak{c}$; see Section 1.3.1. At the same time, (1) implies that the Ricci scalar is $(m+1)\mathfrak{c}/2$. Hence we would have to have either $\mathfrak{c} = 0$ or $m = 1$.

Thus a Kähler manifold (M, h) can have constant sectional curvature in only two ways: either h is flat, or the real dimension of M is 2. This rigidity indicates that in the Kähler category, the weaker concept of constant holomorphic sectional curvature is more appropriate.

In analogy with the constant sectional curvature case we have a classification theorem for Kähler metrics of constant holomorphic sectional curvature [Kobayashi–Nomizu 1996]:

Any simply connected complete Kähler manifold of constant holomorphic sectional curvature $\mathfrak{c}$ *is holomorphically isometric to one of three standard models*:

$\mathfrak{c} > 0$: *the Fubini–Study metric on* $\mathbb{C}P^n$ (see below),
$\mathfrak{c} = 0$: *the standard Euclidean metric on* $\mathbb{C}^n$,
$\mathfrak{c} < 0$: *the Bergmann metric on the unit ball in* $\mathbb{C}^n$.

4.2.2. Killing fields of the Fubini–Study metric. The Fubini–Study metric is a Kähler metric on $\mathbb{C}P^m$ of constant holomorphic sectional curvature $\mathfrak{c} > 0$. Complex projective space is obtained from $\mathbb{C}^{m+1} \setminus 0$ by quotienting out the equivalence relation $\zeta \sim \lambda\zeta$, where $0 \neq \lambda \in \mathbb{C}$. Denote the equivalence class of ζ by $[\zeta]$. $\mathbb{C}P^m$ is covered by the charts $U^j := \{[\zeta] \in \mathbb{C}P^m : \zeta^j \neq 0\}$, $j = 0, 1, \ldots, m$, with holomorphic coordinate mapping

$$[\zeta] \mapsto \frac{1}{\zeta^j}(\zeta^0, \ldots, \widehat{\zeta^j}, \ldots, \zeta^m) =: (z^1, \ldots, z^m) = z \in \mathbb{C}^m.$$

In these coordinates, the Fubini–Study metric of constant holomorphic sectional curvature $\mathfrak{c} > 0$ has components

$$h_{\alpha\bar\beta} := h(Z_\alpha, Z_{\bar\beta}) = \frac{2}{\mathfrak{c}}\Big(\frac{1}{\rho}\delta_{\alpha\beta} - \frac{1}{\rho^2}\,\bar z_\alpha z_\beta\Big),$$

where $\bar z_\alpha := \delta_{\alpha\beta}\,\bar z^\beta$, $z_\beta := \delta_{\beta\tau}\,z^\tau$, and $\rho := 1 + z^\alpha \bar z_\alpha$.

Conventional wisdom in complex manifold theory prompts us to construct some explicit Killing vector field of h by considering $\xi := \pi_*(P^i{}_j\,\zeta^j\,\partial_{\zeta^i})$. Here, $\pi : \mathbb{C}^{m+1} \setminus 0 \to \mathbb{C}P^m$ is the natural projection and $P \in \mathfrak{u}(m+1)$, the Lie algebra of the unitary group $U(m+1)$. Note that $U(m+1)$ is the group of holomorphic isometries of Euclidean $\mathbb{C}^{m+1} \setminus 0$.

For concreteness, we compute ξ in the local coordinates of the chart U^0. We have $\pi(\zeta^0, \zeta^1, \ldots, \zeta^m) := (1/\zeta^0)(\zeta^1, \ldots, \zeta^m) =: (z^1, \ldots, z^m)$, from which it follows that

$$\pi_*(\partial_{\zeta^0}) = -\frac{1}{\zeta^0}\,z^\alpha\,\partial_{z^\alpha} \quad \text{and} \quad \pi_*(\partial_{\zeta^\alpha}) = \frac{1}{\zeta^0}\,\partial_{z^\alpha}$$

because the differential of π is simply the matrix

$$\pi_* = \frac{1}{\zeta^0}\begin{pmatrix} -z^1 & 1 & & 0 \\ \vdots & & \ddots & \\ -z^m & & 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}\pi_*(P^i{}_j\,\zeta^j\,\partial_{\zeta^i}) &= P^0{}_j\,\zeta^j\Big(-\frac{1}{\zeta^0}\,z^\alpha\,\partial_{z^\alpha}\Big) + P^\alpha{}_j\,\zeta^j\Big(\frac{1}{\zeta^0}\,\partial_{z^\alpha}\Big)\\ &= (-P^0{}_0 - P^0{}_\beta\,z^\beta)\,z^\alpha\,\partial_{z^\alpha} + (P^\alpha{}_0 + P^\alpha{}_\beta\,z^\beta)\,\partial_{z^\alpha}.\end{aligned}$$

With the skew-Hermitian property $P^t = -\bar P$, and introducing the decomposition

$$P = \begin{pmatrix} E & \bar C^t \\ C & Q \end{pmatrix}, \quad \text{where} \begin{cases} E := P^0{}_0 \text{ is pure imaginary,} \\ C = (C^\alpha) := (P^\alpha{}_0) \in \mathbb{C}^m, \\ Q = (Q^\alpha{}_\beta) := (P^\alpha{}_\beta) \in \mathfrak{u}(m), \end{cases}$$

we see that

$$\xi := \pi_*(P^i{}_j\,\zeta^j\,\partial_{\zeta^i}) = \big(Q^\alpha{}_\beta\,z^\beta + C^\alpha + (\bar C \cdot z + \bar E)\,z^\alpha\big)\,\partial_{z^\alpha}.$$

The real and imaginary parts of ξ give two real vector fields. A straightforward calculation shows that only $\operatorname{Re}\xi$ is Killing. The failure of $\operatorname{Im}\xi$ to be Killing persists even for $\mathbb{C}P^1$. In that case, the skew-Hermitian Q is simply a pure imaginary number, C is a single complex constant, and

$$\mathcal{L}_{\operatorname{Im}\xi} h = \left(\frac{2}{\rho^2}\left(1-\frac{2}{\rho}\right)\operatorname{Im}(E+\bar{Q}) + \frac{8}{\rho^3}\operatorname{Im}(E\bar{C})\right)(dx^1\otimes dx^1 + dx^{\bar{1}}\otimes dx^{\bar{1}}),$$

where $\rho := 1+|z|^2$. (We hasten to add that for $\mathbb{C}P^m$ with $m>1$, the $dx^\alpha dx^{\bar\beta}$ and $dx^{\bar\alpha}dx^\beta$ components of $\mathcal{L}_{\operatorname{Im}\xi}h$ do not vanish.)

Thus, our construction of ξ gives rise to the real Killing fields

$$W := \operatorname{Re}\xi = \tfrac{1}{2}(\xi^\alpha\,\partial_{z^\alpha} + \overline{\xi^\alpha}\,\partial_{\bar{z}^\alpha}) = \tfrac{1}{2}\big((\operatorname{Re}\xi^\alpha)\,\partial_{x^\alpha} + (\operatorname{Im}\xi^\alpha)\,\partial_{x^{\bar\alpha}}\big),$$

where

$$\xi^\alpha := Q^\alpha{}_\beta\, z^\beta + C^\alpha + (\bar{C}\cdot z + \bar{E})\, z^\alpha.$$

4.2.3. Non-Riemannian Einstein metrics on $\mathbb{C}P^m$. Theorem 9 (page 238) assures us that the Randers metric $F=\alpha+\beta$ generated by the navigation data h, W is a globally defined Einstein metric on $\mathbb{C}P^m$, provided that the Killing field W satisfies $|W|<1$. Since, with $\rho := 1+z^\gamma\bar{z}_\gamma$, the Fubini–Study metric

$$h := \frac{2}{\mathfrak{c}}\left(\frac{1}{\rho}\,\delta_{\alpha\beta} - \frac{1}{\rho^2}\,\bar{z}_\alpha z_\beta\right)(dz^\alpha\otimes d\bar{z}^\beta + d\bar{z}^\beta\otimes dz^\alpha)$$

does *not* have constant sectional curvature for $m>1$, Theorem 10 (page 239) ensures that F will not be of constant flag curvature. Moreover, the Ricci scalar of F equals that of h, which has the constant value $(m+1)\mathfrak{c}/2$ (Section 4.2.1).

Example. To make explicit the Riemannian metric a underlying α and the 1-form b that gives β, it is necessary to have available the covariant description $W^\flat$ of the Killing field W. If we write $W = W^B\,\partial_{x^B}$ and $H_{AB} := h(\partial_{x^A},\partial_{x^B})$, then $W^\flat := (H_{AB}W^B)\,dx^A$. Using $H_{\alpha\beta} = 2\operatorname{Re} h_{\alpha\bar\beta} = H_{\bar\alpha\bar\beta}$ and $H_{\alpha\bar\beta} = 2\operatorname{Im} h_{\alpha\bar\beta} = -H_{\bar\alpha\beta}$, a computation tells us that

$$W^\flat = (\operatorname{Re}\xi_{\bar\alpha})\,dx^\alpha + (\operatorname{Im}\xi_{\bar\alpha})\,dx^{\bar\alpha} = \tfrac{1}{2}(\overline{\xi_{\bar\alpha}}\,dz^\alpha + \xi_{\bar\alpha}\,d\bar{z}^\alpha).$$

Here, $\xi_{\bar\alpha} := h_{\bar\alpha B}\,\xi^B = h_{\bar\alpha\beta}\,\xi^\beta$ has the formula

$$\xi_{\bar\alpha} = \frac{2}{\mathfrak{c}\rho^2}\big(\rho(Q_{\alpha\beta}\,z^\beta + C_\alpha) + (\bar{C}\cdot z - C\cdot\bar{z} + \bar{E} - Q_{\beta\gamma}\,\bar{z}^\beta z^\gamma)\,z_\alpha\big),$$

where indices on Q, C, z are raised and lowered by the Kronecker delta.

As long as

$$|W|^2 := h(W,W) = \tfrac{1}{2}\operatorname{Re}(\xi^\alpha\,\overline{\xi_{\bar\alpha}}) < 1,$$

the Randers metric F with defining data

$$a := \frac{1}{\lambda}h + \frac{1}{\lambda^2}W^\flat\otimes W^\flat, \qquad b := -\frac{1}{\lambda}W^\flat,$$

where $\lambda := 1 - |W|^2$, will be strongly convex. Note that $W^\flat \otimes W^\flat$ does have $dz^\alpha dz^\beta$ and $d\bar{z}^\alpha d\bar{z}^\beta$ components, whereas h doesn't; thus the Riemannian metric a is not Hermitian unless $W = 0$.

For any choice of the constant quantities Q, C, E, the resulting function $|W|^2$ is continuous on the compact $\mathbb{C}P^m$, and is therefore bounded. Normalising these quantities by a common positive number if necessary, the strong convexity criterion $|W|^2 < 1$ can always be met. ♢

4.3. Rigidity and a Ricci-flat example. In this final set of examples, we consider Einstein–Randers metrics on compact boundaryless manifolds M, with an eye toward those with nonpositive *constant* Ricci scalar. The information obtained will complement the Ricci-positive example presented in Section 4.2.3.

We begin by observing that any infinitesimal homothety of (M, h) is necessarily Killing; that is, $\sigma = 0$. The proof follows from a divergence lemma (for compact boundaryless manifolds) and the trace of the $\mathcal{L}_W$ Equation:

$$\begin{aligned} 0 &= \int_M W^i{}_{:i}\, dV_h \qquad \text{by the divergence lemma [Bao et al. 2000]} \\ &= \int_M -\tfrac{1}{2} n\sigma\, dV_h \qquad \text{by tracing } W_{i:j} + W_{j:i} = -\sigma\, h_{ij}, \end{aligned}$$

where $dV_h := \sqrt{h}\, dx$. In conjunction with Theorem 9 (page 238), we have:

LEMMA 14. *Let (M, h) be a compact boundaryless Riemannian manifold. Every infinitesimal homothety W must be a Killing field; equivalently, $\sigma = 0$. In particular, if h is Einstein with Ricci scalar $\mathcal{R}ic$, then the navigation data (h, W) generates an Einstein Randers metric F with Ricci scalar $\mathcal{R}ic$.*

4.3.1. Killing fields versus eigenforms. Let W be any Killing vector field on a Riemannian Einstein manifold (M, h) with constant Ricci scalar $\mathcal{R}ic$. Let $W^\flat := W_i\, dx^i$ denote the 1-form dual to W. The action of the Laplace–Beltrami operator $\Delta := d\delta + \delta d$ on $W^\flat$ is given by the Weitzenböck formula

$$\Delta W^\flat = (-W_i{}^{:j}{}_{:j} + {}^h\mathrm{Ric}_i{}^j\, W_j)\, dx^i. \tag{$\dagger$}$$

See, for example, [Bao et al. 2000]. Given that ${}^h\mathrm{Ric}_{ij} = \mathcal{R}ic\, h_{ij}$, we have

$${}^h\mathrm{Ric}_i{}^j\, W_j = \mathcal{R}ic\, W_i.$$

By a Ricci identity, $W^j{}_{:j:i} - W^j{}_{:i:j} = -{}^h\mathrm{Ric}_i{}^s\, W_s$. Since W is Killing, $W_{i:j} + W_{j:i}$ vanishes. Thus $W^j{}_{:j} = 0$ and $-W^j{}_{:i} = W_i{}^{:j}$, which reduce that Ricci identity to

$$-W_i{}^{:j}{}_{:j} = {}^h\mathrm{Ric}_i{}^s\, W_s = \mathcal{R}ic\, W_i.$$

(Note for later use that if W is parallel but $\not\equiv 0$, then $\mathcal{R}ic = 0$.)

The Weitzenböck formula ($\dagger$) now becomes

$$\Delta W^\flat = 2\, \mathcal{R}ic\, W^\flat. \tag{$*$}$$

Therefore, whenever h is Einstein with *constant* Ricci scalar $\mathcal{R}ic$, all its Killing 1-forms must be eigenforms of Δ, with eigenvalue $2\,\mathcal{R}ic$.

Now suppose M is compact and boundaryless, so that integration by parts can be carried out freely without generating any boundary terms. Let $\langle\ ,\ \rangle$ denote the L_2 inner product on k-forms; namely,

$$\langle \omega, \eta \rangle := \frac{1}{k!} \int_M h^{i_1 j_1} \cdots h^{i_k j_k}\, \omega_{i_1 \cdots i_k} \eta_{j_1 \cdots j_k} \sqrt{h}\, dx.$$

Since the codifferential δ is the L_2 adjoint of d, we have

$$2\,\mathcal{R}ic\, \langle W^\flat, W^\flat \rangle = \langle \Delta W^\flat, W^\flat \rangle = \langle \delta W^\flat, \delta W^\flat \rangle + \langle dW^\flat, dW^\flat \rangle \geqslant 0. \qquad (\ddagger)$$

In particular, if $\mathcal{R}ic$ is negative, then W must be zero.

On the other hand, using (†) and ${}^h\mathrm{Ric}_{ij} = \mathcal{R}ic\, h_{ij}$, we get

$$\langle \Delta W^\flat, W^\flat \rangle = \int_M (W_{i:j} W^{i:j} + \mathcal{R}ic\, |W|^2) \sqrt{h}\, dx.$$

This enables us to make a Bochner type argument: if $\mathcal{R}ic = 0$, so that $W^\flat$ is harmonic by $(*)$, then $W_{i:j}$ must vanish identically.

4.3.2. Digression on Berwald spaces. In anticipation of our discussion of nonpositive Ricci curvature, we review Berwald spaces. (See [Szabó 1981; 2003] for a complete classification of such spaces.)

Let (M, F) be an arbitrary Finsler space. M need not be compact boundaryless, and F need not be Einstein or of Randers type. Let G^i denote the geodesic spray coefficients of F. Then (M, F) is a *Berwald space* if the Berwald connection coefficients $(G^i)_{y^j y^k}$ do not depend on y. In particular, all Riemannian and locally Minkowski spaces are Berwald; for explicit examples belonging to neither of these two camps, see [Bao et al. 2000].

Now suppose $F = \alpha + \beta$ is Randers but not necessarily Einstein, and has navigation data (h, W). It is known that F is Berwald if and only if the defining 1-form b is parallel. This elegant theorem is due to the efforts of the Japanese school. See [Bao et al. 2000] for an account of the history and references therein; see also the errata for a proof by Mike Crampin.

Decompose $b_{i|j} = \frac{1}{2}\,\mathrm{lie}_{ij} + \frac{1}{2}\,\mathrm{curl}_{ij}$ into its symmetric and skew-symmetric parts. Look back at the expression for lie and curl at the end of Section 3.1.3. (We reiterate here that they were derived under no assumptions on F.) Observe that $W_{i:j} = 0$ implies $b_{i|j} = 0$. We prove the converse: suppose $b_{i|j} = 0$.

- We have $0 = 2b_{j|k} W^j W^k = \mathrm{lie}_{jk} W^j W^k = -(1/\lambda)\, W_{j:k} W^j W^k$ by referring to Section 3.1.3. Hence $W_{j:k} W^j W^k = 0$.
- Using this, a similar calculation gives $0 = 2b_{j|k} W^j = -(2/\lambda)\, W^j W_{j:k}$ and $0 = 2b_{j|k} W^k = -(2/\lambda)\, W_{j:k} W^k$. That is, $W^j W_{j:k} = 0 = W_{j:k} W^k$.
- The formulae for lie_{jk} and curl_{jk} now simplify to $0 = \mathrm{lie}_{jk} = -\mathcal{L}_{jk}$ and $0 = \mathrm{curl}_{jk} = -(1/\lambda)\mathcal{C}_{jk}$. Hence $W_{j:k} = \frac{1}{2}(\mathcal{L}_{jk} + \mathcal{C}_{jk}) = 0$.

LEMMA 15. *Let F be any Randers metric, with defining data (a, b) and navigation data (h, W). The following three conditions are equivalent*:

- *F is Berwald.*
- *b is parallel with respect to a.*
- *W is parallel with respect to h.*

4.3.3. A rigidity theorem. We now focus on Einstein Randers metrics of nonpositive *constant* Ricci scalar, and show that there is considerable rigidity.

We begin by addressing the Ricci-flat case. We saw at the end of 4.3.1 that $\mathcal{R}ic = 0$ implies that W is parallel. Conversely, if W is parallel and not identically zero, then $\mathcal{R}ic = 0$ (see parenthetic remark just before $(*)$ on page 251). Whence, in conjunction with Lemma 15 and Proposition 8 (page 234), we obtain:

PROPOSITION 16. *Let F be an Einstein Randers metric on a compact boundaryless manifold M.*

- *If $\mathcal{R}ic = 0$, then F must be Berwald.*
- *If F is non-Riemannian and Berwald, then $\mathcal{R}ic = 0$.*

The second conclusion is false if we remove the stipulation that F be non-Riemannian. A Riemannian metric is always Berwald and Randers, and being Einstein certainly does not mandate it to be Ricci-flat.

Next, we turn our attention to compact boundaryless Einstein Randers spaces of constant negative Ricci scalar. In this case, by (‡) on page 252, W must be identically zero. Equivalently, $F = h$ is Riemannian.

PROPOSITION 17. *Let F be an Einstein Randers metric with constant negative Ricci scalar on a compact boundaryless manifold M. Then F is Riemannian.*

Together, these two propositions imply the following rigidity theorem.

THEOREM 18 (RICCI RIGIDITY). *Suppose (M, F) is a connected compact boundaryless Einstein Randers manifold with constant Ricci scalar $\mathcal{R}ic$.*

- *If $\mathcal{R}ic < 0$, then (M, F) is Riemannian.*
- *If $\mathcal{R}ic = 0$, then (M, F) is Berwald.*

Note that locally Minkowskian spaces, being Berwald and of zero flag curvature, are obvious examples of the second camp. The following arguments show that there exist Ricci-flat non-Riemannian Berwald–Randers metrics which are not locally Minkowskian.

EXAMPLE. Take any $K3$ surface[†], namely a complex surface with zero first Chern class and no nontrivial global holomorphic 1-forms. All $K3$ surfaces admit Kähler

[†] According to [Weil 1979, v. 2, p. 546], $K3$ surfaces are named after Kummer, Kodaira, Kähler, and "the beautiful mountain K2 in Kashmir" — the second tallest peak in the world. One may conjecture that with this last reference Weil was implying that such surfaces are as hard to conquer as the K2. . .

metrics (a result due to Todorov and to Siu), and hence Ricci-flat Kähler metrics, by Yau's proof of the Calabi conjecture. Since $\chi(K3) = 24$ by Riemann–Roch, these metrics are not flat by virtue of the Gauss–Bonnet–Chern theorem. See [Besse 1987] for details and references therein. It is futile to consider the Killing fields of such Ricci-flat metrics because, by an argument involving Serre duality, the isometry groups in question are all discrete. To circumvent this difficulty, set $M := K3 \times S^1$ (a compact boundaryless real 5-manifold) and consider the product metric h on M; it can be checked that h is also Ricci-flat but not flat. The vector field $W := 0 \oplus \partial/\partial t$ on M is parallel, hence Killing, with respect to h. Theorem 9 (page 238) tells us that the Randers metric F on M with navigation data (h, W) is Ricci-flat, while Proposition 8 (page 234) guarantees that it is not Riemannian. Theorem 10 (page 239) ensures that F is not of constant (zero) flag curvature; hence it could not be locally Minkowskian. $\diamondsuit$

Theorem 18 generalises a result of Akbar-Zadeh's for Finsler metrics of constant flag curvature [Akbar-Zadeh 1988]:

Suppose (M, F) *is a connected compact boundaryless Finsler manifold of constant flag curvature* λ.

- *If* $\lambda < 0$, *then* (M, F) *is Riemannian.*
- *If* $\lambda = 0$, *then* (M, F) *is locally Minkowski.*

The Ricci rigidity theorem is a straightforward extension of Akbar-Zadeh's result when $\mathcal{R}ic < 0$. To appreciate the generalisation when $\mathcal{R}ic = 0$, it is helpful to note that locally Minkowski spaces are precisely Berwald spaces of constant flag curvature $K = 0$; see, for instance, [Bao et al. 2000].

Akbar-Zadeh's theorem holds for all compact boundaryless Finsler spaces of constant flag curvature, while the Ricci rigidity theorem above is restricted to the Randers setting. So, towards a complete generalisation of Akbar-Zadeh's result: What should the conclusions be if we replace 'Randers' by 'Finsler' in the Ricci rigidity theorem?

5. Open Problems

5.1. Randers and beyond. Table 2 summarises some key information about Randers metrics, which are the simplest members in the much larger family of strongly convex (α, β) metrics. Matusmoto's slope-of-a-mountain metric (Section 1.1.1) is a prime example from this family. Formal discussions of (α, β) metrics are given in [Matsumoto 1986; Antonelli et al. 1993]; see also [Shen 2004] in this volume.

For strongly convex (α, β) metrics, how should the entries of the table be modified?

Also, Randers metrics exhibit three-dimensional rigidity (Section 3.3.2), and their Ricci scalars obey a Schur lemma (Section 3.3.1).

Property	Characterisation with (a, b)	Description with (h, W)
Strong convexity	$\|b\| < 1$	$\|W\| < 1$
Berwald	${}^a\nabla b = 0$	${}^h\nabla W = 0$
Constant flag curvature	$\mathcal{L}_{b^\sharp} a = \sigma(a - bb) - (b\theta + \theta b)$ ${}^a R = \text{poly}(K, \sigma, a, b, \text{curl})$ (Theorem 5)	h is a space form, $\mathcal{L}_W h = -\sigma h$
Einstein	$\mathcal{L}_{b^\sharp} a = \sigma(a - bb) - (b\theta + \theta b)$ ${}^a\text{Ric} = \text{poly}(\mathcal{R}ic, \sigma, a, b, \text{curl}, {}^a\nabla\theta)$ (Theorem 6)	h is Einstein, $\mathcal{L}_W h = -\sigma h$

Table 2. Summary of information about Randers metrics

- Does the passage from Randers metrics to (α, β) metrics allow us to construct, in three dimensions, a Ricci-constant metric which is not of constant flag curvature?
- Does every Einstein (α, β) metric ($\mathcal{R}ic$ a function of x only) in dimension $\geqslant 3$ have to be Ricci-constant?

Finally, fans of Randers metrics can aim to append an extra row to the above table, characterising Randers metrics of scalar curvature (Section 1.2.1).

5.2. Chern's question. Professor S.-S. Chern has openly asked the following question on many occasions:

Does every smooth manifold admit a Finsler Einstein metric?

Topological obstructions prevent some manifolds, such as $S^2 \times S^1$, from admitting Riemannian Einstein metrics; see Section 1.3.3 and references therein. By the navigation description of Theorem 9 (page 238), any manifold that admits an Einstein Randers metric must also admit a Riemannian Einstein metric. Thus the same topological obstructions confront Einstein metrics of Randers type.

As a prelude to answering Chern's question, it would be prudent to first settle the issue for concrete examples such as $S^2 \times S^1$. The discussion above shows that in searching for a Finsler Einstein metric on this 3-manifold, we must look beyond those of Randers type.

5.3. Geometric flows. On the slit tangent bundle $TM \setminus 0$, there are two interesting curvature invariants: the Ricci scalar and the S-curvature. They open the door to evolution equations which may be used to deform Finsler metrics. In this frame of mind, we wonder:

- Would a flow driven by the Ricci scalar, such as[†] $\partial_t \log F = -\mathcal{R}ic$, enable us to prove the existence of Finsler metrics with coveted curvature properties?
- Can deformations tailored to the S-curvature be used to ascertain the existence of Landsberg metrics ($\dot{A} = 0$) which are not of Berwald type ($P = 0$)?

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[†]The equation for $\partial_t \log F$ has its genesis in $\partial_t g_{ij} = -2\,\mathrm{Ric}_{ij}$, which is formally identical to the unnormalised Ricci flow of Richard Hamilton for Riemannian metrics (see [Cao–Chow 1999] and references therein). In the Finsler setting, this equation makes sense on $TM \setminus 0$. Contracting with y^i, y^j (and using Sections 1.1.1 and 1.3.1) gives $\partial_t F^2 = -2F^2\,\mathcal{R}ic$. Since F is strictly positive on the slit tangent bundle, the stated equation follows readily.

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