Trends in Commutative Algebra
MSRI Publications
Volume **51**, 2004

Modules and Cohomology over Group Algebras: One Commutative Algebraist's Perspective

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Abstract. This article explains basic constructions and results on group algebras and their cohomology, starting from the point of view of commutative algebra. It provides the background necessary for a novice in this subject to begin reading Dave Benson's article in this volume.

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Introduction

The available accounts of group algebras and group cohomology [Benson 1991a; 1991b; Brown 1982; Evens 1991] are all written for the mathematician on the street. This one is written for commutative algebraists by one of their own. There is a point to such an exercise: though group algebras are typically noncommutative, module theory over them shares many properties with that over commutative rings. Thus, an exposition that draws on these parallels could benefit an algebraist familiar with the commutative world. However, such an endeavour is not without its pitfalls, for often there are subtle differences between the two situations. I have tried to draw attention to similarities and to

Mathematics Subject Classification: Primary 13C15, 13C25. Secondary 18G15, 13D45.
Part of this article was written while the author was funded by a grant from the NSF.

discrepancies between the two subjects in a series of commentaries on the text that appear under the rubric Ramble[1].

The approach I have adopted toward group cohomology is entirely algebraic. However, one cannot go too far into it without some familiarity with algebraic topology. To gain an appreciation of the connections between these two subjects, and for a history of group cohomology, one might read [Benson and Kropholler 1995; Mac Lane 1978].

In preparing this article, I had the good fortune of having innumerable 'chalk-and-board' conversations with Lucho Avramov and Dave Benson. My thanks to them for all these, and to the Mathematical Sciences Research Institute for giving me an opportunity to share a roof with them, and many others, during the Spring of 2003. It is also a pleasure to thank Kasper Andersen, Graham Leuschke, and Claudia Miller for their remarks and suggestions.

1. The Group Algebra

Let G be a group, with identity element 1, and let k be a field. Much of what is said in this section is valid, with suitable modifications, more generally when k is a commutative ring. Let $k[G]$ denote the k-vector space with basis the elements of G; thus $k[G] = \bigoplus_{g\in G} kg$. The product on G extends to an associative multiplication on $k[G]$: for basis elements g and h, one has $g \cdot h = gh$, where the product on the right is taken in G, while the product of arbitrary elements is specified by the distributive law and the rule $a \cdot g = g \cdot a$ for $a \in k$. The identity element 1 is the identity in $k[G]$. The k-linear ring homomorphism $\eta\colon k \to k[G]$ with $\eta(1) = 1$ makes $k[G]$ a k-algebra. This is the *group algebra* of G with coefficients in k.

Note that $k[G]$ is commutative if and only if the group G is abelian. Moreover, it is finite-dimensional as a k-vector space precisely when G is finite.

An important part of the structure on $k[G]$ is the augmentation of k-algebras $\varepsilon\colon k[G] \to k$ defined by $\varepsilon(g) = 1$ for each $g \in G$. Through ε one can view k as a $k[G]$-bimodule. The kernel of ε, denoted $\mathrm{I}(G)$, is the k-subspace of $k[G]$ with basis $\{g-1 \mid g \in G\}$; it is a two-sided ideal, called the *augmentation ideal* of G. For every pair of elements g, h in G, the following relations hold in the group algebra:

$$g^{-1} - 1 = g^{-1}(1 - g),$$
$$gh - 1 = g(h - 1) + (g - 1) = (g - 1)h + (h - 1).$$

Thus, if a subset $\{g_\lambda\}_{\lambda\in\Lambda}$ of G, with Λ an index set, generates the group, the subset $\{g_\lambda - 1\}_{\lambda\in\Lambda}$ of $k[G]$ generates $\mathrm{I}(G)$ both as a left ideal and as a right ideal.

[1]This word has at least two meanings: "a leisurely walk", or "to talk or write in a discursive, aimless way"; you can decide which applies. By the by, its etymology, at least according to www.dictionary.com, might amuse you.

(1.1) **Functoriality.** The construction of the group algebra is functorial: given a group homomorphism $\varphi\colon G_1 \to G_2$, the k-linear map

$$k[\varphi]\colon k[G_1] \to k[G_2], \quad \text{where } g \mapsto \varphi(g),$$

is a homomorphism of k-algebras, compatible with augmentations. Its kernel is generated both as a left ideal and as a right ideal by the set $\{g-1 \mid g \in \operatorname{Ker}\varphi\}$.

For example, when N is a normal subgroup of a group G, the canonical surjection $G \to G/N$ induces the surjection of k-algebras $k[G] \to k[G/N]$. Since its kernel is generated by the set $\{n-1 \mid n \in N\}$, there is a natural isomorphism of k-algebras

$$k[G/N] \cong k \otimes_{k[N]} k[G] = \frac{k[G]}{\mathrm{I}(N)k[G]}.$$

Let me illustrate these ideas on a few simple examples.

(1.2) **Cyclic groups.** The group algebra of the infinite cyclic group is $k[x^{\pm 1}]$, the algebra of Laurent polynomials in the variable x. Here x is a generator of the group; its inverse is x^{-1}. The augmentation maps x to 1, and the augmentation ideal is generated, as an ideal, by $x-1$.

In view of (1.1), the group algebra of the cyclic group of order d is $k[x]/(x^d-1)$, and the augmentation ideal is again generated by $x-1$.

(1.3) **Products of groups.** Let G_1 and G_2 be groups. By (1.1), for $n = 1, 2$ the canonical inclusions $\iota_n\colon G_n \to G_1 \times G_2$ induce homomorphisms of k-algebras $k[\iota_n]\colon k[G_n] \to k[G_1 \times G_2]$. Since the elements in the image of $k[\iota_1]$ commute with those in the image of $k[\iota_2]$, one obtains a homomorphism of augmented k-algebras

$$\begin{aligned} k[G_1] \otimes_k k[G_2] &\to k[G_1 \times G_2], \\ g_1 \otimes_k g_2 &\mapsto (g_1, g_2). \end{aligned}$$

This is an isomorphism since it maps the basis $\{g_1 \otimes_k g_2 \mid g_i \in G_i\}$ of the k-vector space $k[G_1] \otimes_k k[G_2]$ bijectively to the basis $\{(g_1, g_2) \mid g_i \in G_i\}$ of $k[G_1 \times G_2]$. For this reason, the group algebra of $G_1 \times G_2$ is usually identified with $k[G_1] \otimes_k k[G_2]$.

(1.4) **Abelian groups.** Let G be a finitely generated abelian group. The structure theorem for such groups tells us that there are nonnegative numbers n and $d_1, \dots, d_m$, with $d_j \geq 2$ and $d_{i+1} \mid d_i$, such that

$$G = \mathbb{Z}^n \oplus \frac{\mathbb{Z}}{(d_1\mathbb{Z})} \oplus \cdots \oplus \frac{\mathbb{Z}}{(d_m\mathbb{Z})}.$$

The description of the group algebra of cyclic groups given in (1.2), in conjunction with the discussion in (1.3), yields

$$k[G] = \frac{k[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1, \dots, y_m]}{(y_1^{d_1}-1, \dots, y_m^{d_m}-1)}$$

The augmentation is given by $x_i \mapsto 1$ and $y_j \mapsto 1$, the augmentation ideal is generated by $\{x_1-1,\dots,x_n-1,y_1-1,\dots,y_m-1\}$.

RAMBLE. Observe: the group algebra in (1.4) above is a complete intersection.

(1.5) **Finite p-groups.** Let R be a ring; it need not be commutative. Recall that the intersection of all its left maximal ideals is equal to the intersection of all its right maximal ideals, and called the Jacobson radical of R. Thus, R has a unique left maximal ideal exactly when it has a unique right maximal ideal, and then these ideals coincide. In this case, one says that R is *local*; note that the corresponding residue ring is a division ring; for details see [Lang 2002, XVII § 6], for example.

Suppose that the characteristic of k is p, with $p \geq 2$. Let G be a finite *p-group*, so that the order of G is a power of p. I claim:

The group algebra $k[G]$ is local with maximal ideal $\mathrm{I}(G)$.

Indeed, it suffices to prove (and the claim is equivalent to): the augmentation ideal $\mathrm{I}(G)$ is nilpotent. Now, since G is a p-group, its centre Z is nontrivial, so (1.1) yields an isomorphism of k-algebras

$$\frac{k[G]}{\mathrm{I}(Z)k[G]} \cong k[G/Z].$$

Since the order of G/Z is strictly less than that of G, one can assume that $\mathrm{I}(G/Z)$ is nilpotent. By the isomorphism above, this entails $\mathrm{I}(G)^n \subseteq \mathrm{I}(Z)k[G]$, for some positive integer n. Now Z is an abelian p-group, so $\mathrm{I}(Z)$ is nilpotent, by (1.4). Since $\mathrm{I}(Z)$ is in the centre of $k[G]$, one obtains that $\mathrm{I}(G)$ is nilpotent, as claimed.

The converse also holds:

(1.6) EXERCISE. Let G be a finite group and p the characteristic of k. Prove that if the ring $k[G]$ is local, then G is a p-group. (Hint: $k[G]$ has finite rank over k, so its nilradical is equal to its Jacobson radical.)

(1.7) **The diagonal map.** Let G be a group and let $G \to G \times G$ be the diagonal homomorphism, given by $g \mapsto (g,g)$. Following (1.3), one identifies the group ring of $G \times G$ with $k[G] \otimes_k k[G]$, and then the diagonal homomorphism induces a homomorphism of augmented k-algebras

$$\Delta\colon k[G] \to k[G] \otimes_k k[G], \quad \text{where } g \mapsto g \otimes_k g.$$

This is called the *diagonal* homomorphism, or *coproduct*, of the group algebra $k[G]$.

There is another piece of structure on the group algebra: the map $G \to G$ given by $g \mapsto g^{-1}$ is an anti-isomorphism of groups, and hence induces an anti-isomorphism of group algebras

$$\sigma\colon k[G] \to k[G],$$

that is to say, σ is an isomorphism of additive groups with $\sigma(rs) = \sigma(s)\sigma(r)$. The map σ is referred to as the *antipode* of the group algebra. It commutes with the diagonal map, in the sense that

$$\sigma^{(G\times G)} \circ \Delta^G = \Delta^G \circ \sigma^G.$$

Here are the salient properties of the diagonal and the antipode:

(a) Δ is a homomorphism of augmented k-algebras;
(b) Δ is co-associative, in that the following diagram commutes:

$$\begin{array}{ccc} k[G] & \xrightarrow{\Delta} & k[G]\otimes_k k[G] \\ \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta\otimes_k 1} \\ k[G]\otimes_k k[G] & \xrightarrow{1\otimes_k\Delta} & k[G]\otimes_k k[G]\otimes_k k[G] \end{array}$$

(c) The following diagram commutes:

$$\begin{array}{ccccc} & & k[G] & & \\ & \swarrow^{\cong} & \downarrow{\scriptstyle \Delta} & \searrow^{\cong} & \\ k\otimes_k k[G] & \xleftarrow{\varepsilon\otimes_k 1} & k[G]\otimes_k k[G] & \xrightarrow{1\otimes_k\varepsilon} & k[G]\otimes_k k \end{array}$$

This property is paraphrased as: ε is a co-unit for Δ.
(d) For each element $r \in k[G]$, if $\Delta(r) = \sum_{i=1}^n (r'_i \otimes_k r''_i)$, then

$$\sum_{i=1}^n \sigma(r'_i)r''_i = \eta(\varepsilon(r)) = \sum_{i=1}^n r'_i\sigma(r''_i)$$

Taking these properties as the starting point, one arrives at the following notion.

(1.8) **Hopf algebras.** An augmented k-algebra H, with unit $\eta\colon k \to H$ and augmentation $\varepsilon\colon H \to k$ with k-linear homomorphisms $\Delta\colon H \to H\otimes_k H$ and $\sigma\colon H \to H$ satisfying conditions (a)–(d) listed above, is said to be a *Hopf algebra*. Among these, (b) and (c) are the defining properties of a *coalgebra* with diagonal Δ; see [Montgomery 1993] or [Sweedler 1969]. Property (a) says that the algebra and coalgebra structures are compatible. At first — and perhaps second and third — glance, property (d) appears mysterious. Here is one explanation that appeals to me: The diagonal homomorphism endows the k-vector space $\operatorname{Hom}_k(H,H)$ with the structure of a k-algebra, with the product of elements f and g given by

$$(f\star g)(r) = \sum_{i=1}^n f(r'_i)g(r''_i), \quad \text{where } \Delta(r) = \sum_{i=1}^n (r'_i\otimes_k r''_i).$$

This is called the *convolution product* on $\operatorname{Hom}_k(H,H)$; its unit is the element $\eta\circ\varepsilon$. Condition (d) asserts that σ is the inverse of the identity on H.

The group algebra is the prototypical example of a Hopf algebra, and many constructions and results pertaining to them are best viewed in that generality; see [Benson 1991a, Chapter 3]. There is another good source of Hopf algebras, close to home: the coordinate rings of algebraic groups. You might, as I did, find it entertaining and illuminating to write down the Hopf structure on the coordinate ring of the circle $x^2 + y^2 = 1$.

If this all too brief foray into Hopf algebras has piqued your curiosity and you wish to know more, you could start by reading Bergman's charming introduction [Bergman 1985]; if you prefer to jump right into the thick of things, then [Montgomery 1993] is the one for you.

2. Modules over Group Algebras

This section is an introduction to modules over group algebras. When G is a *finite* group, the k-algebra $k[G]$ is finite-dimensional, that is to say, of finite rank over k. Much of the basic theory for modules over finite group algebras is just a specialization of the theory for finite-dimensional algebras. For example, I hinted in Exercise (1.6) that for finite group algebras, the nilradical coincides with the Jacobson radical; this holds, more generally, for any finite-dimensional k-algebra. Here I will focus on two crucial concepts: the Jordan–Hölder theorem and the Krull–Schmidt property.

(2.1) **The Jordan–Hölder theorem.** Let R be a ring and M an R-module. It is clear that when M is both artinian and noetherian it has a *composition series*: a series of submodules $0 = M_l \subset M_{l-1} \subset \cdots \subset M_1 \subset M_0 = M$ with the property that the subfactors M_i/M_{i+1} are *simple*, that is to say, they have no proper submodules. It turns out that if $0 = M'_{l'} \subset M'_{l'-1} \subset \cdots \subset M'_1 \subset M'_0 = M$ is another composition series, then $l = l'$ and, for $1 \leq i, j \leq l$, the factors M_i/M_{i-1} are a permutation of the factors M'_j/M'_{j-1}. This is a consequence of the Jordan–Hölder theorem, which says that for each R-module, any two series (not necessarily composition series) of submodules can be refined to series of the same length and with the same subfactors.

Suppose that R is artinian; for example, R may be a finite-dimensional k-algebra, or, more specifically, a finite group algebra. In this case every finite, by which I mean 'finitely generated', module over it is both artinian and noetherian and so has a composition series. Here is one consequence: since every simple module is a quotient of R, all the simple modules appear in a composition series for R, and so there can only be finitely many of them.

(2.2) **Indecomposable modules.** Recall that a module is said to be *indecomposable* if it has no nontrivial direct summands. It is clear that a simple module is indecomposable, but an indecomposable module may be far from simple — in either sense of the word. For example, over a commutative ring, the only simple modules are the residue fields, whereas it is usually not possible to classify all

the indecomposable modules; I will pick up on this point a few paragraphs down the road. For now, here are a couple of remarks that are useful to keep in mind when dealing with indecomposability; see the discussion in (2.10).

In this sequel, when I say $(R, \mathfrak{m}, k)$ is a local ring, I mean that R is local, with maximal ideal $\mathfrak{m}$ and residue ring k.

(2.3) Exercise. Let $(R, \mathfrak{m}, k)$ be a commutative local ring. Prove that if M is indecomposable, then $\operatorname{socle}(M) \subseteq \mathfrak{m}M$.

(2.4) Exercise. Let R be a commutative local Gorenstein ring and M an indecomposable R-module. Prove that if $\operatorname{socle}(R) \cdot M \neq 0$, then $M \cong R$.

(2.5) **The Krull–Schmidt property.** Let R be a ring. It is not hard to see that each finite R-module can be broken down into a finite direct sum of indecomposables. The ring R has the *Krull–Schmidt property* if for each finite R-module such a decomposition is unique up to a permutation of the indecomposable factors: if

$$\bigoplus_{i=1}^{m} M_i \;\cong\; \bigoplus_{j=1}^{n} N_j,$$

with each M_i and N_j indecomposable, then $m = n$, and, with a possible rearrangement of the N_j, one has $M_i \cong N_i$ for each i.

For example, complete commutative noetherian local rings have this property; see [Swan 1968, (2.22)]. In the present context, the relevant result is that artinian rings have the Krull–Schmidt property [Benson 1991a, (1.4.6)]. When G is a finite group, $k[G]$ is artinian; in particular, it has the Krull–Schmidt property.

The Krull–Schmidt property is of great help in studying modules over group algebras, for it allows one to focus on the indecomposables. The natural question arises: when does the group algebra have only finitely many isomorphism classes of indecomposable modules? In other words, when is the group algebra of *finite representation type*? This is the case, for example, when every indecomposable module is simple, for there are only finitely many of them; see (2.1). There is an important context when this happens: when the characteristic of k is coprime to the order of the group. This is a consequence of Maschke's Theorem:

(2.6) Theorem (Maschke). *Let G be a finite group such that $|G|$ is coprime to the the characteristic of k. Each short exact sequence of $k[G]$-modules splits.*

Proof. Let $0 \to L \to M \xrightarrow{\pi} N \to 0$ be an exact sequence of $k[G]$-modules. Since k is a field, π admits a k-linear section; let $\sigma\colon N \to M$ be one such. It is not hard to verify that the map

$$\widetilde{\sigma}\colon N \to M, \qquad \text{where } \widetilde{\sigma}(n) = \frac{1}{|G|} \sum_{g \in G} g\sigma(g^{-1}n) \quad \text{for all } n \in N,$$

is $k[G]$-linear, and that $\pi \circ \widetilde{\sigma} = \mathrm{id}^N$. Thus, the exact sequence splits, as desired. □

This theorem has a perfect converse: if each short exact sequence of $k[G]$-modules splits, the characteristic of k is coprime to $|G|$. In fact, it suffices that the exact sequence $0 \to \mathrm{I}(G) \to k[G] \xrightarrow{\varepsilon} k \to 0$ splits. The proof is elementary, and is recommended as an exercise; I will offer a solution in the proof of Theorem (3.1).

A group algebra can have finite representation type even if not every indecomposable module is simple:

(2.7) **Finite cyclic groups.** In describing this example, it is convenient to let p denote 1 when the characteristic of k is 0, and the characteristic of k otherwise.

Let G be a finite cyclic group. Write $|G|$ as $p^n q$, where n is a nonnegative integer and p and q are coprime. Let $R = k[x]/(x^{p^n q} - 1)$, the group algebra. The binomial theorem in characteristic p yields $x^{p^n q} - 1 = (x^q - 1)^{p^n}$, so the Jacobson radical of R is $(x^q - 1)$. In $k[x]$, the polynomial $x^q - 1$ breaks up into a product of distinct irreducible polynomials:

$$x^q - 1 = \prod_{i=1}^{d} f_i(x), \quad \text{with} \quad \sum_{i=1}^{d} \deg(f_i(x)) = q.$$

Since the ideals $(f_i(x)^{p^n})$, where $1 \leq i \leq d$, in $k[x]$ are pairwise comaximal, the Chinese Remainder Theorem yields

$$R \cong \prod_{i=1}^{d} R_i, \quad \text{where } R_i = \frac{k[x]}{(f_i(x)^{p^n})}.$$

This implies that each R-module M decomposes uniquely as $M = \bigoplus_{i=1}^{d} M_i$, where M_i is an R_i-module. Furthermore, it is easy to see that $R_i/(f_i(x)^s)$, for $1 \leq s \leq p^n$, is a complete list of indecomposable modules over R_i, and that each M_i has a unique decomposition into a direct sum of such modules. This is exactly as predicted by the Krull–Schmidt theory. The upshot is that we know 'everything' about the modules over the group algebras of finite cyclic groups.

All this is subsumed in the structure theory of modules over principal ideal rings. By the by, the finite cyclic groups are the source of group algebras of finite representation type, in the following sense; see [Benson 1991a, (4.4)] for the appropriate references.

(2.8) Theorem. *If k is an infinite field of characteristic p and G a finite group, then $k[G]$ has finite representation type exactly when G has cyclic Sylow p-subgroups.* □

In some cases of infinite representation type, it is still possible to classify all the indecomposable modules. The Klein group is one such. Let me give you a flavour of the modules that arise over its group algebra. For the calculations, it is helpful to recall a result on syzygies of indecomposable modules.

(2.9) Let $(R, \mathfrak{m}, k)$ be a commutative artinian local ring and E the injective hull of the R-module k. Let M be a finite R-module. Write $\Omega^1 M$ for the first

syzygy of M, and $\Omega^{-1}M$ for the first co-syzygy of M. These are defined by exact sequences

$$(\dagger) \qquad 0 \to \Omega^1 M \to R^b \to M \to 0 \quad \text{and} \quad 0 \to M \to E^c \to \Omega^{-1} M \to 0,$$

with $b = \operatorname{rank}_k(M/\mathfrak{m}M)$ and $c = \operatorname{rank}_k \operatorname{socle}(M)$.

The conclusion of the following exercise is valid for the syzygy module even when R is a Gorenstein ring of higher (Krull) dimension, as long as M is also maximal Cohen–Macaulay; this was first proved by J. Herzog [1978].

EXERCISE. Assume that R is Gorenstein. Prove that when M is indecomposable, so are $\Omega^1 M$ and $\Omega^{-1} M$.

I cannot resist giving a sketch of the argument: Suppose $\Omega^1 M = U \oplus V$, with U and V nonzero. Since R is self-injective, neither U nor V can be free: if U is free, then it is injective and hence splits from R^b in the exact sequence $(\dagger)$ above, and that cannot happen. Now, $\operatorname{Hom}_R(-, R)$ applied to $(\dagger)$ yields an exact sequence

$$0 \to M^* \to R^b \to U^* \oplus V^* \to 0.$$

This presents M^* as the first syzygy of $U^* \oplus V^*$ (why?); that is,

$$M^* = \Omega^1(U^* \oplus V^*) = \Omega^1(U^*) \oplus \Omega^1(V^*).$$

Note that the modules $\Omega^1(U^*)$ and $\Omega^1(V^*)$ are nonzero: if $\Omega^1(U^*) = 0$, then $\operatorname{pdim}_R(U^*)$ is finite, so U^* is free, and hence U is free, a contradiction. It follows that the same is true even after we dualize them. Applying $\operatorname{Hom}_R(-, R)$ to the equality above gives us

$$M^{**} = \Omega^1(U^*)^* \oplus \Omega^1(V^*)^*$$

Since $M \cong (M^*)^*$, one obtains that M is indecomposable.

Now we turn to indecomposable modules over the Klein group.

(2.10) **The Klein group.** Let k be a field of characteristic 2 and let G be $\mathbb{Z}_2 \times \mathbb{Z}_2$, the Klein group. Let R denote its group algebra over k, so $R = k[y_1, y_2]/(y_2^2 - 1, y_2^2 - 1)$.

This k-algebra looks more familiar once we change variables: setting $x_i = y_i - 1$ one sees that $R = k[x_1, x_2]/(x_1^2, x_2^2)$; a local zero dimensional complete intersection with maximal ideal $\mathfrak{m} = (x_1, x_2)$. Note that R is Gorenstein, so $R \cong \operatorname{Hom}_k(R, k)$ and, for any R-module M, one has $M^* \cong \operatorname{Hom}_k(M, k)$, where $(-)^* = \operatorname{Hom}_R(M, R)$. I will use these remarks without ado.

For each positive integer n, let M_n denote $\Omega^n(k)$, the n-th syzygy of k. I claim that in the infinite family $\{\dots, M_2, M_1, k, (M_1)^*, (M_2)^*, \dots\}$ no two modules are isomorphic and that each is indecomposable.

Indeed, a repeated application of Exercise (2.9) yields that each M_n is indecomposable, and hence also that $(M_n)^*$ is indecomposable, since $(M_n)^{**} \cong M_n$.

As to the remaining assertion: for $i = 1, 2$, let $R_i = k[x_i]/(x_i^2)$. The minimal R_i-free resolution of k is

$$F_i = \quad \cdots \xrightarrow{x_i} R_i \xrightarrow{x_i} R_i \xrightarrow{x_i} R_i \to 0$$

Since $R = R_1 \otimes_k R_2$, the complex of R-modules $F_1 \otimes_k F_2$ is the minimal free resolution of the R-module k. It follows that the n-th Betti number of k is $n+1$. Thus, for any positive integer n, the n-th syzygy M_n of k is defined by an exact sequence

$$(\dagger) \qquad 0 \to M_n \to R^n \xrightarrow{\partial_{n-1}} R^{n-1} \to \cdots \to R^2 \xrightarrow{\partial_1} R \to k \to 0,$$

with $\partial_i(R^{i+1}) \subseteq \mathfrak{m}R^i$ for each i. It follows that $\operatorname{rank}_k M_n = 2n+1$, and hence also that $\operatorname{rank}_k (M_n)^* = 2n+1$. Therefore, to settle the claim that the modules in question are all distinct, it remains to verify that the R-modules M_n and $(M_n)^*$ are not isomorphic. These modules appear in exact sequences

$$0 \to M_n \to R^n \xrightarrow{\partial_{n-1}} R^{n-1} \qquad \text{and} \qquad 0 \to (M_n)^* \to R^{n+1} \xrightarrow{\partial_{n+1}^*} R^{n+2}.$$

The one on the right is obtained from

$$R^{n+2} \xrightarrow{\partial_{n+1}} R^{n+1} \to M_n \to 0,$$

keeping in mind that $R^* \cong R$. Since $\partial_{n-1}(R^n) \subseteq \mathfrak{m}R^{n-1}$ and $\partial_{n+1}^*(R^{n+1}) \subseteq \mathfrak{m}R^{n+2}$, the desired conclusion is a consequence of:

EXERCISE. Let $(R, \mathfrak{m}, k)$ be a local ring. If $0 \to K \to R^b \xrightarrow{f} R^c$ is an exact sequence of R-modules with $f(R^b) \subseteq \mathfrak{m}R^c$, then

$$\operatorname{socle}(K) = \operatorname{socle}(R^b) = \operatorname{socle}(R)^b.$$

This completes the justification that the given family consists of nonisomorphic indecomposables. In this process we found that $\operatorname{rank}_k M_n = 2n+1 = \operatorname{rank}_k (M_n)^*$. It turns out that the M_n, their k-duals, and k are the only indecomposables of odd rank; here is a sketch of the proof. Exercise: fill in the details.

Let M be an indecomposable R-module with $\operatorname{rank}_k M = 2n+1$ for some integer n. In particular, $M \not\cong R$, and so Exercise (2.4) tells us that $(xy)M = 0$, so $\mathfrak{m}^2 M = 0$ and hence $\mathfrak{m}M \subseteq \operatorname{socle}(M)$; the opposite inclusion also holds, by Exercise (2.3), hence $\mathfrak{m}M = \operatorname{socle}(M)$. Thus, one has an exact sequence of R-modules

$$0 \to \operatorname{socle}(M) \to M \to M/\mathfrak{m}M \to 0$$

Now we use Exercise (2.9); in the notation there, from the exact sequence above one deduces that either $b \leq n$ or $c \leq n$. In the former case $\operatorname{rank}_k(\Omega^1 M) \leq 2n-1$ and in the latter $\operatorname{rank}_k(\Omega^{-1} M) \leq 2n-1$. In any case, the ranks of $\Omega^1 M$ and

$\Omega^{-1}M$ are odd. Now an induction on rank yields that M belongs to the family of indecomposable R-modules that we have already constructed.

At this point, we know all the indecomposable R-modules of odd rank. The ones of even rank are harder to deal with. To get an idea of what goes on here, solve:

EXERCISE. Prove that every rank 2 indecomposable R-module is isomorphic to a member of the family of cyclic R-modules

$$V_{(\alpha_1,\alpha_2)} = \frac{R}{(\alpha_1 x_1 + \alpha_2 x_2, xy)}, \quad \text{where } (\alpha_1,\alpha_2) \neq (0,0).$$

Moreover, $V_{(\alpha_1,\alpha_2)} \cong V_{(\beta_1,\beta_2)}$ if and only if (α_1,α_2) and(β_1,β_2) are proportional.

Thus, the nonisomorphic indecomposable R-modules of rank 2 are parametrized by the projective line over k; it turns out that this is the case in any even rank, at least when k is algebraically closed. This classification of the indecomposable modules over the Klein group goes back to Kronecker; see [Alperin 1986] or [Benson 1991a, (4.3)] for a modern treatment.

This discussion shows that while the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in characteristic 2 is not of finite type, in any given rank all but finitely many of its indecomposable modules are contained in a one-parameter family. More generally, by allowing for finitely many one-parameter families in each rank, one obtains the notion of a *tame* algebra. Tame group algebras $k[G]$ are completely classified: the characteristic of k is 2, and the Sylow 2-subgroups of G are isomorphic to one of the following groups: Klein, dihedral, semidihedral, or generalized quaternion. See [Benson 1991a, (4.4.4)]. The significance of this result lies in that every finite-dimensional k-algebra that is neither of finite type nor tame is *wild*, which implies that the set of isomorphism classes of its finite-rank indecomposable modules contains representatives of the indecomposable modules over a tensor algebra in two variables.

RAMBLE. There is a significant parallel between module theory over finite group algebras and over artinian commutative Gorenstein rings; see the discussion around Theorem (3.6). In fact, this parallel extends to the category of maximal Cohen–Macaulay modules over commutative complete local Gorenstein rings. For example, analogous to Theorem (2.8), among this class of rings those of *finite Cohen–Macaulay type* (which means that there are only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules) have been completely classified, at least when the ring contains a field. A systematic exposition of this result can be found in [Yoshino 1990]. The next order of complexity beyond finite Cohen–Macaulay type is bounded Cohen–Macaulay type, which is a topic of current research: see [Leuschke and Wiegand $\geq$ 2004].

The rest of this section describes a few basic constructions, like tensor products and homomorphisms, involving modules over group algebras.

(2.11) **Conjugation.** Over a noncommutative ring, the category of left modules can be drastically different from that of right modules. For example, there exist rings over which every left module has a finite projective resolution, but not every right module does. Thus, in general, one has to be very careful vis-à-vis left and right module structures.

However, in the case of group algebras, each left module can be endowed with a natural structure of a right module, and vice versa. More precisely, if M is a *left* $k[G]$-module, then the k-vector space underlying M may be viewed as a *right* $k[G]$-module by setting

$$m \cdot g = g^{-1}m \quad \text{for each } g \in G \text{ and } m \in M.$$

For this reason, when dealing with modules over group algebras, one can afford to be lax about whether they are left modules or right modules. This also means, for instance, that a left module is projective (or injective) if and only if the corresponding right module has the same property.

This is similar to the situation over commutative rings: each left module N over a commutative ring R is a right module with multiplication

$$n \cdot r = rn \quad \text{for each } r \in R \text{ and } n \in N.$$

There is an important distinction between the two situations: over R, the module N becomes an R-bimodule with right module structure as above. However, over $k[G]$, the module M with prescribed right module structure is not a bimodule.

(2.12) **Tensor products.** Over an arbitrary ring, one cannot define the tensor product of two left modules. However, if M and N are two left modules over a group algebra $k[G]$, one can view M as a right module via conjugation (2.11) and make sense of $M \otimes_{k[G]} N$. But then this tensor product is *not* a $k[G]$-module, because M and N are not bimodules. In this respect, the group ring behaves like any old ring.

There is another tensor product construction, a lot more important when dealing with group algebras than the one above, that gives us back a $k[G]$-module. To describe it, we return briefly to the world of arbitrary k-algebras.

Let R and S be k-algebras and let M and N be (left) modules over R and S, respectively. There is a natural left $(R \otimes_k S)$-module structure on $M \otimes_k N$ with

$$(r \otimes_k s) \cdot (m \otimes_k n) = rm \otimes_k sn.$$

Now let M and N be left $k[G]$-modules. The preceding recipe provides an action of $k[G] \otimes_k k[G]$ on $M \otimes_k N$. This restricts, via the diagonal map (1.7), to a left $k[G]$-module structure on $M \otimes_k N$. Going through the definitions one finds that

$$g \cdot (m \otimes_k n) = gm \otimes_k gn,$$

for all $g \in G$, $m \in M$ and $n \in N$. It is worth remarking that the 'twisting' map

$$M \otimes_k N \xrightarrow{\cong} N \otimes_k M,$$
$$(m \otimes_k n) \mapsto (n \otimes_k m),$$

which is bijective, is $k[G]$-linear.

RAMBLE. To a commutative algebraist, the tensor product $M \otimes_k N$ has an unsettling feature: it is taken over k, rather than over $k[G]$. However, bear in mind that the $k[G]$-module structure on $M \otimes_k N$ uses the diagonal homomorphism. The other possibilities, namely acquiring the structure from M or from N, don't give us anything nearly as useful. For instance, $M \otimes_k N$ viewed as a $k[G]$-module via its left-hand factor is just a direct sum of copies of M.

(2.13) **Homomorphisms.** Let M and N be left $k[G]$-modules. One can then consider $\mathrm{Hom}_{k[G]}(M, N)$, the k-vector space of $k[G]$-linear maps from M to N. Like the tensor product over $k[G]$, this is not, in general, a $k[G]$-module. Note that since the $k[G]$-module k is cyclic with annihilator $\mathrm{I}(G)$, and $\mathrm{I}(G)$ is generated as an ideal by elements $g - 1$, one has

$$\mathrm{Hom}_{k[G]}(k, M) = \{m \in M \mid gm = m\}.$$

The k-subspace on the right is of course M^G, the set of G-invariant elements in M.

As with $M \otimes_k N$, one can endow the k-vector space $\mathrm{Hom}_k(M, N)$ with a canonical left $k[G]$-structure. This is given by the following prescription: for each $g \in G$, $\alpha \in \mathrm{Hom}_k(M, N)$, and $m \in M$, one has

$$(g \cdot \alpha)(m) = g\alpha(g^{-1}m).$$

In particular, $g \cdot \alpha = \alpha$ if and only if $\alpha(gm) = g\alpha(m)$; that is to say,

$$\mathrm{Hom}_{k[G]}(M, N) = \mathrm{Hom}_k(M, N)^G.$$

Thus the homomorphisms functor $\mathrm{Hom}_{k[G]}(M, N)$ is recovered as the k-subspace of G-invariant elements in $\mathrm{Hom}_k(M, N)$. This identification leads to the following Hom-Tensor adjunction formula:

$$\mathrm{Hom}_{k[G]}(L \otimes_k M, N) \cong \mathrm{Hom}_{k[G]}(L, \mathrm{Hom}_k(M, N)).$$

This avatar of Hom-Tensor adjunction is very useful in the study of modules over group algebras; see, for example, the proof of (3.2).

RAMBLE. Let G be a finite group such that the characteristic of k is coprime to $|G|$, and let $0 \to L \to M \to N \to 0$ be an exact sequence $k[G]$-modules. Applying $\mathrm{Hom}_{k[G]}(k, -)$ to it yields, in view of Maschke's theorem (2.6), an exact sequence

$$0 \to L^G \to M^G \to N^G \to 0.$$

This is why invariant theory in characteristics coprime to $|G|$ is so drastically different from that in the case where the characteristic of k divides $|G|$.

(2.14) **A technical point.** Let M be a left $k[G]$-module and set $M^*=\operatorname{Hom}_k(M,k)$. One has two choices for a left $k[G]$-module structure on M^*: one given by specializing the discussion in (2.13) to the case where $N = k$, and the other by conjugation — see (2.11) — from the natural *right* module structure on M^*. A direct calculation reveals that they coincide. What is more, these modules have the property that the canonical maps of k-vector spaces

$$\begin{aligned} M &\to M^{**} & N \otimes_k M^* &\to \operatorname{Hom}_k(M,N)\\ m &\mapsto \big(f \mapsto f(m)\big) & n \otimes_k f &\mapsto \big(m \mapsto f(m)n\big) \end{aligned}$$

are $k[G]$-linear. These maps are bijective when $\operatorname{rank}_k M$ is finite.

Ramble. Most of what I said from (2.11) onward applies, with appropriate modifications, to arbitrary Hopf algebras. For example, given modules M and N over a Hopf algebra H, the tensor product $M \otimes_k N$ is also an H-module with

$$h \cdot (m \otimes_k n) = \sum_{i=1}^{n} h'_i m \otimes_k h''_i n, \quad \text{where } \Delta(h) = \sum_{i=1}^{n} h'_i \otimes_k h''_i.$$

There are exceptions; for example, over a group algebra $M \otimes_k N \cong N \otimes_k M$; see (2.12). This holds over H only when $\sum_{i=1}^{n} h'_i \otimes_k h''_i = \sum_{i=1}^{n} h''_i \otimes_k h'_i$, that is to say, when the diagram

$$\begin{array}{ccc} & H & \\ {\scriptstyle\Delta}\swarrow & & \searrow{\scriptstyle\Delta} \\ H \otimes_k H & \xrightarrow[\tau]{} & H \otimes_k H \end{array}$$

commutes, where $\tau(h' \otimes_k h'') = (h'' \otimes_k h')$. Such an H is said to be *cocommutative*.

3. Projective Modules

The section focuses on projective modules over group algebras. First, I address the question: When is every module over the group algebra projective? In other words, when is the group algebra *semisimple*? Here is a complete answer, at least in the case of a finite group.

(3.1) Theorem. *Let G be a finite group. The following conditions are equivalent*:

(i) *The group ring $k[G]$ is semisimple.*
(ii) *k, viewed as a $k[G]$-module via the augmentation, is projective.*
(iii) *The characteristic of k is coprime to $|G|$.*

PROOF. (i) $\implies$ (ii) is a tautology.

(ii) $\implies$ (iii): As k is projective, the augmentation homomorphism $\varepsilon\colon k[G] \to k$, being a surjection, has a $k[G]$-linear section $\sigma\colon k \to k[G]$. Write $\sigma(1) = \sum_{g\in G} a_g g$, with a_g in k. Fix an element $h \in G$. Note that $\sigma(1) = \sigma(h \cdot 1) = h \cdot \sigma(1)$, where the first equality holds because $k[G]$ acts on k via ε, the second by the $k[G]$-linearity of σ. This explains the first equality below:

$$\sum_{g\in G} a_g g = \sum_{g\in G} a_g(hg) = \sum_{g\in G} a_{h^{-1}g} g.$$

The second is just a reindexing. The elements of G are a basis for the group algebra, so the equality above entails $a_{h^{-1}} = a_1$. This holds for each $h \in G$, so

$$1 = \varepsilon(\sigma(1)) = a_1 \sum_{g\in G} \varepsilon(g) = a_1 \sum_{g\in G} 1 = a_1|G|.$$

In particular, the characteristic of k is coprime to $|G|$.

(iii) $\implies$ (i): Let M be a $k[G]$-module, and pick a surjection $P \twoheadrightarrow M$ with P projective. Maschke's theorem (2.6) provides that every short exact sequence of $k[G]$-modules splits; equivalently, that every surjective homomorphism is split. In particular, $P \twoheadrightarrow M$ splits, so M is a direct summand of P, and hence projective. □

EXERCISE. A commutative ring is semisimple if and only if it is a product of fields.

The last result dealt with modules en masse; now the focus is shifted to individual modules.

Stability properties of projective modules. The gist of the following paragraphs is that many of the standard functors of interest preserve projectivity. A crucial, and remarkable, result in this direction is

(3.2) THEOREM. *Let G be a group and P a projective $k[G]$-module. For any $k[G]$-module X, the $k[G]$-modules $P \otimes_k X$ and $X \otimes_k P$ are projective.*

Take note that the tensor product is over k, as it must be, for such a conclusion is utterly wrong were it over $k[G]$. This theorem underscores the point raised in (2.12) about the importance of this tensor product in the module theory of group algebras; the other results in this section are all formal consequences of this one.

RAMBLE. There is another way to think about Theorem (3.2): one may view the entire category of $k[G]$-modules as a 'ring' with direct sum and tensor product over k playing the role of addition and multiplication respectively; the unit is k, and the commutativity of the tensor product means that this is even a 'commutative' ring. (With suitable compatibility conditions, such data define a *symmetric monoidal category*.) In this language, the theorem above is equivalent to the statement that the subcategory of projective modules is an ideal.

PROOF OF THEOREM (3.2). I will prove that $P \otimes_k X$ is projective. A similar argument works for $X \otimes_k P$; alternatively, note that it is isomorphic to $P \otimes_k X$, by (2.12).

One way to deduce that $P \otimes_k X$ is projective is to invoke the following isomorphism from (2.13), which is natural on the category of left $k[G]$-modules:

$$\operatorname{Hom}_{k[G]}(P \otimes_k X, -) \cong \operatorname{Hom}_{k[G]}(P, \operatorname{Hom}_k(X, -)).$$

Perhaps the following proof is more illuminating: by standard arguments one reduces to the case where $P = k[G]$. Write $X^\natural$ for the k-vector space underlying X. Now, by general principles, the inclusion of k-vector spaces $X^\natural \subset k[G] \otimes_k X$, defined by $x \mapsto 1 \otimes_k x$, induces a $k[G]$-linear map

$$k[G] \otimes_k X^\natural \to k[G] \otimes_k X, \quad \text{where } g \otimes_k x \mapsto g(1 \otimes_k x) = g \otimes_k gx.$$

The action of $k[G]$ on $k[G] \otimes_k X^\natural$ is *via the left-hand factor*. An elementary calculation verifies that the map below, which is $k[G]$-linear, is its inverse:

$$k[G] \otimes_k X \to k[G] \otimes_k X^\natural, \quad \text{where } g \otimes_k x \mapsto g \otimes_k (g^{-1}x).$$

Therefore, the $k[G]$-modules $k[G] \otimes_k X$ and $k[G] \otimes_k X^\natural$ are isomorphic. It remains to note that the latter module is a direct sum of copies of $k[G]$. □

One corollary of Theorem (3.2) is the following recognition principle for semisimplicity of the group algebra; it extends to arbitrary groups the equivalence of conditions (i) and (ii) in Theorem (3.1).

(3.3) LEMMA. *Let G be a group. The following conditions are equivalent.*

(i) *$k[G]$ is semisimple;*
(ii) *the $k[G]$-module k is projective.*

PROOF. The nontrivial implication is that (ii) $\implies$ (i). As to that, it follows from Theorem (3.2) that $k \otimes_k M$ is projective for each $k[G]$-module M, so it remains to check that the canonical isomorphism $k \otimes_k M \to M$ is $k[G]$-linear. Note that this is something that needs checking for the $k[G]$-action on $k \otimes_k M$ is via the diagonal; see (2.12). □

RAMBLE. Lemma (3.3), although not its proof, is reminiscent of a phenomenon encountered in the theory of commutative local rings: Over such a ring, the residue field is often a 'test' module. The Auslander–Buchsbaum–Serre characterization of regularity is no doubt the most celebrated example. It says that a noetherian commutative local ring R, with residue field k, is regular if and only if the R-module k has finite projective dimension.

There are analogous results that characterize complete intersections (Avramov and Gulliksen) and Gorenstein rings (Auslander and Bridger).

There is however an important distinction between a group algebra over k and a local ring with residue field k: over the latter, k is the only simple module, whilst the former can have many others. From this perspective, Lemma (3.3)

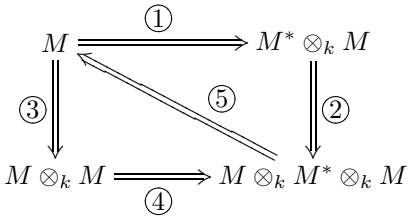
is rather surprising. The point is that an arbitrary finite-dimensional algebra is semisimple if and only if *every* simple module is projective; the nontrivial implication holds because each finite module has a composition series.

(3.4) THEOREM. *Let G be a finite group. For each finite $k[G]$-module M, the following $k[G]$-modules are projective simultaneously*: M, $M \otimes_k M$, $M^* \otimes_k M$, $M \otimes_k M^*$, $\operatorname{Hom}_k(M, M)$, *and* M^*.

PROOF. It suffices to verify: M, $M \otimes_k M$, and $M^* \otimes_k M$ are simultaneously projective.

Indeed, applied to M^* that would imply, in particular, that M^* and $(M^*)^* \otimes_k M^*$ are simultaneously projective. Now, $(M^*)^* \cong M$, since $\operatorname{rank}_k M$ is finite, and $M \otimes_k M^* \cong M^* \otimes_k M$, by the discussion in (2.12). Thus, one obtains the simultaneous projectivity of all the modules in question, except for $\operatorname{Hom}_k(M, M)$. However, the finiteness of $\operatorname{rank}_k M$ implies this last module is isomorphic to $M \otimes_k M^*$.

As to the desired simultaneous projectivity, it is justified by the diagram



where $X \implies Y$ should be read as 'if X is projective, then so is Y'. Implications (1)–(4) hold by Theorem (3.2). As to (5), the natural maps of k-vector spaces

$$M \to \operatorname{Hom}_k(M, M) \otimes_k M \to M$$
$$m \mapsto 1 \otimes_k m \text{ and } \alpha \otimes_k m \mapsto \alpha(m)$$

are $k[G]$-linear, and exhibit M as a direct summand of $\operatorname{Hom}_k(M, M) \otimes_k M$. However, as remarked before, the $k[G]$-modules $\operatorname{Hom}_k(M, M)$ and $M \otimes_k M^*$ are isomorphic, so M is a direct summand of $M \otimes_k M^* \otimes_k M$. □

Projective versus Injectives. So far, I have focused on projective modules, without saying anything at all about injective, or flat, modules. Now, a commutative algebraist well knows that projective modules and injective modules are very different beasts. There is, however, one exception.

(3.5) EXERCISE. Let R be a commutative noetherian local ring. Prove that when R is zero-dimensional and Gorenstein, an R-module is projective if and only if it is injective. Conversely, if there is a nonzero R-module that is both projective and injective, then R is zero-dimensional and Gorenstein.

The preceding exercise should be compared with the next two results.

(3.6) Theorem. *Let G be a finite group and M a finite $k[G]$-module. The following conditions are equivalent*:

(i) *M is projective*;
(ii) *the flat dimension of M is finite*;
(iii) *M is injective*;
(iv) *the injective dimension of M is finite*.

These equivalences hold for any $k[G]$-module, finite or not; see [Benson 1999].

The preceding theorem has an important corollary.

(3.7) Corollary. *The group algebra of a finite group is self-injective.* □

There are many other proofs, long and short, of this corollary; see [Benson 1991a, (3.1.2)]. Moreover, it is an easy exercise (do it) to deduce Theorem (3.6) from it.

Ramble. Let G be a finite group. Thus, the group algebra $k[G]$ is finite-dimensional and, by the preceding corollary, injective as a module over itself. These properties may tempt us commutative algebraists to proclaim: $k[G]$ is a zero-dimensional Gorenstein ring. And, for many purposes, this is a useful point of view, since module theory over a group algebra resembles that over a Gorenstein ring; Theorem (3.6) is one manifestation of this phenomenon. By the by, there are diverse extensions of the Gorenstein property for commutative rings to the noncommutative setting: Frobenius rings, quasi-Frobenius rings, symmetric rings, self-injective rings, etc.

The proof of Theorem (3.6) is based on Theorem (3.4) and an elementary observation about modules over finite-dimensional algebras.

(3.8) Lemma. *Let R be a k-algebra with $\operatorname{rank}_k R$ finite. For each finite left R-module M, one has $\operatorname{pdim}_R M = \operatorname{fdim}_R M = \operatorname{injdim}_{R^{\mathrm{op}}} M^*$.*

Proof. Since $\operatorname{rank}_k M$ is finite, $(M^*)^* \cong M$, so it suffices to prove the equivalence of the conditions

(i) M is projective;
(ii) M is flat;
(iii) the right R-module M^* is injective.

The implication (i) $\implies$ (ii) is immediate and hold for all rings. The equivalence (ii) $\iff$ (iii) is a consequence of the standard adjunction isomorphism

$$\operatorname{Hom}_k(- \otimes_R M, k) \cong \operatorname{Hom}_R(-, M^*)$$

and is valid for arbitrary k-algebras.

(iii) $\implies$ (i): Since M is finite over R, one can construct a surjective map $\pi\colon R^n \twoheadrightarrow M$. Dualizing this yields an inclusion $\pi^*\colon M^* \hookrightarrow (R^n)^*$ of right R-modules. This map is split because M^* is injective, and hence π^{**} is split. Since $\operatorname{rank}_k R$ and $\operatorname{rank}_k M$ are both finite, $\pi^{**} = \pi$, so that π is split as well. Thus, M is projective, as claimed. □

PROOF OF THEOREM (3.6). Theorem (3.4) yields that M is projective if and only if M^* is projective, while the lemma above implies that M^* is projective if and only if $(M^*)^*$ is injective, i.e., M is injective. This settles (i) $\iff$ (iii).

That (i) $\implies$ (ii) needs no comment. The lemma above contains (ii) $\iff$ (iv); moreover, it implies that to verify (ii) $\implies$ (i), one may assume $\operatorname{pdim}_R M$ finite, that is to say, there is an exact sequence

$$0 \to P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to P_0 \to M \to 0,$$

where each P_i is finite and projective; see (6.6). If $n \geq 1$, then, since P_n is injective by the already verified implication (i) $\implies$ (iii), the homomorphism ∂_n splits, and one obtains an exact sequence

$$0 \to \partial_{n-1}(P_{n-1}) \to P_{n-2} \to \cdots \to P_0 \to M \to 0.$$

In this sequence $\partial_{n-1}(P_{n-1})$, being a direct summand of P_{n-1}, is projective, and hence injective. An iteration of the preceding argument yields that M is a direct summand of P_0, and hence projective. □

RAMBLE. The small finitistic left global dimension of a ring R is defined as

$$\sup\{\operatorname{pdim}_R M \mid M \text{ a finite left } R\text{-module with } \operatorname{pdim}_R M < \infty.\}$$

One way of rephrasing Theorem (3.6) is to say that this number is zero when R is a finite group algebra. Exercise: Prove that a similar result holds also for modules over commutative artinian rings. However, over arbitrary finite-dimensional algebras, the small finitistic global dimension can be any nonnegative integer. A conjecture of Bass [1960] and Jans [1961], which remains open, asserts that this number is finite; look up [Happel 1990] for more information on this topic.

Hopf algebras. Theorem (3.2) holds also for modules over any finite-dimensional Hopf algebra; the proof via the adjunction isomorphism does not work, but the other one does. However, I found it a nontrivial task to pin down the details, and I can recommend it to you as a good way to gain familiarity with Hopf algebras. Given this, it is not hard to see that for *cocommutative* Hopf algebras, the analogues of theorems (3.4) and (3.6), and Corollary (3.7), all hold; the cocommutativity comes in because in the proof of (3.4) I used the fact that tensor products are symmetric; confer with the discussion in (2.14).

4. Structure of Projectives

So far, I have not addressed the natural question: what are the projective modules over the group algebra? In this section, I tabulate some crucial facts concerning these. Most are valid for arbitrary finite-dimensional algebras and are easier to state in that generality; [Alperin 1986] is an excellent reference for this circle of ideas.

(4.1) **Projective covers.** Let R be a ring and M a finite R-module. A *projective cover* of M is a surjective homomorphism $\pi\colon P \to M$ with P a projective R-module and such that each homomorphism $\sigma\colon P \to P$ that fits in a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \sigma\ } & P \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M & \end{array}$$

is bijective, and hence an automorphism. It is clear that projective covers, when they exist, are unique up to isomorphism. Thus, one speaks of *the* projective cover of M. Often P, rather than π, is thought as being the projective cover of M, although this is an abuse of terminology.

Among surjective homomorphisms $\kappa\colon Q \to M$ with Q a projective R-module, projective covers can be characterized by either of the properties:

(i) $Q/JQ \cong M/JM$, where J is the Jacobson radical of R;
(ii) Q is minimal with respect to direct sum decompositions.

When R is a noetherian ring over which every finite R-module has a projective cover, it is easy to see that a projective resolution

$$\mathbf{P}\colon \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} P_0 \to 0$$

of M so constructed that P_n is a projective cover of $\operatorname{Ker}(\partial_{n-1})$ is unique up to isomorphism of complexes of R-modules. Such a $\mathbf{P}$ is called the *minimal projective resolution* of M. Following conditions (i) and (ii) above, the minimality can also be characterized by either the property that $\partial(\mathbf{P}) \subseteq J\mathbf{P}$, or that $\mathbf{P}$ splits off from any projective resolution of M.

Projective covers exist for each finite M in two cases of interest: when R is a finite-dimensional k-algebra, and when R is a (commutative) local ring. This is why these two classes of rings have a parallel theory of minimal resolutions.

(4.2) **Simple modules.** Let R be a finite-dimensional k-algebra with Jacobson radical J, and let $\mathscr{P}$ and $\mathscr{S}$ be the isomorphism classes of indecomposable projective R-modules and of simple R-modules, respectively.

(a) The Krull–Schimdt property holds for R, so every P in $\mathscr{P}$ occurs as a direct summand of R, and there is a unique decomposition

$$R \cong \bigoplus_{P\in\mathscr{P}} P^{e_R(P)}, \quad \text{with } e_R(P) \geq 1.$$

In particular, R has only finitely many indecomposable projective modules.

(b) The simple R-modules are precisely the indecomposable modules of the semisimple ring $\widetilde{R} = R/J$ (verify this) so property (a) specialized to $\widetilde{R}$ reads

$$\widetilde{R} \cong \bigoplus_{S\in\mathscr{S}} S^{e_{\widetilde{R}}(S)}, \quad \text{with } e_{\widetilde{R}}(S) \geq 1.$$

(c) The ring $\widetilde{R}$ in (b), being semisimple, is a direct sum of matrix rings over finite-dimensional division algebras over k; see [Lang 2002, XVII]. Moreover, when k is algebraically closed, these division algebras coincide with k (why?), and we obtain that $e_{\widetilde{R}}(S) = \operatorname{rank}_k S$ for each $S \in \mathscr{S}$.

(d) From (a)–(c) one obtains that the assignment $P \mapsto P/JP$ is a bijection between $\mathscr{P}$ and $\mathscr{S}$; in other words, there are as many indecomposable projective R-modules as there are simple R-modules. Moreover, $e_R(P) = e_{\widetilde{R}}(P/JP)$. When k is algebraically closed, combining the last equality with that in (c) and the decomposition in (a) yields

$$\operatorname{rank}_k R = \sum_{P \in \mathscr{P}} \operatorname{rank}_k(P/JP) \operatorname{rank}_k P.$$

I will illustrate the preceding remarks by describing the indecomposable projective modules over certain finite group algebras.

(4.3) **Cyclic groups.** This example builds on the description in (2.7) of modules over the group algebra of a finite cyclic group G. We saw there that

$$k[G] \cong \prod_{i=1}^{d} \frac{k[x]}{(f_i(x)^{p^n})}.$$

This is the decomposition that for general finite-dimensional algebras is a consequence of the Krull–Schmidt property; see (4.2.a). For each $1 \le i \le d$, set $P_i = k[x]/(f_i(x)^{p^n})$. These $k[G]$-modules are all projective, being summands of $k[G]$, indecomposable (why?), and no two of them are isomorphic (count ranks, or look at their annihilators). Moreover, as a consequence of the decomposition above, any projective $k[G]$-module is a direct sum of the P_i. Thus, there are exactly d distinct isomorphism classes of indecomposable projective R-modules.

Over any commutative ring, the only simple modules are the residue fields. Thus, the simple modules over $k[G]$ are $k[x]/(f_i(x))$ where $1 \le i \le d$; in particular, there are as many as there are indecomposable projectives, exactly as (4.2.d) predicts.

Now I will describe the situation over finite abelian groups. Most of what I have to say can be deduced from:

(4.4) Lemma. *Let R and S be finite-dimensional k-algebras, and set $T = R \otimes_k S$. Let M and N be R-modules. If S is local with residue ring is k, and the induced map $k \to S \to k$ is the identity, then*

(a) *$M \cong N$ as R-modules if and only if $M \otimes_k S \cong N \otimes_k S$ as T-modules;*
(b) *the R-module M is indecomposable if and only if the T-module $M \otimes_k S$ is;*
(c) *M is projective if and only if the T-module $M \otimes_k S$ is projective.*

In particular, the map $P \mapsto P \otimes_k S$ induces a bijection between the isomorphism classes of indecomposable projective modules over R and over T.

PROOF. To begin with, note that $M \otimes_k S$ and $N \otimes_k S$ are both left R-modules and also right S-modules, with the obvious actions. Moreover, because of our hypothesis that the residue ring of S is k, one has isomorphisms of R-modules

$$M \cong (M \otimes_k S) \otimes_S k \qquad \text{and} \qquad N \cong (N \otimes_k S) \otimes_S k.$$

Now, the nontrivial implication in (a) and in (c) — the one concerning descent — is settled by applying $- \otimes_S k$. As to (b), the moot point is the ascent, so assume the R-module M is indecomposable and that $M \otimes_k S \cong U \oplus V$ as T-modules. Applying $- \otimes_S k$, one obtains isomorphisms of R-modules

$$M \cong (M \otimes_k S) \otimes_S k \cong (U \otimes_S k) \oplus (V \otimes_S k)$$

Since M is indecomposable, one of $U \otimes_S k$ or $V \otimes_S k$ is zero; say, $U \otimes_S k$ is 0, that is to say, $U = U\mathfrak{n}$, where $\mathfrak{n}$ is the maximal ideal of S. This implies $U = 0$, because, S being local and finite-dimensional over k, the ideal $\mathfrak{n}$ is nilpotent. □

(4.5) **Finite abelian groups.** Again, we adopt that convention that p is the characteristic of k when the latter is positive, and 1 otherwise.

Let G a finite abelian group, and write $|G|$ as $p^n q$, where n is a nonnegative integer and p and q are coprime. Via the fundamental theorem on finitely generated abelian groups this decomposition of $|G|$ translates into one of groups: $G = A \oplus B$, where A and B are abelian, $|A| = p^n$, and $|B| = q$. Hence, $k[G] \cong k[A] \otimes_k k[B]$.

Now, $A \cong \mathbb{Z}/(p^{e_1}\mathbb{Z}) \oplus \cdots \oplus \mathbb{Z}/(p^{e_m}\mathbb{Z})$, for nonnegative integers $e_1, \dots, e_m$, so

$$k[A] \cong \frac{k[y_1, \dots, y_m]}{(y_1^{p^{e_1}} - 1, \dots, y_m^{p^{e_m}} - 1)}$$

The binomial theorem in characteristic p yields $y_i^{p^{e_i}} - 1 = (y_i - 1)^{p^{e_i}}$ for each i. Thus, it is clear that $k[A]$ is an artinian local ring with residue field k.

In the light of this and Lemma (4.4), to find the indecomposable projectives over $k[G]$, it suffices to find those over $k[B]$.

When B is cyclic, this information is contained in (4.3). The general case is more delicate. First, since $|B|$ is coprime to p, every $k[G]$-module is projective, so the indecomposables among them are precisely the simple $k[B]$-modules; see Theorem (3.1). Now, as noted before, over any commutative ring the only simple modules are the residue fields. Thus, the problem is to find the maximal ideals of $k[B]$. Writing B as $\mathbb{Z}/(q_1\mathbb{Z}) \oplus \cdots \oplus \mathbb{Z}/(q_n\mathbb{Z})$, one has

$$k[B] \cong \frac{k[x_1, \dots, x_n]}{(x_1^{q_1} - 1, \dots, x_m^{q_n} - 1)}.$$

If k is algebraically closed, there are $q_1 \cdots q_n$ distinct maximal ideals, and hence as many distinct indecomposable projectives. The general situation is trickier.

By the by, if you use the method outlined above for constructing projective modules over a cyclic group, the outcome will appear to differ from that given by (4.3). Exercise: Reconcile them.

(4.6) **p-groups.** As always, free $k[G]$-modules are projective. When the characteristic of k is p and G is a p-group, these are the only projectives over $k[G]$. This is thus akin to the situation over commutative local rings, and the proof over this latter class of rings given in [Matsumura 1989] carries over; the key ingredient is that, as noted in (1.5), the group algebra of a p-group is an artinian local ring.

In general, the structure of projective modules over the group algebra is a lot more complicated. However, the triviality of the projectives in the case of p-groups also has implications for the possible ranks of indecomposable projectives over the group algebra of an arbitrary group G.

(4.7) **Sylow subgroups.** Let p^d be the order of a p-Sylow subgroup of G. If a finite $k[G]$-module P is projective, then p^d divides $\operatorname{rank}_k P$.

Indeed, for each p-Sylow subgroup $H \subseteq G$, the restriction of P to the subring $k[H]$ of $k[G]$ is a projective module, and hence a free module. Thus, by the preceding remark, $\operatorname{rank}_k P$ is divisible by $\operatorname{rank}_k k[H]$, that is to say, by $|H|$.

The numerological restrictions in (4.2) and (4.7) can be very handy when hunting for projective modules over finite group algebras. Here is a demonstration.

(4.8) **Symmetric group on three letters.** The symmetric group on three letters, Σ_3, is generated by elements a and b, subject to the relations

$$a^2 = 1, \quad b^3 = 1, \quad \text{and} \quad ba = ab^2.$$

Thus, $\Sigma_3 = \{1, b, b^2, a, ab, ab^2\}$. It has three 2-Sylow subgroups: $\{1, a\}$, $\{1, ab\}$, and $\{1, ba\}$, and one 3-Sylow subgroup: $\{1, b, b^2\}$.

Let p be the characteristic of the field k; we allow the possibility that $p = 0$.

Case (α). If $p \neq 2, 3$, every $k[\Sigma_3]$-module is projective, by Theorem (3.1).

Case (β). Suppose $p = 3$. By (4.7), the rank of each finite projective $k[G]$-module is divisible by 3, since the latter is the order of the 3-Sylow subgroup. Moreover, (4.2.d) implies that the number of indecomposable projectives equals the number of simple modules, and the latter is at least 2, for example, by Exercise (1.6). These lead us to the conclusion that there are exactly two indecomposable projectives, each having rank 3.

One way to construct them is as follows: Let $H = \{1, a\}$, a 2-Sylow subgroup of Σ_3. There are two nonisomorphic $k[H]$-module structures on k: the trivial one, given by the augmentation map, and the one defined by character $\sigma\colon H \to k$ with $\sigma(a) = -1$; denote the latter ${}^{\sigma}k$. Plainly, both these $k[H]$-modules are simple and hence, by Theorem (3.1), projective. Consequently, base change along the canonical inclusion $k[H] \to k[\Sigma_3]$ gives us two projective $k[\Sigma_3]$-modules,

$$k[\Sigma_3] \otimes_{k[H]} k \quad \text{and} \quad k[\Sigma_3] \otimes_{k[H]} {}^{\sigma}k.$$

They both have rank 3. I leave it to you to verify that they are not isomorphic. Hint: calculate the Σ_3-invariants.

CASE (γ). The situation gets even more interesting when $p = 2$. I claim that there are two indecomposable projective $k[G]$-modules, of ranks 2 and 4, when $x^2 + x + 1$ is irreducible in k, and three of them, each of rank 2, otherwise.

Indeed, let $H = \{1, b, b^2\}$; this is a cyclic group of order 3. Hence, by (4.3), when $x^2 + x + 1$ is irreducible in $k[x]$, the group algebra $k[H]$ has 2 (nonisomorphic) simple modules, of ranks 1 and 2, and when x^2+x+1 factors in $k[x]$, there are 3 simple modules, each of rank 1. As the characteristic of k does not divide $|H|$, all these simple modules are projective, so base change along the inclusion $k[H] \subset k[\Sigma_3]$ gives rise to the desired number of projective modules, and of the right ranks, over $k[G]$. Note that, by (4.7), projective modules of rank 2 are indecomposable. Thus, to be sure that these are the projectives one seeks, one has to verify that in the former case the rank 4 module is indecomposable, and in the latter that the three rank 2 modules are nonisomorphic. Once again, I will let you check this.

5. Cohomology of Supplemented Algebras

This section collects basic facts concerning the cohomology of supplemented algebras. To begin with, recall that in the language of Cartan and Eilenberg [1956] a *supplemented k-algebra* is a k-algebra R with unit $\eta\colon k \to R$ and an augmentation $\varepsilon\colon R \to k$ such that $\varepsilon \circ \eta$ is the identity on k.

Group algebras are supplemented, but there are many more examples. Take, for instance, any positively (or negatively) graded k-algebra with degree 0 component equal to k. Or, for that matter, take the power series ring $k[\![x_1, \cdots, x_n]\!]$, with η the canonical inclusion, and ε the evaluation at 0. More generally, thanks to Cohen's Structure Theorem, if a complete commutative local ring R, with residue field k, contains a field, then R is a supplemented k-algebra.

Let R be a supplemented k-algebra, and view k as an R-module via the augmentation. Let M be a (left) R-module. The *cohomology of R with coefficients in M* is the graded k-vector space $\operatorname{Ext}^*_R(k, M)$. The cohomology of R with coefficients in k, that is to say, $\operatorname{Ext}^*_R(k, k)$, is usually called the *cohomology of R*.

The k-vector space structure on $\operatorname{Ext}^*_R(k, k)$ can be enriched to that of a supplemented k-algebra, and then $\operatorname{Ext}^*_R(k, M)$ can be made into a *right* module over it. There are two ways to introduce these structures: via Yoneda splicing and via compositions. They yield the same result, up to a sign; see (5.2). I have opted for composition products because it is this description that I use to calculate group cohomology in the sequel.

(5.1) **Composition products.** Let P be a projective resolution of k. Composition endows the complex of k-vector spaces $\operatorname{Hom}_R(P, P)$ with a product structure, and this product is compatible with the differential, in the sense that, for every pair of homogenous elements f, g in $\operatorname{Hom}_R(P, P)$, one has

$$\partial(fg) = \partial(f)g + (-1)^{|f|} f\partial(g).$$

In other words, $\operatorname{Hom}_R(P,P)$ is a differential graded algebra (DGA). One often refers to this as the *endomorphism DGA* of P. It is not hard to verify that the multiplication of $\operatorname{Hom}_R(P,P)$ descends to homology, that is to say, to $\operatorname{Ext}^*_R(k,k)$. This is the *composition product* on cohomology, and it makes it a graded k-algebra. It is even supplemented, since $\operatorname{Ext}^0_R(k,k) = k$.

Let F be a projective resolution of M. The endomorphism DGA $\operatorname{Hom}_R(P,P)$ acts on the complex $\operatorname{Hom}_R(P,F)$ via composition on the right, and, once again, this action is compatible with the differentials. Thus, $\operatorname{Hom}_R(P,F)$ becomes a DG *right* module over $\operatorname{Hom}_R(P,P)$. These structures are inherited by the corresponding homology vector spaces; thus does $\operatorname{Ext}^*_R(k,M)$ become a right $\operatorname{Ext}^*_R(k,k)$-module.

One has to check that the composition products defined do not depend on the choice of resolutions; [Bourbaki 1980, (7.2)] justifies this, and much more.

(5.2) Remark. As mentioned before, one can introduce products on $\operatorname{Ext}^*_R(k,k)$ also via Yoneda multiplication, and, *up to a sign*, this agrees with the composition product; [Bourbaki 1980, (7.4)] has a careful treatment of these issues. The upshot is that one can set up an isomorphism of k-algebras between the Yoneda Ext-algebra and Ext-algebra with composition products. Thus, one has the freedom to use either structure, as long as it is done consistently.

(5.3) **Graded-commutativity.** Let E be a graded algebra. Elements x and y in E are said to commute, in the graded sense of the word, if

$$xy = (-1)^{|x||y|}\, yx.$$

If every pair of its elements commute, E is said to be graded-commutative. When E is concentrated in degree 0 or in even degrees, it is graded-commutative precisely when it is commutative in the usual sense.

An exterior algebra on a finite-dimensional vector space sitting in odd degrees is another important example of a graded-commutative algebra. More generally, given a graded vector space V, with $V_i = 0$ for $i < 0$, the tensor product of the symmetric algebra on V_{even} and exterior algebra on V_{odd}, that is to say, the k-algebra

$$\operatorname{Sym}(V_{\text{even}}) \otimes_k \bigwedge V_{\text{odd}},$$

is graded-commutative. If the characteristic of k happens to be 2, then $\operatorname{Sym}(V)$ is also graded-commutative even when $V_{\text{odd}} \neq 0$. This fails in odd characteristics, the point being that, in a graded-commutative algebra, for an element x of odd degree, $x^2 = -x^2$, so that $x^2 = 0$ when 2 is invertible in E.

A graded-commutative algebra with the property that $x^2 = 0$ whenever the degree of x is odd is said to be *strictly graded-commutative*. An exterior algebra (with generators in odd degrees) is one example. Here is one more, closer to home: for a homomorphism of commutative rings $R \to S$, the graded S-module $\operatorname{Tor}^R_*(S,S)$ is strictly graded-commutative, with the pitchfork product (homology product) defined by Cartan and Eilenberg; see [Mac Lane 1995, VIII § 2].

(5.4) **Functoriality.** The product in cohomology is functorial, in that, given a homomorphism of supplemented k-algebras $\varphi\colon R \to R'$, the induced map of graded k-vector spaces

$$\operatorname{Ext}^*_\varphi(k,k)\colon \operatorname{Ext}^*_{R'}(k,k) \to \operatorname{Ext}^*_R(k,k)$$

is a homomorphism of supplemented k-algebras.

Now let R and S be supplemented k-algebras. The tensor product $R \otimes_k S$ is also a supplemented k-algebra, and the canonical maps

$$R \xleftarrow{1\otimes\varepsilon^S} R \otimes_k S \xrightarrow{\varepsilon^R\otimes 1} S$$

respect this structure. By functoriality of products, the diagram above induces homomorphisms of supplemented k-algebras

$$\operatorname{Ext}^*_R(k,k) \xrightarrow{\operatorname{Ext}^*_{1\otimes\varepsilon^S}(k,k)} \operatorname{Ext}^*_{R\otimes_k S}(k,k) \xleftarrow{\operatorname{Ext}^*_{\varepsilon^R\otimes 1}(k,k)} \operatorname{Ext}^*_S(k,k).$$

It is not hard to check that the images of these maps commute, in the graded sense, so one has a diagram of supplemented k-algebras:

$$(*)\qquad \begin{array}{ccccc} \operatorname{Ext}^*_R(k,k) & \xrightarrow{\operatorname{Ext}^*_{\mathrm{id}\otimes\varepsilon^S}(k,k)} & \operatorname{Ext}^*_{R\otimes_k S}(k,k) & \xleftarrow{\operatorname{Ext}^*_{\varepsilon^R\otimes\mathrm{id}}(k,k)} & \operatorname{Ext}^*_S(k,k) \\ & \searrow^{\mathrm{id}\otimes 1} & \uparrow & \swarrow_{1\otimes\mathrm{id}} & \\ & & \operatorname{Ext}^*_R(k,k)\otimes_k \operatorname{Ext}^*_S(k,k) & & \end{array}$$

I should point out that the tensor product on the lower row is the *graded* tensor product and the multiplication on it is defined accordingly, that is,

$$(r\otimes_k s)\cdot(r'\otimes_k s') = (-1)^{|s||r'|}(rr'\otimes_k ss').$$

Under suitable finiteness hypotheses — for example, if R and S are noetherian — the vertical map in $(*)$ is bijective. However, this is not of importance to us.

The cohomology of Hopf algebras. The remainder of this section deals with the cohomology of Hopf algebras. So let H be a Hopf algebra, with diagonal Δ and augmentation ε; see (1.8). The main example to keep in mind is the case when H is the group algebra of a group, with the diagonal defined in (1.7).

One crucial property of the cohomology algebra of H, which distinguishes it from the cohomology of an arbitrary supplemented algebra, is the following.

(5.5) Proposition. *The cohomology algebra* $\operatorname{Ext}^*_H(k,k)$ *is graded-commutative.*

Note that H is not assumed to be cocommutative. This is a striking result, and its proof is based on the diagram of k-algebra homomorphisms

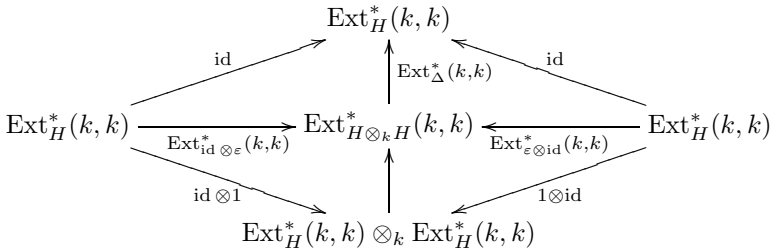
$$(5\text{–}1)\qquad \operatorname{Ext}^*_H(k,k)\otimes_k \operatorname{Ext}^*_H(k,k) \to \operatorname{Ext}^*_{H\otimes_k H}(k,k) \xrightarrow{\operatorname{Ext}^*_\Delta(k,k)} \operatorname{Ext}^*_H(k,k),$$

where the one on the left is the vertical map in (5.4.1), with R and S equal to H, and the one on the right is induced by the diagonal homomorphism.

(5.6) PROPOSITION. *The composition of homomorphisms in* (5.5.5–1) *is the product map*; *that is to say*, $(x \otimes_k y) \mapsto xy$ *for* x *and* y *in* $\operatorname{Ext}^*_H(k,k)$.

In particular, *the product map of* $\operatorname{Ext}^*_H(k,k)$ *is a homomorphism of* k*-algebras*.

PROOF. The diagram in question expands to the following commutative diagram of homomorphisms of k-algebras, where the lower half is obtained from (5.4.1), the upper half is induced by property (c) of Hopf algebras — see (1.8) — to the effect that ε is a co-unit for the diagonal.



Let x and y be elements in $\operatorname{Ext}^*_H(k,k)$. The element x goes to $x \otimes_k 1$ under the map heading southeast, and to x under the map heading northeast. The commutativity of the diagram thus implies that $x \otimes_k 1 \mapsto x$ under the composed vertical map. A similar diagram chase reveals that $1 \otimes_k y \mapsto y$. Since the vertical maps are homomorphisms of k-algebras, one has

$$x \otimes_k y = (x \otimes_k 1) \cdot (1 \otimes_k y) \mapsto xy.$$

This is the conclusion we seek. □

The proof of Proposition (5.5) uses also the following elementary exercise, of which there are versions for groups, for coalgebras, etc.

(5.7) EXERCISE. A graded k-algebra R is graded-commutative precisely when the product map $R \otimes_k R \to R$ with $r \otimes s \mapsto rs$ is a homomorphism of rings.

Now one can prove that the cohomology algebra is graded-commutative.

PROOF OF PROPOSITION (5.5). By the preceding proposition, the product map $\operatorname{Ext}^*_H(k,k) \otimes_k \operatorname{Ext}^*_H(k,k) \to \operatorname{Ext}^*_H(k,k)$ given by $x \otimes_k y \mapsto xy$ is a homomorphism of rings (for a general algebra it is only k-linear). To complete the proof one has to do Exercise (5.7). □

6. Group Cohomology

In this section we return to group algebras.

(6.1) **Cohomology.** Let G be a group and let M be a $k[G]$-module. Recall that $k[G]$ is a supplemented algebra. The *cohomology of* G *with coefficients in* M is the graded k-vector space

$$\mathrm{H}^*(G, M) = \operatorname{Ext}^*_{k[G]}(k, M).$$

There is no ambiguity concerning the field k since $\operatorname{Ext}^*_{k[G]}(k,M)$ is isomorphic to $\operatorname{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z},M)$; see [Evens 1961, (1.1)]. The *cohomology of* G is $\mathrm{H}^*(G,k)$.

Standard properties of Ext-modules carry over to the situation on hand. For instance, each short exact sequence of $k[G]$-modules $0 \to L \to M \to N \to 0$ engenders a long exact sequence of k-vector spaces

$$0 \to \mathrm{H}^0(G,L) \to \mathrm{H}^0(G,M) \to \mathrm{H}^0(G,N) \to \mathrm{H}^1(G,L) \to \mathrm{H}^1(G,M) \to \cdots .$$

Note that $\mathrm{H}^n(G,-) = 0$ for $n \geq 1$ if and only if k is projective. Therefore, one has the following cohomological avatar of Maschke's theorem (3.1):

(6.2) Theorem. *Let G be a finite group. Then $\mathrm{H}^n(G,-) = 0$ for each integer $n \geq 1$ if and only if the characteristic of k is coprime to $|G|$.* □

As is typical in homological algebra, low degree cohomology modules have nice interpretations. For a start, $\operatorname{Ext}^0_{k[G]}(k,M) = \operatorname{Hom}_{k[G]}(k,M)$, so (2.13) yields

$$\mathrm{H}^0(G,M) = M^G.$$

Thus, one can view the functors $\mathrm{H}^n(G,-)$ as the derived functors of invariants.

The degree 1 component of $\mathrm{H}^*(G,M)$ is also pretty down to earth. Recall that a map $\theta\colon G \to M$ is said to be a *derivation*, or a *crossed homomorphism*, if it satisfies the Leibniz formula: $\theta(gh) = \theta(g) + g\theta(h)$, for every g,h in G. The asymmetry in the Leibniz rule is explained when one views M, which is *a priori* only a left $k[G]$-module, as a $k[G]$-bimodule with trivial right action: $m \cdot g = m$. Using the k-vector space structure on M one can add derivations, and multiply them with elements in k, so they form a k-vector space; this is denoted $\operatorname{Der}(G;M)$. This vector space interests us because of the following

(6.3) Lemma. *The k-vector spaces $\operatorname{Hom}_{k[G]}(\mathrm{I}(G),M)$ and $\operatorname{Der}(G;M)$ are isomorphic via the maps*

$$\begin{aligned} \operatorname{Hom}_{k[G]}(\mathrm{I}(G),M) &\to \operatorname{Der}(G;M) & \operatorname{Der}(G;M) &\to \operatorname{Hom}_{k[G]}(\mathrm{I}(G),M)\\ \alpha &\mapsto \big(g \mapsto \alpha(g-1)\big), & \theta &\mapsto \big(g-1 \mapsto \theta(g)\big). \end{aligned}$$

The proof is an elegant computation and is best rediscovered on one's own. As to its bearing on $\mathrm{H}^1(G,M)$: applying $\operatorname{Hom}_{k[G]}(-,M)$ to the exact sequence

$$0 \to \mathrm{I}(G) \to k[G] \to k \to 0$$

of $k[G]$-modules leads to the exact sequence of k-vector spaces

$$0 \to M^G \to M \to \operatorname{Der}(G;M) \to \mathrm{H}^1(G,M) \to 0.$$

In this sequence, each $m \in M$ maps to a derivation: $g \mapsto (g-1)m$; these are the *inner derivations* from G to M, and their set is denoted by $\operatorname{IDer}(G;M)$. Thus,

$$\mathrm{H}^1(G,M) = \operatorname{Der}(G;M)/\operatorname{IDer}(G;M).$$

Let us specialize to the case when $M = k$. The Leibniz rule for a derivation $\theta\colon G \to k$ then reads: $\theta(gh) = \theta(g) + \theta(h)$, so $\operatorname{Der}(G;k)$ coincides with group

homomorphisms from G to k. Moreover, every inner derivation from G to k is trivial. The long and short of this discussion is that $\mathrm{H}^1(G,k)$ is precisely the set of additive characters from G to k.

There are other descriptions, some of a more group theoretic flavour, for $\mathrm{H}^1(G,M)$; for those the reader may look in [Benson 1991a].

The discussion in Section 5 on products on cohomology applies in the special case of the cohomology of group algebras. In particular, since $k[G]$ is a Hopf algebra, Proposition (5.5) specializes thus:

(6.4) THEOREM. *The cohomology algebra* $\mathrm{H}^*(G,k)$ *is graded-commutative.* □

(6.5) **Künneth formula.** Let G_1 and G_2 be groups. Specializing (5.4.1) to the case where $R = k[G_1]$ and $S = k[G_2]$, one obtains a homomorphism of k-algebras

$$\mathrm{H}^*(G_1,k) \otimes_k \mathrm{H}^*(G_2,k) \to \mathrm{H}^*(G_1 \times G_2, k).$$

This map is bijective whenever the group algebras are noetherian. This is the case when, for example, G_i is finite, or finitely generated and abelian.

(6.6) **Resolutions.** If one wants to compute cohomology from first principles, one has to first obtain a projective resolution of k over $k[G]$. In this regard, it is of interest to get as economical a resolution as possible. Fortunately, any finitely generated module over $k[G]$ has a minimal projective resolution; we discussed this point already in (4.1); unfortunately, writing down this minimal resolution is a challenge. In this the situation over group algebras is similar to that over commutative local rings. What is more difficult is calculating products from these minimal resolutions.

There is a canonical resolution for k over $k[G]$ called the *Bar resolution*; while it is never minimal, it has the merit that there is a simple formula for calculating the product of cohomology classes. The are many readable sources for this, such as [Benson 1991a, (3.4)], [Evens 1991, (2.3)], and [Mac Lane 1995, IV § 5], so I will not reproduce the details here.

7. Finite Generation of the Cohomology Algebra

In the preceding section, we noted that the cohomology algebra of a finite group is graded-commutative. From this, the natural progression is to the following theorem, contained in [Evens 1991], [Golod 1959], and [Venkov 1959].

(7.1) THEOREM. *Let G be a finite group. The k-algebra $\mathrm{H}^*(G,k)$ is finitely generated, and hence noetherian.* □

This result, and its analogues for other types of groups, is the starting point of Benson's article [2004]; see the discussion in Section 4 of it. There are many ways of proving Theorem (7.1), some more topological than others; one that is entirely algebraic is given in [Evens 1961, (7.4)].

In this section I prove the theorem in some special cases. But first:

RAMBLE. Theorem (7.1) has an analogue in commutative algebra: Gulliksen [1974] proves that when a commutative local ring R, with residue field k, is a complete intersection, the cohomology algebra $\operatorname{Ext}^*_R(k,k)$ is noetherian. There is a perfect converse: Bøgvad and Halperin [1986] have proved that if the k-algebra $\operatorname{Ext}^*_R(k,k)$ is noetherian, then R must be complete intersection.

There are deep connections between the cohomology of modules over complete intersections and over group algebras. This is best illustrated by the theory of support varieties. In group cohomology it was initiated by Quillen [1971a; 1971b], and developed in depth by Benson and Carlson, among others; see [Benson 1991b] for a systematic introduction. In commutative algebra, support varieties were introduced by Avramov [1989]; see also [Avramov and Buchweitz 2000].

As always, there are important distinctions between the two contexts. For example, the cohomology algebra of a complete intersection ring is generated by its elements of degree 1 and 2, which need not be the case with group algebras. More importantly, once the defining relations of the complete intersection are given, one can write down the cohomology algebra; the prescription for doing so is given in [Sjödin 1976]. Computing group cohomology is an entirely different cup of tea. Look up [Carlson 2001] for more information on the computational aspects of this topic.

Now I describe the cohomology algebra of finitely generated abelian groups. In this case, the group algebra is a complete intersection — see (1.4) — so one may view the results below as being about commutative rings or about finite groups.

(7.2) PROPOSITION. *For each positive integer n, the cohomology of $\mathbb{Z}^n$ is the exterior algebra on an n-dimensional vector space concentrated in degree* 1.

PROOF. As noted in (1.2), the group algebra of $\mathbb{Z}$ is $k[x^{\pm 1}]$, with augmentation defined by $\varepsilon(x) = 1$. The augmentation ideal is generated by $x-1$, and since this element is regular, the Koszul complex

$$0 \to k[x^{\pm 1}] \xrightarrow{x-1} k[x^{\pm 1}] \to 0,$$

is a free resolution of k. Applying $\operatorname{Hom}_{k[x^{\pm 1}]}(-,k)$ yields the complex with trivial differentials: $0 \to k \to k \to 0$, and situated in cohomological degrees 0 and 1. Thus, $\mathrm{H}^0(\mathbb{Z},k) = k = \mathrm{H}^1(\mathbb{Z},k)$. Moreover, $\mathrm{H}^1(\mathbb{Z},k) \cdot \mathrm{H}^1(\mathbb{Z},k) = 0$, by degree considerations, so that the cohomology algebra is the exterior algebra $\wedge_k k$, where the generator for k sits in degree 1.

For $\mathbb{Z}^n$, one uses the Künneth formula (6.5) to calculate group cohomology:

$$\mathrm{H}^*(\mathbb{Z}^n, k) = \mathrm{H}^*(\mathbb{Z},k)^{\otimes n} = \wedge_k k^n,$$

where the generators of k^n are all in (cohomological) degree 1. □

The next proposition computes the cohomology of cyclic p-groups. It turns out that one gets the same answer for all but one of them; the odd man out is the group of order two.

(7.3) PROPOSITION. *Let k be a field of characteristic p, and let $G = \mathbb{Z}/p^e\mathbb{Z}$, for some integer $e \geq 1$.*

(i) *When $p = 2$ and $e = 1$, $\mathrm{H}^*(G,k) = \mathrm{Sym}(ke_1^*)$, with $|e_1^*| = 1$.*
(ii) *Otherwise $\mathrm{H}^*(G,k) = \bigwedge(ke_1^*) \otimes_k \mathrm{Sym}(ke_2^*)$, with $|e_1^*| = 1$ and $|e_2^*| = 2$.*

PROOF. The group algebra of G is $k[x]/(x^{p^e} - 1)$, and its augmentation ideal is $(x-1)$. Note that $x^{p^e} - 1 = (x-1)^{p^e}$, so the substitution $y = x - 1$ presents the group algebra in the more psychologically comforting, to this commutative algebraist, form $k[y]/(y^{p^e})$. Write R for this algebra; it is a 0-dimensional hypersurface ring — the simplest example of a complete intersection — with socle generated by the element y^{p^e-1}. The R-module k has minimal free resolution

$$P\colon\ \cdots \to Re_3 \xrightarrow{y} Re_2 \xrightarrow{y^{p^e-1}} Re_1 \xrightarrow{y} Re_0 \to 0.$$

This is an elementary instance of the periodic minimal free resolution, of period 2, of the residue field of hypersurfaces constructed by Tate [1957]; see also [Eisenbud 1980]. Applying $\mathrm{Hom}_R(-,k)$ to the resolution above results in the complex

$$\mathrm{Hom}_R(P,k)\colon\ 0 \to ke_0^* \xrightarrow{0} ke_1^* \xrightarrow{0} ke_2^* \xrightarrow{0} ke_3^* \xrightarrow{0} \cdots$$

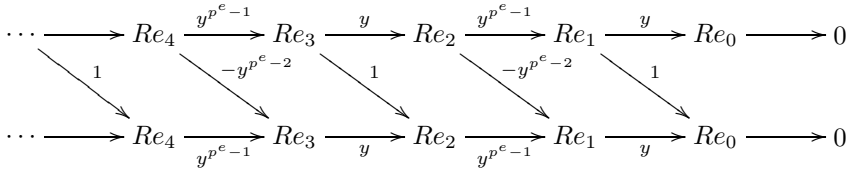
Thus, one obtains $\mathrm{H}^n(G,k) = k$ for each integer $n \geq 0$.

Multiplicative structure. Next we calculate the products in group cohomology, and for this I propose to use compositions in $\mathrm{Hom}_R(P,P)$; see (5.1). More precisely: since P is a complex of free modules, the canonical map

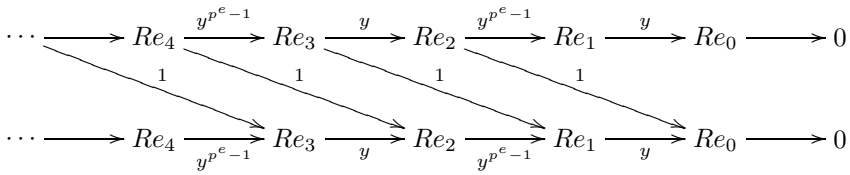
$$\mathrm{Hom}_R(P,\varepsilon)\colon\ \mathrm{Hom}_R(P,P) \to \mathrm{Hom}_R(P,k)$$

is an isomorphism in homology. Given two cycles in $\mathrm{Hom}_R(P,k)$, I will lift them to cycles in $\mathrm{Hom}_R(P,P)$, compose them there, and then push down the resultant cycle to $\mathrm{Hom}_R(P,k)$; this is their product.

For example, the cycle e_1^* of degree -1 lifts to the cycle α in $\mathrm{Hom}_R(P,P)$ given by



It is a lifting of e_1^* since $\varepsilon(\alpha(e_1)) = 1$, and a cycle since $\partial\alpha = -\alpha\partial$. Similarly, the cycle e_2^* lifts to the cycle β given by



This is all one needs in order to compute the entire cohomology rings of G. As indicated before, there are two cases to consider.

When $p = 2$ and $e = 1$, one has $y^{p^e-2} = 1$, so that $\varepsilon(\alpha^n(e_n)) = 1$ for each positive integer n. Therefore, $(e_1^*)^n = e_n^*$, and since the e_n^* form a basis for the graded k-vector space $\mathrm{H}^*(G,k)$, one obtains $\mathrm{H}^*(G,k) = k[e_1^*]$, as desired.

Suppose that either $p \geq 3$ or $e \geq 2$. In this case

$$\varepsilon(\alpha^{n+1}(e_{n+1})) = 0, \quad \varepsilon(\beta^n(e_{2n})) = 1, \quad \text{and} \quad \varepsilon(\alpha\beta^{n-1}(e_{2n-1})) = 1,$$

for each positive integer n. Passing to $\mathrm{Hom}_R(P,k)$, these relations translate to

$$(e_1^*)^{n+1} = 0, \quad (e_2^*)^n = e_{2n}^*, \quad e_1^*(e_2^*)^{n-1} = e_{2n-1}^*.$$

In particular, the homomorphism of k-algebras $k[e_1^*, e_2^*] \to \mathrm{H}^*(G,k)$ is surjective; here, $k[e_1^*, e_2^*]$ is the graded-polynomial algebra on e_1^* and e_2^*, that is to say, it is the tensor product of the exterior algebra on e_1^* and the usual polynomial algebra on e_2^*. This map is also injective: just compare Hilbert series.

This completes our calculation of the cohomology of cyclic p-groups. □

(7.4) **Finitely generated abelian groups.** Let the characteristic of k be p, and let the group G be finitely generated and abelian. By the fundamental theorem of finitely generated abelian groups, there are integers n and $e_1, \dots, e_m$, such that

$$G \cong \mathbb{Z}^n \oplus \frac{\mathbb{Z}}{(p^{e_1}\mathbb{Z})} \oplus \cdots \oplus \frac{\mathbb{Z}}{(p^{e_m}\mathbb{Z})} \oplus G'.$$

where G' is a finite abelian group whose order is coprime to p. By the Künneth formula (6.5), the group cohomology of G is the k-algebra

$$\mathrm{H}^*(\mathbb{Z}^n, k) \otimes_k \mathrm{H}^*(\mathbb{Z}/p^{e_1}\mathbb{Z}, k) \otimes_k \cdots \otimes_k \mathrm{H}^*(\mathbb{Z}/p^{e_m}\mathbb{Z}, k) \otimes_k \mathrm{H}^*(G', k).$$

Note that $\mathrm{H}^*(G', k) = k$, by Theorem (6.2); the remaining terms of the tensor product above are computed by propositions (7.2) and (7.3).

To give a flavour of the issues that may arise in the nonabelian case, I will calculate the cohomology of Σ_3. This gives me also an excuse to introduce an important tool in this subject:

(7.5) **The Lyndon–Hochschild–Serre spectral sequence.** Let G be a finite group and M a $k[G]$-module. Let N be a normal subgroup in G.

Via the canonical inclusion of k-algebras $k[N] \subseteq k[G]$, one can view M also as an $k[N]$-module. Since N is a normal subgroup, the k-subspace M^N of N-invariant elements of M is stable under multiplication by elements in G (check!) and hence it is a $k[G]$-submodule of M. Furthermore, $\mathrm{I}(N) \cdot M^N = 0$, so that M^N has the structure of a module over $k[G]/\mathrm{I}(N)k[G]$, that is to say, of a $k[G/N]$-module; see (1.1). It is clear from the definitions that $(M^N)^{G/N} = M^G$. In other words, one has an isomorphism of functors

$$\mathrm{Hom}_{k[G/N]}(k, \mathrm{Hom}_{k[N]}(k, -)) \cong \mathrm{Hom}_{k[G]}(k, -).$$

The functor on the left is the composition of two functors: $\operatorname{Hom}_{k[N]}(k,-)$ and $\operatorname{Hom}_{k[G/N]}(k,-)$. Thus standard homological algebra provides us with a spectral sequence that converges to its composition, that is to say, to $\mathrm{H}^*(G,M)$. In our case, the spectral sequence sits in the first quadrant and has second page

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(G/N, \mathrm{H}^q(N,M))$$

and differential

$$\partial_r^{p,q} \colon \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}\,.$$

This is the *Lyndon–Hochschild–Serre spectral sequence* associated to N.

Here are two scenarios where the spectral sequence collapses.

(7.6) Suppose the characteristic of k does not divide $[G:N]$, the index of N in G. In this case, $\mathrm{H}^p(G/N,-) = 0$ for $p \geq 1$, by Maschke's theorem (6.2), so that the spectral sequence in (7.5) collapses to yield an isomorphism

$$\mathrm{H}^*(G,M) \cong \mathrm{H}^0(G/N, \mathrm{H}^*(N,M)) = \mathrm{H}^*(N,M)^{G/N}.$$

In particular, with $M = k$, one obtains that $\mathrm{H}^*(G,k) \cong \mathrm{H}^*(N,k)^{G/N}$; this isomorphism is compatible with the multiplicative structures. Note that the object on the right is the ring of invariants of the action of G/N on the group cohomology of N. Thus does invariant theory resurface in group cohomology.

(7.7) Suppose the characteristic of k does not divide $|N|$. Then $\mathrm{H}^q(N,M) = 0$ for $q \geq 1$, and once again the spectral sequence collapses to yield an isomorphism

$$\mathrm{H}^*(G,M) \cong \mathrm{H}^*(G/N, M^N).$$

The special case $M = k$ reads $\mathrm{H}^*(G,k) = \mathrm{H}^*(G/N,k)$.

As an application we calculate the cohomology of Σ_3:

(7.8) **The symmetric group on three elements.** In the notation in (4.8), set $N = \{1, b, b^2\}$; this is a normal subgroup of Σ_3, and the quotient group Σ_3/N is (isomorphic to) $\mathbb{Z}/2\mathbb{Z}$. We use the Hochschild–Serre spectral sequence generated by N in order to calculate the cohomology of Σ_3. There are three cases.

Case (α). When $p \neq 2, 3$, Maschke's theorem (6.2) yields

$$\mathrm{H}^n(\Sigma_3,k) \cong \begin{cases} k & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Case (γ). If $p = 2$, then

$$\mathrm{H}^*(\Sigma_3,k) = k[e_1^*], \quad \text{where } |e_1^*| = 1;$$

the polynomial ring on the variable e_1 of degree 1. Indeed, the order of N is 3, so (7.7) yields that $\mathrm{H}^*(\Sigma_3,k) = \mathrm{H}^*(\mathbb{Z}/2\mathbb{Z},k)$. Proposition (7.3) does the rest.

CASE (β). Suppose that $p = 3$. One obtains from (7.6) that

$$\mathrm{H}^*(\Sigma_3, k) = \mathrm{H}^*(N, k)^{\mathbb{Z}/2\mathbb{Z}}.$$

The group N is cyclic of order 3, so its cohomology is $k[e_1^*, e_2^*]$, with $|e_1^*| = 1$ and $|e_2^*| = 2$; see Proposition (7.3). The next step is to compute the ring of invariants. The action of y, the generator of $\mathbb{Z}/2\mathbb{Z}$, on $\mathrm{H}^*(N, k)$ is compatible with products, so it is determined entirely by its actions on e_1^* and on e_2^*. I claim that

$$y(e_1^*) = -e_1^* \quad \text{and} \quad y(e_2^*) = -e_2^*.$$

Using the description of $\mathrm{H}^1(N, k)$ given in (6.3), it is easy to verify the assertion on the left; the one of the right is a little harder. Perhaps the best way to get this is to observe that the action of y on $\mathrm{H}^*(N, k)$ is compatible with the *Bockstein* operator on cohomology and that this takes e_1^* to e_2^*; see [Evens 1961, (3.3)]. At any rate, given this, it is not hard to see that

$$\mathrm{H}^*(\Sigma_3, k) = \bigwedge(ke_1^*e_2^*) \otimes_k \mathrm{Sym}(k(e_2^*)^2),$$

the tensor product of an exterior algebra on an element of degree 3 and a symmetric algebra on an element of degree 4.

Hopf algebras. In this article I have indicated at various points that much of the module theory over group algebras extends to Hopf algebras. I wrap up by mentioning a perfect generalization of Theorem (7.1), due to E. Friedlander and Suslin [1997]: If a finite-dimensional Hopf algebra H is cocommutative, its cohomology algebra $\mathrm{Ext}_H^*(k, k)$ is finitely generated.

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