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On the Number of Mutually Touching Cylinders

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ABSTRACT. In a three-dimensional arrangement of 25 congruent nonoverlapping infinite circular cylinders there are always two that do not touch each other.

1. Introduction

The following problem was posed by Littlewood [1968]:

What is the maximum number of congruent infinite circular cylinders that can be arranged in $\mathbb{R}^3$ *so that any two of them are touching? Is it* 7?

This problem is still open. The analogous problem concerning circular cylinders of finite length became known as a mathematical puzzle due to a the popular book [Gardner 1959]: Find an arrangement of 7 cigarettes so that any two touch each other. The question whether 7 is the largest such number is open. For constructions and for a more detailed account on both of these problems see the research problem collection [Moser and Pach $\geq$ 2005].

A very large bound for the maximal number of cylinders in Littlewood's original problem was found by the author in 1981 (an outline proof was presented at the Discrete Geometry meeting in Oberwolfach in that year). The bound was expressed in terms of various Ramsey constants, and so large that it merely showed the existence of a finite bound. In this paper we use a different approach to show that at most 24 cylinders can be arranged so that any two of them are touching:

THEOREM 1. *In an arrangement of* 25 *congruent nonoverlaping infinite circular cylinders there are always two that do not touch each other.*

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In Section 2, we introduce the necessary terminology to talk about relative positions of the cylinders. In Section 3 we prove Theorem 1. We will describe a four-cylinder arrangement in which the cylinders cannot be mutually touching and show that in a family of 25 mutually touching cylinders there are always four cylinders of this type.

One of the needed lemmas can be stated and proved independently from the cylinder problem. To ease the description of the proof of Theorem 1 we place this lemma separately, in Section 4.

2. Terminology

The term *cylinder* will always refer to a circular cylinder infinite at both ends. More precisely, the *cylinder of radius r and axis l* is the set of those points in $\mathbb{R}^3$ that are at a distance of at most r from a given line l. If $r = 1$, we speak of *unit* cylinders. Two cylinders are *nonoverlapping* if they do not have common interior points. Two cylinders are *touching* if they do not overlap, but have at least one common boundary point.

Consider a family of mutually touching cylinders. For reference choose one of the cylinders, say c, and assign a positive direction to its axis l. We say that a cylinder *lies in front of* another cylinder with respect to the directed axis l if the first cylinder can be shifted parallel to l in the positive direction to infinity without crossing the other cylinder. This relation is not transitive, so it does not give rise to an ordering among the cylinders.

There is another natural way of describing a relative position among mutually touching cylinders. We say that a cylinder is (*clockwise*) *to the right* of another if a clockwise rotation by α (with $0 < \alpha \leq \pi$) around l takes the plane separating the second cylinder from c to the plane separating the first cylinder from c. To avoid ambiguity, we say that counterclockwise rotation around the axis l is the one which matches the right-hand rule with the thumb pointing in the positive direction of the axis l. The relation of "being to the right" clearly defines an order among cylinders that are touching c, in such a way that their contact points, if looked at from the direction of the axis of c, belong to a circular arc less than π. We will refer to this order as *the clockwise order* with respect to l.

3. Proof of Theorem 1

Assume we have an arrangement of 25 mutually touching cylinders so that one of the cylinders is c with directed axis l. Most likely the first thing one notices while studying cylinder arrangements is that no two of the cylinders are parallel. Otherwise the number of cylinders is at most four.

Most of our conclusions will come from studying the *front view*, which is what we see by looking at the cylinder packing from the positive direction of l. We intentionally use the term "front view" instead of "projection", since we would like to keep track of the relation of "being in front". Let the unit disc d be the image of cylinder c. The images of the other cylinders are strips of width 2, all touching disc d at different points. A simple integral averaging argument shows that among these 24 contact points in the front view one can choose 5 along an arc on the boundary of d with central angle at most $\pi/3$.

Label the corresponding cylinders c_1, c_2, c_3, c_4, c_5 in clockwise order, so that cylinder c_5 is rightmost.

LEMMA 1. *In any oriented complete graph with vertices labelled* $1, 2, 3, 4, 5$ *one can choose three vertices* $i < j < k$ *so that either* $i \to j \to k$ *or* $i \leftarrow j \leftarrow k$ *holds.*

PROOF. If the conclusion is not true, we may assume that $2 \to 3 \leftarrow 4$ or $2 \leftarrow 3 \to 4$ holds. Consider the first case: If $2 \leftarrow 4$, then either $1 \leftarrow 2 \leftarrow 4$ or $1 \to 2 \to 3$ holds, a contradiction. If $2 \to 4$ then either $3 \leftarrow 4 \leftarrow 5$ or $2 \to 4 \to 5$ holds, a contradiction. The second case is handled in the same way. □

Consider the abstract complete graph whose vertices are the cylinders c_1, c_2, c_3, c_4, c_5. Orient the edges according to the "being in front" relation. According to Lemma 1 three of the cylinders, say c_1, c_2, c_3, are such that (i) c_1 is in front of c_2 which is in front of c_3, or (ii) c_1 is behind c_2 which is behind c_3.

We will show that cylinders c, c_1, c_2 and c_3 cannot be mutually touching. In this respect case (ii) can be reduced to case (i) by reflecting the cylinders along a plane passing through the axis of the cylinder c. Indeed such plane reflection preserves the relation of "being in front", but reverses the clockwise order. The impossibility of case (i) is stated as a separate lemma below. Its proof completes the proof of Theorem 1.

LEMMA 2 (A FORBIDDEN ARRANGEMENT OF FOUR CYLINDERS). *If a packing of four cylinders* c, c_1, c_2, c_3 *satisfies the conditions listed below, two of them must be disjoint.*

Contact condition: *Cylinders* c_1, c_2, c_3 *are touching* c *so that their contact points if looked at from the direction of the axis of* c *belong to a circular arc of length at most* $\pi/3$.

Clockwise order condition: *Cylinders* c_1, c_2, c_3 *are labelled according to their clockwise order with respect to the directed axis* l *of* c *so that* c_3 *is the rightmost one.*

"Being in front" condition: *Cylinder* c_1 *is in front of cylinder* c_2 *which is in front of cylinder* c_3 *with respect to the directed axis* l *of* c.

PROOF. Assume to the contrary that cylinders c, c_1, c_2, c_3 are mutually touching and satisfy all three conditions. Let strips s_1, s_2 and s_3 be the images of cylinders c_1, c_2 and c_3 in front view. Assume that strip s_3 is horizontal. Let the unit disc d with center O be the image of cylinder c. According to the contact condition and the clockwise order condition, the elevation angle of s_2 is positive and smaller than $\pi/3$. See Figure 1, left.

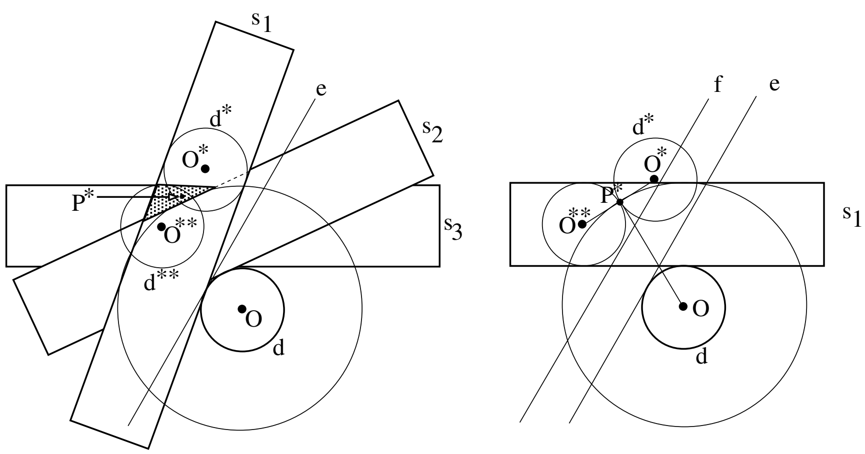

Figure 1.

Denote by P the contact point of cylinders c_1 and c_3, and by P^* the image of P in front view. P^* certainly belongs to both s_1 and s_3, but not to strip s_2, since c_2 is in front of c_3. Since strip s_1 is obtained from s_2 by a counterclockwise rotation around O, P^* lies to the left of strip s_2.

Let the unit discs d^* and d^{**} with centers O^* and O^{**} be the images in front view of the unit spheres inscribed in c_1 and c_3 respectively and containing P. Strip s_1 contains d^*, and is tangent to d. There are two such strips, but since P^* does not belong to s_2, the strip that is clockwise to the right of the other must be also to the right of s_2, thus it cannot be the same as s_1. Thus the position of d^* determines s_1.

Discs d^* and d^{**} are symmetrical with respect to point P^*. First fix P^* and move d^* horizontally to the right so that it has P^* on its boundary. Simultaneously move d^{**} so that P^* remains the symmetry center of d^* and d^{**}. Then move P^*, along with d^* and d^{**} horizontally to the right until P^* gets onto the circle centered at O of radius 3 (see Figure 1, right).

Notice that in the new position, (i) distance O^*O^{**} is 2 and the distance P^*O is 3, (ii) P^* is the midpoint of O^*O^{**} and (iii) O^{**} is on the left of the vertical line through O. Let e be the support line of d whose slope is $\sqrt{3}$. Lemma 3 of Section 4 states that in this new position, d^* lies to the left of line e, without touching e (except when $O^{**}O = 4$). This means that d^*, before it was moved, was to the left of line e, without touching e. Thus strip s_1 is obtained from s_3 by a counterclockwise rotation by an angle greater than $\pi/3$, contradicting Contact condition of Lemma 3. $\square$

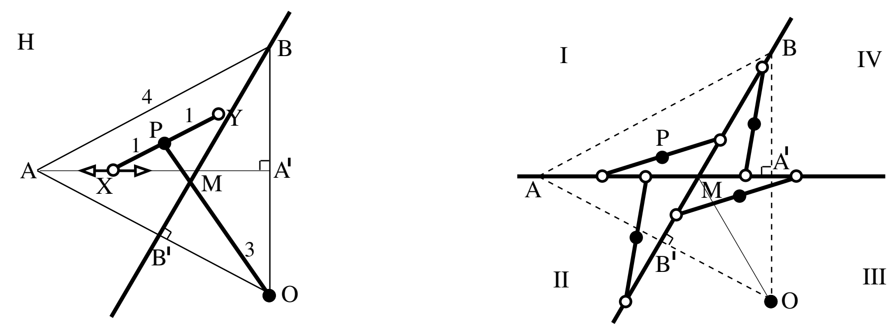

Figure 2.

4. T-linkages

By a *T-linkage* we will mean a mobile structure consisting of a bar of length 3 connected at its endpoint to the midpoint of a bar of length 2, so they can rotate about the contact point.

Lemma 3. *Let AOB be an equilateral triangle of side length* 4. *Assume that a T-linkage is attached to O by the free endpoint of its longer bar* (see Figure 2, left). *As one endpoint of the shorter bar moves along the interior of median AA', the other endpoint of the shorter bar and A stay in the same open halfplane bounded by the line of median BB'.*

Proof. Denote by H the open halfplane bounded by line BB' and containing A. Denote by M the intersection of AA' and BB'. A simple computation shows that when one endpoint of the shorter bar of the T-linkage coincides with M then the other one belongs to H. Thus, if Lemma 3 were not true then by a continuity argument the T-linkage would have a position with endpoints of the shorter bar on lines AA' and BB' respectively. We will prove that such a position does not exists. In fact we show more:

Claim. *If X is a point on line AA' different from both A and A' and if Y is a point on line BB' such that $XY = 2$, the distance from O to the midpoint of XY is smaller than* 3.

We distinguish four cases depending on which of the angles determined by lines of AA' and BB' contains the segment XY. Figure 2, right, shows how the angles are labelled I, II, III, IV. It suffices to check the cases when XY belongs to angles I or II. Indeed the cases of angles II and IV are the same by symmetry. Furthermore, if segment XY belongs to the angle III then reflecting XY around M we get a segment whose midpoint is farther from O than the midpoint of XY.

Case 1: *XY lies in angle I.* Let k be the circumcircle of the triangle XMY (see Figure 3, left). Since MO is the angle bisector of $\angle B'MA'$ the line of MO and

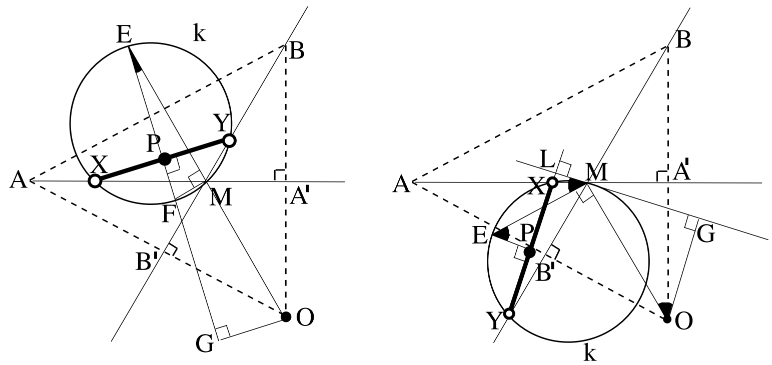

Figure 3.

the perpendicular bisector of XY intersect each other on k, say at E. Denote by F the diagonally opposite point of E on k.

Let G be the perpendicular projection of O onto line EF. Denote by P the midpoint of XY. We will express PO^2 in terms of the angle $\alpha = \angle PEM$ (with $-\pi/6 \le \alpha \le \pi/6$) and show that PO^2 is smaller than 9. Since $\angle XMY = 2\pi/3$ we have $EF = 4/\sqrt{3}$. Since $EP = \sqrt{3}$ and $MO = 4/\sqrt{3}$ we get

$$OE = EF\cos\alpha + MO = \frac{4}{\sqrt{3}}(\cos\alpha + 1).$$

Computing the parallel and perpendicular components of PO with respect to line EF we get

$$\begin{aligned} PO^2 &= OE^2\sin^2\alpha + (OE^2\cos\alpha - EP)^2 = OE^2 - 2\,OE\cos\alpha\sqrt{3} + 3 \\ &= \tfrac{16}{3}(\cos\alpha+1)^2 - 8(\cos\alpha+1)\cos\alpha + 3 = \tfrac{1}{3}(-8\cos^2\alpha + 8\cos\alpha + 25) \\ &= -\tfrac{1}{24}\left(\cos\alpha - \tfrac{1}{2}\right)^2 + 9 < 9, \end{aligned}$$

as claimed.

Case 2: *XY lies in angle II.* Let k be the circumcircle of triangle XMY (see Figure 3, right). The line perpendicular to MO and the perpendicular bisector of XY intersect each other on k, say at E. Let L be the perpendicular projection of M onto line XY. Let G be the perpendicular projection of O onto line LM.

Denote by P the midpoint of XY. We will express PO^2 in terms of the directed angle $\alpha = \angle PEM = \angle EML = \angle GOM$ (with $-\pi/3 \le \alpha \le \pi/3$) and show that PO^2 is smaller than 9. It is easy to see that $MO = 4/\sqrt{3}$, $EM = 4/\sqrt{3}\cos\alpha$ and $EP = 1/\sqrt{3}$. Computing the parallel and perpendicular components of PO with respect to line XY we get

$$\begin{aligned} PO^2 &= (EM\cos\alpha - EP + MO\sin\alpha)^2 + (-EM\sin\alpha + MO\cos\alpha)^2 \\ &= \tfrac{1}{3}\big((4\cos^2\alpha - 1 + 4\sin\alpha)^2 + (-4\cos\alpha\sin\alpha + 4\cos\alpha)^2\big) \\ &= \tfrac{1}{3}(17 + 8\cos^2\alpha - 8\sin\alpha) = \tfrac{1}{3}(25 - 8\sin^2\alpha - 8\sin\alpha) \\ &= -\tfrac{2}{3}(1 + 2\sin\alpha)^2 + 9 \le 9. \end{aligned}$$

Equality holds only if $\alpha = -\pi/6$, that is, when X coincides with A. Thus the Claim holds. □

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