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Shelling and the h-Vector of the (Extra)ordinary Polytope

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Abstract. Ordinary polytopes were introduced by Bisztriczky as a (nonsimplicial) generalization of cyclic polytopes. We show that the colex order of facets of the ordinary polytope is a shelling order. This shelling shares many nice properties with the shellings of simplicial polytopes. We also give a shallow triangulation of the ordinary polytope, and show how the shelling and the triangulation are used to compute the toric h-vector of the ordinary polytope. As one consequence, we get that the contribution from each shelling component to the h-vector is nonnegative. Another consequence is a combinatorial proof that the entries of the h-vector of any ordinary polytope are simple sums of binomial coefficients.

1. Introduction

This paper has a couple of main motivations. The first comes from the study of toric h-vectors of convex polytopes. The h-vector played a crucial role in the characterization of face vectors of simplicial polytopes [Billera and Lee 1981; McMullen and Shephard 1971; Stanley 1980]. In the simplicial case, the h-vector is linearly equivalent to the face vector, and has a combinatorial interpretation in a shelling of the polytope. The h-vector of a simplicial polytope is also the sequence of Betti numbers of an associated toric variety. In this context it generalizes to nonsimplicial polytopes. However, for nonsimplicial polytopes, we do not have a good combinatorial understanding of the entries of the h-vector. (Chan [1991] gives a combinatorial interpretation for the h-vector of cubical polytopes.)

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The definition of the (toric) h-vector for general polytopes (and even more generally, for Eulerian posets) first appeared in [Stanley 1987]. Already there Stanley raised the issue of computing the h-vector from a shelling of the polytope. Associated with any shelling, $F_1, F_2, \ldots, F_n$, of a polytope P is a partition of the faces of P into the sets $\mathcal{G}_j$ of faces of F_j not in $\bigcup_{i<j} F_i$. The h-vector can be decomposed into contributions from each set $\mathcal{G}_j$. When P is simplicial, the set $\mathcal{G}_j$ is a single interval $[G_j, F_j]$ in the face lattice of P, and the contribution to the h-vector is a single 1 in position $|G_j|$. For nonsimplicial polytopes, the set $\mathcal{G}_j$ is not so simple. It is not clear whether the contribution to the h-vector from $\mathcal{G}_j$ must be nonnegative, and, if it is, whether it counts something natural. (Tom Braden [2003] has announced a positive answer to this question, based on [Barthel et al. 2002; Karu 2002].) Another issue is the relation of the h-vector of a polytope P to the h-vector of a triangulation of P. This is addressed in [Bayer 1993; Stanley 1992].

A problem in studying nonsimplicial polytopes is the difficulty of generating examples with a broad range of combinatorial types. Bisztriczky [1997] discovered the fascinating "ordinary" polytopes, a class of generally nonsimplicial polytopes, which includes as its simplicial members the cyclic polytopes. These polytopes have been studied further in [Dinh 1999; Bayer et al. 2002; Bayer 2004]. The last of these articles showed that ordinary polytopes have surprisingly nice h-vectors, namely, the h-vector is the sum of the h-vector of a cyclic polytope and the shifted h-vector of a lower-dimensional cyclic polytope. These h-vectors were calculated from the flag vectors, and the calculation did not give a combinatorial explanation for the nice form that came out. So we were motivated to find a combinatorial interpretation for these h-vectors, most likely through shellings or triangulations of the polytopes.

This paper is organized as follows. In the second part of this introduction we give the main definitions. The brief Section 2 warms up with the natural triangulation of the multiplex. Section 3 is devoted to showing that the colex order of facets is a shelling of the ordinary polytope. The proof, while laborious, is constructive, explicitly describing the minimal new faces of the polytope as each facet is shelled on. We then turn in Section 4 to h-vectors of multiplicial polytopes in general, and of the ordinary polytope in particular. Here a "fake simplicial h-vector" arises in the shelling of the ordinary polytope. In Section 5, the triangulation of the multiplex is used to triangulate the boundary of the ordinary polytope. This triangulation is shown to have a shelling compatible with the shelling of Section 3. The shelling and triangulation together explain combinatorially the h-vector of the ordinary polytope.

About the title. Bisztriczky chose the name "ordinary polytope" to invoke the idea of ordinary curves. The name is, of course, a bit misleading, since it applies to a truly extraordinary class of polytopes. We feel that these polytopes are extraordinary because of their special structure, but we hope that they will also turn out to be extraordinary for their usefulness in understanding general convex polytopes.

Definitions. For common polytope terminology, refer to [Ziegler 1995].

The *toric h-vector* was defined by Stanley for Eulerian posets, including the face lattices of convex polytopes.

Definition 1 [Stanley 1987]. Let P be a $(d-1)$-dimensional polytopal sphere. The h-vector and g-vector of P are encoded as polynomials:

$$h(P,x)=\sum_{i=0}^{d} h_i x^{d-i} \quad \text{and} \quad g(P,x)=\sum_{i=0}^{\lfloor d/2\rfloor} g_i x^i,$$

with the relations $g_0 = h_0$ and $g_i = h_i - h_{i-1}$ for $1 \le i \le d/2$. Then the h-polynomial and g-polynomial are defined by the recursion

(i) $g(\varnothing,x)=h(\varnothing,x)=1$, and
(ii) $h(P,x)=\sum_{\substack{G \text{ face of } P\\ G\neq P}} g(G,x)(x-1)^{d-1-\dim G}$.

It is easy to see that the h-vector depends linearly on the flag vector. In the case of simplicial polytopes, the formulas reduce to the well-known transformation between f-vector and h-vector.

Definition 2 [Ziegler 1995]. Let $\mathcal{C}$ be a pure d-dimensional polytopal complex. If $d=0$, a *shelling* of $\mathcal{C}$ is any ordering of the points of $\mathcal{C}$. If $d>0$, a *shelling* of $\mathcal{C}$ is a linear ordering $F_1, F_2, \dots, F_s$ of the facets of $\mathcal{C}$ such that for $2 \le j \le s$, the intersection $F_j \cap \bigl(\bigcup_{i<j} F_i\bigr)$ is nonempty and is the union of ridges (that is, $(d-1)$-dimensional faces) of $\mathcal{C}$ that form the initial segment of a shelling of F_j.

Definition 3 [Bayer 1993]. A triangulation Δ of a polytopal complex $\mathcal{C}$ is *shallow* if and only if every face σ of Δ is contained in a face of $\mathcal{C}$ of dimension at most $2\dim\sigma$.

Theorem 1.1 [Bayer 1993]. *If Δ is a simplicial sphere forming a shallow triangulation of the boundary of the convex d-polytope P, then $h(\Delta,x)=h(P,x)$.*

Note: Theorem 4 in [Bayer 1993] gives $h(P,x)=h(\Delta,x)$ for a shallow subdivision Δ of the solid polytope P. The proof goes through for shallow subdivisions of the boundary, because it is based on the uniqueness of low-degree acceptable functions [Stanley 1987], which holds for lower Eulerian posets.

Definition 4 [Bisztriczky 1996]. A d-dimensional *multiplex* is a polytope with an ordered list of vertices, x_0, x_1, $\dots$, x_n, with facets F_0, F_1, $\dots$, F_n given by

$$F_i = \operatorname{conv}\{x_{i-d+1}, x_{i-d+2}, \dots, x_{i-1}, x_{i+1}, x_{i+2}, \dots, x_{i+d-1}\},$$

with the conventions that $x_i = x_0$ if $i < 0$, and $x_i = x_n$ if $i > n$.

Given an ordered set $V = \{x_0, x_1, \dots, x_n\}$, a subset $Y \subseteq V$ is called a *Gale subset* if between any two elements of $V \setminus Y$ there is an even number of elements of Y. A polytope P with ordered vertex set V is a *Gale polytope* if the set of vertices of each facet is a Gale subset.

Definition 5 [Bisztriczky 1997]. An *ordinary polytope* is a Gale polytope such that each facet is a multiplex with the induced order on the vertices.

Cyclic polytopes can be characterized as the simplicial Gale polytopes. Thus the only simplicial ordinary polytopes are cyclics. In fact, these are the only ordinary polytopes in even dimensions. However, the odd-dimensional, nonsimplicial ordinary polytopes are quite interesting.

We use the following notational conventions. Vertices are generally denoted by integers i rather than by x_i. Where it does not cause confusion, a face of a polytope or a triangulation is identified with its vertex set, and $\max F$ denotes the vertex of maximum index of the face F. Interval notation is used to denote sets of consecutive integers, so $[a, b] = \{a, a+1, \dots, b-1, b\}$. If X is a set of integers and c is an integer, write $X + c = \{x + c : x \in X\}$.

2. Triangulating the Multiplex

Multiplexes have minimal triangulations that are particularly easy to describe.

Theorem 2.1. *Let $M^{d,n}$ be a multiplex with ordered vertices* 0, 1, $\dots$, n. *For $0 \le i \le n-d$, let T_i be the convex hull of $[i, i+d]$. Then $M^{d,n}$ has a shallow triangulation as the union of the $n-d+1$ d-simplices T_i.*

Proof. The proof is by induction on n. For $n = d$, the multiplex $M^{d,d}$ is the simplex T_0 itself. Assume $M^{d,n}$ has a triangulation into simplices T_i, for $0 \le i \le n-d$. Consider the multiplex $M^{d,n+1}$ with ordered vertices 0, 1, $\dots$, $n+1$. Then $M^{d,n+1} = \operatorname{conv}(M^{d,n} \cup \{n+1\})$, where $n+1$ is a point beyond facet F_n of $M^{d,n}$, beneath the facets F_i for $0 \le i \le n-d+1$, and in the affine hulls of the facets F_i for $n-d+2 \le i \le n-1$. (See [Bisztriczky 1996].) Thus, $M^{d,n+1}$ is the union of $M^{d,n}$ and $\operatorname{conv}(F_n \cup \{n+1\}) = T_{n+1-d}$, and $M^{d,n} \cap T_{n+1-d} = F_n$. By the induction assumption, the simplices T_i, with $0 \le i \le n+1-d$, form a triangulation of $M^{d,n+1}$.

The dual graph of the triangulation is simply a path. (The dual graph is the graph having a vertex for each d-simplex, and an edge between two vertices if the corresponding d-simplices share a $(d-1)$-face.) The ordering T_0, T_1, T_2, $\dots$,

T_{n-d} is a shelling of the simplicial complex that triangulates $M^{d,n}$. So the h-vector of the triangulation is $(1, n-d, 0, 0, \ldots)$. This is the same as the g-vector of the boundary of the multiplex, which is the h-vector of the solid multiplex. So by [Bayer 1993], the triangulation is shallow. □

Note, however, that $M^{d,n}$ is not *weakly neighborly* for $n \geq d+2$ (as observed in [Bayer et al. 2002]). This means that it has nonshallow triangulations. This is easy to see because the vertices 0 and n are not contained in a common proper face of $M^{d,n}$.

Consider the induced triangulation of the boundary of $M^{d,n}$. For notational purposes we consider T_0 and T_n separately. All facets of T_0 except $[1, d]$ are boundary facets of $M^{d,n}$. Write $T_{0\backslash 0} = [0, d-1] = F_0$, and $T_{0\backslash j} = [0, d] \setminus \{j\}$ for $1 \leq j \leq d-1$. Write $T_{n-d\backslash n} = [n-d+1, n] = F_n$, and $T_{n\backslash j} = [n-d, n] \setminus \{j\}$ for $n-d+1 \leq j \leq n-1$. For $1 \leq i \leq n-d-1$, the facets of T_i are $T_{i\backslash j} = [i, i+d] \setminus \{j\}$. Two of these facets ($j = i$ and $j = i+d$) intersect the interior of $M^{d,n}$. For $1 \leq j \leq n-1$, the facet F_j is triangulated by $T_{i\backslash j}$ for $j-d+1 \leq i \leq j-1$ (and $0 \leq i \leq n-d$). The facet order $F_0, F_1, \ldots, F_n$, is a shelling of the multiplex $M^{d,n}$. The $(d-1)$-simplices $T_{i\backslash j}$ in the order $T_{0\backslash 0}$, $T_{0\backslash 1}$, $T_{0\backslash 2}$, $T_{1\backslash 2}$, $\ldots$, $T_{n-d-1\backslash n-2}$, $T_{n-d\backslash n-2}$, $T_{n-d\backslash n-1}$, $T_{n-d+1\backslash n}$ (increasing order of j and, for each j, increasing order of i), form a shelling of the triangulated boundary of $M^{d,n}$.

3. Shelling the Ordinary Polytope

Shelling is used to calculate the h-vector, and hence the f-vector of simplicial complexes (in particular, the boundaries of simplicial polytopes). This is possible because (1) the h-vector has a simple expression in terms of the f-vector and vice versa; (2) in a shelling of a simplicial complex, among the faces added to the subcomplex as a new facet is shelled on, there is a unique minimal face; (3) the interval from this minimal new face to the facet is a Boolean algebra; and (4) the numbers of new faces given by (3) match the coefficients in the f-vector/h-vector formula. These conditions all fail for shellings of arbitrary polytopes. However, some hold for certain shellings of ordinary polytopes.

As mentioned earlier, noncyclic ordinary polytopes exist only in odd dimensions. Furthermore, three-dimensional ordinary polytopes are quite different combinatorially from those in higher dimensions. We thus restrict our attention to ordinary polytopes of odd dimension at least five. It turns out that these are classified by the vertex figure of the first vertex.

THEOREM 3.1 [Bisztriczky 1997; Dinh 1999]. *For each choice of integers $n \geq k \geq d = 2m+1 \geq 5$, there is a unique combinatorial type of ordinary polytope $P = P^{d,k,n}$ such that the dimension of P is d, P has $n+1$ vertices, and the first vertex of P is on exactly k edges. The vertex figure of the first vertex of $P^{d,k,n}$ is the cyclic $(d-1)$-polytope with k vertices.*

We use the following description of the facets of $P^{d,k,n}$ by Dinh. For any subset $X \subseteq \mathbb{Z}$, let $\mathrm{ret}_n(X)$ (the *retraction* of X) be the set obtained from X by replacing every negative element by 0 and replacing every element greater than n by n.

THEOREM 3.2 [Dinh 1999]. *Let $\mathcal{X}_n$ be the collection of sets*

$$X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1], \tag{3–1}$$

where $i \in \mathbb{Z}$, $1 \le r \le m$, Y is a paired $(d-2r-1)$-element subset of $[i+2r+1, i+k-2]$, and $|ret_n(X)| \ge d$. The set of facets of $P^{d,k,n}$ is

$$\mathcal{F}(P^{d,k,n}) = \{ret_n(X) : X \in \mathcal{X}_n\}.$$

It is easy to check that when $n = k$, $|\mathrm{ret}_n(X)| = d$ for all $X \in \mathcal{X}_n$, and that $\mathrm{ret}_n(\mathcal{X}_n)$ is the set of d-element Gale subsets of $[0, k]$, that is, the facets of the cyclic polytope $P^{d,k,k}$.

Note that $\mathcal{X}_{n-1} \subseteq \mathcal{X}_n$. We wish to describe $\mathcal{F}(P^{d,k,n})$ in terms of $\mathcal{F}(P^{d,k,n-1})$; for this we need the following shift operations. If $F = \mathrm{ret}_{n-1}(X) \in \mathcal{F}(P^{d,k,n-1})$, let the right-shift of F be $\mathrm{rsh}(F) = \mathrm{ret}_n(X+1)$. Note that $\mathrm{rsh}(F)$ may or may not contain 0. In either case, $\mathrm{rsh}(F) \cap [1, n] = F+1$, so $|\mathrm{rsh}(F)| \ge |F| \ge d$, If $F = \mathrm{ret}_n(X) \in \mathcal{F}(P^{d,k,n})$, let the left-shift of F be $\mathrm{lsh}(F) = \mathrm{ret}_{n-1}(X-1)$. Note that $\mathrm{lsh}(F) \setminus \{0\} = (F-1) \cap [1, n]$; $\mathrm{lsh}(F)$ contains 0 if $0 \in F$ or $1 \in F$.

LEMMA 3.3. *If $n \ge k+1$ and $F \in \mathcal{F}(P^{d,k,n})$ with $\max F \ge k$, then* $\mathrm{lsh(F)} \in \mathcal{F}(\mathrm{P^{d,k,n-1}})$.

PROOF. Let $F = \mathrm{ret}_n(X)$, with $X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1]$. Then $X-1$ also has the form of equation (3–1) (for $i-1$). The set $\mathrm{lsh}(F)$ is the vertex set of a facet of $P^{d,k,n-1}$ as long as $|\mathrm{lsh}(F)| \ge d$. We check this in three cases.

Case 1. If $k \le i+k+2r-1 \le n$, then $i+2r-1 \ge 0$, so $Y \subseteq [i+2r+1, i+k-2] \subseteq [2, i+k-2]$. Then

$$\mathrm{lsh}(F) \supseteq \max\{i+2r-2, 0\} \cup (Y-1) \cup [i+k-1, i+k+2r-2],$$

so $|\mathrm{lsh}(F)| \ge 1+(d-2r-1)+2r = d$.

Case 2. If $i+k \ge n$, then $i \ge n-k \ge 1$. Also, $|F| \ge d$ implies $\max Y \le n-1$. So

$$\mathrm{lsh}(F) = [i-1, i+2r-2] \cup (Y-1) \cup \{n-1\},$$

so $|\mathrm{lsh}(F)| = 2r+(d-2r-1)+1 = d$.

Case 3. If $i+k < n < i+k+2r-1$, then $i+2r-1 \ge n-k \ge 1$, and

$$F = [\max\{0, i\}, i+2r-1] \cup Y \cup [i+k, n],$$

so

$$\begin{aligned} |F| &= (i+2r-\max\{0, i\})+(d-2r-1)+(n-i-k+1) \\ &= d+n-k-\max\{i, 0\} \ge d+1. \end{aligned}$$

Then $|\mathrm{lsh}(F)| \ge |F|-1 \ge d$.

Thus, $\mathrm{lsh}(F)$ is a facet of $P^{d,k,n-1}$. □

Identify each facet of the ordinary polytope $P^{d,k,n}$ with its ordered list of vertices. Then order the facets of $P^{d,k,n}$ in colex order. This means, if $F = i_1 i_2 \dots i_p$ and $G = j_1 j_2 \dots j_q$, then $F \prec_c G$ if and only if for some $t \geq 0$, $i_{p-t} < j_{q-t}$ while for $0 \leq s < t$, $i_{p-s} = j_{q-s}$.

Lemma 3.4. *If $n \geq k+1$ and F_1 and F_2 are facets of $P^{d,k,n}$ with $\max F_i \geq k$, then $F_1 \prec_c F_2$ implies $lsh(F_1) \prec_c lsh(F_2)$.*

Proof. Suppose $F_1 \prec_c F_2$, and let q be the maximum vertex in F_2 not in F_1. Then $\mathrm{lsh}(F_1) \prec_c \mathrm{lsh}(F_2)$ as long as $q \geq 2$, for in that case $q-1 \in \mathrm{lsh}(F_2) \backslash \mathrm{lsh}(F_1)$, while $[q, n-1] \cap \mathrm{lsh}(F_1) = [q, n-1] \cap \mathrm{lsh}(F_2)$. (If $q = 1$, then q shifts to 0 in $\mathrm{lsh}(F_2)$, but 0 may be in $\mathrm{lsh}(F_1)$ as a shift of a smaller element.) So we prove $q \geq 2$. Write

$$F_2 = \mathrm{ret}_n([i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1])$$

and

$$F_1 = \mathrm{ret}_n([i', i'+2r'-1] \cup Y' \cup [i'+k, i'+k+2r'-1]).$$

Since $\max F_2 \geq k$, $i+2r-1 \geq 0$, so $Y \cup [i+k, i+k+2r-1] \subseteq [2, n]$, Thus, if $q \in Y \cup [i+k, i+k+2r-1]$, then $q \geq 2$. Otherwise

$$Y \cup [i+k, i+k+2r-1]) = Y' \cup [i'+k, i'+k+2r'-1]),$$

but $Y \neq Y'$. This can only happen when $Y \cup [i+k, i+k+2r-1])$ is an interval; in this case $i+k+2r-1 \geq n+1$. Then $q = i+2r-1 = (i+k+2r-1)-k \geq n+1-k \geq 2$. □

Proposition 3.5. *Let $n \geq k+1$. The facets of $P^{d,k,n}$ are*

$$\{F : F \in \mathcal{F}(P^{d,k,n-1}) \text{ and } \max F \leq n-2\} \cup \{rsh(F) : F \in \mathcal{F}(P^{d,k,n-1}) \text{ and } \max F \geq n-2\}.$$

Proof. If $\max X \leq n-2$, then $\mathrm{ret}_n(X) = \mathrm{ret}_{n-1}(X)$; in this case, letting $F = \mathrm{ret}_n(X)$, $F \in \mathcal{F}(P^{d,k,n-1})$ if and only if $F \in \mathcal{F}(P^{d,k,n})$. If $F \in \mathcal{F}(P^{d,k,n-1})$ with $\max F \geq n-2$, then $\mathrm{rsh}(F) \in \mathcal{F}(P^{d,k,n})$ with $\max \mathrm{rsh}(F) \geq n-1$. Now suppose that $G = \mathrm{ret}_n(X) \in \mathcal{F}(P^{d,k,n})$ with $\max G \geq n-1$. Let $F = \mathrm{lsh}(G) = \mathrm{ret}_{n-1}(X-1) \in \mathcal{F}(P^{d,k,n-1})$; then $\max F \geq n-2$. By definition, $\mathrm{rsh}(F) = \mathrm{ret}_n((X-1)+1) = \mathrm{ret}_n(X) = G$. □

Theorem 3.6. *Let $F_1, F_2, \dots, F_v$ be the facets of $P^{d,k,n}$ in colex order.*

(i) *$F_1, F_2, \dots, F_v$ is a shelling of $P^{d,k,n}$.*
(ii) *For each j there is a unique minimal face G_j of F_j not contained in $\bigcup_{i=1}^{j-1} F_i$.*
(iii) *For each j, $2 \leq j \leq v-1$, G_j contains the vertex of F_j of maximum index, and is contained in the $d-1$ highest vertices of F_j.*
(iv) *For each j, the interval $[G_j, F_j]$ is a Boolean lattice.*

Note that this theorem is not saying that the faces of $P^{d,k,n}$ in the interval $[G_j, F_j]$ are all simplices.

Proof. We construct explicitly the faces G_j in terms of F_j. The reader may wish to refer to the example that follows the proof.

Cyclic polytopes. We start with the cyclic polytopes. (For the cyclics, the theorem is generally known, or at least a shorter proof based on [Billera and Lee 1981] is possible, but we will need the description of the faces G_j later.)

Let $F_1, F_2, \ldots, F_v$ be the facets, in colex order, of $P^{d,k,k}$, the cyclic d-polytope with vertex set $[0, k]$. Each facet F_j can be written as

$$F_j = I_j^0 \cup I_j^1 \cup I_j^2 \cup \cdots \cup I_j^p \cup I_j^k,$$

where I_j^0 is the interval of F_j containing 0, if $0 \in F_j$, and $I_j^0 = \varnothing$ otherwise; I_j^k is the interval of F_j containing k, if $k \in F_j$, and $I_j^k = \varnothing$ otherwise; and the I_j^ℓ are the other (even) intervals of F_j with the elements of I_j^ℓ preceding the elements of $I_j^{\ell+1}$. (For example, in $P^{7,9,9}$, $F_6 = \{0,1,2,4,5,7,8\}$, $I_6^0 = \{0,1,2\}$, $I_6^1 = \{4,5\}$, $I_6^2 = \{7,8\}$, and $I_6^9 = \varnothing$.) For the interval $[a, b]$, write $E([a, b])$ for the integers in the even positions in the interval, that is, $E([a, b]) = [a, b] \cap \{a+2i+1 : i \in \mathbb{N}\}$. Let $G_j = \bigcup_{\ell=1}^p E(I_j^\ell) \cup I_j^k$. Since $I_j^0 = F_j$ if and only if $j = 1$, $G_1 = \varnothing$, and for all $j > 1$, G_j contains the maximum vertex of F_j. Since F_j is a simplex, $[G_j, F_j]$ is a Boolean lattice.

To show that $F_1, F_2, \ldots, F_v$ is a shelling of $P^{d,k,k}$ we show that G_j is not in a facet before F_j and that every ridge of $P^{d,k,k}$ in F_j that does not contain G_j is contained in a previous facet. For $j > 0$ the face G_j consists of the right end-set I_j^k (if nonempty) and the set $\bigcup_{j=1}^p E(I_j^\ell)$ of singletons. Note that G_j satisfies condition (c) of the theorem (which here just says that the lowest vertex of F_j is not in G_j), unless $j = v$, in which case $G_v = F_v$. Any facet F of $P^{d,k,k}$ containing G_j must satisfy Gale's evenness condition and therefore must contain an integer adjacent to each element of $\bigcup_{j=1}^p E(I_j^\ell)$. If any element of the form $\max I_j^\ell + 1$ is in F, then F occurs after F_j in colex order. This implies that any F_i previous to F_j and containing G_j also contains $\bigcup_{\ell=1}^p I_j^\ell \cup I_j^k$. But F_j is the first facet in colex order that contains $\bigcup_{\ell=1}^p I_j^\ell \cup I_j^k$. So G_j is not in a facet before F_j.

Now let $g \in G_j$; we wish to show that $F_j \setminus \{g\}$ is in a previous facet. If $g \in E(I_j^\ell)$ for $\ell > 0$, let $F = F_j \setminus \{g\} \cup \{\min I_j^\ell - 1\}$. Then F satisfies Gale's evenness condition and is a facet before F_j. Otherwise $g \in I_j^k \setminus E(I_j^k)$; in this case let $F = F_j \setminus \{g\} \cup \{\max I_j^0 + 1\}$ (where we let $\max I_j^0 + 1 = 0$ if $I_j^0 = \varnothing$). Again F satisfies Gale's evenness condition and is a facet before F_j.

Thus the colex order of facets is a shelling order for the cyclic polytope $P^{d,k,k}$, and we have an explicit description for the minimal new face G_j as F_j is shelled on.

General ordinary. Now we prove the theorem for general $P^{d,k,n}$ by induction on $n \geq k$, for fixed k. Among the facets of $P^{d,k,n}$, first in colex order are those with maximum vertex at most $n-2$. These are also the first facets in colex order of $P^{d,k,n-1}$. Thus the induction hypothesis gives us that this initial segment is a partial shelling of $P^{d,k,n}$, and that assertions 2–4 hold for these facets.

Later facets. It remains to consider the facets of $P^{d,k,n}$ ending in $n-1$ or n. These facets come from shifting facets of $P^{d,k,n-1}$ ending in $n-2$ or $n-1$. Our strategy here will be to prove statement (b) of the theorem for these facets. The intersection of F_j with $\bigcup_{i=1}^{j-1} F_i$ is then the antistar of G_j in F_j, and so it is the union of $(d-2)$-faces that form an initial segment of a shelling of F_j. This will prove that the colex order $F_1, F_2, \ldots, F_v$ is a shelling of $P^{d,k,n}$.

Note that there is nothing to show for the last facet of $P^{d,k,n}$ in colex order. It is $F_v = [n-d+1, n]$, and is the only facet (other than the first) whose vertex set forms a single interval. Assume from now on that j is fixed, with $j \leq v-1$. Later we will describe recursively the minimal new face G_j as F_j is shelled on. It will always be the case that $\max F_j \in G_j$. We will prove that G_j is truly a new face (is not contained in a previous facet), and that every ridge not containing all of G_j is contained in a previous facet.

Ridges not containing the last vertex. It is convenient to start by showing that every ridge of $P^{d,k,n}$ contained in F_j and not containing $\max F_j$ is contained in an earlier facet. This case does not use the recursion needed for the other parts of the proof. Write

$$X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1]$$

and $F_j = \mathrm{ret}_n(X) = \{z_1, z_2, \ldots, z_p\}$ with $0 \leq z_1 < z_2 < \cdots < z_p \leq n$. The facet F_j is a $(d-1)$-multiplex, so its facets are of the form

$$F_j(\hat{z}_t) = \{z_\ell : 1 \leq \ell \leq p,\ 0 < |\ell - t| \leq d-2\}$$

for $2 \leq t \leq p-1$, $F_j(\hat{z}_1) = \{z_1, z_2, \ldots, z_{d-1}\}$, and $F_j(\hat{z}_p) = \{z_{p-d+2}, \ldots, z_{p-1}, z_p\}$. If $F_j(\hat{z}_t)$ does not contain $\max F_j = z_p$, then $t \leq p-d+1$ and this implies $i \leq z_t \leq i+2r-1$. Consider such a z_t.

The first ridge. For $t = 1$, there are three cases to consider.

Case 1. Suppose $z_1 \geq 1$. Then $F_j(\hat{z}_1) = [i, i+2r-1] \cup Y$. Let I be the right-most interval of $F_j(\hat{z}_1)$. Let $Z = (I-k) \cup F_j(\hat{z}_1)$, and $F = \mathrm{ret}_n(Z)$. Since $i \geq 1$ and $\max F_j(\hat{z}_1) \leq i+k-2$, the interval $I-k$ contributes at least one new element to F, so $|F| \geq d$.

Case 2. Suppose $z_1 = 0$ and the right-most interval of $F_j(\hat{z}_1)$ is odd. In this case the left-most interval of F_j must also be odd, so $i < 0$, and $F_j(\hat{z}_1)$ contains $i+k$ but not $i+k-1$. Let $F = F_j(\hat{z}_1) \cup \{i+k-1\}$.

Case 3. Suppose $z_1 = 0$ and the right-most interval of $F_j(\hat{z}_1)$ is even (and then so is the left-most interval). Then $F_j(\hat{z}_1) = [0, i+2r-1] \cup Y \cup [i+k, k-1]$ (where the last interval is empty if $i = 0$). Let

$$F = F_j(\hat{z}_1) \cup \{i+2r\} = \{0\} \cup [1, i+2r] \cup Y \cup [i+k, k-1].$$

(When $i = 0$ and $r = (d-1)/2$, this gives $F = [0, d-1]$.) In all cases F is a facet of $P^{d,k,n}$ containing $F_j(\hat{z}_1)$. It does not contain $\max F_j$, so $F \prec_c F_j$.

Deleting a later vertex. Now assume $2 \le t \le p-d+1$; then $z_t \ge \max\{i+1, 1\}$. Here

$$F_j(\hat{z}_t) = [\max\{i, 0\}, z_t - 1] \cup [z_{t+1}, i+2r-1] \cup Y \cup [i+k, z_t - 1 + k],$$

and $|F_j(\hat{z}_t)| = z_t - \max\{i, 0\} + d - 2 \ge d-1$. Also note that $z_t - 1 + k$ is the $(d-2)$nd element of $\{z_1, z_2, \ldots, z_p\}$ after z_t, so $z_t - 1 + k = z_{t+d-2} < z_p = \max F_j$.

Case 1. If $z_t - i$ is even, let $F = F_j(\hat{z}_t) \cup \{i+2r\}$. Then $F = \mathrm{ret}_n(Z)$, where

$$Z = [i, z_t - 1] \cup [z_t + 1, i+2r] \cup Y \cup [i+k, z_t - 1 + k],$$

and $|F| \ge d$.

Case 2. If $z_t - i$ is odd and $\max([i, i+2r-1] \cup Y) < i+k-2$, let $F = \mathrm{ret}_n(Z)$, where

$$Z = [i-1, z_t - 1] \cup [z_t + 1, i+2r-1] \cup Y \cup [i+k-1, z_t - 1 + k].$$

Then $F \supseteq F_j(\hat{z}_t) \cup \{i+k-1\}$, so $|F| \ge d$.

Case 3. Finally, suppose $z_t - i$ is odd and $\max Y = i+k-2$. Let $[q, i+k-2]$ be the right-most interval of Y, and let $F = \mathrm{ret}_n(Z)$, where

$$Z = [q-k, z_t - 1] \cup [z_t + 1, i+2r-1] \cup (Y \setminus [q, i+k-2]) \cup [q, z_t - 1 + k].$$

Then $F \supseteq F_j(\hat{z}_t) \cup \{i+k-1\}$, so $|F| \ge d$.

In all cases, F is a facet of $P^{d,k,n}$ containing $F_j(\hat{z}_t)$ and $\max F_j \notin F$, so F occurs before F_j in colex order.

Determining the minimal new face. We now describe the faces G_j recursively. (We are still assuming that $\max F_j \ge n-1$.) Let G be the face of $\mathrm{lsh}(F_j)$ that is the minimal new face when $\mathrm{lsh}(F_j)$ is shelled on, in the colex shelling of the polytope $P^{d,k,n-1}$. Let $G_j = G+1$; this is a subset of the last $d-1$ vertices of F_j and contains $\max F_j$. By [Bayer et al. 2002, Theorem 2.6] and [Bisztriczky 1996], G_j is a face of F_j. For any facet F_i of $P^{d,k,n}$, $G_j \subseteq F_i$ if and only if $G \subseteq \mathrm{lsh}(F_i)$. So by the induction hypothesis, G_j is not contained in a facet occurring before F_j in colex order.

Ridges in previous facets. It remains to show that any ridge of $P^{d,k,n}$ contained in F_j but not containing all of G_j is contained in a facet prior to F_j. Note that we have already dealt with those ridges not containing $\max F_j$. Now let $g \in G$, $g_j = g+1 \in G_j$, and assume $g_j \neq \max F_j$. The only ridge of $P^{d,k,n}$ contained in F_j, containing $\max F_j$, and not containing g_j is $F_j(\hat{g}_j)$.

Let H be the unique ridge of $P^{d,k,n-1}$ in $\operatorname{lsh}(F_j)$ containing $\max(\operatorname{lsh}(F_j))$, but not containing g. By the induction hypothesis, H is contained in a facet F of $P^{d,k,n-1}$ occurring before $\operatorname{lsh}(F_j)$ in colex order. Suppose $F_j(\hat{g}_j)$ is contained in a facet F_ℓ of $P^{d,k,n}$ occurring after F_j in colex order. Then H is contained in $\operatorname{lsh}(F_\ell)$. Thus the ridge H of $P^{d,k,n-1}$ is contained in three different facets: F (occurring before $\operatorname{lsh}(F_j)$ in colex order), $\operatorname{lsh}(F_j)$, and $\operatorname{lsh}(F_\ell)$ (occurring after $\operatorname{lsh}(F_j)$ in colex order). This contradiction shows that the ridge $F_j(\hat{g}_j)$ can only be contained in a facet of $P^{d,k,n}$ occurring before F_j in colex order.

Boolean intervals. Finally to verify assertion 4 of the theorem, observe that every facet F_j is a $(d-1)$-dimensional multiplex. The face G_j of F_j contains the maximum vertex u of F_j. The vertex figure of the maximum vertex in any multiplex is a simplex [Bisztriczky 1996]. The interval $[G_j, F_j]$ is an interval in $[u, F_j]$, which is the face lattice of a simplex, so $[G_j, F_j]$ is a Boolean lattice. □

A nonrecursive description of the faces G_j, generalizing that for the cyclic case in the proof, is as follows. Write the facet F_j as a disjoint union, $F_j = A_j^0 \cup I_j^1 \cup I_j^2 \cup \cdots \cup I_j^p \cup I_j^n$, where I_j^n is the interval of F_j containing n if $n \in F_j$, and $I_j^n = \varnothing$ otherwise; the I_j^ℓ $(1 \le \ell \le p)$ are even intervals of F_j written in increasing order; and A_j^0 is

- the interval containing 0, if $\max F_j \le k-1$;
- the union of the interval containing $\max F_j - k$ and the interval containing $\max F_j - k+2$ (if the latter exists), if $k \le \max F_j \le n-1$;
- the interval containing $n-k$, if $\max F_j = n$ and $n-k \in F_j$;
- $\varnothing$, if $\max F_j = n$ and $n-k \notin F_j$.

Then $G_j = \bigcup_{\ell=1}^p E(I_j^\ell) \cup I_j^n$. The vertices of G_j are among the last d vertices of F_j and so are affinely independent [Bisztriczky 1996]; thus G_j is a simplex.

Example. Table 1 gives the faces F_j and G_j for the colex shelling of the ordinary polytope $P^{5,6,8}$.

Let us look at what happens when facet F_{13} is shelled on. The ridges of $P^{5,6,8}$ contained in F_{13} are 0123, 0236, 01367, 012678, 12378, 2368, and 3678. The first ridge, 0123, is contained in $F_1 = 01234$. The ridge 0236 is $F_{13}(\hat{z}_2) = F_{13}(\hat{1})$, and $\max([i, i+2r-1] \cup Y) = 3 < 4 = i+k-2$, so we find that 0236 is contained in $F_4 = 02356$. The ridge 01367 is $F_{13}(\hat{z}_3) = F_{13}(\hat{2})$, so we find that 01367 is contained in $F_6 = 013467$. This facet $F_{13} = 0123678$ is shifted from the facet 012567 of $P^{5,6,7}$, which in turn is shifted from the facet 01456 of the cyclic

j	F_j	G_j	j	F_j	G_j
1	01234	∅	9	23 56 8	68
2	012 45	5	10	3456 8	468
3	0 2345	35	11	1234 78	78
4	0 23 56	6	12	12 45 78	578
5	0 3456	46	13	0123 678	678
6	01 34 67	7	14	34 678	4678
7	01 4567	57	15	012 5678	5678
8	2345 8	8	16	45678	45678

Table 1. Shelling of $P^{5,6,8}$

polytope $P^{5,6,6}$. When 01456 occurs in the shelling of the cyclic polytope, its minimal new face is its right interval, 456. In $P^{5,6,8}$, then, the minimal new face when F_{13} is shelled on is 678. The other ridges of F_{13} not containing 678 are 12378 and 2368. The interval $[G_{13}, F_{13}]$ contains the triangle 678, the 3-simplex 3678, the 3-multiplex 012678, and F_{13} itself (which is a pyramid over 012678).

Note that for the multiplex, $M^{d,n} = P^{d,d,n}$, this theorem gives a shelling different from the one mentioned in Section 2. In the standard notation for the facets of the multiplex (see Definition 4), the colex shelling order is F_0, F_1, ..., F_{n-d}, F_{n-1}, F_{n-2}, ..., F_{n-d+1}, F_n. The statements of this section hold also for even-dimensional multiplexes.

4. The h-Vector from the Shelling

The h-vector of a simplicial polytope can be obtained easily from any shelling of the polytope. For P a simplicial polytope, and $\cup [G_j, F_j]$ the partition of a face lattice of P arising from a shelling, $h(P, x) = \sum_j x^{d-|G_j|}$. For general polytopes, the (toric) h-vector can also be decomposed according to the shelling partition. For a shelling, F_1, F_2, ..., F_n, of a polytope P, write $\mathcal{G}_j$ for the set of faces of F_j not in $\bigcup_{i<j} F_i$. Then $h(P, x) = \sum_{j=1}^{n} h(\mathcal{G}_j, x)$, where $h(\mathcal{G}_j, x) = \sum_{G \in \mathcal{G}_j} g(G, x)(x-1)^{d-1-\dim G}$. However, in general we do not know that the coefficients of $h(\mathcal{G}_j, x)$ count anything natural, nor even that they are nonnegative. Stanley raised this issue in [Stanley 1987, Section 6]. It has apparently been settled in [Braden 2003].

We turn now to h-vectors of ordinary polytopes. In [Bayer 2004] we used the flag vector of the ordinary polytope to compute its toric h-vector.

Theorem 4.1 [Bayer 2004]. *For $n \ge k \ge d = 2m+1 \ge 5$ and $1 \le i \le m$,*

$$h_i(P^{d,k,n}) = \binom{k-d+i}{i} + (n-k)\binom{k-d+i-1}{i-1}.$$

We did not understand why the h-vector turned out to have such a nice form. Here we show how the h-vector can be computed from the colex shelling. Properties 2 and 4 of Theorem 3.6 are critical.

In [Bayer 2004] we showed that the flag vector of a multiplicial polytope depends only on the f-vector. However, for our purposes here it is more useful to write the h-vector in terms of the f-vector and the flag vector entries of the form f_{0i}. We introduce a modified f-vector. Let $\bar{f}_{-1} = f_{-1} = 1$, $\bar{f}_0 = f_0$, and $\bar{f}_{d-1} = f_{d-1} + (f_{0,d-1} - df_{d-1})$; and for $1 \le j \le d-2$, let

$$\bar{f}_j = f_j + (f_{0,j+1} - (j+2)f_{j+1}) + (f_{0,j} - (j+1)f_j).$$

(Thus, $\bar{f}_1 = f_1 + (f_{02} - 3f_2) + (f_{01} - 2f_1) = f_1 + (f_{02} - 3f_2)$.)

THEOREM 4.2. *If P is a multiplicial d-polytope, then*

$$h(P,x) = \sum_{i=0}^{d} h_i(P)x^{d-i} = \sum_{i=0}^{d} \bar{f}_{i-1}(P)(x-1)^{d-i}.$$

PROOF. As observed in the proof of Theorem 2.1, the g-polynomial of an e-dimensional multiplex M with $n+1$ vertices is $g(M,x) = 1 + (n-e)x$. So for a multiplicial d-polytope P,

$$\begin{aligned}
h(P,x) &= \sum_{\substack{G \text{ face of } P\\ G \ne P}} g(G,x)(x-1)^{d-1-\dim G}\\
&= \sum_{\substack{G \text{ face of } P\\ G \ne P}} (1 + (f_0(G) - 1 - \dim G)x)(x-1)^{d-1-\dim G}\\
&= \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} + \sum_{i=1}^{d-1}(f_{0i} - (i+1)f_i)x(x-1)^{d-1-i}\\
&= \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} + \sum_{i=1}^{d-1}(f_{0i} - (i+1)f_i)[(x-1)^{d-i} + (x-1)^{d-1-i}]\\
&= (x-1)^d + f_0(x-1)^{d-1}\\
&\quad + \sum_{i=2}^{d-1}(f_{i-1} + (f_{0i} - (i+1)f_i) + (f_{0,i-1} - if_{i-1}))(x-1)^{d-i}\\
&\quad + (f_{d-1} + (f_{0,d-1} - df_{d-1}))\\
&= \sum_{i=0}^{d} \bar{f}_{i-1}(P)(x-1)^{d-i}. \qquad \square
\end{aligned}$$

Simplicial polytopes are a special case of multiplicial polytopes. Clearly, when P is simplicial, $\bar{f}(P) = f(P)$, and we recover the definition of the simplicial h-vector in terms of the f-vector. The multiplicial h-vector formula can be thought of as breaking into two parts: one involving the f-vector, and matching the simplicial h-vector formula; the other involving the "excess vertex counts," $f_{0,j} - (j+1)f_j$.

In the simplicial case the sum of the entries in the h-vector is the number of facets. For multiplicial polytopes $\sum_{i=0}^{d} h_i(P) = \bar{f}_{d-1}(P) = f_{d-1} + (f_{0,d-1} - df_{d-1})$.

In general, applying the simplicial h-formula to a nonsimplicial f-vector produces a vector with no (known) combinatorial interpretation. This vector is neither symmetric nor nonnegative in general. We will see that in the case of ordinary polytopes something special happens. Write $h'(P,x)$ for the h-polynomial that P would have if it were simplicial.

Definition 6. The *h'-polynomial* of a multiplicial d-polytope P is given by

$$h'(P,x) = \sum_{i=0}^{d} h'_i(P)x^{d-i} = \sum_{i=0}^{d} f_{i-1}(P)(x-1)^{d-i}.$$

(The *h'-vector* is then the vector of coefficients of the h'-polynomial.)

Theorem 4.3. *Let $P^{d,k,n}$ be an ordinary polytope. Let $\bigcup_{j=1}^{v}[G_j,F_j]$ be the partition of the face lattice of $P^{d,k,n}$ associated with the colex shelling of $P^{d,k,n}$. Then for all i, $0 \le i \le d$, $h'(P^{d,k,n},x) = \sum_{j=1}^{v} x^{d-|G_j|}$.*

Furthermore, if $C^{d,k}$ is the cyclic d-polytope with $k+1$ vertices, then for all i, $0 \le i \le d$, $h'_i(P^{d,k,n}) \ge h_i(C^{d,k})$, with equality for $i > d/2$.

Proof. Direct evaluation gives $h'_0(P) = h'_d(P) = 1$. Let $F_1, F_2, \ldots, F_v$ be the colex shelling of $P^{d,k,n}$. By Theorem 3.6, part 2, the set of faces of $P^{d,k,n}$ has a partition as $\bigcup_{j=1}^{v}[G_j,F_j]$. By Theorem 3.6, part 4, the interval $[G_j,F_j]$ has exactly

$$\binom{d-1-\dim G_j}{\ell-\dim G_j}$$

faces of dimension ℓ for $\dim G_j \le \ell \le d-1$. Let $k_i = \bigl|\{j : \dim G_j = i-1\}\bigr|$. Then $f_\ell = \sum_{i=0}^{l+1}\binom{d-i}{\ell-i+1}k_i$. These are the (invertible) equations that give f_ℓ in terms of h'_i, so for all i, $h'_i = k_i = \bigl|\{j : \dim G_j = i-1\}\bigr|$.

The second part we prove by induction on $n \ge k$. We will also need the following statement, which we prove in the course of the induction as well. If F_j is a facet of $P^{d,k,n}$ with $\max F_j = n-2$, then $|G_j| \le (d-1)/2$. The base case of the induction is the cyclic polytope, $C^{d,k} = P^{d,k,k}$. We need to show that if F_j is a facet of $C^{d,k}$ with $\max F_j = k-2$, then $|G_j| \le (d-1)/2$. This follows from the description of G_j in the proof of Theorem 3.6, because in this case, in $F_j = I_j^0 \cup I_j^1 \cup I_j^2 \cup \cdots \cup I_j^p \cup I_j^k$, the set I_j^k is empty and $|G_j| = \frac{1}{2}\bigl|\bigcup_{\ell=1}^{p} I_j^\ell\bigr| \le \frac{1}{2}(d-1)$ (since d is odd).

Recall from the proof of Theorem 3.6 that for each facet F_j of $P^{d,k,n}$, G_j is the same size as the minimum new face G of the corresponding facet of $P^{d,k,n-1}$; that facet is the same (as vertex set) as F_j, if $\max F_j \le n-2$, and is $\operatorname{lsh}(F_j)$, if $\max F_j \ge n-1$. From Proposition 3.5 we see that each facet of $P^{d,k,n-1}$ with maximum vertex $n-2$ gives rise to two facets of $P^{d,k,n}$, while all others give rise to exactly one facet each. Thus, for all i,

$$h'_i(P^{d,k,n}) = h'_i(P^{d,k,n-1}) \\ + \big|\{j : F_j \text{ is a facet of } P^{d,k,n} \text{ with } \max F_j = n-1 \text{ and } |G_j| = i\}\big|.$$

Thus, for all i, $h'_i(P^{d,k,n}) \geq h'_i(P^{d,k,n-1})$, so by induction, $h'_i(P^{d,k,n}) \geq h'_i(C^{d,k})$. Furthermore, if $\max F_j = n-1$, then $\max(\mathrm{lsh}(F_j)) = (n-1)-1$, so by the induction hypothesis, $|G_j| \leq (d-1)/2$. So for $i > d/2$, $h'_i(P^{d,k,n}) = h'_i(P^{d,k,n-1}) = h_i(C^{d,k})$. □

Note that for the multiplex $M^{d,n}$ (d odd or even), $h'(M^{d,n}) = (1, n-d+1, 1, 1, \ldots, 1, 1)$, while $h(M^{d,n}) = (1, n-d+1, n-d+1, \ldots, n-d+1, 1)$.

Now for multiplicial polytopes, we consider the remaining part of the h-vector, coming from the parameters $f_{0,j} - (j+1)f_j$. This is

$$h(P,x) - h'(P,x) \\ = (f_{0,d-1} - df_{d-1}) + \sum_{i=2}^{d-1} \left((f_{0,i} - (i+1)f_i) + (f_{0,i-1} - if_{i-1})\right)(x-1)^{d-i}.$$

So

$$h(P,x+1) - h'(P,x+1) \\ = (f_{0,d-1} - df_{d-1}) + \sum_{i=2}^{d-1} \left((f_{0,i} - (i+1)f_i) + (f_{0,i-1} - if_{i-1})\right) x^{d-i} \\ = \sum_{i=2}^{d-1} (f_{0,i} - (i+1)f_i)(x+1)x^{d-1-i}.$$

So

$$\sum_{i=2}^{d-1} (h_i(P) - h'_i(P))(x+1)^{d-1-i} = \sum_{i=2}^{d-1} (f_{0,i} - (i+1)f_i) x^{d-1-i}.$$

For the ordinary polytope, this equation can be applied locally to give the contribution to $h(P^{d,k,n}, x) - h'(P^{d,k,n}, x)$ from each interval $[G_j, F_j]$ of the shelling partition. For each j, and each $i \geq \dim G_j$, let $b_{j,i} = \sum (f_0(H) - (i+1))$, where the sum is over all i-faces H in $[G_j, F_j]$. Let $b_j(x) = \sum_{i=\dim G_j}^{d-1} b_{j,i} x^{d-1-i}$. Write $b_j(x)$ in the basis of powers of $(x+1)$: $b_j(x) = \sum a_{j,i}(x+1)^{d-1-i}$. Then $a_{j,i} = h_i(\mathcal{G}_j) - h'_i(\mathcal{G}_j)$, the contribution to $h_i(P^{d,k,n}) - h'_i(P^{d,k,n})$ from faces in the interval $[G_j, F_j]$. Note that for fixed j, $\sum_i a_{j,i} = b_j(0) = f_0(F_j) - d$. We will return to the coefficients $a_{j,i}$ after triangulating the ordinary polytope.

Example. The h-vector of $P^{5,6,8}$ is $h(P^{5,6,8}) = (1, 4, 7, 7, 4, 1)$. The sum of the h_i is 24, which counts the 16 facets plus one for each of the four 6-vertex facets, plus two for each of the two 7-vertex facets. Referring to Table 1, we see that $h'(P^{5,6,8}) = (1, 4, 5, 3, 2, 1)$; from this we compute $f(P^{5,6,8}) = (9, 31, 52, 44, 16)$. The nonzero $a_{j,i}$ here are $a_{6,2} = a_{7,3} = a_{11,2} = a_{12,3} = 1$ and $a_{13,3} = a_{15,4} = 2$.

In this case each interval $[G_j, F_j]$ contributes to $h_i(P^{d,k,n}) - h'_i(P^{d,k,n})$ for at most one i, but this is not true in general.

5. Triangulating the Ordinary Polytope

Triangulations of polytopes or of their boundaries can be used to calculate the h-vector of the polytope if the triangulation is shallow [Bayer 1993]. The solid ordinary polytope need not have a shallow triangulation, but its boundary does have a shallow triangulation. The triangulation is obtained simply by triangulating each multiplex as in Section 2. This triangulation is obtained by "pushing" the vertices in the order 0, 1, ..., n. (See [Lee 1991] for pushing (placing) triangulations.)

Theorem 5.1. *The boundary of the ordinary polytope $P^{d,k,n}$ has a shallow triangulation. The facets of one such triangulation are the Gale subsets of $[i, i+k]$ (where $0 \le i \le n-k$) of size d containing either 0 or n or the set $\{i, i+k\}$.*

Proof. First we show that each such set is a consecutive subset of some facet of $P^{d,k,n}$. Suppose Z is a Gale subset of $[i, i+k]$ of size d containing $\{i, i+k\}$. Write $Z = [i, i+a-1] \cup Y \cup [i+k-b+1, i+k]$, where $a \ge 1$, $b \ge 1$, and

$$Y \cap \{i+a, i+k-b\} = \varnothing.$$

Since Z is a Gale subset, $|Y|$ is even; let $r = (d-1-|Y|)/2$. Since $|Z| = d$, $a+b = 2r+1$, so a and b are each at most $2r$. Define $X = [i+a-2r, i+a-1] \cup Y \cup [i+k-b+1, i+k-b+2r]$. Note that $i+k-b+1 = (i+a-2r)+k$. Then $\operatorname{ret}_n(X)$ is the vertex set of a facet of $P^{d,k,n}$, and Z is a consecutive subset of $\operatorname{ret}_n(X)$.

Now suppose that Z is a Gale subset of $[0, k]$ of size d containing 0, but not k. Write $Z = \{0\} \cup Y \cup [j-2r+1, j]$, where $j < k$, $r \ge 1$, and $j-2r \notin Y$. Then $|Y| = d-2r-1$, and $Z = \operatorname{ret}_n(X)$, where $X = [j-2r+1-k, j-k] \cup Y \cup [j-2r+1, j]$. So Z itself is the vertex set of a facet of $P^{d,k,n}$. The case of sets containing n but not $n-k$ works the same way.

Next we show that all consecutive d-subsets of facets F of $P^{d,k,n}$ are of one of these types. Let $F = \operatorname{ret}_n(X)$, where

$$X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1],$$

with Y a paired subset of size $d-2r-1$ of $[i+2r+1, i+k-2]$. Suppose first that $i+2r-1 \ge 0$ and $i+k \le n$. Let Z be a consecutive d-subset of F. Since $|Y| = d-2r-1$, $|[i, i+2r-1] \cap F| \le 2r$, and $|[i+k, i+k+2r-1] \cap F| \le 2r$, it follows that $i+2r-1$ and $i+k$ must both be in Z. Thus we can write $Z = [i+2r-a, i+2r-1] \cup Y \cup [i+k, i+k+b-1]$, with $a+b = 2r+1$, $i+2r-a \ge 0$, and $i+k+b-1 \le n$. Let $\ell = i+2r-a$. Then $i+k+b-1 = \ell+k$, so $0 \le \ell \le n-k$, and Z is a Gale subset of $[\ell, \ell+k]$ containing $\{\ell, \ell+k\}$.

If $i+2r-1<0$, then $i+k+2r-1<k\le n$, and

$$F=\{0\}\cup Y\cup[i+k,i+k+2r-1].$$

Then $|F|=d$ and F itself is a Gale subset of $[0,k]$ of size d containing 0. Similarly for the case $i+k>n$.

The sets described are exactly the $(d-1)$-simplices obtained by triangulating each facet of $P^{d,k,n}$ according to Theorem 2.1. The fact that this triangulation is shallow follows from the corresponding fact for this triangulation of a multiplex. □

Let $\mathcal{T}=\mathcal{T}(P^{d,k,n})$ be this triangulation of $\partial P^{d,k,n}$. Since $\mathcal{T}$ is shallow, we have $h(P^{d,k,n},x)=h(\mathcal{T},x)$. We calculate $h(\mathcal{T},x)$ by shelling $\mathcal{T}$.

Theorem 5.2. *Let F_1, F_2, …, F_v be the colex order of the facets of $P^{d,k,n}$. For each j, if $F_j=\{z_1,z_2,\dots,z_{p_j}\}$ ($z_1<z_2<\cdots<z_{p_j}$), and $1\le\ell\le p_j-d+1$, let $T_{j,\ell}=\{z_\ell,z_{\ell+1},\dots,z_{\ell+d-1}\}$. Then $T_{1,1}$, $T_{1,2}$, …, T_{1,p_1-d+1}, $T_{2,1}$, …, T_{2,p_2-d+1}, …, $T_{v,1}$, …, T_{v,p_v-d+1} is a shelling of $\mathcal{T}(P^{d,k,n})$.*

Let $U_{j,\ell}$ be the minimal new face when $T_{j,\ell}$ is shelled on. As vertex sets, $U_{j,p_j-d+1}=G_j$.

Proof. Throughout the proof, write $F_j=\{z_1,z_2,\dots,z_{p_j}\}$ ($z_1<z_2<\cdots<z_{p_j}$). We first show that G_j is the unique minimal face of T_{j,p_j-d+1} not contained in $\bigcup_{i=1}^{j-1}\bigcup_{\ell=1}^{p_i-d+1}T_{i,\ell})\cup\big(\bigcup_{\ell=1}^{p_j-d}T_{j,\ell}\big)$. The set G_j is not contained in a facet of $P^{d,k,n}$ earlier than F_j. So G_j does not occur in a facet of $\mathcal{T}$ of the form $T_{i,\ell}$ for $i<j$. Also, $\max F_j\in G_j$, so G_j does not occur in a facet of $\mathcal{T}$ of the form $T_{j,\ell}$ for $\ell\le p_j-d$. Thus G_j does not occur in a facet of $\mathcal{T}$ before T_{j,p_j-d+1}.

We show that for $z_q\in G_j$, $T_{j,p_j-d+1}\setminus\{z_q\}$ is contained in a facet of $\mathcal{T}$ occurring before T_{j,p_j-d+1}. There is nothing to check for $j=v$, because $p_v-d+1=1$ and so $T_{v,1}=F_v$ is the last simplex in the purported shelling order. So we may assume that $j<v$ and thus G_j is contained in the last $d-1$ vertices of F_j.

Case 1. If $p_j>d$ and $q=p_j$ (giving the maximal element of F_j), then $T_{j,p_j-d+1}\setminus\{z_{p_j}\}\subset T_{j,p_j-d}$.

Case 2. Suppose $p_j-d+2\le q\le p_j-1$. Then

$$T_{j,p_j-d+1}\setminus\{z_q\}\subseteq\{z_{q-d+2},\dots,z_{q-1},z_{q+1}\dots,z_{p_j}\}=H.$$

This is a ridge of $P^{d,k,n}$ in F_j not containing G_j, and hence H is contained in a previous facet F_ℓ of $P^{d,k,n}$. Since H is a ridge in both F_j and F_ℓ, H is obtained from each facet by deleting a single element from a consecutive string of vertices in the facet. So $|H|\le|F_\ell\cap[z_{q-d+2},z_{p_j}]|\le|H|+1$, and so $d-1\le|F_\ell\cap[z_{p_j-d+1},z_{p_j}]|\le d$. So $T_{j,p_j-d+1}\setminus\{z_q\}$ is contained in a consecutive set of d elements of F_ℓ, and hence in a $(d-1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to F_ℓ. This simplex occurs before T_{j,p_j-d+1} in the specified shelling order.

Case 3. Otherwise $p_j=d$ (so $p_j-d+1=1$) and $q=d$. Then $T_{j,1}=F_j$ and $H=T_{j,1}\setminus\{z_d\}$ is a ridge of $P^{d,k,n}$ in F_j not containing $\max F_j$, so H is contained in a previous facet F_ℓ of $P^{d,k,n}$. As in Case 2, $d-1 \le |F_\ell \cap [z_1, z_{d-1}]| \le d$. So $T_{j,1} \setminus \{z_d\}$ is contained in a consecutive set of d elements of F_ℓ, and hence in a $(d-1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to F_ℓ. This simplex occurs before T_{j,p_j-d+1} in the specified shelling order.

So in the potential shelling of $\mathcal{T}$, G_j is the unique minimal new face as T_{j,p_j-d+1} is shelled on. Write $U_{j,p_j-d+1} = G_j$. At this point we need a clearer view of the simplex $T_{j,\ell}$. Recall that F_j is of the form $\mathrm{ret}_n(X)$, where $X = [i, i+2r-1] \cup Y \cup [i+k, i+k+2r-1]$, with Y a subset of size $d-2r-1$. If $i+2r-1 < 0$ or $i+k > n$, then $p_j = |F_j| = d$, and $T_{j,1} = T_{j,p_j-d+1} = F_j$; we have already completed this case. So assume $i+2r-1 \ge 0$ and $i+k \le n$. A consecutive string of length d in $\mathrm{ret}_n(X)$ must then be of the form $[i+s, i+2r-1] \cup Y \cup [i+k, i+k+s]$ for some s, $0 \le s \le 2r-1$. (All such strings—with appropriate Y—having $i+s \ge 0$ and $i+k+s \le n$ occur as $T_{j,\ell}$.) In particular, for $\ell < p_j - d+1$, $T_{j,\ell} = T_{j,\ell+1} \setminus \{\max T_{j,\ell+1}\} \cup \{\min T_{j,\ell+1} - 1\}$ and $\max T_{j,\ell} = \min T_{j,\ell} + k$.

Now define $U_{j,\ell}$ for $\ell \le p_j - d$ recursively by $U_{j,\ell} = U_{j,\ell+1} \setminus \{z\} \cup \{z-k, z-1\}$, where $z = \max T_{j,\ell+1}$. By the observations above, $U_{j,\ell} \subseteq T_{j,\ell}$. We prove by downward induction that $U_{j,\ell}$ is not contained in a facet F_i of $P^{d,k,n}$ before F_j, that $U_{j,\ell}$ is not contained in a facet of $\mathcal{T}$ occurring before $T_{j,\ell}$, and that any ridge of $\mathcal{T}$ in $T_{j,\ell}$ not containing all of $U_{j,\ell}$ is in an earlier facet of $\mathcal{T}$. The base case of the induction is $\ell = p_j - d+1$, and this case has been handled above.

Note that $\{z-k, z-1\}$ is a diagonal of the 2-face $\{z-k-1, z-k, z-1, z\}$ of $P^{d,k,n}$ [Dinh 1999]. So if F_i is a facet of $P^{d,k,n}$ containing $U_{j,\ell}$, then F_i contains $\{z-k-1, z-k, z-1, z\}$. Thus F_i contains $U_{j,\ell+1}$, so, by the induction assumption, $i \ge j$. Therefore, for $i < j$, and any r, $T_{i,r}$ does not contain $U_{j,\ell}$. For $r < \ell$, $T_{j,r}$ does not contain $z-1 = \max T_{j,\ell}$, so $T_{j,r}$ does not contain $U_{j,\ell}$.

Now we will show that for any $g \in U_{j,\ell}$, $T_{j,\ell} \setminus \{g\}$ is in a previous facet of $\mathcal{T}$.

Case 1. If $g = z-1 = \max T_{j,\ell}$ and $\ell \ge 2$, then $T_{j,\ell} \setminus \{g\} \subset T_{j,\ell-1}$.

Case 2. If $g = z-1 = \max T_{j,\ell}$ and $\ell = 1$, then $T_{j,\ell} \setminus \{g\}$ is the leftmost ridge of $P^{d,k,n}$ in F_j and, in particular, does not contain $\max F_j$. So $H = T_{j,\ell} \setminus \{g\}$ is contained in a previous facet F_e of $P^{d,k,n}$. As in the $\ell = p_j - d+1$ case, $F_e \cap [\min T_{j,\ell}, \max T_{j,\ell}]$ is contained in a consecutive set of d elements of F_e, and hence in a $(d-1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to F_e. So $T_{j,\ell} \setminus \{g\}$ is contained in a previous facet of $\mathcal{T}$.

Case 3. Suppose $g < z-1$ and $g \in U_{j,\ell} \cap U_{j,\ell+1}$. Since $\{z-1, z\} \subset T_{j,\ell+1}$, $T_{j,\ell+1}$ contains at most $d-3$ elements less than g. The ridge H of $P^{d,k,n}$ in F_j containing $T_{j,\ell+1} \setminus \{g\}$ consists of the $d-2$ elements of F_j below g and the (up to) $d-2$ elements of F_j above g. In particular, H contains $\min T_{j,\ell+1} - 1 = \min T_{j,\ell}$. So $T_{j,\ell} \setminus \{g\} \subset H$. Since $\dim T_{j,\ell} \setminus \{g\} = d-2$, H is the (unique) smallest face of $P^{d,k,n}$ containing $T_{j,\ell+1} \setminus \{g\}$. By the induction hypothesis $T_{j,\ell+1} \setminus \{g\}$ is contained in a previous facet $T_{i,r}$ of $\mathcal{T}$; here $i < j$ because $\max T_{j,\ell+1} \in T_{j,\ell+1} \setminus \{g\}$. The $(d-2)$-simplex $T_{j,\ell+1} \setminus \{g\}$ is then contained in a ridge of $P^{d,k,n}$ contained in F_i, but this ridge must be H, by the uniqueness of H. So

$T_{j,\ell}\setminus\{g\} \subset H = F_i \cap F_j$. As in earlier cases, $F_i \cap [\min T_{j,\ell}, \max T_{j,\ell}]$ is contained in a consecutive set of d elements of F_i, and hence in a $(d-1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to F_i. So $T_{j,\ell}\setminus\{g\}$ is contained in a previous facet of $\mathcal{T}$.

Case 4. Finally, let $g = z-k$, which is $\min T_{j,\ell}+1$. Then $T_{j,\ell}$ contains $d-2$ elements above g. Let H be the ridge of $P^{d,k,n}$ in F_j containing $T_{j,\ell}\setminus\{g\}$. Then $\max H = \max T_{j,\ell} < \max F_j$, so H does not contain G_j. So H is in a previous facet F_i of $P^{d,k,n}$. As in earlier cases, $F_i \cap [\min T_{j,\ell}, \max T_{j,\ell}]$ is contained in a consecutive set of d elements of F_i, and hence in a $(d-1)$-simplex of $\mathcal{T}(P^{d,k,n})$ belonging to F_i. So $T_{j,\ell}\setminus\{g\}$ is contained in a previous facet of $\mathcal{T}$.

Thus $T_{1,1}, T_{1,2}, \ldots, T_{1,p_1-d+1}, T_{2,1}, \ldots, T_{2,p_2-d+1}, \ldots, T_{v,1}, \ldots, T_{v,p_v-d+1}$ is a shelling of $\mathcal{T}(P^{d,k,n})$. □

COROLLARY 5.3. *Let $n \geq k \geq d = 2m+1 \geq 5$. Let $\cup[G_j, F_j]$ be the partition of the face lattice of $P^{d,k,n}$ from the colex shelling, and let $\cup[U_{j,\ell}, T_{j,\ell}]$ be the partition of the face lattice of $\mathcal{T}(P^{d,k,n})$ from the shelling of Theorem* 5.2. *Then*

(i) *For each i, $h_i(P^{d,k,n}) \geq h'_i(P^{d,k,n})$.*
(ii) *The contribution to $h_i(P^{d,k,n}) - h'_i(P^{d,k,n})$ from the interval $[G_j, F_j]$ is*

$$a_{j,i} = \bigl|\{\ell : |U_{j,\ell}| = i,\ 1 \leq \ell \leq p_\ell - d\}\bigr| \geq 0.$$

PROOF. The h-vector of $\mathcal{T}$ counts the sets $U_{j,\ell}$ of each size. Among these are all the sets G_j counted by the h'-vector of $P^{d,k,n}$. Thus

$$\begin{aligned} h_i(\mathcal{T}(P^{d,k,n})) &= \bigl|\{(j,\ell) : |U_{j,\ell}| = i\}\bigr| \\ &\geq \bigl|\{(j,\ell) : |U_{j,\ell}| = i \text{ and } \ell = p_j - d + 1\}\bigr| = h'_i(P^{d,k,n}). \end{aligned}$$

Recall that we write $\mathcal{G}_j$ for the set of faces of F_j not in $\bigcup_{i<j} F_i$; here $\mathcal{G}_j$ is the set of faces in $[G_j, F_j]$. Write also $\mathcal{T}G_j$ for the set of faces of $\mathcal{T}$ that are contained in F_j but not in $\bigcup_{i<j} F_i$. By [Bayer 1993, Corollary 7], since $\mathcal{T}$ is a shallow triangulation of $\partial P^{d,k,n}$, $g(G,x) = \sum (x-1)^{d-1-\dim\sigma}$, where the sum is over all faces σ of $\mathcal{T}$ that are contained in G but not in any proper subface of G. Thus

$$\begin{aligned} h(\mathcal{G}_j, x) &= \sum_{G \in [G_j, F_j]} g(G,x)(x-1)^{d-1-\dim G} \\ &= \sum_{\sigma \in \mathcal{T}G_j} (x-1)^{d-1-\dim\sigma} = \sum_{\ell=1}^{p_\ell-d+1} x^{d-|U_{j,\ell}|} \end{aligned}$$

Since $h'(\mathcal{G}_j, x) = x^{d-|G_j|} = x^{d-|U_{j,p_j-d+1}|}$,

$$\sum_i a_{j,i} x^i = h(\mathcal{G}_j, x) - h'(\mathcal{G}_j, x) = \sum_{\ell=1}^{p_\ell-d} x^{d-|U_{j,\ell}|},$$

or

$$a_{j,i} = \bigl|\{\ell : |U_{j,\ell}| = i,\ 1 \leq \ell \leq p_\ell - d\}\bigr| \geq 0. \qquad \square$$

(j,ℓ)	$T_{j,\ell}$	$U_{j,\ell}$	(j,ℓ)	$T_{j,\ell}$	$U_{j,\ell}$
1, 1	01234	∅	11, 1	1234 7	27
2, 1	012 45	5	11, 2	234 78	78
3, 1	0 2345	35	12, 1	12 45 7	257
4, 1	0 23 56	6	12, 2	2 45 78	578
5, 1	0 3456	46	13, 1	0123 6	126
6, 1	01 34 6	16	13, 2	123 67	267
6, 2	1 34 67	7	13, 3	23 678	678
7, 1	01 456	156	14, 1	34 678	4678
7, 2	1 4567	57	15, 1	012 56	1256
8, 1	2345 8	8	15, 2	12 567	2567
9, 1	23 56 8	68	15, 3	2 5678	5678
10, 1	3456 8	468	16, 1	45678	45678

Table 2. Shelling of triangulation of $P^{5,6,8}$

Example. Table 2 gives the shelling of the triangulation of $P^{5,6,8}$. (Refer back to Table 1 for the shelling of $P^{5,6,8}$ itself.) Among the rows $(6,1)$, $(7,1)$, $(11,1)$, $(12,1)$, $(13,1)$, $(13,2)$, $(15,1)$, $(15,2)$ (rows (j,ℓ) that are not the last row for that j), count the $U_{j,\ell}$ of cardinality i to get $h_i(P^{5,6,8}) - h'_i(P^{5,6,8})$. Note that $U_{13,3} = G_{13}$ (from Table 1), and that $U_{13,2} = U_{13,3} \setminus \{8\} \cup \{2,7\}$. The ridges in $T_{13,2}$ are 1236, 1237, 1267, 1367, and 2367. The first ridge, 1236, falls under Case 1 of the proof of Theorem 5.2; it is contained in the previous facet, $T_{13,1}$. The next ridge, 1237, falls under Case 3; it is contained in the ridge 12378 of $P^{5,6,8}$ in $F_{13} = 0123678$, and 12378 also contains the ridge 2378 in $T_{13,3}$. The induction assumption says that 2378 is contained in an earlier facet, in this case $T_{11,2}$, and 12378 is contained in F_{11}. Finally, the ridge 1237 is contained in the simplex $T_{11,1}$, part of the triangulation of F_{11}. The last ridge of $T_{13,2}$ not containing 267 is 1367. It falls under Case 4. The set 1367 is contained in the ridge 01367 of $P^{5,6,8}$, contained in F_{13}. This ridge is also contained in the earlier facet F_6. The ridge 1367 of the triangulation is contained in the simplex $T_{6,2}$.

Theorem 5.4. *Let $n \geq d+k-1$. For $1 \leq i \leq d-1$, $h_i(P^{d,k,n}) - h_i(P^{d,k,n-1})$ is the number of facets $T_{j,\ell}$ of $\mathcal{T}(P^{d,k,n})$ such that $\max F_j = n-1$ and $|U_{j,\ell}| = i$. For $1 \leq i \leq m$, this is $\binom{k-d+i-1}{i-1}$.*

Proof. Refer to Proposition 3.5 for a description of the facets of $P^{d,k,n}$ in terms of those of $P^{d,k,n-1}$. For $n \geq d+k-1$, for every facet $P^{d,k,n}$ ending in n, the translation $F-1$ is a facet of $P^{d,k,n-1}$. (For smaller n, a facet of $P^{d,k,n}$ may end in 0, in which case lsh(F) is a proper subset of $F-1$.) The same holds for the simplices $T_{j,\ell}$ triangulating these facets, and for the sets $U_{j,\ell}$. The facets of $P^{d,k,n}$ ending in $n-2$ are facets of $P^{d,k,n-1}$, and the same holds for the corresponding $T_{j,\ell}$ and $U_{j,\ell}$. The contributions to $h(P^{d,k,n})$ from facets ending in any element but $n-1$ thus total $h(P^{d,k,n-1})$. So for $1 \le i \le d-1$, $h_i(P^{d,k,n}) - h_i(P^{d,k,n-1})$ is the number of facets $T_{j,\ell}$ of $\mathcal{T}(P^{d,k,n})$ such that $\max F_j = n-1$ and $|U_{j,\ell}| = i$.

Now consider the set $\mathcal{S}$ of facets $T_{j,\ell}$ of $\mathcal{T}(P^{d,k,n})$ with $\max F_j = n-1$. For each $T \in \mathcal{S}$, T is a set of d elements occurring consecutively in some F_j with maximum element $n-1$. So T can be written as

$$T = [b, n-k-1] \cup [n-k+1, c] \cup Y \cup [e, b+k], \tag{5–1}$$

where

(i) $n-k-d+1 \le b \le n-k-1$;
(ii) $n-k \le c \le b+d-1$ and $c-n+k$ is even (here $c = n-k$ means $[n-k+1, c] = \varnothing$);
(iii) Y is a paired subset of $[c+2, e-1]$;
(iv) $e = b+k-1$ if $n-k-b$ is odd, and $e = b+k$ if $n-k-b$ is even; and
(v) $|T| = d$.

In these terms, the minimum new face U when T is shelled on is $U = [b+1, n-k-1] \cup E(Y) \cup \{b+k\}$.

We give a bijection between the facets T in $\mathcal{S}$ with $|U| = i$ (where $1 \le i \le m$) and the $(k-d)$-element subsets of $[1, k-d+i-1]$. Let T be as in Equation 5–1. Then $i = |U| = n-k-b+|Y|/2$. For each $x \ge c+1$, let $y(x)$ be the number of pairs in Y with both elements less than x. Let $a_1 = n-k-b = i-|Y|/2$. Write $[c+1, e-1] \setminus Y = \{x_1, x_2, \dots, x_{k-d}\}$, with the x_ℓs increasing. (This set has $k-d$ elements because $d = (c-b)+|Y|+(b+k-e+1)$, so $|[c+1, e-1] \setminus Y| = e-c-1-|Y| = k-d$.) Set

$$A(T) = \{a_1 + y(x_\ell) + \ell - 1 : 1 \le \ell \le k-d\}.$$

To see that this is a subset of $[1, k-d+i-1]$, note that the elements of $A(T)$ form an increasing sequence with minimum element a_1 and maximum element $a_1 + y(x_{k-d}) + (k-d-1) \le a_1 + |Y|/2 + (k-d-1) = k-d+i-1$.

For the inverse of this map, write a $(k-d)$-element subset of $[1, k-d+i-1]$ as $A = \{a_1, a_2, \dots, a_{k-d}\}$, with the a_ℓs increasing. Then $1 \le a_1 \le i$. Let

$$x_1 = n-k+d-2i+a_1-\chi(a_1 \text{ odd}).$$

Set

$$T(A) = [n-k-a_1, n-k-1] \cup [n-k+1, x_1-1] \cup Y \cup [n-a_1-\chi(a_1 \text{ odd}), n-a_1],$$

where

$$Y = ([x_1, n-a_1-1-\chi(a_1 \text{ odd})] \setminus \{x_1 + 2(a_\ell - a_1) - (\ell-1) : 1 \le \ell \le k-d\}).$$

We check that this gives a set of the required form.

(1) Since $1 \le a_1 \le i \le d-1$ $n-k-d+1 \le n-k-a_1 \le n-k-1$.

(2) $x_1 - 1 - n + k = d-2i-1+(a_1 - \chi(a_1 \text{ odd}))$, which is nonnegative and even; $x_1 - 1 = (n-k-a_1+d-1) - (2i-2a_1+\chi(a_1 \text{ odd})) \le n-k-a_1+d-1$.

(3) Y is clearly a subset of $[x_1+1, n-a_1-\chi(a_1 \text{ odd})-1]$. To see that Y is paired note that the difference between two consecutive elements in the removed set is $(x_1+2(a_{\ell+1}-a_1)-\ell)-(x_1+2(a_\ell-a_1)-(\ell-1)) = 2(a_{\ell+1}-a_\ell)-1$.

(4) This condition holds by definition.

(5) To check the cardinality of $T(A)$, observe that

$$\begin{aligned} x_1+2(a_{k-d}-a_1)-(k-d-1) &\le x_1+2(k-d+i-1)-2a_1-(k-d-1) \\ &= x_1+k-d+2i-2a_1-1 = n-a_1-\chi(a_1 \text{ odd})-1. \end{aligned}$$

So

$$\{x_1+2(a_\ell-a_1)-(\ell-1) : 1 \le \ell \le k-d\} \subseteq [x_1+1, n-a_1-1-\chi(a_1 \text{ odd})],$$

and

$$|Y| = (n-a_1-\chi(a_1 \text{ odd})-x_1)-(k-d) = 2i-2a_1.$$

So $|T(A)| = x_1-(n-k-a_1)+|Y|+\chi(a_1 \text{ odd}) = d$.

Also, in this case $U = [n-k-a_1+1, n-k-1] \cup E(Y) \cup \{n-a_1\}$, so $|U| = i$.

It is straightforward to check that these maps are inverses. The main point is that, if $a_\ell = a_1+y(x_\ell)+\ell-1$, then

$$\begin{aligned} x_1+2(a_\ell-a_1)-(\ell-1) &= x_1+2(y(x_\ell)+\ell-1)-(\ell-1) \\ &= x_1+2y(x_\ell)+\ell-1 = x_\ell. \end{aligned}$$ □

Example. Consider the ordinary polytope $P^{7,9,15}$. There are six facets with maximum vertex 14; they are (with sets G_j underlined) $\{4,5,7,8,9,10,13,\underline{14}\}$, $\{4,5,7,8,10,\underline{11},13,\underline{14}\}$, $\{4,5,8,\underline{9},10,\underline{11},13,\underline{14}\}$, $\{2,3,4,5,7,8,11,\underline{12},13,\underline{14}\}$, $\{2,3,4,5,8,\underline{9},11,\underline{12},13,\underline{14}\}$, and $\{0,1,2,3,4,5,9,\underline{10},11,\underline{12},13,\underline{14}\}$. Among the 6-simplices occurring in the triangulation of these facets, six have $|U_{j,\ell}| = 3$. Table 3 gives the bijection from this set of simplices to the 2-element subsets of $[1,4]$.

$T_{j,\ell}$	b	c	e	Y	a_1	x_1, x_2	$y(x_i)$	$A(T_{j,\ell})$
$4,\underline{5},7,8,10,\underline{11},\underline{13}$	4	8	13	10, 11	2	9, 12	0, 1	$\{2,4\}$
$5,8,\underline{9},10,\underline{11},13,\underline{14}$	5	6	13	8, 9, 10, 11	1	7, 12	0, 2	$\{1,4\}$
$3,\underline{4},\underline{5},7,8,11,\underline{12}$	3	8	11	$\varnothing$	3	9, 10	0, 0	$\{3,4\}$
$4,\underline{5},7,8,11,\underline{12},\underline{13}$	4	8	13	11, 12	2	9, 10	0, 0	$\{2,3\}$
$5,8,\underline{9},11,\underline{12},13,\underline{14}$	5	6	13	8, 9, 11, 12	1	7, 10	0, 1	$\{1,3\}$
$5,9,\underline{10},11,\underline{12},13,\underline{14}$	5	6	13	9, 10, 11, 12	1	7, 8	0, 0	$\{1,2\}$

Table 3. Bijection with 2-element subsets of $\{1,2,3,4\}$

Again, the results of this section hold for even-dimensional multiplexes as well.

6. Afterword

The story of the combinatorics of simplicial polytopes is a beautiful one. There one finds an intricate interplay among the face lattice of the polytope, shellings, the Stanley–Reisner ring and the toric variety, tied together with the h-vector. The cyclic polytopes play a special role, serving as the extreme examples, and providing the environment in which to build representative polytopes for each h-vector (the Billera–Lee construction [Billera and Lee 1981]). In the general case of arbitrary convex polytopes, the various puzzle pieces have not interlocked as well. In this paper we made progress on putting the puzzle together for the special class of ordinary polytopes. Since the ordinary polytopes generalize the cyclic polytopes, a natural next step would be to mimic the Billera–Lee construction, or Kalai's extension of it [1988], on the ordinary polytopes, as a way of generating multiplicial flag vectors. It would also be interesting to see if there is a ring associated with these polytopes, particularly one having a quotient with Hilbert function equal to the h'-polynomial. Another open problem is to determine the best even-dimensional analogues of the ordinary polytopes. They may come from taking vertex figures of odd-dimensional ordinary polytopes, or from generalizing Dinh's combinatorial description of the facets of ordinary polytopes. Looking beyond ordinary and multiplicial polytopes, we should ask what other classes of polytopes have shellings with special properties that relate to the h-vector?

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