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Lectures on the Geometry of Syzygies

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WITH A CHAPTER BY JESSICA SIDMAN

Abstract. The theory of syzygies connects the qualitative study of algebraic varieties and commutative rings with the study of their defining equations. It started with Hilbert's work on what we now call the Hilbert function and polynomial, and is important in our day in many new ways, from the high abstractions of derived equivalences to the explicit computations made possible by Gröbner bases. These lectures present some highlights of these interactions, with a focus on concrete invariants of syzygies that reflect basic invariants of algebraic sets.

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These notes illustrate a few of the ways in which properties of syzygies reflect qualitative geometric properties of algebraic varieties. Chapters 1, 3 and 4 were written by David Eisenbud, and closely follow his lectures at the introductory workshop. Chapter 2 was written by Jessica Sidman, from the lecture she gave enlarging on the themes of the first lecture and providing examples. The lectures may serve as an introduction to the book *The Geometry of Syzygies* [Eisenbud 1995]; in particular, the book contains proofs for the many of the unproved assertions here.

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1. Hilbert Functions and Syzygies

1.1. Counting functions that vanish on a set. Let $\mathbb{K}$ be a field and let $S = \mathbb{K}[x_0, \dots, x_r]$ be a ring of polynomials over $\mathbb{K}$. If $X \subset \mathbb{P}^r$ is a projective variety, the dimension of the space of forms (homogeneous polynomials) of each degree d vanishing on X is an invariant of X, called the Hilbert function of the ideal I_X of X. More generally, any finitely generated graded S-module $M = \bigoplus M_d$ has a *Hilbert function* $H_M(d) = \dim_{\mathbb{K}} M_d$. The *minimal free resolution* of a finitely generated graded S-module M provides invariants that refine the information in the Hilbert function. We begin by reviewing the origin and significance of Hilbert functions and polynomials and the way in which they can be computed from free resolutions.

Hilbert's interest in what is now known as the Hilbert function came from invariant theory. Given a group G acting on a vector space with basis $z_1, \dots, z_n$, it was a central problem of nineteenth century algebra to determine the set of polynomial functions $p(z_1, \dots, z_n)$ that are invariant under G in the sense that $p(g(z_1, \dots, z_n)) = p(z_1, \dots, z_n)$. The invariant functions form a graded subring, denoted T^G, of the ring $T = \mathbb{K}[z_1, \dots, z_n]$ of all polynomials; the problem of invariant theory was to find generators for this subring.

For example, if G is the full symmetric group on $z_1, \dots, z_n$, then T^G is the polynomial ring generated by the elementary symmetric functions $\sigma_1, \dots, \sigma_n$, where

$$\sigma_i = \sum_{j_1 < \cdots < j_i} \prod_{t=1}^{i} z_{j_t};$$

see [Lang 2002, V.9] or [Eisenbud 1995, Example 1.1 and Exercise 1.6]. The result that first made Hilbert famous [1890] was that over the complex numbers ($\mathbb{K} = \mathbb{C}$), if G is either a finite group or a classical group of matrices (such as GL_n) acting algebraically — that is, via matrices whose entries are rational functions of the entries of the matrix representing an element of G — then the ring T^G is a finitely generated $\mathbb{K}$-algebra.

The homogeneous components of any invariant function are again invariant, so the ring T^G is naturally graded by (nonnegative) degree. For each integer d the homogeneous component $(T^G)_d$ of degree d is contained in T_d, a finite-dimensional vector space, so it too has finite dimension.

How does the number of independent invariant functions of degree d, say $h_d = \dim_{\mathbb{K}}(T^G)_d$, change with d? Hilbert's argument, reproduced in a similar case below, shows that the generating function of these numbers, $\sum_0^\infty h_d t^d$, is a rational function of a particularly simple form:

$$\sum_0^\infty h_d t^d = \frac{p(t)}{\prod_0^s (1 - t_i^{\alpha})},$$

for a polynomial p and positive integers α_i.

A similar problem, which will motivate these lectures, arises in projective geometry: Let $X \subset \mathbb{P}^r = \mathbb{P}^r_{\mathbb{K}}$ be a projective algebraic variety (or more generally a projective scheme) and let $I = I_X \subset S = \mathbb{K}[x_0, \ldots, x_r]$ be the homogeneous ideal of forms vanishing on X. An easy discrete invariant of X is given by the vector-space dimension $\dim_{\mathbb{K}} I_d$ of the degree d component of I. Again, we may ask how this "number of forms of degree d vanishing on X" changes with d. This number is usually expressed in terms of its complement in $\dim S_d$. We write $S_X := S/I$ for the homogeneous coordinate ring of X and we set $H_X(d) = \dim_{\mathbb{K}}(S_X)_d = \dim_{\mathbb{K}} S_d - \dim_{\mathbb{K}} I_d = \binom{r+d}{r} - \dim_{\mathbb{K}} I_d$. We call $H_X(d)$ the *Hilbert function of* X. Using Hilbert's ideas we will see that $H_X(d)$ agrees with a polynomial $P_X(d)$, called the Hilbert polynomial of X, when d is sufficiently large. Further, its generating function $\sum_d \mathrm{H}_X(d)t^d$ can be written as a rational function in t, t^{-1} as above with denominator $(1-t)^{r+1}$. Hilbert proved both the Hilbert Basis Theorem (polynomial rings are Noetherian) and the Hilbert Syzygy Theorem (modules over polynomial rings have finite free resolutions) in order to deduce this. As a first illustration of the usefulness of syzygies we shall see how these results fit together.

This situation of projective geometry is a little simpler than that of invariant theory because the generators x_i of S have degree 1, whereas in the case of invariants we have to deal with graded rings generated by elements of different degrees (the α_i). For simplicity we will henceforward stick to the case of degree-1 generators. See [Goto and Watanabe 1978a; 1978b] for more information.

Hilbert's argument *requires* us to generalize to the case of modules. If M is any finitely generated graded S-module (such as the ideal I or the homogeneous coordinate ring S_X), then the d-th homogeneous component M_d of M is a finite-dimensional vector space. We set $H_M(d) := \dim_{\mathbb{K}} M_d$. The function H_M is called the *Hilbert function* of M.

THEOREM 1.1. *Let $S = \mathbb{K}[x_0, \ldots, x_r]$ be the polynomial ring in $r+1$ variables over a field $\mathbb{K}$. Let M be a finitely generated graded S-module.*

(i) *$H_M(d)$ is equal to a finite sum of the form $\sum_i \pm\binom{r+d-e_i}{r}$, and thus $H_M(d)$ agrees with a polynomial function $P_M(d)$ for $d \geq \max_i e_i - r$.*

(ii) *The generating function $\sum_d H_M(d)t^d$ can be expressed as a rational function of the form*

$$\frac{p(t, t^{-1})}{(1-t)^{r+1}}$$

for some polynomial $p(t, t^{-1})$.

PROOF. First consider the case $M = S$. The dimension of the d-th graded component is $\dim_{\mathbb{K}} S_d = \binom{r+d}{r}$, which agrees with the polynomial in d

$$\frac{(r+d)\cdots(1+d)}{r\cdots 1} = \frac{d^r}{r!} + \cdots + 1$$

for $d \geq -r$. Further,

$$\sum_0^\infty H_S(d)t^d = \sum_0^\infty \binom{r+d}{r} t^d = \frac{1}{(1-t)^{r+1}}$$

proving the theorem in this case.

At this point it is useful to introduce some notation: If M is any module we write $M(e)$ for the module obtained by "shifting" M by e positions, so that $M(e)_d = M_{e+d}$. Thus for example $S(-e)$ is the free module of rank 1 generated in degree e (note the change of signs!) Shifting the formula above we see that $H_{S(-e)}(d) = \binom{r+d-e}{r}$.

We immediately deduce the theorem in case $M = \bigoplus_i S(-e_i)$ is a free graded module, since then

$$H_M(d) = \sum_i H_{S(-e_i)}(d) = \sum_i \binom{r+d-e_i}{r}.$$

This expression is equal to a polynomial for $d \geq \max_i e_i - r$, and

$$\sum_{d=-\infty}^{\infty} H_M(d) = \frac{\sum_i t^{e_i}}{(1-t)^{r+1}}.$$

Hilbert's strategy for the general case was to compare an arbitrary module M to a free module. For this purpose, we choose a finite set of homogeneous generators m_i in M. Suppose $\deg m_i = e_i$. We can define a map (all maps are assumed homogeneous of degree 0) from a free graded module $F_0 = \bigoplus S(-e_i)$ onto M by sending the i-th generator to m_i. Let $M_1 := \ker F_0 \to M$ be the kernel of this map. Since $H_M(d) = H_{F_0}(d) - H_{M_1}(d)$, it suffices to prove the desired assertions for M_1 in place of M.

To use this strategy, Hilbert needed to know that M_1 would again be finitely generated, and that M_1 was in some way closer to being a free module than was M. The following two results yield exactly this information.

Theorem 1.2 (Hilbert's Basis Theorem). *Let S be the polynomial ring in $r+1$ variables over a field $\mathbb{K}$. Any submodule of a finitely generated S-module is finitely generated.*

Thus the module $M_1 = \ker F_0 \to M$, as a submodule of F_0, is finitely generated. To define the sense in which M_1 might be "more nearly free" than M, we need the following result:

Theorem 1.3 (Hilbert's Syzygy Theorem). *Let S be the polynomial ring in $r+1$ variables over a field $\mathbb{K}$. Any finitely generated graded S-module M has a finite free resolution of length at most $r+1$, that is, an exact sequence*

$$0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0,$$

where the modules F_i are free and $n \leq r+1$.

We will not prove the Basis Theorem and the Syzygy Theorem here; see the very readable [Hilbert 1890], or [Eisenbud 1995, Corollary 19.7], for example. The Syzygy Theorem is true without the hypotheses that M is finitely generated and graded (see [Rotman 1979, Theorem 9.12] or [Eisenbud 1995, Theorem 19.1]), but we shall not need this.

If we take

$$\boldsymbol{F}: \quad 0 \longrightarrow F_n \xrightarrow{\phi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \longrightarrow M \longrightarrow 0$$

to be a free resolution of M with the smallest possible n, then n is called the *projective dimension* of M. Thus the projective dimension of M is zero if and only if M is free. If M is not free, and we take $M_1 = \operatorname{im}\phi_1$ in such a minimal resolution, we see that the projective dimension of M_1 is strictly less than that of M. Thus we could complete the proof of Theorem 1.1 by induction.

However, given a finite free resolution of M we can compute the Hilbert function of M, and its generating function, directly. To see this, notice that if we take the degree d part of each module we get an exact sequence of vector spaces. In such a sequence the alternating sum of the dimensions is zero. With notation as above we have $H_M(d) = \sum (-1)^i H_{F_i}(d)$. If we decompose each F_i as $F_i = \sum_j S(-j)^{\beta_{i,j}}$ we may write this more explicitly as

$$H_M(d) = \sum_i (-1)^i \sum_j \beta_{i,j} \binom{r+d-j}{r}.$$

The sums are finite, so this function agrees with a polynomial in d for $d \geq \max\{j - r \mid \beta_{i,j} \neq 0 \text{ for some } i\}$. Further,

$$\sum_d H_M(d) t^d = \frac{\sum_i (-1)^i \sum_j \beta_{i,j} t^j}{(1-t)^{r+1}}$$

as required for Theorem 1.1.

Conversely, given the Hilbert function of a finitely generated module, one can recover some information about the $\beta_{i,j}$ in any finite free resolution $\boldsymbol{F}$. For this we use the fact that $\binom{r+d-j}{r} = 0$ for all $d < j$. We have

$$H_M(d) = \sum_i (-1)^i \sum_j \beta_{i,j} \binom{r+d-j}{r} = \sum_j \left(\sum_i (-1)^i \beta_{i,j} \right) \binom{r+d-j}{r}.$$

Since $\boldsymbol{F}$ is finite there is an integer d_0 such that $\beta_{i,j} = 0$ for $j < d_0$. If we put $d = d_0$ in the expression for $H_M(d)$ then all the $\binom{r+d-j}{r}$ vanish except for $j = d_0$, and because $\binom{r+d_0-d_0}{r} = 1$ we get $\sum_i (-1)^i \beta_{i,d_0} = H_M(d_0)$. Proceeding inductively we arrive at the proof of:

PROPOSITION 1.4. *Let M be a finitely generated graded module over $S = \mathbb{K}[x_0, \dots, x_r]$, and suppose that $\boldsymbol{F}$ is a finite free resolution of M with graded*

Betti numbers $\beta_{i,j}$. *If* $\beta_{i,j} = 0$ *for all* $j < d_0$, *then the numbers* $B_j = \sum_i (-1)^i \beta_{i,j}$ *are inductively determined by the formulas*

$$B_{d_0} = H_M(d_0)$$

and

$$B_j = H_M(j) - \sum_{k<j} B_k \binom{j+d-k}{r}.$$

1.2. Meaning of the Hilbert function and polynomial. The Hilbert function and polynomial are easy invariants to define, so it is perhaps surprising that they should be so important. For example, consider a variety $X \subset \mathbb{P}^r$ with homogeneous coordinate ring S_X. The restriction map to X gives an exact sequence of sheaves $0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbb{P}^r} \to \mathscr{O}_X \to 0$. Tensoring with the line bundle $\mathscr{O}_{\mathbb{P}^r}(d)$ and taking cohomology we get a long exact sequence beginning

$$0 \to \mathrm{H}^0 \mathscr{I}_X(d) \to \mathrm{H}^0 \mathscr{O}_{\mathbb{P}^r}(d) \to \mathrm{H}^0 \mathscr{O}_X(d) \to \mathrm{H}^1 \mathscr{I}_X(d) \to \cdots.$$

The term $\mathrm{H}^0 \mathscr{O}_{\mathbb{P}^r}(d)$ may be identified with the vector space S_d of forms of degree d in S. The space $\mathrm{H}^0 \mathscr{I}_X(d)$ is thus the space of forms of degree d that induce 0 on X, that is $(I_X)_d$. Further, by Serre's vanishing theorem [Hartshorne 1977, Ch. III, Theorem 5.2], $\mathrm{H}^1 \mathscr{I}_X(d) = 0$ for large d. Thus for large d

$$(S_X)_d = S_d/(I_X)_d = \mathrm{H}^0 \mathscr{O}_X(d).$$

Applying Serre's theorem again, we see that all the higher cohomology of $\mathscr{O}_X(d)$ is zero for large d. Taking dimensions, we see that for large d the Hilbert function of S_X equals the Euler characteristic

$$\chi(\mathscr{O}_X(d)) := \sum_i (-1)^i \dim_{\mathbb{K}} \mathrm{H}^i(\mathscr{O}_X(d)).$$

The Hilbert function equals the Hilbert polynomial for large d; and the Euler characteristic is a polynomial for all d. Thus we may interpret the Hilbert polynomial as the Euler characteristic, and the difference from the Hilbert function (for small d) as an effect of the nonvanishing of higher cohomology.

For a trivial case, take X to be a set of points. Then $\mathscr{O}_X(d)$ is isomorphic to $\mathscr{O}_X$ whatever the value of d, and its global sections are spanned by the characteristic functions of the individual points. Thus $\chi(\mathscr{O}_X(d)) = P_X(d)$ is a constant function of d, equal to the number of points in X.

In general the Riemann–Roch Theorem gives a formula for the Euler characteristic, and thus the Hilbert polynomial, in terms of geometric data on X. In the simplest interesting case, where X is a smooth curve, the Riemann–Roch theorem says that

$$\chi(\mathscr{O}_X(d)) = P_X(d) = d + 1 - g,$$

where g is the genus of X.

These examples only suggest the strength of the invariants $P_X(d)$ and $H_X(d)$. To explain their real role, we recall some basic definitions. A *family of algebraic* *sets parametrized by a variety* T is simply a map of algebraic sets $\pi : \mathscr{X} \to T$. The subschemes $X_t = \pi^{-1}(t)$ for $t \in T$ are called the fibers of the family. Of course we are most interested in families where the fibers vary continuously in some reasonable sense! Of the various conditions we might put on the family to ensure this, the most general and the most important is the notion of flatness, due to Serre: the family $\pi : \mathscr{X} \to T$ is said to be flat if, for each point $p \in T$ and each point $x \in \mathscr{X}$ mapping to t, the pullback map on functions $\pi^* : \mathscr{O}_{T,t} \to \mathscr{O}_{\mathscr{X},x}$ is flat. This means simply that $\mathscr{O}_{\mathscr{X},x}$ is a flat $\mathscr{O}_{T,t}$-module; tensoring it with short exact sequences of $\mathscr{O}_{T,t}$-modules preserves exactness. More generally, a sheaf $\mathscr{F}$ on $\mathscr{X}$ is said to be flat if the $\mathscr{O}_{T,t}$-module $\mathscr{F}_x$ (the stalk of $\mathscr{F}$ at x) is flat for all x mapping to t. The same definitions work for the case of maps of schemes.

The condition of flatness for a family $\mathscr{X} \to T$ has many technical advantages. It includes the important case where $\mathscr{X}$, T, and all the fibers X_t are smooth and of the same dimension. It also includes the example of a family from which algebraic geometry started, the family of curves of degree d in the projective plane, even though the geometry and topology of such curves varies considerably as they acquire singularities. But the geometric meaning of flatness in general could well be called obscure.

In some cases flatness is nonetheless easy to understand. Suppose that $\mathscr{X} \subset \mathbb{P}^r \times T$ and the map $\pi : \mathscr{X} \to T$ is the inclusion followed by the projection onto T (this is not a very restrictive condition: any map of projective varieties, for example, has this form). In this case each fiber X_t is naturally contained as an algebraic set in $\mathbb{P}^r$.

We say in this case that $\pi : \mathscr{X} \to T$ is a *projective family.* Corresponding to a projective family $\mathscr{X} \to T$ we can look at the family of cones

$$\widetilde{\mathscr{X}} \subset \mathbb{A}^{r+1} \times T \to T$$

obtained as the affine set corresponding to the (homogeneous) defining ideal of $\mathscr{X}$. The fibers $\tilde{X}_t$ are then all affine cones.

THEOREM 1.5. *Let* $\pi : \mathscr{X} \to T$ *be a projective family, as above. If* T *is a reduced algebraic set then* $\pi : \mathscr{X} \to T$ *is flat if and only if all the fibers* X_t *of* $\mathscr{X}$ *have the same Hilbert polynomial. The family of affine cones over* X_t *is flat if and only if all the* $\tilde{X}_t$ *have the same Hilbert function.*

These ideas can be generalized to the flatness of families of sheaves, giving an interpretation of the Hilbert function and polynomial of modules.

1.3. Minimal free resolutions. As we have defined it, a free resolution $\boldsymbol{F}$ of M does not seem to offer any easy invariant of M beyond the Hilbert function, since $\boldsymbol{F}$ depends on the choice of generators for M, the choice of generators for $M_1 = \ker F_0 \to M$, and so on. But this dependence on choices turns out to be very weak. We will say that $\boldsymbol{F}$ is a *minimal free resolution* of M if at each stage we choose the minimal number of generators.

Proposition 1.6. *Let S be the polynomial ring in $r+1$ variables over a field $\mathbb{K}$, and M a finitely generated graded S-module. Any two minimal free resolutions of M are isomorphic. Moreover, any free resolution of M can be obtained from a minimal one by adding "trivial complexes" of the form*

$$G_i = S(-a) \xrightarrow{1} S(-a) = G_{i-1}$$

for various integers i and a.

The proof is an exercise in the use of Nakayama's Lemma; see for example [Eisenbud 1995, Theorem 20.2].

Thus the ranks of the modules in the minimal free resolution, and even the numbers $\beta_{i,j}$ of generators of degree j in F_i, are invariants of M. Theorem 1.1 shows that these invariants are at least as strong as the Hilbert function of M, and we will soon see that they contain interesting additional information.

The numerical invariants in the minimal free resolution of a module in non-negative degrees can be described conveniently using a piece of notation introduced by Bayer and Stillman: the Betti diagram. This is a table displaying the numbers $\beta_{i,j}$ in the pattern

$$\begin{array}{llll} \cdots & & & \\ \beta_{0,0} & \beta_{1,1} & \cdots & \beta_{i,i} \\ \beta_{0,1} & \beta_{1,2} & \cdots & \beta_{i,i+1} \\ \cdots & & & \end{array}$$

with $\beta_{i,j}$ in the i-th column and $(j-i)$-th row. Thus the i-th column corresponds to the i-th free module in the resolution, $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. The utility of this pattern will become clearer later in these notes, but it was introduced partly to save space. For example, suppose that a module M has all its minimal generators in degree j, so that $\beta_{0,j} \neq 0$ but $\beta_{0,m} = 0$ for $m < j$. The minimality of $\boldsymbol{F}$ then implies that $\beta_{1,j} = 0$; otherwise, there would be a generator of F_1 of degree j, and it would map to a nonzero scalar linear combination of the generators of F_0. Since this combination would go to 0 in M, one of the generators of M would be superfluous, contradicting minimality. Thus there is no reason to leave a space for $\beta_{1,j}$ in the diagram. Arguing in a similar way we can show that $\beta_{i,m} = 0$ for all $m < i + j$. Thus if we arrange the $\beta_{i,j}$ in a Betti diagram as above we will be able to start with the j-th row, simply leaving out the rest.

To avoid confusion, we will label the rows, and sometimes the columns of the Betti diagram. The column containing $\beta_{i,j}$ (for all j) will be labeled i while the row with $\beta_{0,j}$ will be labeled j. For readability we often replace entries that are zero with $-$, and unknown entries with $*$, and we suppress rows in the region where all entries are 0. Thus for example if I is an ideal with 2 generators of degree 4 and one of degree 5, and relations of degrees 6 and 7, then the free resolution of S/I has the form

$$0 \to S(-6) \oplus S(-7) \to S^2(-4) \oplus S(-5) \to S$$

and Betti diagram

	0	1	2
0	1	–	–
1	–	–	–
2	–	–	–
3	–	2	–
4	–	1	1
5	–	–	1

An example that makes the space-saving nature of the notation clearer is the Koszul complex (the minimal free resolution of $S/(x_0, \dots, x_r)$ — see [Eisenbud 1995, Ch. 17]), which has Betti diagram

	0	1	$\cdots$	i	$\cdots$	$r+1$
0	1	$r+1$	$\cdots$	$\binom{r+1}{i}$	$\cdots$	1

1.4. Four points in $\mathbb{P}^2$. We illustrate what has gone before by describing the Hilbert functions, polynomials, and Betti diagrams of each possible configuration $X \subset \mathbb{P}^2$ of four distinct points in the plane. We let $S = \mathbb{K}[x_0, x_1, x_2]$ be the homogeneous coordinate ring of the plane. We already know that the Hilbert polynomial of a set of four points, no matter what the configuration, is the constant polynomial $P_X(d) \equiv 4$. In particular, the family of 4-tuples of points is flat over the natural parameter variety

$$T = \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \setminus \text{diagonals}.$$

We shall see that the Hilbert function of X depends only on whether all four points lie on a line. The graded Betti numbers of the minimal resolution, in contrast, capture all the remaining geometry: they tell us whether any three of the points are collinear as well.

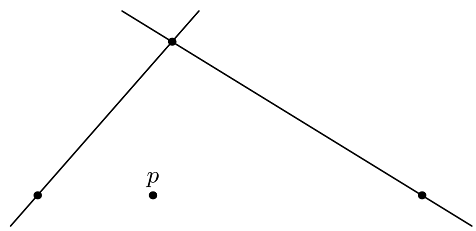
PROPOSITION 1.7. (i) *If X consists of four collinear points*, $H_{S_X}(d)$ *has the values* $1, 2, 3, 4, 4, \dots$ *at* $d = 0, 1, 2, 3, 4, \dots$

(ii) *If $X \subset \mathbb{P}^2$ consists of four points not all on a line*, $H_{S_X}(d)$ *has the values* $1, 3, 4, 4, \dots$ *at* $d = 0, 1, 2, 3, \dots$. *In classical language*: *X imposes 4 conditions on degree d curves for $d \geq 2$.*

PROOF. Let $H_X := H_{S_X}(d)$. In case (i), H_X has the same values that it would if we considered X to be a subset of $\mathbb{P}^1$. But in $\mathbb{P}^1$ the ideal of any d points is generated by one form of degree d, so the Hilbert function $H_X(d)$ for four collinear points X takes the values $1, 2, 3, 4, 4, \dots$ at $d = 0, 1, 2, 3, 4, \dots$.

In case (ii) there are no equations of degree $d \leq 1$, so for $d = 0, 1$ we get the claimed values for $H_X(d)$. In general, $H_X(d)$ is the number of independent functions induced on X by ratios of forms of degree d (see the next lecture) so $H_X(d) \leq 4$ for any value of d.

To see that $H_X(2) = 4$ it suffices to produce forms of degree 2 vanishing at all $X \setminus p$ for each of the four points p in X, since these forms must be linearly independent modulo the forms vanishing on X. But it is easy to draw two lines going through the three points of $X \setminus p$ but not through p:



and the union of two lines has as equation the quadric given by the product of the corresponding pair of linear forms. A similar argument works in higher degree: just add lines to the quadric that do not pass through any of the points to get curves of the desired higher degree. □

In particular we see that the set of lines through a point in affine 3-space (the cones over the sets of four points) do not form a flat family; but the ones where not all the lines are coplanar do form a flat family. (For those who know about schemes: the limit of a set of four noncoplanar lines as they become coplanar has an embedded point at the vertex.)

When all four points are collinear it is easy to compute the free resolution: The ideal of X contains the linear form L that vanishes on the line containing the points. But S/L is the homogeneous coordinate ring of the line, and in the line the ideal of four points is a single form of degree 4. Lifting this back (in any way) to S we see that I_X is generated by L and a quartic form, say f. Since L does not divide f the two are relatively prime, so the free resolution of $S_X = S/(L, f)$ has the form

$$0 \to S(-5) \xrightarrow{\begin{pmatrix} f \\ -L \end{pmatrix}} S(-1) \oplus S(-4) \xrightarrow{(L,\, f)} S,$$

with Betti diagram

	0	1	2
0	1	1	–
1	–	–	–
2	–	–	–
3	–	1	1

We now suppose that the points of X are not all collinear, and we want to see that the minimal free resolutions determine whether three are on a line. In fact, this information is already present in the number of generators required by I_X. If three points of X lie on a line $L = 0$, then by Bézout's theorem any conic vanishing on X must contain this line, so the ideal of X requires at least one cubic generator.

On the other hand, any four noncollinear points lie on an irreducible conic (to see this, note that any four noncollinear points can be transformed into any other four noncollinear points by an invertible linear transformation of $\mathbb{P}^2$; and we can choose four noncollinear points on an irreducible conic.) From the Hilbert function we see that there is a two dimensional family of conics through the points, and since one is irreducible, any two distinct quadrics Q_1, Q_2 vanishing on X are relatively prime. It is easy to see from relative primeness that the syzygy $(Q_2, -Q_1)$ generates all the syzygies on (Q_1, Q_2). Thus the minimal free resolution of $S/(Q_1, Q_2)$ has Betti diagram

	0	1	2
0	1	–	–
1	–	2	–
2	–	–	1

It follows that $S/(Q_1, Q_2)$ has the same Hilbert function as $S_X = S/I_X$. Since $I_X \supset (Q_1, Q_2)$ we have $I_X = (Q_1, Q_2)$.

In the remaining case, where precisely three of the points of X lie on a line, we have already seen that the ideal of X requires at least one cubic generator. Corollary 2.3 makes it easy to see from this that the Betti diagram of a minimal free resolution must be

	0	1	2
0	1	–	–
1	–	2	1
2	–	1	1

2. Points in the Plane and an Introduction to Castelnuovo–Mumford Regularity

2.1. Resolutions of points in the projective plane. This section gives a detailed description of the numerical invariants of a minimal free resolution of a finite set of points in the projective plane. To illustrate both the potential and the limitations of these invariants in capturing the geometry of the points we compute the Betti diagrams of all possible configurations of five points in the plane. In contrast to the example of four points worked out in the previous section, it is not possible to determine whether the points are in linearly general position from the Betti numbers alone. The presentation in this section is adapted from Chapter 3 of [Eisenbud $\geq$ 2004], to which we refer the reader who wishes to find proofs omitted here.

Let $X = \{p_1, \dots, p_n\}$ be a set of distinct points in $\mathbb{P}^2$ and let I_X be the homogeneous ideal of X in $S = \mathbb{K}[x_0, x_1, x_2]$. Considering this situation has the virtue of simplifying the algebra to the point where one can describe a resolution of I_X quite explicitly while still retaining a lot of interesting geometry.

Fundamentally, the algebra is simple because the resolution of I_X is very short. In particular:

LEMMA 2.1. *If $I_X \subseteq \mathbb{K}[x_0, x_1, x_2]$ is the homogeneous ideal of a finite set of points in the plane, then a minimal resolution of I_X has length one.*

PROOF. Recall that if

$$0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow S/I_X \longrightarrow 0$$

is a resolution of S/I_X then we get a resolution of I_X by simply deleting the term F_0 (which of course is just S). We will proceed by showing that S/I_X has a resolution of length two.

From the Auslander–Buchsbaum formula (see Theorem 3.1) we know that the length of a minimal resolution of S/I_X is:

$$\operatorname{depth} S - \operatorname{depth} S/I_X.$$

Since S is a polynomial ring in three variables, it has depth three. Our hypothesis is that S/I_X is the coordinate ring of a finite set points taken as a reduced subscheme of $\mathbb{P}^2$. The Krull dimension of S/I_X is one, and hence $\operatorname{depth} S/I_X \leq 1$. Furthermore, since I_X is the ideal of all homogeneous forms in S that vanish on X, the irrelevant ideal is not associated. Therefore, we can find an element of S with positive degree that is a nonzerodivisor on S/I_X. We conclude that S/I_X has a free resolution of length two. □

We see now that a resolution of I_X has the form

$$0 \longrightarrow \bigoplus_{i=1}^{t_1} S(-b_i) \xrightarrow{M} \bigoplus_{i=1}^{t_0} S(-a_i) \longrightarrow I_X \longrightarrow 0.$$

We can complete our description of the shape of the resolution via the following theorem:

THEOREM 2.2 (HILBERT–BURCH). *Suppose that an ideal I in a Noetherian ring R admits a free resolution of length one*:

$$0 \longrightarrow F \xrightarrow{M} G \longrightarrow I \longrightarrow 0.$$

If the rank of the free module F is t, then the rank of G is $t+1$, and there exists a nonzerodivisor $a \in R$ such that I is $aI_t(M)$; in fact, regarding M as a matrix with respect to given bases of F and G, the generator of I that is the image of the i-th basis vector of G is $\pm a$ times the determinant of the submatrix of M formed by deleting the i-th row. Moreover, the depth of $I_t(M)$ is two.

Conversely, given a nonzerodivisor a of R and a $(t+1) \times t$ matrix M with entries in R such that the depth of $I_t(M)$ is at least 2, the ideal $I = aI_t(M)$ admits a free resolution as above.

We will not prove the Hilbert–Burch Theorem here, or its corollary stated below; our main concern is with their consequences. (Proofs can be found in [Eisenbud ≥ 2004, Chapter 3]; alternatively, see [Eisenbud 1995, Theorem 20.15] for Hilbert–Burch and [Ciliberto et al. 1986] for the last statement of Corollary 2.3.)

As we saw in Section 1.1, the Hilbert function and the Hilbert polynomial of S/I_X are determined by the invariants of a minimal free resolution. So, for example, we expect to be able to compute the degree of X from the degrees of the entries of M. When X is a complete intersection this is already familiar to us from Bézout's theorem. In this case M is a 2×1 matrix whose entries generate I_X. Bézout's theorem says that the product of the degrees of the entries of M gives the degree of X.

The following corollary of the Hilbert–Burch Theorem generalizes Bézout's theorem and describes the relationships between the degrees of the generators of I_X and the degrees of the generators of the module of their syzygies. Since the map given by M has degree zero, the (i,j) entry of M has degree $b_j - a_i$. Let $e_i = b_i - a_i$ and $f_i = b_i - a_{i+1}$ denote the degrees of the entries on the two main diagonals of M. Schematically:

$$\begin{pmatrix} e_1 & & & & & \\ f_1 & e_2 & & & & \\ & f_2 & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & f_{t-1} & e_t \\ & & & & & f_t \end{pmatrix}$$

Corollary 2.3. *Assume that $a_1 \geq a_2 \geq \cdots \geq a_{t+1}$ and $b_1 \geq b_2 \geq \cdots \geq b_t$. Then, for $1 \leq i \leq t$, we have*

$$e_i \geq 1, \qquad f_i \geq 1, \qquad a_i = \sum_{j<i} e_j + \sum_{j \geq i} f_j, \qquad b_i = a_i + e_i.$$

Moreover,

$$n = \deg X = \sum_{i \leq j} e_i f_j. \tag{2–1}$$

The last equality is due to Ciliberto, Geramita, and Orecchia [Ciliberto et al. 1986].

From Corollary 2.3 we can already bound the number of minimal generators of I_X given a little bit of information about the geometry of X.

Corollary 2.4 [Burch 1967]. *If X lies on a curve of degree d, then I_X requires at most $d + 1$ generators.*

Proof. Since X lies on a curve of degree d there is an element of I_X of degree d. Therefore, at least one of the a_i's must be at most d. By Corollary 2.3, each a_i is the sum of t integers that are all at least 1. Therefore, $t \leq a_i \leq d$, which implies that $t + 1$, the number of generators of I_X, is at most $d + 1$. □

Using the information above one can show that for small values of n there are very few possibilities for the invariants of the resolution of I_X.

2.2. Resolutions of five points in the plane. We now show how to use these ideas to compute all possible Betti diagrams of X when X is a set of five distinct points in the projective plane. As before, let $S = \mathbb{K}[x_0, x_1, x_2]$, and let I_X be the saturated homogeneous ideal of X. In keeping with conventions, we give the Betti diagrams of the quotient S/I_X. From these computations it will be easy to determine the Hilbert function $H_X(d)$ as well.

First, we organize the possible configurations of the points into four categories based on their geometry:

(1) The five points are all collinear.
(2) Precisely four of the points are collinear.
(3) Some subset of three of the points lies on a line but no subset of four of the points lies on a line.
(4) The points are linearly general.

Case (1): Corollary 2.4 implies that I_X has at most two generators. Thus, $t = 1$, and the points are a complete intersection. By Bézout's Theorem, the generators of I_X have degrees $a_1 = 5$ and $a_2 = 1$. Furthermore, we see that I_X is resolved by a Koszul complex and hence $b_1 = 6$. The Betti diagram of the resolution of 5 collinear points is

	0	1	2
0	1	1	–
1	–	–	–
2	–	–	–
3	–	–	–
4	–	1	1

From Section 1.1 we see that the Hilbert function is given by

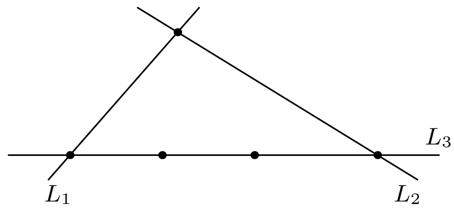
$$\binom{2+d}{2} - \binom{2+d-1}{2} - \binom{2+d-5}{2} + \binom{2+d-6}{2}.$$

Thus, $H_X(d)$ has values $1, 2, 3, 4, 5, 5, \ldots$ at $d = 1, 2, 3, 4, 5, \ldots$.

We claim that $t = 2$ in the remaining cases. Since the degree of X is prime, Bézout's theorem tells us that the points are a complete intersection if and only if they are collinear. Hence, $t \geq 2$. Since there is a 6-dimensional space of conics in three variables, any set of five points must lie on a conic. Thus Corollary 2.4 implies that I_X has at most three generators, so $t = 2$. We conclude that in Cases (2)–(4), the invariants of the resolution satisfy the relationships $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2$, and

$$a_1 = f_1 + f_2,$$
$$a_2 = e_1 + f_2.$$
$$a_3 = e_1 + e_2.$$

Case (2): If precisely four of the points are collinear, we see from the picture below that there are two conics containing the points: $L_1 \cup L_3$ and $L_2 \cup L_3$.



Since these conics are visibly different, their defining equations must be linearly independent. We conclude that I_X must have two minimal generators of degree two, and hence that

$$a_2 = e_1 + f_2 = 2,$$
$$a_3 = e_1 + e_2 = 2.$$

By Corollary 2.3, $e_1, e_2, f_2 \geq 1$, which implies that $e_1 = e_2 = f_2 = 1$. We also know that

$$5 = e_1 f_1 + e_1 f_2 + e_2 f_2 = f_1 + 1 + 1.$$

Hence, $f_1 = 3$, $a_1 = 4$, $b_1 = a_1 + e_1 = 5$, and $b_2 = a_2 + e_2 = 3$. In this case, the points have Betti diagram

	0	1	2
0	1	–	–
1	–	2	1
2	–	–	–
3	–	1	1

and Hilbert function

$$\binom{2+d}{2} - 2\binom{2+d-2}{2} - \binom{2+d-4}{2} + \binom{2+d-3}{2} + \binom{2+d-5}{2},$$

taking on the values $1, 3, 4, 5, 5, \ldots$ at $d = 0, 1, 2, 3, 4, \ldots$.

Case (3): We will show that the points lie on a unique reducible conic. By assumption there is a line L containing three of the points. Any conic containing all five must vanish at these three points and hence will vanish identically on L. Therefore, L must be a component of any conic that contains X. There are precisely two points not on L, and they determine a line L' uniquely. The union of L and L' is the unique conic containing these points.

If the five points lie on a unique conic, we can determine all of the remaining numerical invariants. We must have $a_3 = e_1 + e_2 = 2$, which implies that $e_1 = e_2 = 1$. We also know that $3 \leq a_2 = e_1 + f_2 = 1 + f_2$, which implies that $f_2 \geq 2$. Since

$$5 = e_1 f_1 + e_1 f_2 + e_2 f_2 = f_1 + f_2 + f_2,$$

we must have $f_1 = 1$ and $f_2 = 2$. Now the invariants a_1, a_2, b_1, and b_2 are completely determined: $a_1 = f_1 + f_2 = 3$, $a_2 = e_1 + f_2 = 3$, $b_1 = a_1 + e_1 = 4$ and $b_2 = a_2 + e_2 = 4$. The Betti diagram is

	0	1	2
0	1	–	–
1	–	1	–
2	–	2	2

We have the Hilbert function

$$\binom{2+d}{2} - \binom{2+d-2}{2} - 2\binom{2+d-3}{2} + 2\binom{2+d-4}{2},$$

which takes on values $1, 3, 5, 5, \ldots$ at $d = 0, 1, 2, 3, \ldots$.

Case (4): We claim that five points in linearly general position also lie on a unique conic. If the points lie on a reducible conic then it is the union of two lines, and one of the lines must contain at least three points. Therefore, if the points are linearly general, any conic containing them must be irreducible. By Bézout's theorem, five points cannot lie on two irreducible conics because the intersection of the conics contains only four points.

As we saw in Case (3), the Betti diagram of I_X was completely determined after we discovered that X lay on a unique conic. We conclude that the Betti numbers are not fine enough to distinguish between the geometric situations presented by Cases (3) and (4).

2.3. An introduction to Castelnuovo–Mumford regularity. Let $S = \mathbb{K}[x_0, \ldots, x_r]$ and let M be a finitely generated graded S-module. One of the ways in which the Betti diagrams for the examples in Section 2.1 differ is in the number of rows. This apparently artificial invariant turns out to be fundamental. In this section we introduce it systematically via the notion of *Castelnuovo–Mumford regularity*. We follow along the lines of [Eisenbud ≥ 2004, Chapter 4].

Definition 2.5. (i) If F is a finitely generated free module, we define $\operatorname{reg} F$, the *regularity of* F, as the maximum degree of a minimal generator of F:

$$\operatorname{reg} F = \max\bigl\{i \mid (F/(x_0, \ldots, x_r)F)_i \neq 0\bigr\}.$$

(The maximum over the empty set is $-\infty$.)

(ii) For an arbitrary finitely generated graded module M, we define the *regularity of* M as

$$\operatorname{reg} M = \max_i \{\operatorname{reg} F_i - i\},$$

where

$$0 \longrightarrow F_m \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

is a minimal free graded resolution of M. We say that M is *d-regular* if $d \geq \operatorname{reg} M$.

Notice that

$$\operatorname{reg} M \geq \operatorname{reg} F_0 - 0 = \operatorname{reg} F_0,$$

where $\operatorname{reg} F_0$ is the maximum degree of a generator of M. The regularity should be thought of as a stabilized version of this "generator degree" which takes into account the nonfreeness of M. One of the most fundamental results about the regularity is a reinterpretation in terms of cohomology. We begin with a special case where the cohomology has a very concrete meaning.

Suppose that we are given a finite set of points $X = \{p_1, \ldots, p_n\} \subset \mathbb{A}^r_{\mathbb{K}}$, where $\mathbb{K}$ is an infinite field. We claim that for any function $\phi : X \to \mathbb{K}$ there is a polynomial $f \in R = \mathbb{K}[x_1, \ldots, x_n]$ such that $f|_X = \phi$. As noted in Section 1.2, the set of all functions from X to $\mathbb{K}$ is spanned by characteristic functions $\phi_1, \ldots, \phi_n$, where

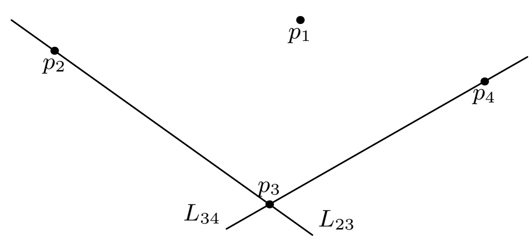
$$\phi_i(p_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

So it is enough to show that the characteristic functions can be given by polynomials. For each $i = 1, \ldots, n$, let L_i be a linear polynomial defining a line containing p_i but not any other point of X. Let $f_i = \prod_{j \neq i} L_i$. The restriction of f_i to X is the function ϕ_i up to a constant scalar.

Definition 2.6. The *interpolation degree* of X is the least integer d such that for each $\phi : X \to \mathbb{K}$ there exists $f \in R$ with $\deg f \leq d$ such that $\phi = f|_X$.

When n is small, one can compute the interpolation degree of X easily from first principles:

Example 2.7. Let X be a set of four points in $\mathbb{A}^2$ in linearly general position. None of the characteristic functions ϕ_i can be the restriction of a linear polynomial; each ϕ_i must vanish at three of the points, but no three are collinear. However, we can easily find quadratic polynomials whose restrictions give the ϕ_i. For instance, let L_{23} be a linear polynomial defining the line joining p_2 and p_3 and let L_{34} be a linear polynomial defining the line joining p_3 and p_4.



The product $L_{23}L_{34}|_X$ equals ϕ_1 up to a constant factor. By symmetry, we can repeat this procedure for each of the remaining ϕ_i. Thus, the interpolation degree of X is two.

It is clear that the interpolation degree of X depends on the cardinality of X. It will also depend on the geometry of the points. To see this, let's compute the interpolation degree of X when X consists of four points that lie on a line L. Any polynomial that restricts to one of the characteristic functions vanishes on three of the points and so intersects the line L at least three times. Therefore, the interpolation degree is at least three. We can easily find a polynomial f_i of degree three that restricts to each ϕ_i: For each $j \neq i$, let L_j be a linear polynomial that vanishes at p_j and no other points of X. Let f_i be the product of these three linear polynomials. Thus, if X consists of four collinear points, its interpolation degree is three.

To study what happens more generally we projectivize. We can view the affine r-space containing our n points as a standard affine open patch of $\mathbb{P}^r$ with coordinate ring $S = \mathbb{K}[x_0, \dots, x_r]$, say, where x_0 is nonzero. A homogeneous polynomial does not have a well-defined value at a point of $\mathbb{P}^r$, so elements of S do not give functions on projective space. However, the notion of when a homogeneous polynomial vanishes at a point is well-defined. This observation shows that we can hope to find homogeneous polynomials that play the role that characteristic functions played for points in affine space and is the basis for the following definition.

DEFINITION 2.8. We say that X imposes *independent conditions* on forms of degree d if there exist $F_1, \dots, F_n \in S_d$ such that $F_j(p_i)$ is nonzero if and only if $i = j$.

We may rephrase the condition in Definition 2.8 as follows: Suppose that F is a form of degree d, say $F = \sum a_\alpha x^\alpha$ with $a_\alpha \in \mathbb{K}$ for each $\alpha \in \mathbb{Z}^{n+1}_{\geq 0}$ such that $|\alpha| = d$. Fix a set of coordinates for each of the points $p_1, \dots, p_n$ and substitute these values into F. Then $F(p_1) = 0, \dots, F(p_n) = 0$ are n equations, linear in the a_α. These equations are the "conditions" that the points $p_1, \dots, p_n$ impose.

We can translate the interpolation degree problem into the projective setting:

PROPOSITION 2.9. *The interpolation degree of X is the minimum degree d such that X imposes independent conditions on forms of degree d.*

PROOF. Let $\{f_1, \dots, f_n\}$ be a set of polynomials in $\mathbb{K}[x_1, \dots, x_r]$ of degree d whose restrictions to X are the characteristic functions of the points. Homogenizing the f_i with respect to x_0 gives us homogeneous forms satisfying the condition of Definition 2.8. So if the interpolation degree of X is d, then X imposes independent conditions of forms of degree d. Furthermore, if X imposes independent conditions on forms of degree d, then there exists a set of forms of degree d that are "homogeneous characteristic functions" and dehomogenizing by setting $x_0 = 1$ gives a set of polynomials of degree at most d whose restrictions to X are characteristic functions for the points in affine space. □

To analyze the new problem in the projective setting we will use methods of coherent sheaf cohomology. (One could use local cohomology with respect to $(x_0, \dots, x_r)$ equally well. See [Eisenbud $\geq$ 2004, Chapter 4] for this point of view.) Let $\mathscr{I}_X$ be the ideal sheaf of X and $\mathscr{O}_X$ its structure sheaf. These sheaves fit into the short exact sequence:

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{O}_{\mathbb{P}^r} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

The following proposition shows that the property that X imposes independent conditions on forms of degree d can be interpreted cohomologically.

PROPOSITION 2.10. *X imposes independent conditions on forms of degree d if and only if $H^1\mathscr{I}_X(d)$ vanishes.*

PROOF. We are interested in whether X imposes independent conditions on forms of degree d so we tensor (or "twist") the short exact sequence by $\mathscr{O}_{\mathbb{P}^r}(d)$. Exactness is clearly preserved on the level of stalks since the stalks of $\mathscr{O}_{\mathbb{P}^r}(d)$ are rank one free modules, and this suffices to show that exactness of the sequence of sheaves is also preserved. We write

$$0 \longrightarrow \mathscr{I}_X(d) \longrightarrow \mathscr{O}_{\mathbb{P}^r}(d) \longrightarrow \mathscr{O}_X(d) \longrightarrow 0.$$

The first three terms in the long exact sequence in cohomology have very concrete interpretations:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0\mathscr{I}_X(d) & \longrightarrow & H^0\mathscr{O}_{\mathbb{P}^r}(d) & \xrightarrow[\rho]{} & H^0\mathscr{O}_X(d) & \longrightarrow & H^1\mathscr{I}_X(d) \longrightarrow \cdots \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & (I_X)_d & \longrightarrow & S_d & \longrightarrow & \mathbb{K}^n & \longrightarrow & \cdots \end{array}$$

Note that the map ρ, which is given by dehomogenizing with respect to x_0 and evaluating the resulting degree d polynomials at the points of X, is surjective if and only if X imposes independent conditions on forms of degree d. The equivalent cohomological condition is that $H^1\mathscr{I}_X(d) = 0$. ☐

The next proposition is a first step in relating vanishings in cohomology with regularity.

PROPOSITION 2.11. *If $\operatorname{reg} I_X \leq d$ then $H^i\mathscr{I}_X(d-i) = 0$.*

PROOF. The point is to construct a short exact sequence of sheaves where $\mathscr{I}_X$ is either the middle or right-hand term and we can say a lot about the vanishing of the higher cohomology of the other two terms. We will get this short exact sequence (with $\mathscr{I}_X$ as the right-hand term) from a free resolution of $\mathscr{I}_X$. Let

$$0 \longrightarrow \bigoplus S(-a_{m,j}) \longrightarrow \cdots \bigoplus S(-a_{1,j}) \longrightarrow \bigoplus S(-a_{0,j}) \longrightarrow I_X \longrightarrow 0$$

be a minimal free resolution of I_X. Its sheafification is exact because localization is an exact functor. Splitting this complex into a series of short exact sequences, and twisting by $d-i$ gives

$$
\begin{array}{ccccccccc}
 & & & 0 & & 0 & & & \\
 & & & & \searrow & \nearrow & & & \\
 & & & & \mathscr{K}_1(d{-}i) & & & & \\
 & & & \nearrow & & \searrow & & & \\
\cdots \longrightarrow & \bigoplus \mathscr{O}_{\mathbb{P}^r}(d{-}i{-}a_{1,j}) & & \longrightarrow & & & \bigoplus \mathscr{O}_{\mathbb{P}^r}(d{-}i{-}a_{0,j}) \longrightarrow & \mathscr{I}_X(d{-}i) \longrightarrow & 0 \\
 & \nearrow & & & & & & & \\
\mathscr{K}_2(d{-}i) & & & & & & & &
\end{array}
$$

From the long exact sequence associated to each short exact sequence in the diagram we see that if $d-i \geq a_{i,j}$ for each i,j, then $H^i\mathscr{I}_X(d-i) = 0$ for each $i > 0$. □

These conditions on the vanishings of the higher cohomology of twists of a sheaf were first captured by Mumford by what we now call the *Castelnuovo–Mumford regularity* of a coherent sheaf. We give the definition for sheaves in terms of the regularity of the cohomology modules

$$H^i_*\mathscr{F} := \bigoplus_{d\geq 0} H^i\mathscr{F}(d),$$

which are finite-dimensional $\mathbb{K}-$vector spaces by a theorem of Serre. (See [Hartshorne 1977, Ch. III, Theorem 5.2] for a proof.)

Definition 2.12. If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^r$,

$$\operatorname{reg}\mathscr{F} = \max_{i>0}\{\operatorname{reg} H^i_*\mathscr{F} + i + 1\}.$$

We say that $\mathscr{F}$ is $d-$*regular* if $d \geq \operatorname{reg}\mathscr{F}$.

One may also reformulate the definition of the regularity of a finitely generated graded module in a similar fashion.

Theorem 2.13. *If H is an Artinian module, define*

$$\operatorname{reg} H = \max\{i \mid H_i \neq 0\}.$$

If M is an arbitrary finitely generated graded S-module define

$$\operatorname{reg} M = \max_{i\geq 0} \operatorname{reg} H^i_{(x_0,\dots,x_r)} M + i.$$

For a proof, one may see [Eisenbud $\geq$ 2004].

The following theorem is Mumford's original definition of regularity.

Theorem 2.14 [Mumford 1966, p. 99]. *If $\mathscr{F}$ is a coherent sheaf on $\mathbb{P}^r$ then $\mathscr{F}$ is d-regular if $H^i\mathscr{F}(d-i) = 0$ for all $i > 0$.*

From the proof of Proposition 2.11 it is clear that if an arbitrary homogeneous ideal $I \subseteq S$ is d-regular, then $\mathscr{I}$ is d-regular. In general, the relationship between the regularity of coherent sheaves and finitely generated modules is a bit technical. (One should expect this since each coherent sheaf on projective space corresponds to an equivalence class of finitely generated graded S-modules.) However, if we work with saturated homogeneous ideals of closed subsets of $\mathbb{P}^r$ the correspondence is quite nice:

Theorem 2.15 [Bayer and Mumford 1993, Definition 3.2]. *If I_Z is the saturated homogeneous ideal of all elements of S that vanish on a Zariski-closed subset Z in $\mathbb{P}^r$ and $\mathscr{I}_Z$ is its sheafification, then* $\operatorname{reg} \mathscr{I}_Z = \operatorname{reg} I_Z$.

(See the technical appendix to Chapter 3 in [Bayer and Mumford 1993] for a proof.)

We return now to the problem of computing the interpolation degree of X. As a consequence of Proposition 2.10, the interpolation degree of X is the minimum d such that $H^1 \mathscr{I}_X(d) = 0$. We claim that $\mathscr{I}_X$ is $(d+1)$-regular if and only if $H^1 \mathscr{I}_X(d) = 0$. If $\mathscr{I}_X$ is $(d+1)$-regular, the vanishing is part of the definition. To see the opposite direction, look at the long exact sequence in cohomology associated to any positive twist of

$$0 \longrightarrow \mathscr{I}_X \longrightarrow \mathscr{O}_{\mathbb{P}^r} \longrightarrow \mathscr{O}_X \longrightarrow 0.$$

The higher cohomology of positive twists of $\mathscr{O}_{\mathbb{P}^r}$ always vanishes, and the higher cohomology of any twist of $\mathscr{O}_X$ vanishes because the support of $\mathscr{O}_X$ has dimension zero. Therefore, $H^i \mathscr{I}_X(k) = 0$ for any positive integer k and all $i \geq 2$.

We conclude that the interpolation degree of X equals d if and only if $\operatorname{reg} \mathscr{I}_X = d+1$, if and only if $\operatorname{reg} I_X = d+1$, if and only if $\operatorname{reg} S/I_X = d$. Thus, the interpolation degree of X is equal to the Castelnuovo–Mumford regularity of its homogeneous coordinate ring S/I_X. In Section 3 we will see many more ways in which regularity and geometry interact.

3. The Size of Free Resolutions

Throughout this section we set $S = \mathbb{K}[x_0, \dots, x_r]$ and let M denote a finitely generated graded S-module. Let

$$\boldsymbol{F}: \quad 0 \longrightarrow F_m \xrightarrow{\phi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

be a minimal free resolution of M, and let $\beta_{i,j}$ be the graded Betti numbers — that is, $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. In this section we will survey some results and conjectures related to the size of $\boldsymbol{F}$.

3.1. Projective dimension, the Auslander–Buchsbaum Theorem, and Cohen–Macaulay modules. The most obvious question is about the length m, usually called the *projective dimension* of M, written $\operatorname{pd} M$. The Auslander–Buchsbaum Theorem gives a very useful characterization. Recall that a *regular sequence of length t on M* is a sequence of homogeneous elements $f_1, f_2, \dots, f_t$ of positive degree in S such that f_{i+1} is a nonzerodivisor on $M/(f_1, \dots, f_i)M$ for each $0 \leq i < t$. (The definition usually includes the condition $(f_1, \dots, f_t)M \neq M$, but this is superfluous because the f_i have positive degree and M is finitely generated.) The *depth of M* is the length of a maximal regular sequence on M (all such maximal regular sequences have the same length). For example, $x_0, \dots, x_r$ is a maximal regular sequence on S and thus on any graded free S-module. In general, the depth of M is at most the (Krull) dimension of M; M is said to be a Cohen–Macaulay module when these numbers are equal, or equivalently when $\operatorname{pd} M = \operatorname{codim} M$.

THEOREM 3.1 (AUSLANDER–BUCHSBAUM). *If $S = \mathbb{K}[x_0, \dots, x_r]$ and M is a finitely generated graded S-module, then the projective dimension of M is $r+1-t$, where t is the length of a maximal regular sequence on M.*

Despite this neat result there are many open problems related to the existence of modules with given projective dimension. Perhaps the most interesting concern Cohen–Macaulay modules:

PROBLEM 3.2. What is the minimal projective dimension of a module annihilated by a given homogeneous ideal I? From the Auslander–Buchsbaum Theorem this number is greater than or equal to $\operatorname{codim} I$; is it in fact equal? That is, does every factor ring S/I have a Cohen–Macaulay module? If S/I does have a Cohen–Macaulay module, what is the smallest rank such a module can have?

If we drop the restriction that M should be finitely generated, then Hochster has proved that Cohen–Macaulay modules ("big Cohen–Macaulay modules") exist for all S/I, and the problem of existence of finitely generated ("small") Cohen–Macaulay modules was posed by him. The problem is open for most S/I of dimension ≥ 3. See [Hochster 1975] for further information.

One of the first author's favorite problems is a strengthening of this one. A module M is said to have *linear resolution* if its Betti diagram has just one row — that is, if all the generators of M are in some degree d, the first syzygies are generated in degree $d+1$, and so on. For example the Koszul complex is a linear resolution of the residue class field $\mathbb{K}$ of S. Thus the following problem makes sense:

PROBLEM 3.3. What is the minimal projective dimension of a module *with linear resolution* annihilated by a given homogeneous ideal I? Is it in fact equal to $\operatorname{codim} I$? That is, does every factor ring S/I have a Cohen–Macaulay module with linear resolution? If so, what is the smallest rank such a module can have?

Cohen–Macaulay modules with linear resolutions are often called *linear Cohen–Macaulay modules* or *Ulrich modules*. They appear, among other places, in the computation of resultants. See [Eisenbud et al. 2003b] for results in this direction and pointers to the literature. This last problem is open even when S/I is a Cohen–Macaulay ring — and the rank question is open even when I is generated by a single element. See [Brennan et al. 1987].

An interesting consequence of the Auslander–Buchsbaum theorem is that it allows us to compare projective dimensions of a module over different polynomial rings. A very special case of this argument gives us a nice interpretation of what it means to be a Cohen–Macaulay module or an Ulrich module. To explain it we need another notion:

Recall that a sequence of homogeneous elements $y_1, \dots, y_d \in S$ is called a *system of parameters* on a graded module M of Krull dimension d if and only if $M/(y_1, \dots, y_d)M$ has (Krull) dimension 0, that is, has finite length. (This happens if and only if $(y_1, \dots, y_d) + \operatorname{ann} M$ contains a power of $(x_0, \dots, x_r)$. For details see [Eisenbud 1995, Ch. 10], for example.) If $\mathbb{K}$ is an infinite field, M is a finitely generated graded module of dimension > 0, and y is a general linear form, then $\dim M/yM = \dim M - 1$. It follows that if M has Krull dimension d then any sufficiently general sequence of linear forms $y_1, \dots, y_d$ is a system of parameters. Moreover, if M is a Cohen–Macaulay module then every system of parameters is a regular sequence on M.

COROLLARY 3.4. *Suppose that $\mathbb{K}$ is an infinite field, and that M is a finitely generated graded S-module of dimension d. Let $y_1, \dots, y_d$ be general linear forms. The module M is a finitely generated module over the subring $T := \mathbb{K}[y_1, \dots, y_d]$. It is a Cohen–Macaulay S-module if and only if it is free as a graded T-module. It is an Ulrich S-module if and only if, for some n, it is isomorphic to T^n as a graded T-module.*

PROOF. Because M is a graded S-module, it is also a graded T-module and M is zero in sufficiently negative degrees. It follows that M can be generated by any set of elements whose images generate $\overline{M} := M/(y_1, \dots, y_d)M$. In particular, M is a finitely generated T module if $\overline{M}$ is a finite-dimensional vector space. Since $y_1, \dots, y_d$ are general, the Krull dimension of $\overline{M}$ is 0. Since $\overline{M}$ is also a finitely generated S-module, it is finite-dimensional as a vector space, proving that M is a finitely generated T-module.

The module M is a Cohen–Macaulay S-module if and only if $y_1, \dots y_d$ is an M-regular sequence, and this is the same as the condition that M be a Cohen–Macaulay T-module. Since T is regular and has the same dimension as M, the Auslander–Buchsbaum formula shows that M is a Cohen–Macaulay T-module if and only if it is a free graded T-module.

For the statement about Ulrich modules we need to use the characterization of Castelnuovo–Mumford regularity by local cohomology; see [Brodmann and Sharp 1998] or [Eisenbud $\geq$ 2004]. Since $y_1, \dots, y_d$ is a system of parameters on M, the ideal $(y_1, \dots, y_d) + \operatorname{ann} M$ has radical $(x_0, \dots, x_r)$ and it follows that the local cohomology modules $\mathrm{H}^i_{(x_0,\dots,x_r)}(M)$ and $\mathrm{H}^i_{(y_1,\dots,y_d)}(M)$ are the same. Thus the regularity of M is the same as a T-module or as an S-module. We can rephrase the definition of the Ulrich property to say that M is Ulrich if and only if M is Cohen–Macaulay, $M_i = 0$ for $i < 0$ and M has regularity 0. Thus M is Ulrich as an S-module if and only if it is Ulrich as a T-module. Since M is a graded free T-module, we see that it is Ulrich if and only if, as a T-module, it is a direct sum of copies of T. □

Since $S/\operatorname{ann} M$ acts on M as endomorphisms, we can say from Corollary 3.4 that there is an Ulrich module with annihilator I if and only if (for some n) the ring S/I admits a faithful representation as $n \times n$ matrices over a polynomial ring. Similarly, there is a Cohen–Macaulay module with annihilator I if S/I has a faithful representation as $\operatorname{End} F$ modules for some graded free module F.

3.2. Bounds on the regularity. The regularity of an arbitrary ideal $I \subset S$ can behave very wildly, but there is evidence to suggest that the regularity of ideals defining (nice) varieties is much lower. Here is a sampling of results and conjectures in this direction. See for example [Bayer and Mumford 1993] for the classic conjectures and [Chardin and D'Cruz 2003] and the papers cited there for a more detailed idea of current research.

Arbitrary ideals: Mayr–Meyer and Bayer–Stillman. Arguments going back to Hermann [1926] give a bound on the regularity of an ideal that depends only on the degrees of its generators and the number of variables — a bound that is extremely large.

Theorem 3.5. *If I is generated by forms of degree d in a polynomial ring in $r+1$ variables over a field of characteristic zero, then* $\operatorname{reg} I \leq (2d)^{2^{r-1}}$.

For recent progress in positive characteristic, see [Caviglia and Sbarra 2003].

An argument of Mayr and Meyer [1982], adapted to the case of ideals in a polynomial ring by Bayer and Stillman [1988], shows that the regularity can really be (roughly) as large as this bound allows: doubly exponential in the number of variables. These examples were improved slightly by Koh [1998] to give the following result.

Theorem 3.6. *For each integer $n \geq 1$ there exists an ideal $I_n \subset \mathbb{K}[x_0, \dots, x_r]$ with $r = 22n - 1$ that is generated by quadrics and has regularity*

$$\operatorname{reg} I_n \geq 2^{2^{n-1}}.$$

See [Swanson 2004] for a detailed study of the primary decomposition of the Bayer–Stillman ideals, which are highly nonreduced. See [Giaimo 2004] for a way of making reduced examples using these ideals as a starting point.

By contrast, for smooth or nearly smooth varieties, there are much better bounds, linear in each of r and d, due to Bertram, Ein and Lazarsfeld [Bertram et al. 1991] and Chardin and Ulrich [2002]: For example:

THEOREM 3.7. *If $\mathbb{K}$ has characteristic 0 and $X \subset \mathbb{P}^r$ is a smooth variety defined scheme-theoretically by equations of degree $\leq d$, then*

$$\operatorname{reg} I_X \leq 1 + (d-1)r.$$

More precisely, if X has codimension c and X is defined scheme-theoretically by equations of degrees $d_1 \geq d_2 \geq \cdots$, then

$$\operatorname{reg} I_X \leq d_1 + \cdots d_c - c + 1.$$

The hypotheses "smooth" and "characteristic 0" are used in the proof through the use of the Kawamata–Viehweg vanishing theorems; but there is no evidence that they are necessary to the statement, which might be true for any reduced algebraic set over an algebraically closed field.

The Bertram–Ein–Lazarsfeld bound is sharp for complete intersection varieties. But one might feel that, when dealing with the defining ideal of a variety that is far from a complete intersection, the degree of the variety is a more natural measure of complexity than the degrees of the equations. This point of view is borne out by a classic theorem proved by Castelnuovo in the smooth case and by Gruson, Lazarsfeld and Peskine [Gruson et al. 1983] in general: If $\mathbb{K}$ is algebraically closed and I is prime defining a projective curve X, then the regularity is linear in the degree of X. (Extending the ground field does not change the regularity, but may spoil primeness.)

THEOREM 3.8. *If $\mathbb{K}$ is algebraically closed, and I is the ideal of an irreducible curve X of degree d in $\mathbb{P}^r$ not contained in a hyperplane, then*

$$\operatorname{reg} I \leq d - r + 2.$$

Giaimo [2003] has proved a generalization of this bound when X is only assumed to be reduced, answering a conjecture of Eisenbud.

On the other hand, it is easy to see (or look up in [Eisenbud $\geq$ 2004, Ch. 4]) that if $X \subset \mathbb{P}^r$ is a scheme not contained in any hyperplane, and S/I_X is Cohen–Macaulay, then

$$\operatorname{reg} I_X \leq \deg X - \operatorname{codim} X + 1.$$

When X is an irreducible curve, this coincides with the Gruson–Lazarsfeld–Peskine Theorem. From this remark and some (scanty) further evidence, Eisenbud and Goto [1984] conjectured that the same bound holds for prime ideals:

CONJECTURE 3.9. Let $\mathbb{K}$ be an algebraically closed field. If $X \subset \mathbb{P}^r$ is an irreducible variety not contained in a hyperplane, then

$$\operatorname{reg} I_X \leq \deg X - \operatorname{codim} X + 1.$$

This is now known to hold for surfaces that are smooth [Bayer and Mumford 1993] and a little more generally [Brodmann 1999; Brodmann and Vogel 1993]; also for toric varieties of codimension two [Peeva and Sturmfels 1998] and a few other classes. Slightly weaker bounds, still linear in the degree, are known for smooth varieties up to dimension six [Kwak 1998; 2000]. Based on a similar analogy and and a little more evidence, Eisenbud has conjectured that the bound of Conjecture 3.9 holds if X is merely reduced and connected in codimension 1.

Both the connectedness and the reducedness hypothesis are necessary, as the following examples show:

Example 3.10 (Two skew lines in $\mathbb{P}^3$). Let

$$I = (s,t) \cap (u,v) = (s,t) \cdot (u,v) \subset S = \mathbb{K}[s,t,u,v]$$

be the ideal of the union X of two skew lines (that is, lines that do not meet) in $\mathbb{P}^3$. The degree of X is of course 2, and X is certainly not contained in a hyperplane. But the Betti diagram of the resolution of S/I is

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

so $\operatorname{reg} I = 2 > \deg X - \operatorname{codim} X + 1$.

Example 3.11 (A multiple line in $\mathbb{P}^3$). Let

$$I = (s,t)^2 + (p(u,v) \cdot s \ + q(u,v) \cdot t) \subset S = \mathbb{K}[s,t,u,v],$$

where $p(u,v)$ and $q(u,v)$ are relatively prime forms of degree $d \geq 1$. The ideal I has degree 2, independent of d, and no embedded components. The scheme X defined by I has degree 2; it is a double structure on the line $V(s,t)$, contained in the first infinitesimal neighborhood $V((s,t)^2)$ of the line in $\mathbb{P}^3$. It may be visualized as the thickening of the line along a "ribbon" that twists d times around the line. But the Betti diagram of the resolution of S/I is

	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	–	–	–
⋮	⋮	⋮	⋮	⋮
$d-1$	–	–	–	–
d	–	1	2	1

so $\operatorname{reg} I = d + 1 > \deg X - \operatorname{codim} X + 1 = 1$.

This last example (and many more) shows that there is no bound on the regularity of a nonreduced scheme in terms of the degree of the scheme alone. But the problem for reduced schemes is much milder, and Bayer and Stillman [1988] have conjectured that the regularity of a reduced scheme over an algebraically closed field should be bounded by its degree (the sum of the degrees of its components). Perhaps the strongest current evidence for this assertion is the recent result of Derksen and Sidman [2002]:

THEOREM 3.12. *If* $\mathbb{K}$ *is an algebraically closed field and* X *is a union of* d *linear subspaces of* $\mathbb{P}^r$, *then* $\operatorname{reg} I_X \le d$

3.3. Bounds on the ranks of the free modules. From the work of Hermann [1926], there are (very large) upper bounds known for the ranks of the free modules F_i in a minimal free resolution

$$\boldsymbol{F}: \quad 0 \longrightarrow F_m \xrightarrow{\phi_m} \ldots \to F_1 \xrightarrow{\phi_1} F_0$$

in terms of the ranks of the modules F_1 and F_0 and the degrees of their generators. However, recent work has focused on *lower* bounds. The only general result known is that of Evans and Griffith [1981; 1985]:

THEOREM 3.13. *If* $F_m \neq 0$ *then* $\operatorname{rank} \operatorname{im} \phi_i \ge i$ *for* $i < m$; *in particular*, $\operatorname{rank} F_i \ge 2i+1$ *for* $i < m-1$, *and* $\operatorname{rank} F_{m-1} \ge n$.

For an example, consider the Koszul complex resolving S/I, where I is generated by a regular sequence of length m. In this case $\operatorname{rank} F_{m-1} = m$, showing that the first statement of Theorem 3.13 is sharp for $i = m-1$. But in the Koszul complex case the "right" bound for the rank is a binomial coefficient. Based on many small examples, Horrocks (see [Hartshorne 1979, problem 24]), motivated by questions on low rank vector bundles, and independently Buchsbaum and Eisenbud [1977], conjectured that something like this should be true more generally:

CONJECTURE 3.14. If M has codimension c, then the i-th map ϕ_i in the minimal free resolution of M has $\operatorname{rank} \phi_i \ge \binom{c-1}{i-1}$, so the i-th free module, has $\operatorname{rank} F_i \ge \binom{c}{i}$. In particular, $\sum \operatorname{rank} F_i \ge 2^c$.

The last statement, slightly generalized, was made independently by the topologist Gunnar Carlsson [1982; 1983] in connection with the study of group actions on products of spheres.

The conjecture is known to hold for resolutions of monomial ideals [Charalambous 1991], for ideals in the linkage class of a complete intersection [Huneke and Ulrich 1987], and for small r (see [Charalambous and Evans 1992] for more information). The conjectured bound on the sum of the ranks holds for almost complete intersections by Dugger [2000] and for graded modules in certain cases by Avramov and Buchweitz [1993].

4. Linear Complexes and the Strands of Resolutions

As before we set $S = \mathbb{K}[x_0, \ldots, x_r]$. The free resolution of a finitely generated graded module can be built up as an iterated extension of *linear complexes*, its *linear strands*. These are complexes whose maps can be represented by matrices of linear forms. In this section we will explain the *Bernstein–Gelfand–Gelfand correspondence* (BGG) between linear free complexes and modules over a certain exterior algebra. We will develop an exterior algebra version of Fitting's Lemma, which connects annihilators of modules over a commutative ring with minors of matrices. Finally, we will use these tools to explain Green's proof of the Linear Syzygy Conjecture of Eisenbud, Koh, and Stillman.

4.1. Strands of resolutions. Let

$$\boldsymbol{F}: \quad 0 \longrightarrow F_m \xrightarrow{\phi_m} F_{m-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0$$

be any complex of free graded modules, and write $F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$. Although we do not assume that $\boldsymbol{F}$ is a resolution, we require it to be a *minimal complex* in the sense that F_i maps to a submodule not containing any minimal generator of F_{i-1}. By Nakayama's Lemma, this condition is equivalent to the condition that $\phi_i F_i \subset (x_0, \dots, x_r)F_{i-1}$ for all $i > 0$.

Under these circumstances $\boldsymbol{F}$ has a natural filtration by subcomplexes as follows. Let $b_0 = \min_{i,j}\{j - i \mid \beta_{i,j} \neq 0\}$. For each i let L_i be the submodule of F_i generated by elements of degree $b_0 + i$. Because F_i is free and has no elements of degree $< b_0 + i$ the module L_i is free, $L_i \cong S(-b_0 - i)^{\beta_{i,b_0+i}}$. Further, since $\phi_i F_i$ does not contain any of the minimal degree elements of F_{i-1} we see that $\phi_i L_i \subset L_{i-1}$; that is, the modules L_i form a subcomplex of $\boldsymbol{F}$. This subcomplex $\boldsymbol{L}$ is a *linear free complex* in the sense that L_i is generated in degree 1 more than L_{i-1}, so that the differential $\psi_i = \phi_i|_{L_i} : L_i \to L_{i-1}$ can be represented by a matrix of linear forms.

We will denote this complex $\mathscr{L}_{b_0}$, and call it the first strand of $\boldsymbol{F}$. Factoring out $\boldsymbol{L}_{b_0}$ we get a new minimal free complex, so we can repeat the process to get a filtration of $\boldsymbol{F}$ by these strands.

We can see the numerical characteristics of the strands of $\boldsymbol{F}$: it follows at once from the definition that the Betti diagrams of the linear strands of $\boldsymbol{F}$ are the rows of the Betti diagram of $\boldsymbol{F}$! This is perhaps the best reason of all for writing the Betti diagram in the form we have given.

Now suppose that $\boldsymbol{F}$ is actually the minimal free resolution of I_X for some projective scheme X. It turns out in many interesting cases that the lengths of the individual strands of $\boldsymbol{F}$ carry much deeper geometric information than does the length of $\boldsymbol{F}$ itself. A first example of this can be seen in the case of the four points, treated in Section 1. The Auslander–Buchsbaum Theorem shows that the resolution of S_X has length exactly r for any finite set of points $X \subset \mathbb{P}^r$. But for four points not all contained in a line, the first linear strand of the minimal resolution of the ideal I_X had length 1 if and only if some three of the points were collinear (else it had length 0). In fact, the line itself was visible in the first strand: in case it had length 1, it had the form

$$0 \to S(-3) \xrightarrow{\phi} S(-2)^2 \longrightarrow 0,$$

and the entries of a matrix representing ϕ were exactly the generators of the ideal of the line.

A much deeper example is represented by the following conjecture of Mark Green. After the genus, perhaps the most important invariant of a smooth algebraic curve is its *Clifford index*. For a smooth curve of genus $g \geq 3$ this may be defined as the minimum, over all degree d maps α (where $d \leq g-1$) from X to a projective space $\mathbb{P}^r$, with image not contained in a hyperplane, as $\deg \alpha - 2r$. The number Cliff X is always nonnegative (Clifford's Theorem). The smaller the Clifford index is, the more special X is. For example, the Clifford index Cliff X is 0 if and only if X is a double cover of $\mathbb{P}^1$. If it is not 0, it is 1 if and only if X is either a smooth plane curve or a triple cover of the line. For "most curves" the Clifford index is simply $d-2$, where d is the smallest degree of a nonconstant map from X to $\mathbb{P}^1$; see [Eisenbud et al. 1989].

A curve X of genus $g \geq 3$ that is not hyperelliptic has a distinguished embedding in the projective space $X \subset \mathbb{P}^{g-1}$ called the *canonical embedding*, obtained by taking the complete linear series of canonical divisors. Any invariant derived from the canonical embedding of a curve is thus an invariant of the abstract (nonembedded) curve. It turns out that the Hilbert functions of all canonically embedded curves of genus g are the same. This is true also of the projective dimension and the regularity of the ideals of such curves. But the graded Betti numbers seem to reflect quite a lot of the geometry of the curve. In particular, Green conjectured that the length of the first linear strand of the resolution of I_X gives precisely the Clifford index:

CONJECTURE 4.1. Suppose that $\mathbb{K}$ has characteristic 0, and let X be a smooth curve of genus g, embedded in $\mathbb{P}^{g-1}$ by the complete canonical series. The length of the first linear strand of the minimal free resolution of I_X is $g-3-\operatorname{Cliff} X$.

The conjecture has been verified by Schreyer [1989] for all curves of genus $g \leq 8$. It was recently proved for a generic curve of each Clifford index by Teixidor [2002] and Voisin [2002a; 2002b] (this may not prove the whole conjecture, because the family of curves of given genus and Clifford index may not be irreducible).

For a version of the conjecture involving high-degree embeddings of X instead of canonical embeddings, see [Green and Lazarsfeld 1988]. See also [Eisenbud 1992; Schreyer 1991] for more information.

4.2. How long is a linear strand? With these motivations, we now ask for bounds on the length of the first linear strand of the minimal free resolution of an arbitrary graded module M. One of the few general results in this direction is due to Green. To prepare for it, we give two examples of minimal free resolutions with rather long linear strands — in fact they will have the maximal length allowed by Green's Theorem:

EXAMPLE 4.2 (THE KOSZUL COMPLEX). The first example is already familiar: the Koszul complex of the linear forms $x_0, \dots, x_s$ is equal to its first linear strand.

Notice that it is a resolution of a module with just one generator, and that the length of the resolution (or its linear strand) is the dimension of the vector space $\langle x_0, \dots, x_s \rangle$ of linear forms that annihilate that generator.

EXAMPLE 4.3 (THE CANONICAL MODULE OF THE RATIONAL NORMAL CURVE). Let X be the *rational normal curve of degree* d, the image in $\mathbb{P}^d$ of the map

$$\mathbb{P}^1 \to \mathbb{P}^d : \qquad (s,t) \mapsto (s^d, s^{d-1}t, \dots, st^{d-1}, t^d).$$

It is not hard to show (see [Harris 1992, Example 1.16] or [Eisenbud 1995, Exercise A2.10]) that the ideal I_X is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \cdots & x_{d-1} \\ x_1 & x_2 & \cdots & x_d \end{pmatrix}$$

and has minimal free resolution $\boldsymbol{F}$ with Betti diagram

	0	1	$\cdots$	$d-2$	$d-1$
0	1	–	$\cdots$	–	–
1	–	$\binom{d}{2}$	$\cdots$	$(d-2)d$	$d-1$

Let ω be the module of twisted global sections of the canonical sheaf $\omega_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(-2 \text{ points})$:

$$\mathrm{H}^0 \mathscr{O}_{\mathbb{P}^1}(-2+d \text{ points}) \oplus \mathrm{H}^0 \mathscr{O}_{\mathbb{P}^1}(-2+2d \text{ points}) \oplus \cdots$$

By duality, ω can be expressed as $\mathrm{Ext}_S^{d-1}(S_X, S(-d-1))$, which is the cokernel of the dual of the last map in the resolution of S_X, twisted by $-d-1$. From the length of the resolution and the Auslander–Buchsbaum Theorem we see that S_X is Cohen–Macaulay, and it follows that the twisted dual of the resolution, $\boldsymbol{F}^*(-d-1)$, is the resolution of ω, which has Betti diagram

	0	1	$\cdots$	$d-2$	$d-1$
1	$d-1$	$(d-2)d$	$\cdots$	$\binom{d}{2}$	–
2	–	–	$\cdots$	–	1

In particular, we see that ω has $d-1$ generators, and the length of the first linear strand of its resolution is $d-2$. Since ω is the module of twisted global sections of a line bundle on X, it is a torsion-free S_X-module. In particular, since the ideal of X does not contain any linear forms, no element of ω is annihilated by any linear form.

These examples hint at two factors that might influence the length of the first linear strand of the resolution of M: the generators of M that are annihilated by linear forms; and the sheer number of generators of M. We can pack both these numbers into one invariant. For convenience we will normalize M by shifting the grading until $M_i = 0$ for $i < 0$ and $M_0 \neq 0$. Let $W = S_1$ be the space of linear forms in S, and let $\mathbb{P} = \mathbb{P}(M_0^*)$ be the projective space of 1-dimensional subspaces of M_0 (we use the convention that $\mathbb{P}(V)$ is the projective space of 1-dimensional quotients of V). Let $A(M) \subset W \times \mathbb{P}$ be the set

$$A(M) := \{(x, \langle m \rangle) \in W \times \mathbb{P} \mid xm = 0\}.$$

The set $A(M)$ contains $0 \times \mathbb{P}$, so it has dimension $\geq \dim_{\mathbb{K}} M_0 - 1$. In Example 4.3 this is all it contains. In Example 4.2 however it also contains $\langle x_0, \dots, x_s \rangle \times \langle 1 \rangle$, a variety of dimension $s+1$, so it has dimension $s+1$. In both cases its dimension is the same as the length of the first linear strand of the resolution. The following result was one of those conjectured by Eisenbud, Koh and Stillman (the "Linear Syzygy Conjecture") based on a result of Green's covering the torsion-free case, and then proved in general by Green [1999]:

Theorem 4.4 (Linear Syzygy Conjecture). *Let M be a graded S-module, and suppose for convenience that $M_i = 0$ for $i < 0$ while $M_0 \neq 0$. The length of the first linear strand of the minimal free resolution of M is at most* $\dim A(M)$.

Put differently, the only way that the length of the first linear strand can be $> \dim_{\mathbb{K}} M_0$ is if there are "many" nontrivial pairs $(x, m) \in W \times M_0$ such that $xm = 0$.

The statement of Theorem 4.4 is only one of several conjectures in the paper of Eisenbud, Koh, and Stillman. For example, they also conjecture that if the resolution of M has first linear strand of length $> \dim_{\mathbb{K}} M_0$, and M is minimal in a suitable sense, then every element of M_0 must be annihilated by some linear form. See [Eisenbud et al. 1988] for this and other stronger forms.

Though it explains the length of the first linear strand of the resolutions of the residue field $\mathbb{K}$ or its first syzygy module $(x_0, \dots, x_r)$, Theorem 4.4 is far from sharp in general. A typical case where one would like to do better is the following: the second syzygy module of $\mathbb{K}$ has no torsion and $\binom{r+1}{2}$ generators, so the theorem bounds the length of its first linear strand by $\binom{r+1}{2} - 1$. However, its first linear strand only has length $r - 1$. We have no theory — not even a conjecture — capable of predicting this.

We will sketch the proof after developing some basic theory connecting the question with questions about modules over exterior algebras.

4.3. Linear free complexes and exterior modules. The Bernstein–Gelfand–Gelfand correspondence is usually thought of as a rather abstract isomorphism between some derived categories. However, it has at its root a very simple observation about linear free complexes: A linear free complex over S is "the same thing" as a module over the exterior algebra on the dual of S_1.

To simplify the notation, we will (continue to) write W for the vector space of linear forms S_1 of S, and we write $V := W^*$ for its vector space dual. We set $E = \bigwedge V$, the exterior algebra on V. Since W consists of elements of degree 1, we regard elements of V as having degree -1, and this gives a grading on E with $\bigwedge^i V$ in degree $-i$. We will use the element $\sum_i x_i \otimes e_i \in W \otimes V$, where $\{x_i\}$ and

$\{e_i\}$ are dual bases of W and V. This element does not depend on the choice of bases — it is the image of $1 \in \mathbb{K}$ under the dual of the natural contraction map

$$V \otimes W \to \mathbb{K} : \qquad v \otimes w \mapsto v(w).$$

Although E is not commutative in the usual sense, it is *strictly commutative* in the sense that

$$ef = (-1)^{(\deg e)(\deg f)} fe$$

for any two homogeneous elements $e, f \in E$. In nearly all respects, E behaves just like a finite-dimensional commutative graded ring. The following simple idea connects graded modules over E with linear free complexes over S:

PROPOSITION 4.5. *Let $\{x_i\}$ and $\{e_i\}$ be dual bases of W and V, and let $P = \bigoplus P_i$ be a graded E-module. The maps*

$$\begin{aligned} \phi_i : S \otimes P_i &\to S \otimes P_{i-1} \\ 1 \otimes p &\mapsto \textstyle\sum_j x_j \otimes e_j p \end{aligned}$$

make

$$\boldsymbol{L}(P) : \qquad \cdots \longrightarrow S \otimes P_i \xrightarrow{\phi_i} S \otimes P_{i-1} \longrightarrow \cdots,$$

into a linear complex of free S-modules. Every complex of free S modules $\boldsymbol{L}$: $\cdots \to F_i \to F_{i-1} \to \cdots$, where F_i is a sum of copies of $S(-i)$ has the form $\boldsymbol{L}(P)$ for a unique graded E-module P.

PROOF. Given an E-module P, we have

$$\phi_{i-1}\phi_i(p) = \phi_{i-1}\left(\sum_j x_j \otimes e_j p\right) = \sum_k \left(\sum_j x_j x_k \otimes e_j e_k p\right).$$

The terms $x_j^2 \otimes e_j^2 p$ are zero because $e_j^2 = 0$. Each other term occurs twice, with opposite signs, because of the skew-commutativity of E, so $\phi_{i-1}\phi_i = 0$ and $\boldsymbol{L}(P)$ is a linear free complex as claimed.

Conversely, given a linear free complex $\boldsymbol{L}$, we set $P_i = F_i/(x_0, \dots, x_r)F_i$. Because $\boldsymbol{L}$ is linear, differentials of $\boldsymbol{L}$ provide maps $\psi_i : P_i \to W \otimes P_{i-1}$. Suppose $p \in P_i$. If $\psi_i(p) = \sum x_j \otimes p_j$ then we define $\mu_i : E \otimes P_i \to E \otimes P_{i-1}$ by $1 \otimes p \mapsto \sum e_j \otimes p_j$. Using the fact that the differentials of $\boldsymbol{L}$ compose to 0, it is easy to check that these "multiplication" maps make P into a graded E-module, and that the two operations are inverse to one another. □

EXAMPLE 4.6. The Koszul complex

$$0 \to \bigwedge^{r+1} S^{r+1} \to \cdots \to \bigwedge^1 S^{r+1} \to S \to 0$$

is a linear free complex over S. To make the maps natural, we should think of S^{r+1} as $S \otimes W$. Applying the recipe in Proposition 4.5 we see that $P_i = \bigwedge^i W$. The module structure on P is that given by contraction, $e \otimes x \mapsto e \neg x$. This module P is canonically isomorphic to $\mathrm{Hom}(E, \mathbb{K})$, which is a left E-module via the right-module structure of E. It is also noncanonically isomorphic to E; to define the isomorphism $E \to P$ we must choose a nonzero element of $\bigwedge^{r+1} W$, an *orientation*, to be the image of $1 \in E$. See [Eisenbud 1995, Ch. 17], for details.

The BGG point of view on linear complexes is well-adapted to studying the linear strand of a resolution, as one sees from the following result. For any finitely generated left module P over E we write $\hat{P}$ for the module $\mathrm{Hom}(P, \mathbb{K})$; it is naturally a right module, but we make it back into a left module via the involution $\iota : E \to E$ sending a homogeneous element $a \in E$ to $\iota(a) = (-1)^{\deg a} a$.

PROPOSITION 4.7. *Let $\boldsymbol{L} = \boldsymbol{L}(P)$ be a finite linear free complex over S corresponding to the graded E-module P.*

1. *$\boldsymbol{L}$ is a subcomplex of the first linear strand of a minimal free resolution if and only if $\hat{P}$ is generated in degree* 0.
2. *$\boldsymbol{L}$ is the first linear strand of a minimal free resolution if and only if $\hat{P}$ has a linear presentation matrix.*

...

∞. *$\boldsymbol{L}$ is a free resolution if and only if $\hat{P}$ has a linear free resolution.*

Here the infinitely many parts of the proposition correspond to the infinitely many degrees in which $\boldsymbol{L}$ could have homology. For the proof, which depends on the Koszul homology formula $\mathrm{H}_i(\boldsymbol{L})_{i+d} = \mathrm{Tor}_d(\mathbb{K}, \hat{P})_{-d-i}$, see [Eisenbud et al. 2003a].

4.4. The exterior Fitting Lemma and the proof of the Linear Syzygy Conjecture. The strategy of Green's proof of the Linear Syzygy Conjecture is now easy to describe. We first reformulate the statement slightly. The algebraic set $A(M)$ consists of $\mathbb{P}$ and the set

$$A' = \big\{(x, \langle y\rangle) \mid 0 \neq x \in W,\ y \in M_0 \text{ and } xm = 0\big\}.$$

Supposing that the first linear strand $\boldsymbol{L}$ of the resolution of M has length k greater than $\dim A'$, we must show that $k < \dim M_0$. We can write $\boldsymbol{L} = \boldsymbol{L}(P)$ for some graded E-module P, and we must show that $P_m = 0$, where $m = \dim_{\mathbb{K}} M_0$.

From Proposition 4.7 we know that $\hat{P}$ is generated in degree 0. Thus to show $P_m = 0$, it is necessary and sufficient that we show that $\hat{P}$ is annihilated by the m-th power of the maximal ideal E_+ of E. In fact, we also know that $\hat{P}$ is linearly presented. Thus we need to use the linear relations on $\hat{P}$ to produce enough elements of the annihilator of P to generate $(E_+)^m$.

If E were a commutative ring, this would be exactly the sort of thing where we would need to apply the classical Fitting Lemma (see for example [Eisenbud 1995, Ch. 20]), which derives information about the annihilator of a module from a free presentation. We will explain a version of the Fitting Lemma that can be used in our exterior situation.

First we review the classical version. We write $I_m(\phi)$ for the ideal of $m \times m$ minors of a matrix ϕ

Lemma 4.8 (Fitting's Lemma). *If*

$$M = \operatorname{coker}(S^n(-1) \xrightarrow{\phi} S^m)$$

is a free presentation, then $I_m(\phi) \subset \operatorname{ann} M$. *In the generic case, when* ϕ *is represented by a matrix of indeterminates, the annihilator is equal to* $I_m(\phi)$.

To get an idea of the analogue over the exterior algebra, consider first the special case of a module with presentation

$$P = \operatorname{coker}\Bigl(E(1) \xrightarrow{\begin{pmatrix} e_1 \\ \vdots \\ e_t \end{pmatrix}} E^t\Bigr),$$

where the e_i are elements of $V = E_{-1}$. If we write $p_1, \dots, p_t$ for the elements of P that are images of the basis vectors of E^t, then the defining relation is $\sum e_i p_i = 0$. We claim that $\prod_i e_i$ annihilates P. Indeed,

$$\Bigl(\prod_i e_i\Bigr) p_j = \Bigl(\prod_{i \neq j} e_i\Bigr) e_j p_j = \Bigl(\prod_{i \neq j} e_i\Bigr) \sum_{i \neq j} -e_i p_i = 0$$

since $e_i^2 = 0$. Once can show that, if the e_i are linearly independent, then $\prod_i e_i$ actually generates $\operatorname{ann} P$. Thus the product is the analogue of the "Fitting ideal" in this case.

In general, if

$$P = \operatorname{coker}(E(1)^s \xrightarrow{\phi} E^t),$$

then the product of the elements of every column of the matrix ϕ annihilates P for the same reason. The same is true of the *generalized columns* of ϕ — that is, the linear combinations with $\mathbb{K}$ coefficients of the columns. In the generic case these products generate the annihilator. Unfortunately it is not clear from this description which — or even how many — generalized columns are required to generate this ideal.

To get a more usable description, recall that the *permanent* $\operatorname{perm} \phi$ of a $t \times t$ matrix ϕ is the sum over permutations σ of the products $\phi_{1,\sigma(1)} \cdots \phi_{t,\sigma(t)}$ (the "determinant without signs"). At least in characteristic zero, the product $\prod e_i$ in our first example is $t!$ times the permanent of the $t \times t$ matrix obtained by repeating the same column t times. More generally, if we make a $t \times t$ matrix ϕ using a_1 copies of a column ϕ^1, a_2 copies of a second column ϕ^2, and in general a_u copies of ϕ^u, so that $\sum a_i = t$, we find that, in the exterior algebra over the integers, the permanent is divisible by $a_1! a_2! \cdots a_u!$. We will write

$$(\phi_1^{(a_1)}, \dots, \phi_u^{(a_u)}) = \frac{1}{a_1! a_2! \cdots a_u!} \operatorname{perm} \phi$$

for this expression, and we call it a $t \times t$ divided permanent of the matrix ϕ. It is easy to see that the divided permanents are in the linear span of the products of the generalized columns of ϕ. This leads us to the desired analogue of the Fitting Lemma:

THEOREM 4.9. *Let P be a module over E with linear presentation matrix $\phi : E^s(1) \to E^t$. The divided permanents $(\phi_1^{(a_1)}, \dots, \phi_u^{(a_u)})$ are elements of the annihilator of P. If the st entries of the matrix ϕ are linearly independent in V, then these elements generate the annihilator.*

The ideal generated by the divided permanents can be described without recourse to the bases above as the image of a certain map $D_t(E^s(1)) \otimes \bigwedge^t E^t \to E$ defined from ϕ by multilinear algebra, where $D_t(F) = (\mathrm{Sym}_t F^*)^*$ is the t-th divided power. This formula first appears in [Green 1999] For the fact that the annihilator is generated by the divided permanents, and a generalization to matrices with entries of any degree, see [Eisenbud and Weyman 2003].

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