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A Survey of Folding and Unfolding in Computational Geometry

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Abstract. We survey results in a recent branch of computational geometry: folding and unfolding of linkages, paper, and polyhedra.

Contents

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1. Introduction

Folding and unfolding problems have been implicit since Albrecht Dürer [1525], but have not been studied extensively in the mathematical literature until recently. Over the past few years, there has been a surge of interest in these problems in discrete and computational geometry. This paper gives a brief survey of most of the work in this area. Related, shorter surveys are [Connelly and Demaine 2004; Demaine 2001; Demaine and Demaine 2002; O'Rourke 2000]. We are currently preparing a monograph on the topic [Demaine and O'Rourke $\geq$ 2005].

In general, we are interested in how objects (such as linkages, pieces of paper, and polyhedra) can be moved or reconfigured (folded) subject to certain constraints depending on the type of object and the problem of interest. Typically the process of *unfolding* approaches a more basic shape, whereas *folding* complicates the shape. We define the *configuration space* as the set of all configurations or states of the object permitted by the folding constraints, with paths in the space corresponding to motions (foldings) of the object.

This survey is divided into three sections corresponding to the type of object being folded: linkages, paper, or polyhedra. Unavoidably, areas with which we are more familiar or for which there is a more extensive literature are covered in more detail. For example, more problems have been explored in linkage and paper folding than in polyhedron folding, and our corresponding sections reflect this imbalance. On the other hand, this survey cannot do justice to the wealth of research on protein folding, so only a partial survey appears in Section 2.5.

2. Linkages

2.1. Definitions and fundamental questions. A *linkage* or *framework* consists of a collection of rigid line segments (*bars* or *links*) joined at their endpoints (*vertices* or *joints*) to form a particular graph. A linkage can be *folded* by moving the vertices in $\mathbb{R}^d$ in any way that preserves the length of every bar. Unless otherwise specified, we assume the vertices to be universal joints, permitting the full angular range of motions. Restricted angular motions will be discussed in Section 2.5.2.

Linkages have been studied extensively in the case that bars are permitted to cross; see, for example, [Hopcroft et al. 1984; Jordan and Steiner 1999; Kapovich and Millson 1995; Kempe 1876; Lenhart and Whitesides 1995; Sallee 1973; Whitesides 1992]. Such linkages can be very complex, even in the plane. Kempe [1876] suggested an incomplete argument to show that a planar linkage can be built so that a vertex traces an arbitrary polynomial curve — there is a linkage that can "sign your name." It was not until recently that Kempe's claim was established rigorously by Kapovich and Millson [2002]. Hopcroft, Joseph, and Whitesides [Hopcroft et al. 1984] showed that deciding whether a planar linkage can reach a particular configuration is PSPACE-complete. Jordan and Steiner [1999] proved that there is a linkage whose configuration space is homeomorphic to an arbitrary compact real algebraic variety with Euclidean topology, and thus planar linkages are equivalent to the theory of the reals (solving systems of polynomial inequalities over reals). On the other hand, for a linkage whose graph is just a cycle, all configurations can be reached in Euclidean space of any dimension greater than 2 by a sequence of simple motions [Lenhart and Whitesides 1995; Sallee 1973], and in the plane there is a simple restriction characterizing which polygons can be inverted in orientation [Lenhart and Whitesides 1995].

Recently there has been much work on the case that the linkage must remain *simple*, that is, never have two bars cross.[1] The remainder of this survey assumes this noncrossing constraint. Such linkage folding has applications in hydraulic tube bending [O'Rourke 2000] and motion planning of robot arms. There are also connections to protein folding in molecular biology, which we touch upon in Section 2.5. See also [Connelly et al. 2003; O'Rourke 2000; Toussaint 1999a] for other surveys on linkage folding without crossings.

Perhaps the most fundamental question one can ask about folding linkages is whether it is possible to fold between any two configurations. That is, is there a folding between any two simple configurations of the same linkage (with matching graphs, combinatorial embeddings, and bar lengths) while preserving the bar lengths and not crossing any bars during the folding? Because folding motions can be reversed and concatenated, this fundamental question is equivalent to whether every simple configuration can be folded into some *canonical configuration*, a configuration whose definition depends on the type of linkage under consideration.

We concentrate here on allowing all continuous motions that maintain simplicity, but we should mention that different applications often further constrain the permissible motions in various ways. For example, hydraulic tube bending allows only one joint to bend at any one time, and moreover the joint angle can never reverse direction. Such constraints often drastically alter what is possible. See, for example, [Arkin et al. 2003].

In the context of linkages whose edges cannot cross, three general types of linkages are commonly studied, characterized by the structure of their associated graphs (see Figure 1): a *polygonal arc* or *open polygonal chain* (a single path); a *polygonal cycle*, *polygon*, or *closed polygonal chain* (a single cycle); and a *polygonal tree* (a single tree).[2] The canonical configuration of an arc is the *straight configuration*, all vertex angles equal to 180°. A canonical configuration

[1] Typically, bars are allowed to touch, provided they do not properly cross. However, insisting that bars only touch at common endpoints does not change the results.

[2] More general graphs have been studied largely in the context of allowing bars to cross, exploring either aspects of the configurations space (e.g., the Kempe work mentioned earlier), or the conditions which render the graph rigid. Graph rigidity is a rich topic, not detailed here, which also plays a role in the noncrossing-bar scenario in Section 2.2.1.

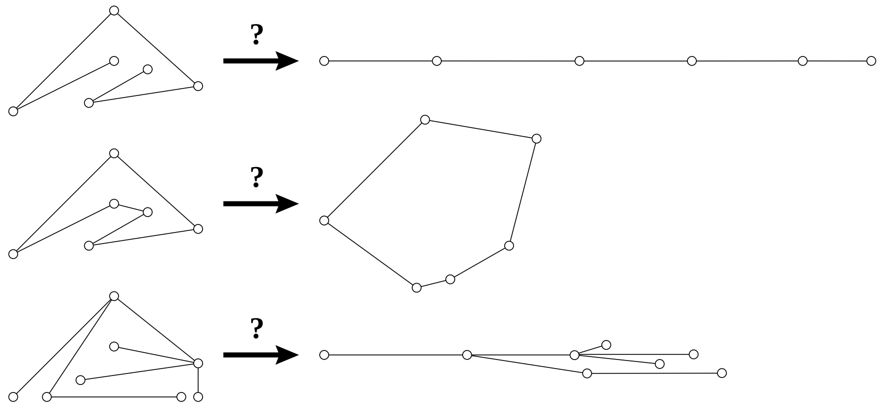


Figure 1. The three common types of linkages and their associated canonical configurations. From top to bottom, a polygonal arc $\stackrel{?}{\to}$ the straight configuration, a polygonal cycle $\stackrel{?}{\to}$ a convex configuration, and a polygonal tree $\stackrel{?}{\to}$ a (nearly) flat configuration.

of a cycle is a *convex configuration*, planar and having all interior vertex angles less than or equal to 180°. It is relatively easy to show that convex configurations are indeed "canonical" in the sense that any one can be folded into any other, a result that first appeared in [Aichholzer et al. 2001]. Finally, a canonical configuration of a tree is a *flat configuration*: all vertices lie on a horizontal line, and all bars point "rightward" from a common root. Again it is easy to fold any flat configuration into any other [Biedl et al. 2002b].

The fundamental questions thus become whether every arc can be straightened, every cycle can be convexified, and every tree can be flattened. The answers to these questions depend on the dimension of the space in which the linkage starts, and the dimension of the space in which the linkage may be folded. Over the past few years, this collection of questions has been completely resolved:

Can all arcs be straightened?

2D: Yes [Connelly et al. 2003]
3D: No [Cantarella and Johnston 1998; Biedl et al. 2001]
4D+: Yes [Cocan and O'Rourke 2001]

Can all cycles be convexified?

2D: Yes [Connelly et al. 2003]
3D: No [Cantarella and Johnston 1998; Biedl et al. 2001]
4D+: Yes [Cocan and O'Rourke 2001]

Can all trees be flattened?

2D: No [Biedl et al. 2002b]
3D: No (from arcs)
4D+: Yes [Cocan and O'Rourke 2001]

The answers for arcs and cycles are analogous to the existence of knots tied from one-dimensional string: nontrivial knots exist only in 3D. In contrast, the situation for trees presents an interesting difference in 2D: while trees in the plane are topologically unknotted, they can be geometrically locked. This observation is some evidence for the belief that the fundamental problems are most difficult in 2D.

The next three subsections describe the historical progress of these results and other results closely related to the fundamental questions. Along the way, Sections 2.3.1–2.3.4 describe several special forms of linkage folding arising out of a problem posed by Erdős in 1935; and Section 2.3.8 considers the generalization of multiple chains. Finally, Section 2.5 discusses the connections between linkage folding and protein folding, and describes the most closely related results and open problems.

2.2. Fundamental questions in 2D. Section 2.2.1 describes the development of the theorems for straightening arcs and convexifying cycles in 2D. Section 2.2.2 discusses the contrary result that not all trees can be flattened.

2.2.1. The carpenter's rule problem: polygonal chains in 2D. The questions of whether every polygonal arc can be straightened and every polygonal cycle can be convexified in the plane have arisen in many contexts over the last quarter of a century.[3] In the discrete and computational geometry community, the arc-straightening problem has become known as the *carpenter's rule problem* because a carpenter's rule folds like a polygonal arc.

Most people's initial intuition is that the answers to these problems are YES, but describing a precise general motion proved difficult. It was not until 2000 that the problems were solved by Connelly, Demaine, and Rote [Connelly et al. 2003], with an affirmative answer. Figure 2 shows an example of the motion resulting from this theorem.

More generally, the result in [Connelly et al. 2003] shows that a collection of nonintersecting polygonal arcs and cycles in the plane may be simultaneously folded so that the outermost arcs are straightened and the outermost cycles are convexified. The "outermost" proviso is necessary because arcs and cycles cannot always be straightened and convexified when they are contained in other cycles. The key idea for the solution, introduced by Günter Rote, is to look for *expansive* motions in which no vertex-to-vertex distance decreases. Bars cannot cross before getting closer, so expansiveness allows us to ignore the difficult nonlocal constraint that bars must not cross. Expansiveness brings the problem into the areas of rigidity theory and tensegrity theory, which study frameworks of rigid bars, unshrinkable *struts*, and unexpandable *cables*. Tools from these

[3]Posed independently by Stephen Schanuel and George Bergman in the early 1970's, Ulf Grenander in 1987, William Lenhart and Sue Whitesides in 1991, and Joseph Mitchell in 1992; see [Connelly et al. 2003].

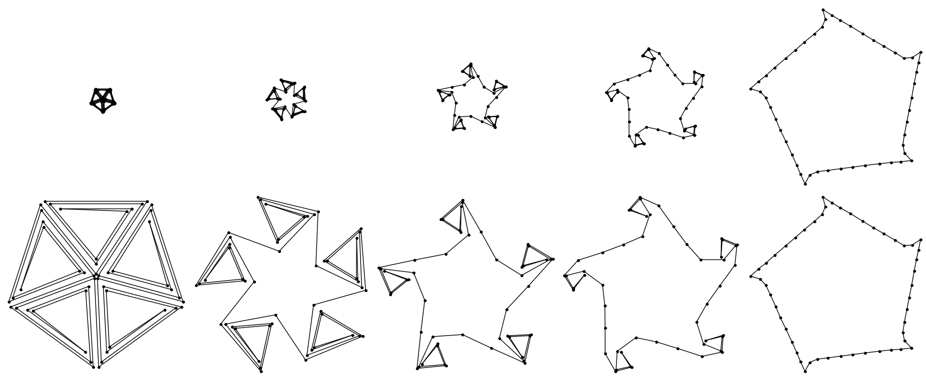


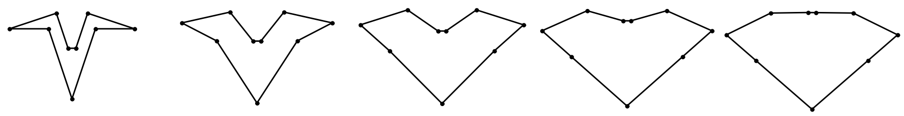
Figure 2. Two views of convexifying a "doubled tree" linkage. The top snapshots are all scaled the same, and the bottom snapshots are rescaled to improve visibility.

areas helped show that, *infinitesimally*, arcs and cycles can be unfolded expansively. These infinitesimal motions are combined by flowing along a vector field defined implicitly by an optimization problem. As a result, the motion is piecewise-differentiable (C^1). In addition, any symmetries present in the initial configuration of the linkage are preserved throughout the motion. Similar techniques show that the area of each cycle increases by this motion and furthermore by any expansive motion [Connelly et al. 2003].

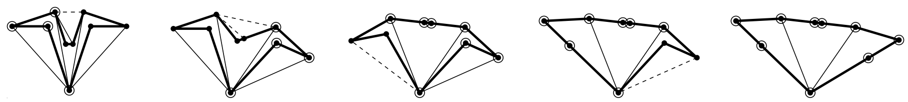
Since the original theorem, two additional algorithms have been developed for unfolding polygonal chains. Figure 3 provides a visual comparison of all three algorithms.

Ileana Streinu [2000] demonstrated another expansive motion for straightening arcs and convexifying polygons that is piecewise-algebraic, composed of a polynomial-length sequence of *mechanisms*, each with a single degree of freedom. In this sense the motion is easier to implement "mechanically." It is also possible to compute the algebraic curves involved, though the running time is exponential in n. This method also elucidates an interesting combinatorial structure to 2D linkage unfolding through "pseudotriangulations," which have subsequently received much attention in computational geometry (see [O'Rourke 2002; Rote 2003], for example).

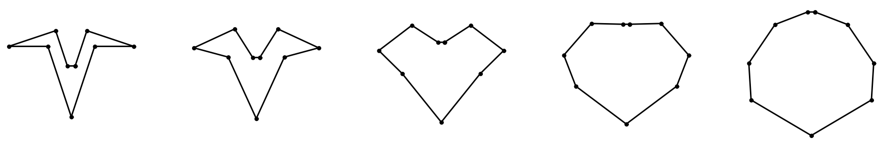
Cantarella, Demaine, Iben, and O'Brien [Cantarella et al. 2004] gave an energy-based algorithm for straightening arcs and convexifying polygons. This algorithm follows the downhill gradient of an appropriate energy function, corresponding roughly to the intuition of filling the polygon with air. The resulting motion is not expansive, essentially averaging out the strut constraints. On the other hand, the existence of the downhill gradient relies on the existence of expansive motions from [Connelly et al. 2003], by showing that the latter decrease energy. The motion avoids self-intersection not through expansiveness but by designing the energy function to approach $+\infty$ near an intersecting configura-



(a) Via convex programming [Connelly et al. 2003]



(b) Via pseudotriangulations [Streinu 2000]. Pinned vertices are circled.



(c) Via energy minimization [Cantarella et al. 2004].

Figure 3. Convexifying a common polygon via all three convexification methods.

tion; any downhill flow avoids such spikes. The result is a C^∞ motion, easily computed as a piecewise-linear motion in angle space. The number of steps in the piecewise-linear motion is polynomial in two quantities: in the number of vertices n, and in the ratio between the maximum edge length and the initial minimum distance between a vertex and an edge.

2.2.2. Trees in 2D. It was shown in [Biedl et al. 2002b] that not all trees can be flattened in the plane. The example there consists of at least 5 *petals* connected at a central high-degree vertex. The version shown in Figure 4 uses 8 petals. Each petal is an arc of three bars, the last of which is "wedged" into the center vertex.

Intuitively, the argument that the tree is locked is as follows. No petal can be straightened unless enough angular room has been made. But no petal can be reduced to occupy less angular space by more than a small positive number unless the petal has already been straightened. This circular dependence implies that no petal can be straightened, so the tree is locked. The details of this argument, in particular obtaining suitable tolerances for closeness, are somewhat intricate [Biedl et al. 2002b]. The key is that each petal occupies a wedge of space whose angle is less than 90°, which is why at least 5 petals are required.

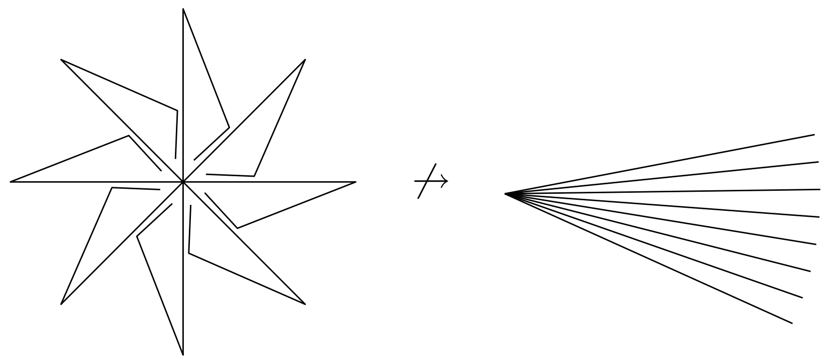


Figure 4. The locked tree on the left, from [Biedl et al. 2002b], cannot be reconfigured into the nearly flat configuration on the right. (Figure 1 of [Biedl et al. 2002b].)

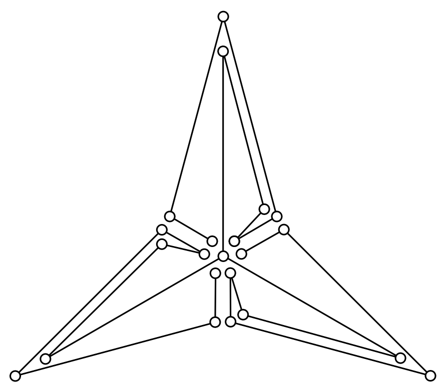


Figure 5. The locked tree from [Connelly et al. 2002]. Based on Figure 1(c) of [Connelly et al. 2002].

This tree remains locked if we replace the central degree-5 (or higher) vertex with multiple degree-3 vertices connected by very short bars [Biedl et al. 2002b, full version]. Connelly, Demaine, and Rote [Connelly et al. 2002] showed that the tree in Figure 5, with a single degree-3 vertex and the remaining vertices having degrees 1 and 2, is locked, proving tightness of the arc-and-cycle result in [Connelly et al. 2003]. In [Connelly et al. 2002] an extension to rigidity/tensegrity theory is given that permits establishing via linear programming that many classes of planar linkages (e.g., trees) are locked. In particular, this method is used to give short proofs that the tree in Figure 4 and the tree with one degree-3 vertex are *strongly locked*, in the sense that sufficiently small perturbations of the vertex positions and bar lengths result in a tree that cannot be moved more than ε in the configuration space for any $\varepsilon > 0$.

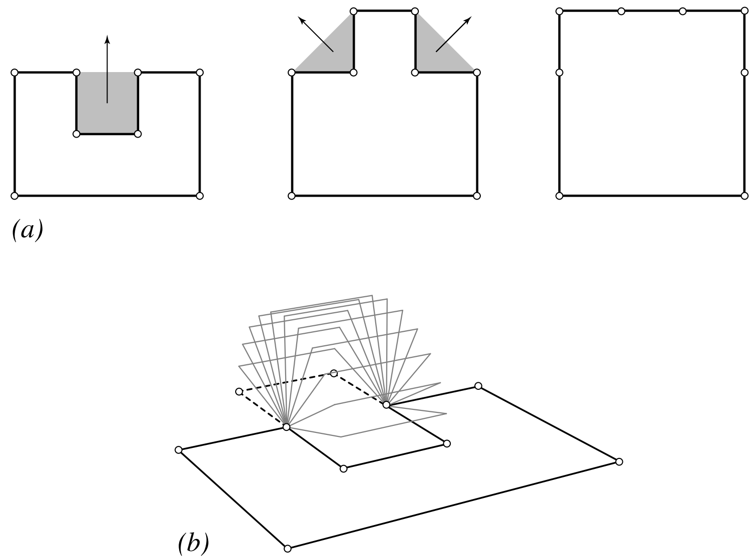


Figure 6. (a) Flipping a polygon until it is convex. Pockets are shaded. (b) The first flip shown in three dimensions.

2.3. Fundamental questions in 3D. Linkage folding in 3D was initiated earlier, by Paul Erdős [1935]. His problem and its solution are described in Section 2.3.1. Sections 2.3.2–2.3.4 consider various extensions of this problem. All of this work deals with linkages that start in the plane, but fold through 3D. The more general situation, an arbitrary linkage starting in 3D, is addressed in Section 2.3.6. As this problem proves unsolvable in general, additional special cases are addressed in Section 2.3.7. Finally, Section 2.3.8 considers the generalized problem of multiple interlocking chains.

2.3.1. Flips for planar polygons in 3D. The roots of linkage folding go back to [Erdős 1935], a problem posed in the *American Mathematics Monthly*. Define a *pocket* of a polygon to be a region bounded by a subchain of the polygon edges, and define the *lid* of the pocket to be the edge of the convex hull connecting the endpoints of that subchain. Every nonconvex polygon has at least one pocket. Erdős defined a *flip* as a rotation of a pocket's chain of edges into 3D about the pocket lid by 180°, landing the subchain back in the plane of the polygon, such that the polygon remains simple (i.e., non-self-intersecting); see Figure 6. He asked whether every polygon may be convexified by a finite number of simultaneous pocket flips.

The answer was provided in a later issue of the *Monthly* [Nagy 1939]. First, Nagy observed that flipping several pockets at once could lead to self-crossing; see Figure 7b. However, restricting to one flip at a time, Nagy proved that a finite number of flips suffice to convexify any polygon; see Figure 6 for a three-step example. This beautiful result has been rediscovered and reproved several

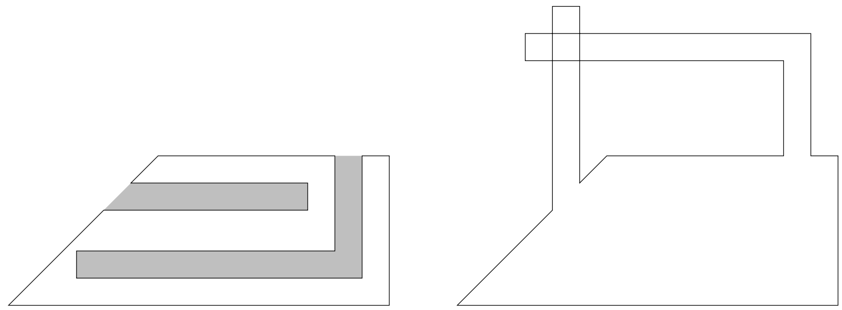


Figure 7. Flipping multiple pockets simultaneously can lead to crossings [Nagy 1939].

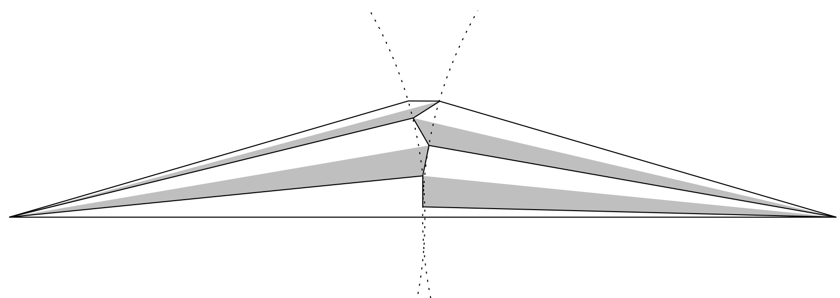


Figure 8. Quadrangles can require arbitrarily many flips to convexify [Grünbaum 1995; Toussaint 1999b; Biedl et al. 2001].

times, as uncovered by Grünbaum and Toussaint and detailed in their histories of the problem [Grünbaum 1995; Toussaint 1999b]; only recently has a subtle oversight in Nagy's proof been corrected.

Unfortunately, the number of required flips can be arbitrarily large in terms of the number of vertices, even for a quadrangle. This fact was originally proved by Joss and Shannon (1973); see [Grünbaum 1995; Toussaint 1999b; Biedl et al. 2001]. Figure 8 shows the construction. By making the vertical edge of the quadrangle very short and even closer to the horizontal edge, the angles after the first flip approach the mirror image of the original quadrangle, and hence the number of required flips approaches infinity.

Mark Overmars[4] posed the still-open problem of bounding the number of flips in terms of natural measures of geometric closeness such as the diameter (maximum distance between two vertices), sharpest angle, or the minimum feature size (minimum distance between two nonincident edges).

Another open problem is to determine the complexity of finding the shortest or longest sequence of flips to convexify a given polygon. Weak NP-hardness has been established for the related problem of finding the longest sequence of *flipturns* [Aichholzer et al. 2002].

[4]Personal communication, February 1998.

2.3.2. Flips in nonsimple polygons. Flips can be generalized to apply to nonsimple polygons: consider two vertices adjacent along the convex hull of the polygon, splitting the polygon into two chains, and rotate one (either) chain by 180° with respect to the other chain about the axis through the two vertices. Simplicity may not be preserved throughout the motion, just as it may not hold in the initial or final configuration. The obvious question is whether every nonsimple polygon can be convexified by a finite sequence of such flips. Grünbaum and Zaks [1998] proved that if at each step we choose the flip that maximizes the resulting sum of distances between all pairs of vertices, then this metric increases at each flip, and the polygon becomes convex after finitely many flips. Without sophisticated data structures, computing these flips requires $\Omega(n^2)$ time per flip. Toussaint [1999b] proved that a different sequence of flips convexifies a nonsimple polygon, and this sequence can be computed in $O(n)$ time per flip. More recently, it has been established[5] that every sequence of flips eventually convexifies a nonsimple polygon. We expect that each flip can be executed in polylogarithmic amortized time using dynamic convex-hull data structures as in [Aichholzer et al. 2002].[6]

2.3.3. Deflations. A *deflation* [Fevens et al. 2001; Wegner 1993; Toussaint 1999b] is the reverse of a flip, in the sense that a deflation of a polygon should result in a simple polygon that can be flipped into the original polygon. More precisely, a deflation is a rotation by 180° about a line meeting the polygon at two vertices and nowhere else, thus separating the chain into two subchains, such that the rotation does not cause any intersections. Hence, after the deflation, this line becomes a line of support (a line extending a convex-hull edge). Wegner [1993] proposed the notion of deflations, and their striking similarity to flips led him to conjecture that every polygon can be deflated only a finite number of times. Surprisingly, this is not true: Fevens, Hernandez, Mesa, Soss, and Toussaint [Fevens et al. 2001] characterized a class of quadrangles whose unique deflation leads to another quadrangle in the class, thus repeating ad infinitum.

2.3.4. Other variations. Erdős flips have inspired several directions of research on related notions, including pivots, pops, and flipturns. See [Toussaint 1999b] for a survey of this area, with more recent work on flipturns in [Ahn et al. 2000; Aichholzer et al. 2002; Biedl 2005].

2.3.5. Efficient algorithms for planar linkages in 3D. Motivated by the inefficiency of the flip algorithm, Biedl et al. [2001] developed an algorithm to convexify planar polygons by motions in 3D using a linear number of simple moves. The essence of this algorithm is to lift the polygon, bar by bar, at all times maintaining a convex chain (or *arch*) lying in a plane orthogonal to the plane containing the polygon; see Figure 9. The details of the algorithm are significantly more involved than the overarching idea.

[5] Personal communication with Therese Biedl, May 2001.

[6] Personal communication with Jeff Erickson.

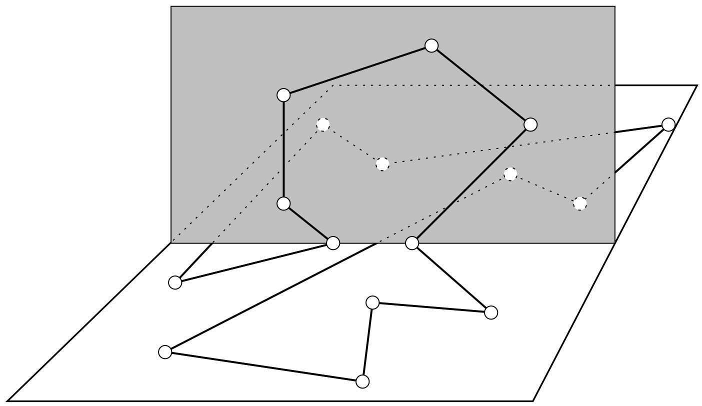


Figure 9. A planar polygon partially lifted into a convex arch lying in a vertical plane (shaded). (Based on Figure 6 of [Biedl et al. 2001].)

A second linear-time algorithm, which is in some ways conceptually simpler, was developed by Aronov, Goodman, and Pollack [Aronov et al. 2002]. Their algorithm at all times maintains the arch as a convex quadrilateral. At each step, the algorithm lifts two edges, forming a "twisted trapezoid," incorporates the trapezoid into the arch, makes the arch planar, and reduces it back to a quadrilateral. Avoiding intersections during the lifting phase requires a delicate argument.

In contrast to convexifying a cycle, it is relatively easy to straighten a polygonal arc lying in a plane, or on the surface of a convex polyhedron, by motions in 3D [Biedl et al. 2001]. For an arc in a plane, the basic idea is to pull the arc up into a vertical line. For a convex surface, the same idea is followed, but with the orientation of the line changing to remain normal to the surface. The algorithm lifts each bar in turn, from one end of the arc to the other, at all times maintaining a prefix of the arc in a line normal to the current facet of the polyhedron. Each lifting motion causes two joint angles to rotate, so that the lifted prefix remains normal to the facet at all times, while the remainder of the chain remains in its original position. Whenever the algorithm reaches a vertex that bridges between two adjacent facets, it rotates the prefix to bring it normal to the next facet. This algorithm also generalizes to flattening planar trees and trees on the surface of a convex polyhedron, via motions in 3D.

2.3.6. Almost knots. What if the linkage starts in an arbitrary position in 3D instead of in a plane? In general, a polygonal arc or an unknotted polygonal cycle in 3D cannot always be straightened or convexified [Cantarella and Johnston 1998; Toussaint 2001; Biedl et al. 2001] (page 170). Figure 10 shows an example of a locked arc in 3D. Provided that each of the two end bars is longer than the sum s of the middle three bar lengths, the ends of the chain cannot get close enough to the middle bars to untangle the chain (sometimes called the "knitting

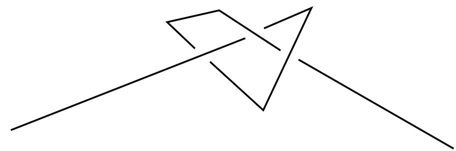


Figure 10. A locked polygonal arc in 3D with 5 bars [Cantarella and Johnston 1998; Biedl et al. 2001].

needles" example). More precisely, because the ends of the chain remain outside a sphere with radius s and centered at one of the middle vertices, we can connect the ends of the chain with an unknotted flexible cord outside the sphere, and any straightening motion unties the resulting knot, which is impossible without crossings [Biedl et al. 2001].

Alt, Knauer, Rote, and Whitesides [Alt et al. 2004] proved that it is PSPACE-hard to decide whether a 3D polygonal arc (or a 2D polygonal tree) can be reconfigured between two specified configurations. On the other hand, it remains open to determine the complexity of deciding whether a polygonal arc can be straightened. The next two sections describe special cases of 3D chains, more general than planar chains, that can be straightened and convexified.

2.3.7. Simple projection. The "almost knottedness" of the example in Section 2.3.6 suggests that polygonal chains having simple orthogonal projections can always be straightened or convexified. This fact is established by two papers [Biedl et al. 2001; Calvo et al. 2001]. In addition, there is a polynomial-time algorithm to decide whether a polygonal chain has a simple projection, and if so find a suitable plane for projection [Bose et al. 1999].

For a polygonal arc with a simple orthogonal projection, the straightening method is relatively straightforward [Biedl et al. 2001]. The basic idea is to process the arc from one end to the other, accumulating bars into a compact "accordion" (x-monotone chain) lying in a plane orthogonal to the projection plane, in which each bar is nearly vertical. Once this accumulation is complete, the planar accordion is unfolded joint-by-joint into a straight arc. We observe that a similar algorithm can be used to fold a polygonal tree with a simple orthogonal projection into a generalized accordion, which can then be folded into a flat configuration.

For a polygonal cycle with a simple orthogonal projection, the convexification method is based on two steps [Calvo et al. 2001]. First, the projection of the polygon is convexified via the results described in Section 2.2.1, by folding the 3D polygon to track the shadow, keeping constant the ascent of each bar. Second, Calvo, Krizanc, Morin, Soss, and Toussaint [Calvo et al. 2001] develop an algorithm for convexifying a polygon with convex projection. The basic idea is to reconfigure the convex projection into a triangle, and stretch each accordion formed by an edge in the projection. In linear time they show how to compute a motion for the second step that consists of $O(n)$ simple moves, each changing at most seven vertex angles.

2.3.8. Interlocked chains in 3D. Although we have settled on page 170 the question of when *one* chain can lock (only in 3D), the conditions that permit pairs of chains to "interlock" are largely unknown. This line of investigation was prompted by a question posed by Anna Lubiw [Demaine and O'Rourke 2001]: into how many pieces must an n-bar 3D chain be cut (at vertices) so that the pieces can be separated and straightened? It is now known that the chain need be fractured into no more than $\lceil n/2 \rceil - 1$ pieces [Demaine et al. 2002b] but this upper bound is likely not tight: the only lower bound known is $\lfloor (n-1)/4 \rfloor$.

A collection of disjoint, noncrossing chains can be *separated* if, for any distance d, there is a non-self-crossing motion that results in every pair of points on different chains being separated by at least d. If a collection cannot be separated, its chains are *interlocked*. Which collections of relatively short chains can interlock was investigated in several papers [Demaine et al. 2003c; Demaine et al. 2002b]. Three typical results (all for chains with universal joints) are as follows:

(i) No pair of 3-bar open chains can interlock, even with an arbitrary number of additional 2-bar open chains.
(ii) A 3-bar open chain can interlock with a 4-bar closed chain. (See Figure 11.)
(iii) A 3-bar open chain can interlock with a 4-bar open chain.

The proof of the first result (for just a pair of 3-bar chains) identifies a plane parallel to and separating the middle bars of each chain, and then nonuniformly scales the coordinate system to straighten the other links while avoiding intersections. The second result uses a topological argument based on "links" (multicomponent knots), in a manner similar to the use of knots in the proof that the chain in Figure 10 is locked. The proof of the third listed result is quite intricate, relying on ad hoc geometric arguments [Demaine et al. 2002b]. There are many open problems here, one of the most intriguing being this: what is the smallest k that permits a k-bar open chain to interlock with a 2-bar open chain? (See [Glass et al. 2004].)

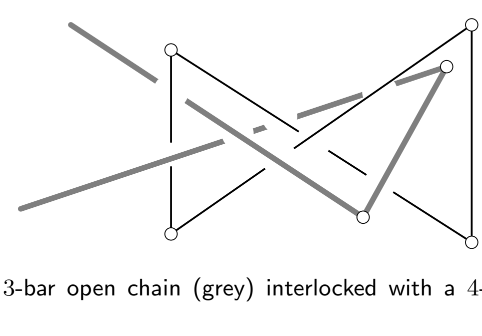


Figure 11. A 3-bar open chain (grey) interlocked with a 4-bar closed chain (black).

2.4. Fundamental questions in 4D and higher dimensions. In all dimensions higher than 3, it is known that all knots are trivial; analogously, all polygonal arcs can be straightened, all polygonal cycles can be convexified, and all polygonal trees can be flattened [Cocan and O'Rourke 2001] (page 170). Intuitively, this result holds because the number of degrees of freedom of any vertex is at least two higher than the dimensionality of the obstacles imposed by any bar. This property allows Cocan and O'Rourke [2001] to establish, for example, that the last bar of a polygonal arc can be unfolded by itself to any target position that is simple.

Cocan and O'Rourke [2001] show how to straighten an arc using $O(n)$ simple moves that can be computed in $O(n^2)$ time and $O(n)$ space. On the other hand, their method for convexifying a polygon requires $O(n^6)$ simple moves and $O(n^6 \log n)$ time to compute.

2.5. Protein folding. Protein folding [Chan and Dill 1993; Hayes 1998; Merz and Le Grand 1994] is an important problem in molecular biology because it is generally believed that the folded structure of a protein (the fundamental building block of life) determines its function and behavior.

2.5.1. Connection to linkages. A protein can be modeled by a linkage in which the vertices represent amino acids and the bars represent bonds connecting them. The bars representing bonds are typically close in length, within a factor less than two. Depending on the level of detail, the protein can be modeled as a tree (more precise) or as a chain (less precise).

An amazing property of proteins is that they fold quickly and consistently to a minimum-energy configuration. Understanding this motion has immediate connections to linkage folding in 3D. A central unsolved theoretical question [Biedl et al. 2001] arising in this context is whether every equilateral polygonal arc in 3D can be straightened. Cantarella and Johnston [Cantarella and Johnston 1998] proved that this is true for arcs of at most 5 bars. More generally, can every equilateral polygonal tree in 3D be flattened?

2.5.2. Fixed-angle linkages. A more accurate mathematical model of foldings of proteins is not by linkages whose vertices are universal joints, but rather by *fixed-angle* linkages in which each vertex forms a fixed angle between its incident bars. This angular constraint roughly halves the number of degrees of freedom in the linkage; the basic motion is rotating a portion of the linkage around a bar of the linkage. Foldings of such linkages have been explored extensively by Soss and Toussaint [Soss and Toussaint 2000; Soss 2001]. For example, they prove in [Soss and Toussaint 2000] that it is NP-complete to decide whether a fixed-angle polygonal arc can be flattened (reconfigured to lie a plane), and in [Soss 2001] that it is NP-complete to decide whether a fixed-angle polygonal arc can be folded into its mirror image.

More positive results analyze the polynomial complexity of determining the maximum extent of a rotation around a bar: Soss and Toussaint [Soss and Toussaint 2000; Soss 2001] prove an $O(n^2)$ upper bound, and Soss, Erickson, and Overmars [Soss 2001; Soss et al. 2003] give a 3SUM-hardness reduction, suggesting an $\Omega(n^2)$ lower bound.

Another line of investigation on fixed-angle chains was opened in [Aloupis et al. 2002a; Aloupis et al. 2002b]. Define a linkage X to be *flat-state connected* if, for each pair of its flat realizations x_1 and x_2, there is a reconfiguration from x_1 to x_2 that avoids self-intersection throughout. In general this motion alters the linkage to nonflat configurations in $\mathbb{R}^3$ intermediate between the two flat states. The main question is to determine whether every fixed-angle open chain is flat-state connected. It has been established that the answer is YES for chains all of whose fixed angles between consecutive bars are nonacute [Aloupis et al. 2002a], and although other special cases have been settled [Aloupis et al. 2002b], the main question remains open.

2.5.3. Producible chains. A connection between fixed-angle nonacute chains and a model of protein production was recently established in [Demaine et al. 2003b]. Here the ribosome — the "machine" that creates protein chains in biological cells — is modeled as a cone, with the fixed-angled chain produced bar-by-bar inside and emerging through the cone's apex. A configuration of a chain is said to be α*-producible* if there exists a continuous motion of the chain as it is created by the above model from within a cone of half-angle $\alpha \le \pi/2$. The main result of [Demaine et al. 2003b] is a theorem that identifies producible with flattenable chains, in this sense: a configuration of a chain whose fixed angles are $\ge \pi - \alpha$, for $\alpha \le \pi/2$, is α-producible if and only if it is flattenable. For example, for $\alpha = 45°$, this theorem says that a fixed-135°-angle chain (which is nonacute) is producible within a 90° cone if and only if that configuration is flattenable.

The proof uses a coiled cannonical configuration of the chain, which can be obtained by time-reversal of the production steps, winding the chain inside the cone. This canonical form establishes that all α-producible chains can be reconfigured to one another. Then it is shown how to produce any flat configuration by rolling the cone around on the plane into which the flat chain is produced. Because locked chains are not flattenable, the equivalence of producible and flattenable configurations shows that cone production cannot lead to locked configurations. This result in turn leads to the conclusion that the producible chains are rare, in a technical sense, suggesting that the entire configuration space for folding proteins might not need to be searched.

2.5.4. The H-P model. So far in this section we have not considered the forces involved in protein folding in nature. There are several models of these forces.

One of the most popular models of protein folding is the hydrophobic-hydrophilic (H-P) model [Chan and Dill 1993; Dill 1990; Hayes 1998], which defines both a geometry and a quality metric of foldings. This model represents a protein as a chain of amino acids, distinguished into two categories, hydrophobic (H) and hydrophilic (P). A folding of such a protein chain in this model is an embedding along edges of the square lattice in 2D or the cubic lattice in 3D without self-intersection. The optimum or minimum-energy folding maximizes the number of hydrophobic (H) nodes that are adjacent in the lattice. Intuitively, this metric causes hydrophobic amino acids to avoid the surrounding water.

This combinatorial model is attractive in its simplicity, and already seems to capture several essential features of protein folding such as the tendency for the hydrophobic components to fold to the center of a globular protein [Chan and Dill 1993]. While a 3D H-P model most naturally matches the physical world, in fact it is more realistic as a 2D model for computationally feasible problem sizes. The reason for this is that the perimeter-to-area ratio of a short 2D chain is a close approximation to the surface-to-volume ratio of a long 3D chain [Chan and Dill 1993; Hayes 1998].

Much work has been done on the H-P model [Berger and Leighton 1998; Chan and Dill 1991; Chan and Dill 1990; Crescenzi et al. 1998; Hart and Istrail 1996; Lau and Dill 1989; Lau and Dill 1990; Lipman and Wilber 1991; Unger and Moult 1993a; Unger and Moult 1993b; Unger and Moult 1993c]. Recently, on the computational side, Berger and Leighton [Berger and Leighton 1998] proved NP-completeness of finding the optimal folding in 3D, and Crescenzi et al. [Crescenzi et al. 1998] proved NP-completeness in 2D. Hart and Istrail [Hart and Istrail 1996] have developed a 3/8-approximation in 3D and a 1/4-approximation in 2D for maximizing the number of hydrophobic-hydrophobic adjacencies.

Aichholzer, Bremner, Demaine, Meijer, Sacristan, and Soss [Aichholzer et al. 2003] have begun exploring an important yet potentially more tractable aspect of protein folding: can we design a protein that folds stably into a desired shape? In the H-P model, a protein folds *stably* if it has a unique minimum-energy configuration. So far, Aichholzer et al. [Aichholzer et al. 2003] have proved the existence of stably folding proteins of all lengths divisible by 4, and for closed chains of all possible (even) lengths. It remains open to characterize the possible shapes (connected subsets of the square grid) attained by stable protein foldings.

3. Paper

Paper folding (origami) has led to several interesting mathematical and computational questions over the past fifteen years or so. A piece of paper, normally a (solid) polygon such as a square or rectangle, can be folded by a continuous motion that preserves the distances on the surface and does not cause the paper to properly self-intersect. Informally, paper cannot tear, stretch, or cross itself, but may otherwise bend freely. (There is a contrast here to folding other materials, such as sheet metal, that must remain piecewise planar throughout the folding process.) Formally, a folding is a continuum of isometric embeddings of the piece of paper in $\mathbb{R}^3$. However, the use of the term “embedding” is weak:

paper is permitted to touch itself provided it does not properly cross itself. In particular, a *flat folding* folds the piece of paper back into the plane, and so the paper must necessarily touch itself. We frequently ignore the continuous motion of a folding and instead concentrate on the final folded state of the paper; in the case of a flat folding, the flat folded state is called a *flat origami*. This concentration on the final folded state was recently justified by a proof that there always exists a continuous motion from a planar polygonal piece of paper to any "legal" folded state [Demaine et al. 2004].

Some of the pioneering work in origami mathematics (see Section 3.3.1) studies the *crease pattern* that results from unfolding a flat origami, that is, the graph of edges on the paper that fold to edges of a flat origami. Stated in reverse, what crease patterns have flat foldings? Various necessary conditions are known [Hull 1994; Justin 1994; Kawasaki 1989], but there is little hope for a polynomial characterization: Bern and Hayes [Bern and Hayes 1996] have shown that this decision problem is NP-hard.

A more recent trend, as in [Bern and Hayes 1996], is to explore *computational origami*, the algorithmic aspects of paper folding. This field essentially began with Robert Lang's work on algorithmic origami design [Lang 1996], starting around 1993. Since then, the field of computational origami has grown significantly, in particular in the past two years by applying computational geometry techniques. This section surveys this work. See also [Demaine and Demaine 2002].

3.1. Categorization. Most results in computational origami fall under one or more of three categories: universality results, efficient decision algorithms, and computational intractability results. This categorization applies more generally to folding and unfolding, but is particularly useful for results in computational origami.

A *universality result* shows that, subject to a certain model of folding, everything is possible. For example, any tree-shaped origami base (Section 3.2.2), any polygonal silhouette (Section 3.2.1), and any polyhedral surface (Section 3.2.1) can be folded out of a sufficiently large piece of paper. Universality results often come with efficient algorithms for finding the foldings; pure existence results are rare.

When universality results are impossible (some objects cannot be folded), the next-best result is an *efficient decision algorithm* to determine whether a given object is foldable. Here "efficient" normally means "polynomial time." For example, there is a polynomial-time algorithm to decide whether a "map" (grid of creases marked mountain and valley) can be folded by a sequence of "simple folds" (Section 3.3.4).

Not all paper-folding problems have efficient algorithms, and this can be proved by a *computational intractability result*. For example, it is NP-hard to tell whether a given crease pattern folds into some flat origami (Section 3.3.2), even when folds are restricted to simple folds (Section 3.3.4). These results imply that there are no polynomial-time algorithms for these problems, unless some of the hardest computational problems known can also be solved in polynomial time, which is generally deemed unlikely.

We further distinguish computational origami results as addressing either *origami design* or *origami foldability*. In origami design, some aspects of the target configuration are specified, and the goal is to design a suitable detailed folded state that can be folded out of paper. In origami foldability, the target configuration is unspecified and arbitrary; rather, the initial configuration is specified, in particular the crease pattern, possibly marked with mountains and valleys, and the goal is to fold something (anything) using precisely those creases. While at first it may seem that understanding origami foldability is a necessary component for origami design, the results indicate that in fact origami design is easier to solve than origami foldability, which is usually intractable.

Our survey of computational origami is divided accordingly into Section 3.2 (origami design) and Section 3.3 (origami foldability).

3.2. Origami design. We define *origami design* loosely as, given a piece of paper, fold it into an object with certain desired properties, e.g., a particular shape. The natural theoretical version of this problem is to ask for an origami with a specific silhouette or three-dimensional shape; this problem can be solved in general (Section 3.2.1), although the algorithms developed so far do not lead to practical foldings. A specific form of this problem has been solved for practical purposes by Lang's tree method (Section 3.2.2), which has brought modern origami design to a new level of complexity. Related to this work is the problem of folding a piece of paper to align a prescribed graph (Section 3.2.3), which can be used for a magic trick involving folding and one complete straight cut.

3.2.1. Silhouettes and polyhedra. A direct approach to origami design is to specify the exact final shape that the paper should take. More precisely, suppose we specify a particular flat silhouette, or a three-dimensional polyhedral surface, and desire a folding of a sufficiently large square of paper into precisely this object, allowing coverage by multiple layers of paper. For what polyhedral shapes is this possible? This problem is implicit throughout origami design, and was first formally posed in [Bern and Hayes 1996]. The surprising answer is "always," as established by Demaine, Demaine, and Mitchell in 1999 [Demaine et al. 1999c; 2000d].

The basic idea of the approach is to fold the piece of paper into a thin strip, and then wrap this strip around the desired shape. This wrapping can be done particularly efficiently using methods in computational geometry. Specifically, three algorithms are described in [Demaine et al. 2000d] for this process. One algorithm optimizes paper usage: the amount of paper required can be made arbitrarily close to the surface area of the shape, but only at the expense of increasing the aspect ratio of the rectangular paper. Another algorithm maximizes

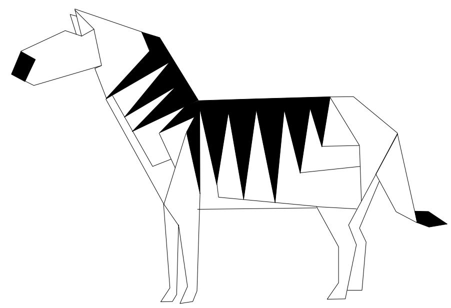


Figure 12. A flat folding of a square of paper, black on one side and white on the other side, designed by John Montroll [Montroll 1991, pp. 94–103]. (Figure 1(b) of [Demaine et al. 2000d].)

the width of the strip subject to some constraints. A third algorithm places the visible *seams* of the paper in any desired pattern forming a decomposition of the sides into convex polygons. In particular, the number and total length of seams can be optimized in polynomial time in most cases [Demaine et al. 2000d].

All of these algorithms allow an additional twist: the paper may be colored differently on both sides, and the shape may be two-colored according to which side should be showing. In principle, this allows the design of two-color models similar to the models in Montroll's *Origami Inside-Out* [Montroll 1993]. An example is shown in Figure 12.

Because of the use of thin strips, none of these methods lead to practical foldings, except for small examples or when the initial piece of paper is a thin strip. Nonetheless, the universality results of [Demaine et al. 2000d] open the door to many new problems. For example, how small a square can be folded into a desired object, e.g., a $k \times k$ chessboard? This optimization problem remains open even in this special case, as do many other problems about finding efficient, practical foldings of silhouettes, two-color patterns, and polyhedra.

3.2.2. Tree method. The *tree method of origami design* is a general approach for "true" origami design (in contrast to the other topics that we discuss, which involve less usual forms of origami). In short, the tree method enables design of efficient and practical origami within a particular class of three-dimensional shapes, most useful for origami design. Some components of this method, such as special cases of the constituent molecules and the idea of disk packing, as well as other methods for origami design, have been explored in the Japanese technical origami community, in particular by Jun Maekawa, Fumiaki Kawahata, and Toshiyuki Meguro. This work has led to several successful designs, but a full survey is beyond the scope of this paper; see [Lang 2003; Lang 1998]. It suffices to say that the explosion in origami design over the last 30 years, during which the majority of origami models have been designed, may largely be due to an understanding of these general techniques.

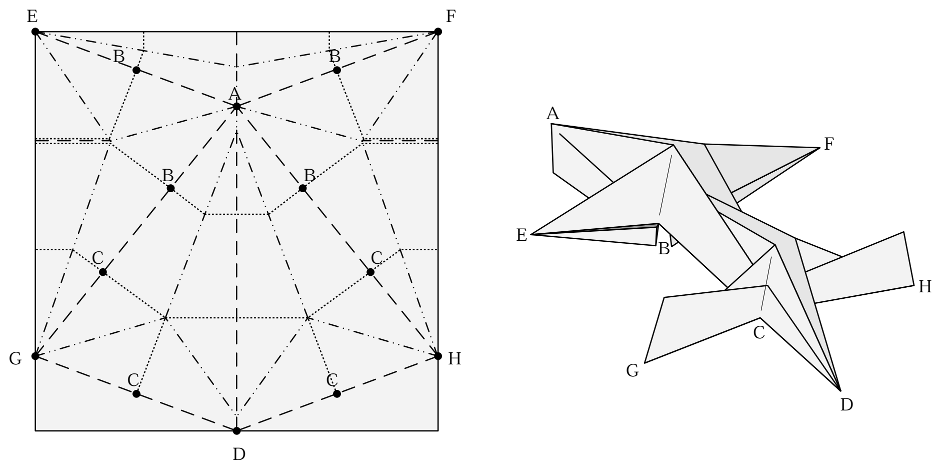


Figure 13. Lang's TreeMaker applied to an 8-vertex tree to produce a lizard base. (Figure 2.1.11 of [Lang 1998].)

Here we concentrate on Robert Lang's work [1994a; 1994b; 1996; 1998; 2003], which is the most extensive. Over the past decade, starting around 1993, Lang has developed the tree method to the point where an algorithm and computer program have been explicitly defined and implemented. Anyone with a Macintosh computer can experiment with the tree method using Lang's program TreeMaker [Lang 1998].

The tree method allows one to design an origami *base* in the shape of a specified tree with desired edge lengths, which can then be folded and shaped into an origami model. See Figure 13 for an example. More precisely, the tree method designs a *uniaxial base* [Lang 1996], which must have the following properties: the base lies above and on the xy-plane, all facets of the base are perpendicular to the xy-plane, the projection of the base to the xy plane is precisely where the base comes in contact with the xy-plane, and this projection is a one-dimensional tree.

It is known that every metric tree (unrooted tree with prescribed edge lengths) is the projection of a uniaxial base that can be folded from, e.g., a square. The tree method gives an algorithm to find the folding that is optimal in the sense that it folds the uniaxial base with the specified projection using the smallest possible square piece of paper (or more generally, using the smallest possible scaling of a given convex polygon). These foldings have led to many impressive origami designs; see [Lang 2003] in particular.

There are two catches to this result. First, it is currently unknown whether the prescribed folding self-intersects, though it is conjectured that self-intersection does not arise, and this conjecture has been verified on extensive examples. Second, the optimization problem is difficult, a fairly general form of nonlinear constrained optimization. So while optimization is possible in principle in finite

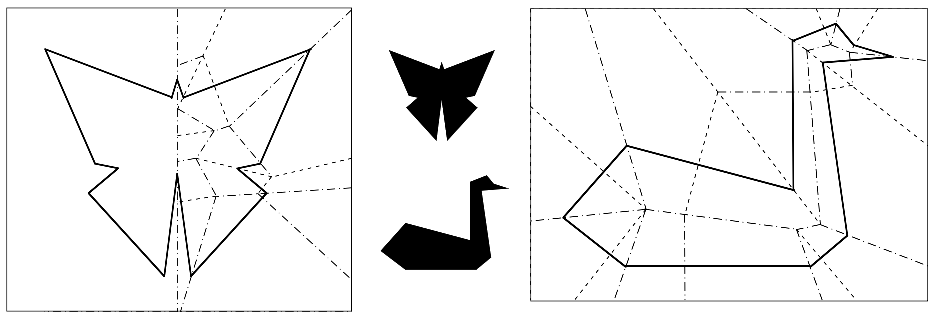


Figure 14. Crease patterns for folding a rectangle of paper flat so that one complete straight cut makes a butterfly (left) or a swan (right), based on [Demaine et al. 2000c; Demaine et al. 1999b].

time, in practice heuristics must be applied; fortunately, such heuristics frequently yield good, practical solutions. Indeed, additional practical constraints can be imposed, such as symmetry in the crease pattern, or the constraint that angles of creases are integer multiples of some value (e.g., 22.5°) subject to some flexibility in the metric tree.

3.2.3. One complete straight cut. Take a piece of paper, fold it flat, make one complete straight cut, and unfold the pieces. What shapes can result? This *fold-and-cut* problem was first formally stated by Martin Gardner [1960], but goes back much further, to a Japanese puzzle book [Sen 1721] and perhaps to Betsy Ross in 1777 [Harper's 1873]; see also [Houdini 1922, pp. 176–177]. A more detailed history can be found in [Demaine et al. 2000c].

More formally, given a planar graph drawn with straight edges on a piece of paper, can the paper be folded flat so as to map the entire graph to a common line, and map nothing else to that line? The surprising answer is that this is always possible, for any collection of line segments in the plane, forming nonconvex polygons, adjoining polygons, nested polygons, etc. There are two solutions to the problem. The first (partial) solution [Demaine et al. 2000c; Demaine et al. 1999b] is based on a structure called the straight skeleton, which captures the symmetries of the graph, thereby exploiting a more global structure of the problem. This solution applies to a large class of instances, which we do not describe in detail here. See Figure 14 for two examples. The second (complete) solution [Bern et al. 2002] is based on disk packing to make the problem more local, and achieves efficient bounds on the number of creases.

While this problem may not seem directly connected to pure paper folding because of the one cut, the equivalent problem of folding a piece of paper to line up a given collection of edges is in fact closely connected to origami design. Specifically, one subproblem that arises in TreeMaker (Section 3.2.2) is that the piece of paper is decomposed into convex polygons, and the paper must be folded flat so as to line up all the edges of the convex polygons, and place the interior

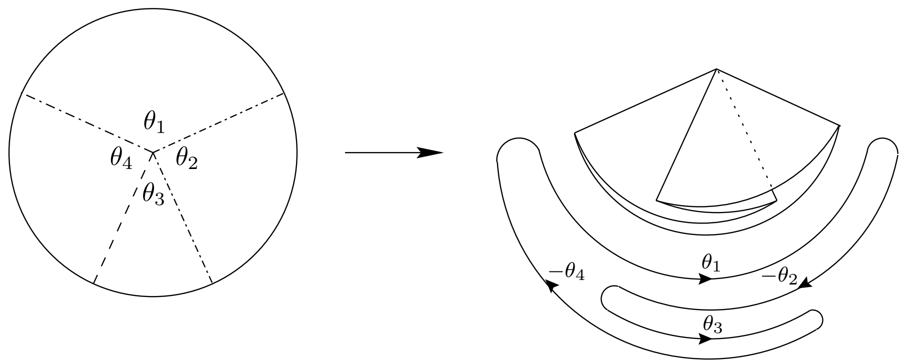


Figure 15. A locally flat-foldable vertex: $\theta_1 + \theta_3 + \cdots = \theta_2 + \theta_4 + \cdots = 180°$.

of these polygons above this line. The fold-and-cut problem is a generalization of this situation to arbitrary graphs: nonconvex polygons, nested polygons, etc. In TreeMaker, there are important additional constraints in how the edges can be lined up, called path constraints, which are necessary to enforce the desired geometric tree. These constraints lead to additional components in the solution called *gussets*.

3.3. Origami foldability. We distinguish origami design from *origami foldability* in which the starting point is a given crease pattern and the goal is to fold an origami that uses precisely these creases. (Arguably, this is a special case of our generic definition of origami design, but we find it a useful distinction.) The most common case studied is when the resulting origami should be flat, i.e., lie in a plane.

3.3.1. Local foldability. For crease patterns with a single vertex, it is relatively easy to characterize flat foldability. Without specified crease directions, a single-vertex crease pattern is flat-foldable precisely if the alternate angles around the vertex sum to 180°; see Figure 15. This is known as Kawasaki's theorem [Bern and Hayes 1996; Hull 1994; Justin 1994; Kawasaki 1989]. When the angle condition is satisfied, a characterization of valid mountain-valley assignments and flat foldings can be found in linear time [Bern and Hayes 1996; Justin 1994], using Maekawa's theorem [Bern and Hayes 1996; Hull 1994; Justin 1994] and another theorem of Kawasaki [Bern and Hayes 1996; Hull 1994; Kawasaki 1989] about constraints on mountains and valleys. In particular, Hull has shown that the number of distinct mountain-valley assignments of a vertex can be computed in linear time [Hull 2003].

A crease pattern is called *locally foldable* if there is a mountain-valley assignment so that each vertex locally folds flat, i.e., a small disk around each vertex folds flat. Testing local foldability is nontrivial because each vertex has flexibility in its assignment, and these assignments must be chosen consistently: no crease should be assigned both mountain and valley by the two incident vertices.

Bern and Hayes [Bern and Hayes 1996] proved that consistency can be resolved efficiently when it is possible: local foldability can be tested in linear time.

3.3.2. Existence of folded states. Given a crease pattern, does it have a flat folded state? Bern and Hayes [Bern and Hayes 1996] have proved that this decision problem is NP-hard, and thus computationally intractable. Because local foldability is easy to test, the only difficult part is global foldability, or more precisely, computing a valid *overlap order* of the crease faces that fold to a common portion of the plane. Indeed, Bern and Hayes [Bern and Hayes 1996] prove that, given a crease pattern and a mountain-valley assignment that definitely folds flat, finding the overlap order of a flat folded state is NP-hard.

3.3.3. Equivalence to continuous folding process. In the previous section we have alluded to the difference between two models of folding: the final folded state (specified by a crease pattern, mountain-valley or angle assignment, and overlap order) and a continuous motion to bring the paper to that folded state. Basically all results, in particular those described so far, have focused on the former model: proving that a folded state exists with the desired properties. Intuitively, by appropriately flexing the paper, any folded state can be reached by a continuous motion, so the two models should be equivalent. Only recently has this been proved, initially for rectangular pieces of paper [Demaine and Mitchell 2001], and recently for general polygonal pieces of paper [Demaine et al. 2004] but overall the number of creases is uncountably infinite. An interesting open problem is whether a finite crease pattern suffices.

The only other paper of which we are aware that explicitly constructs continuous folding processes is [Demaine and Demaine 1997]. This paper proves that every convex polygon can be folded into a uniaxial base via Lang's universal molecule [Lang 1998] without gussets. Furthermore, unlike [Demaine and Mitchell 2001], no additional creases are introduced during the motion, and each crease face remains flat. This result can be used to animate the folding process.

3.3.4. Map folding: sequence of simple folds. In contrast to the complex origami folds arising from reaching folded states [Demaine and Demaine 1997; Demaine and Mitchell 2001], we can consider the less complex model of simple folds. A *simple fold* (or *book fold*) is a fold by $\pm 180^\circ$ along a single line. Examples are shown in Figure 16. This model is closely related to "pureland origami", introduced by Smith [1976; 1980; 1988; 1993].

We can ask the same foldability questions for a sequence of simple folds. Given a crease pattern, can it be folded flat via a sequence of simple folds? What if a particular mountain-valley assignment is imposed?

An interesting special case of these problems is *map folding* (see Figure 16): given a rectangle of paper with horizontal and vertical creases, each marked mountain or valley, can it be folded flat via a sequence of simple folds? Traditionally, map folding has been studied from a combinatorial point of view; see,

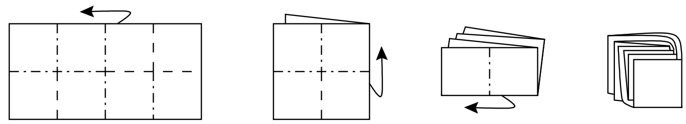


Figure 16. Folding a 2×4 map via a sequence of 3 simple folds.

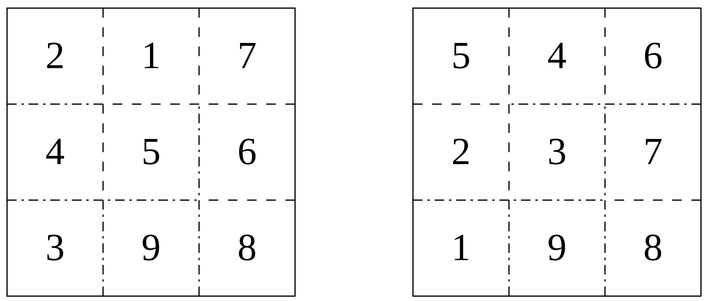


Figure 17. Two maps that cannot be folded by simple folds, but can be folded flat. (These are challenging puzzles.) The numbering indicates the overlap order of faces. (Figure 12 of [Arkin et al. 2004].)

e.g., [Lunnon 1968; Lunnon 1971]. Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, and Skiena [Arkin et al. 2004] have shown that deciding foldability of a map by simple folds can be solved in polynomial time. If the simple folds are required to fold all layers at once, the running time is at most $O(n \log n)$, and otherwise the running time is linear.

Surprisingly, slight generalizations of map folding are (weakly) NP-complete [Arkin et al. 2004]. Deciding whether a rectangle with horizontal, vertical, and diagonal ($\pm 45°$) creases can be folded via a sequence of simple folds is NP-complete. Alternatively, if the piece of paper is more general, a polygon with horizontal and vertical sides, and the creases are only horizontal and vertical, the same problem is NP-complete.

These hardness results are *weak* in the sense that they leave open the existence of a *pseudopolynomial-time* algorithm, whose running time is polynomial in the total length of creases. Another intriguing open problem, posed by Jack Edmonds, is the complexity of deciding whether a map has some flat folded state, as opposed to a folding by a sequence of simple folds. Examples of maps in which these two notions of foldability differ are shown in Figure 17.

3.4. Flattening polyhedra. When one flattens a cardboard box for recycling, generally the surface is cut open. Suppose instead of allowing cuts to a polyhedral surface in order to flatten it, we treat it as a piece of paper and fold as in origami. We run into the same dichotomy as in Section 3.3.2: do we want a continuous motion of the polyhedron, or does a description of the final folded state suffice? If we start with a convex polyhedron, and each face of the crease pattern must remain rigid during the folding, then Connelly's extension [Connelly 1980] of

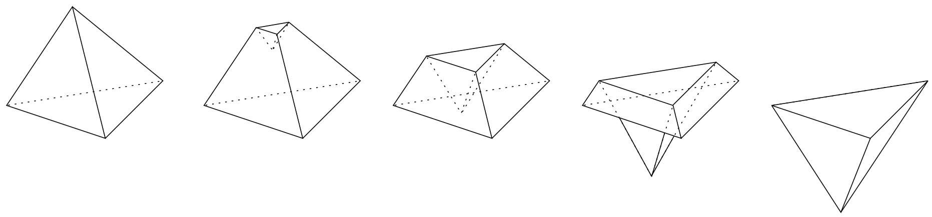


Figure 18. Inverting a tetrahedral cone by a continuous isometric motion. Based on Figure 2.5 of [Connelly 1993].

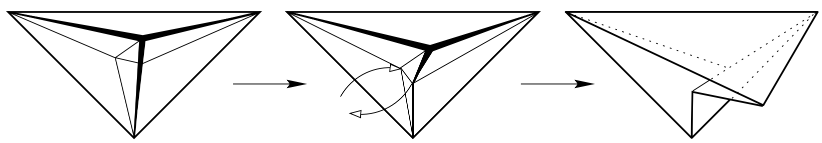


Figure 19. Flattening a tetrahedron, from left to right. Note that the faces are not flat in the middle picture.

Cauchy's rigidity theorem [1813] (see also [Cromwell 1997, pp. 219–247]) says that the polyhedron cannot fold at all. Even if we start with a nonconvex polyhedron and keep each face of the crease pattern rigid, the Bellows Theorem [Connelly et al. 1997] says that the volume of the polyhedron cannot change, so foldings are limited. However, if we allow the paper to curve (e.g., introduce new creases) during the motion, as in origami, then folding becomes surprisingly flexible. For example, a cone can be inverted [Connelly 1993]; see Figure 18.

A natural question is whether every polyhedron can be *flattened*: folded into a flat origami. Intuitively, this can be achieved by applying force to the polyhedral model, but in practice this can easily lead to tearing. There is an interesting connection of this problem to a higher-dimensional version of the fold-and-cut problem from Section 3.2.3. Given any polyhedral complex, can $\mathbb{R}^3$ be folded (through $\mathbb{R}^4$) "flat" into $\mathbb{R}^3$ so that the surface of the polyhedral complex maps to a common plane, and nothing else maps to that plane? While the applicability of four dimensions is difficult to imagine, the problem's restriction to the surface of the complex is quite practical, e.g., in packaging: *flatten* the polyhedral complex into a flat folded state, without cutting or stretching the paper.

The flattening problem remains open if we desire a continuous folding process into the flat state. If we instead focus on the existence of a flat folded state of a polyhedron, then much more is known. Demaine, Demaine, and Lubiw[7] have shown how to flatten several classes of polyhedra, including convex polyhedra and orthogonal polyhedra. See Figure 19 for an example. Recently, Demaine,

Demaine, Hayes, and Lubiw[8] have shown that all polyhedra have flat folded states. They conjecture further that every polyhedral complex can be flattened.

A natural question is whether the methods of Demaine and Mitchell [Demaine and Mitchell 2001] and [Demaine et al. 2004] described in Section 3.3.3 can be generalized to show that these folded states induce continuous folding motions as in Figure 18.

4. Polyhedra

A standard method for building a model of a polyhedron is to cut out a flat *net* or *unfolding*, fold it up, and glue the edges together so as to make precisely the desired surface. Given the polyhedron of interest, a natural problem is to find a suitable unfolding. On the other hand, given a polygonal piece of paper, we might ask whether it can be folded and its edges can be glued together so as to form a convex polyhedron. These two questions are addressed in Sections 4.1 and 4.2, respectively. Section 4.3 extends different forms of the latter question to nonconvex polyhedra. Section 4.4 connects these problems to linkage and paper folding.

4.1. Unfolding polyhedra. A classic open problem is whether (the surface of) every convex polyhedron can be cut along some of its edges and unfolded into one flat piece without overlap [Shephard 1975; O'Rourke 2000]. Such *edge-unfoldings* go back to Dürer [1525], and have important practical applications in manufacturing, such as sheet-metal bending [O'Rourke 2000; Wang 1997]. It seems folklore that the answer to this question should be YES, but the evidence for a positive answer is actually slim. Only very simple classes of polyhedra are known to be edge-unfoldable; for example, pyramids, prisms, "prismoids,"[9] and other more specialized classes [Demaine and O'Rourke $\geq$ 2005]. In contrast, experiments by Schevon [Schevon 1989; O'Rourke 2000] suggest that a random edge-unfolding of a random polytope overlaps with probability 1. Of course, such a result would not preclude, for every polytope, the existence of at least one nonoverlapping edge-unfolding, or even that a large but subconstant fraction of the polytope's edge-unfoldings do not overlap. However, the unlikeliness of finding an unfolding by chance makes the search more difficult.

An easier version of this edge-unfolding problem is the *fewest-nets* problem: prove an upper bound on the number of pieces required by a multipiece non-overlapping edge unfolding of a convex polyhedron. The obvious upper bound is the number F of faces in the polyhedron; the original problem asks whether an upper bound of 1 is possible. The first bound of cF for $c < 1$ was obtained by

[7] Manuscript, March 2001.

[8] Manuscript in preparation.

[9] The convex hull of two equiangular convex polygons, oriented so that corresponding edges are parallel.

Michael Spriggs,[10] who established $c = 2/3$. The smallest value of c obtained so far[11] is $1/2$. Proving an upper bound that is sublinear in F would be a significant advancement.

We can also examine to what extent edge unfoldings can be generalized to nonconvex polyhedra. In particular, define a polyhedron to be *topologically convex* if its 1-skeleton (graph) is the 1-skeleton of a convex polyhedron. Does every topologically convex polyhedron have an edge-unfolding? In particular, every polyhedron composed of convex faces and homeomorphic to a sphere is topologically convex; can they all be edge-unfolded? This problem was posed by Schevon [Schevon 1987].

Bern, Demaine, Eppstein, Kuo, Mantler, and Snoeyink [Bern et al. 2003] have shown that the answer to both of these questions is NO: there is a polyhedron composed of triangles and homeomorphic to a sphere that has no (one-piece, nonoverlapping) edge-unfolding. The polyhedron is shown in Figure 20. It consists of four "hats" glued to the faces of a regular tetrahedron, such that only the peaks of the hats have positive curvature, that is, have less than 360° of incident material. This property limits the unfoldings significantly, because (1) any set of cuts must avoid cycles in order to create a one-piece unfolding, and (2) a leaf in a forest of cuts can only lie at a positive-curvature vertex of the polyhedron: a leaf at a negative-curvature vertex (more than 360° of incident material) would cause local overlap.

The complexity of deciding whether a given topologically convex polyhedron can be edge-unfolded remains open.

Another intriguing open problem in this area is whether every polyhedron homeomorphic to a sphere has *some* one-piece unfolding, not necessarily using cuts along edges. It is known that every convex polyhedron has an unfolding in this model, allowing cuts across the faces of the polytope. Specifically, the *star unfolding* [Agarwal et al. 1997; Aronov and O'Rourke 1992] cuts the shortest paths from a common source point to each vertex of the polytope, and the *source unfolding* [Mitchell et al. 1987] cuts the points with more than one shortest path to a common source. Both of these unfoldings avoid overlap, the star unfolding being the more difficult case to establish [Aronov and O'Rourke 1992]. The source unfolding (but not the star unfolding) also generalizes to unfold convex polyhedra in higher dimensions [Miller and Pak 2003].

But many nonconvex polyhedra also have such unfoldings. For example, Figure 20 illustrates one for the polyhedron described above. Biedl, Demaine, Demaine, Lubiw, Overmars, O'Rourke, Robbins, and Whitesides [Biedl et al. 1998] have shown how to unfold many orthogonal polyhedra, even with holes and knotted topology, although it remains open whether all orthogonal polyhedra

[10] Personal communication, August 2003.

[11] Personal communication from Vida Dujmović, Pat Morin, and David Wood, February 2004.

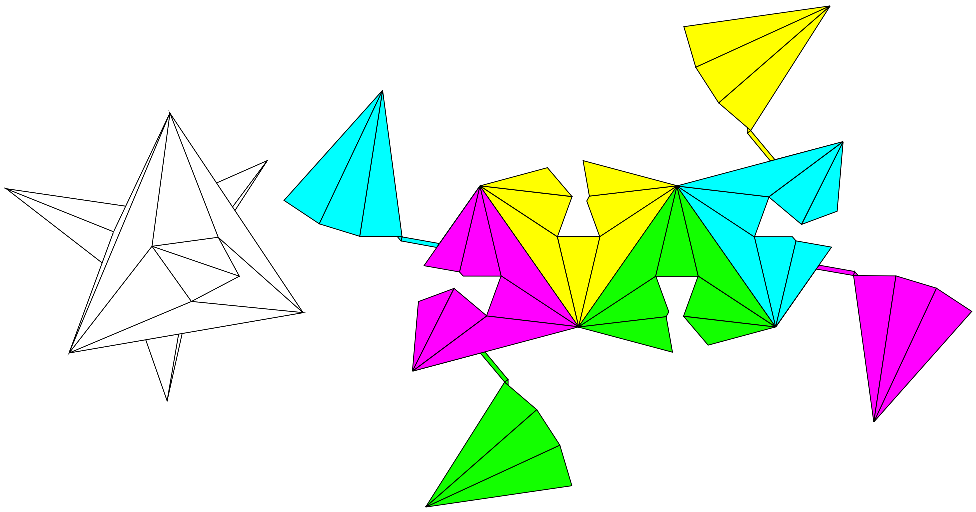


Figure 20. (Left) Simplicial polyhedron with no edge-unfolding. (Right) An unfolding when cuts are allowed across faces.

can be unfolded. The only known scenario that prevents unfolding altogether [Bern et al. 2003] is a polyhedron with a single vertex of negative curvature (see Figure 21), but this requires the polyhedron to have boundary (edges incident to only one face).

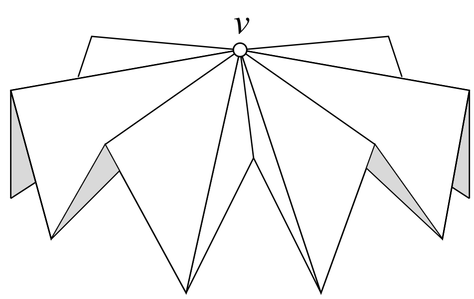


Figure 21. A polyhedron with boundary that has no one-piece unfolding even when cuts are allowed across faces. Vertex v has negative curvature, that is, more than 360° of incident material. (Based on Figure 9 of [Bern et al. 2003].)

A recent approach to unfolding both convex and nonconvex polyhedra in any dimension is the notion of "vertex-unfolding" [Demaine et al. 2003a]; see Figure 22. Specifically, a *vertex-unfolding* may cut only along edges of the polyhedron (like an edge-unfolding) but permits the facets to remain connected only at vertices (instead of along edges as in edge-unfolding). Thus, a vertex-unfolding is connected, but its interior may be disconnected, "pinching" at a vertex. This notion also generalizes to polyhedra in any dimension. Demaine, Eppstein, Erickson, Hart, and O'Rourke [Demaine et al. 2003a] proved that every simplicial

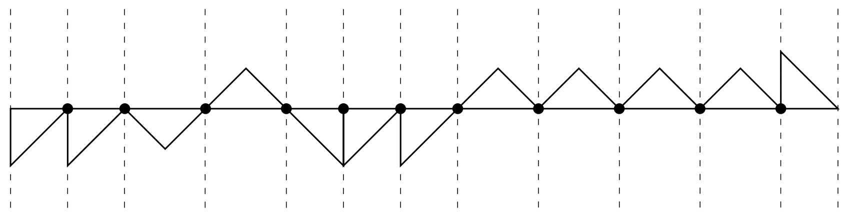


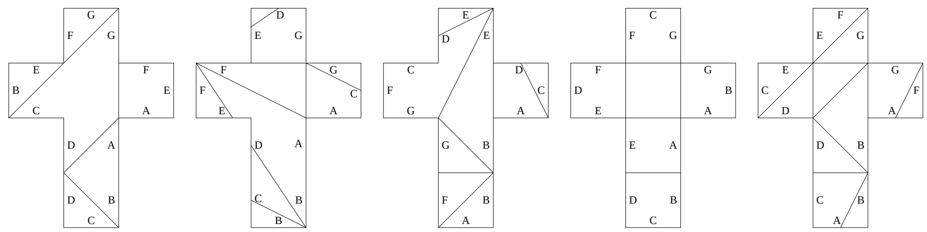
Figure 22. Vertex-unfolding of a triangulated cube with hinge points aligned. (Based on Figure 2 of [Demaine et al. 2003a].)

manifold in any dimension has a nonoverlapping vertex-unfolding. In particular, this result covers triangulated polyhedra in 3D, possibly with boundary, but it remains open to what extent vertex-unfoldings exist for polyhedra with nontriangular faces. For example, does every convex polyhedron in 3D have a vertex-unfolding?

4.2. Folding polygons into convex polyhedra. In addition to unfolding polyhedra into simple planar polygons, we can consider the reverse problem of folding polygons into polyhedra. More precisely, when can a polygon have its boundary glued together, with each portion gluing to portions of matching length, and the resulting topological object be folded into a *convex* polyhedron? (There is almost too much flexibility with nonconvex polyhedra for this problem, but see Section 4.3 for related problems of interest in this context.) A particular kind of gluing is an *edge-to-edge* gluing, in which each entire edge of the polygon is glued to precisely one other edge of the polygon. The existence of such a gluing requires a perfect pairing of edges with matching lengths.

4.2.1. Edge-to-edge gluings. Introducing this area, Lubiw and O'Rourke [Lubiw and O'Rourke 1996] showed how to test in polynomial time whether a polygon has an edge-to-edge gluing that can be folded into a convex polyhedron, and how to list all such edge-to-edge gluings in exponential time. A key tool in their work is a theorem of A. D. Aleksandrov [Alexandrov 1950]. The theorem states that a topological gluing can be realized geometrically by a convex polyhedron precisely if the gluing is topologically a sphere, and at most 360° of material is glued to any one point — that is, every point should have nonnegative curvature.

Based on this tool, Lubiw and O'Rourke use dynamic programming to develop their algorithms. There are $\Omega(n^2)$ subproblems corresponding to gluing subchains of the polygon, assuming that the two ends of the subchain have already been glued together. These subproblems are additionally parameterized by how much angle of material remains at the point to which the two ends of the chain glue in order to maintain positive curvature. It is this parameterization that forces enumeration of all gluings to take exponential time. But for the decision problem of the existence of any gluing, the remaining angle at the ends only needs to be bounded, and only polynomially many subproblems need to be considered, resulting in an $O(n^3)$ algorithm.



(a) Doubly covered quadrangle (b) Tetrahedron (c) Pentahedron (d) Cube (e) Octahedron

Figure 23. The five edge-to-edge gluings of the Latin cross [Lubiw and O'Rourke 1996].

A particularly surprising discovery from this work [Lubiw and O'Rourke 1996] is that the well-known "Latin cross" unfolding of the cube can be folded into exactly five convex polyhedra by edge-to-edge gluing: a doubly covered (flat) quadrangle, an (irregular) tetrahedron, a pentahedron, the cube, and an (irregular) octahedron. See Figure 23 for crease patterns and gluing instructions. These foldings are the subject of a video [Demaine et al. 1999a].

4.2.2. Non-edge-to-edge gluings. More recently, Demaine, Demaine, Lubiw, and O'Rourke [Demaine et al. 2000b; Demaine et al. 2002a] have extended this work in various directions, in particular to non-edge-to-edge gluings.

In contrast to edge-to-edge gluings, any convex polygon can be glued into a continuum of distinct convex polyhedra, making it more difficult for an algorithm to enumerate all gluings of a given polygon. Fortunately, there are only finitely many *combinatorially distinct* gluings of any polygon. For convex polygons, there are only polynomially many combinatorially distinct gluings, and they can be enumerated for a given convex polygon in polynomial time. This result generalizes to any polygon in which there is a constant bound on the sharpest angle. For general nonconvex polygons, there can be exponentially many ($2^{\Theta(n)}$) combinatorially distinct gluings, but only that many. Again this corresponds to an algorithm running in $2^{O(n)}$ time. Because of the exponential worst-case lower bound on the number of combinatorially distinct gluings, we are justified both here and in the enumeration algorithm of [Lubiw and O'Rourke 1996] to spend exponential time. It remains open whether there is an output-sensitive algorithm, whose running time is polynomial in the number of resulting gluings, or in the number of gluings desired by the user. For non-edge-to-edge gluings, it even remains open whether there is a polynomial-time algorithm to decide whether a gluing exists.

The algorithms for enumerating all non-edge-to-edge gluings have been implemented independently by Anna Lubiw (July 2000) and by Koichi Hirata [Hirata 2000] (June 2000). These programs have been applied to the example of the

Latin cross. There are surprisingly many more, but still finitely many, non-edge-to-edge gluings: a total of 85 distinct gluings (43 modulo symmetry). A manual reconstruction of the polyhedra resulting from these gluings reveals 23 distinct shapes: the cube, seven different tetrahedra, three different pentahedra, four different hexahedra, six different octahedra, and two flat quadrangles [Demaine et al. 2000a; Demaine and O'Rourke $\geq$ 2005].

Alexander, Dyson, and O'Rourke [Alexander et al. 2002] performed a case study of all the gluings of the square, reconstructing all the incongruent polyhedra that result. This situation is complicated by the existence of entire continua of gluings and polyhedra. Nonetheless, the entire configuration space of the polyhedra can be characterized, as shown in Figure 24. Although in this case it is connected, there are convex polygons of n vertices whose space of all gluings into polyhedra has $\Omega(n^2)$ connected components [Demaine and O'Rourke $\geq$ 2005]. Although it is almost certain that all of these gluings lead to distinct polyhedra, it seems difficult to establish this property without a method for reconstructing the three-dimensional structure, the topic of the next section.

4.2.3. Constructing polyhedra. Another intriguing open problem in this area [Demaine et al. 2002a] remains relatively unexplored: Aleksandrov's theorem implies that any valid gluing (homeomorphic to a sphere and having nonnegative curvature everywhere) can be folded into a unique convex polyhedron, but how efficiently can this polyhedron be constructed? The key difficulty here is to determine the dihedral angles of the polyhedron, that is, by how much each crease is folded. Finding a (superset of) the creases is straightforward:[12] every edge of the polyhedron is a shortest path between two positive-curvature vertices, so compute all-pairs shortest paths in the polyhedral metric defined by the gluing [Chen and Han 1996; Kaneva and O'Rourke 2000; Kapoor 1999].

Sabitov [Sabitov 1996] recently presented a finite algorithm for this reconstruction problem, reducing the problem to finding roots of a collection of polynomials of exponentially high degree. The algorithm is based on another his results [Sabitov 1998; Sabitov 1996] that expresses the volume of a triangulated polyhedron as the root of a polynomial in the edge lengths, independent of how the polyhedron is geometrically embedded in 3-space. (This result was also used to settle the famous Bellows Conjecture [Connelly et al. 1997].) Sabitov's algorithm was recently extended and its bounds improved by Fedorchuk and Pak [Fedorchuk and Pak 2004] to express the internal vertex-to-vertex diagonal lengths as roots of a polynomial of degree 4^m for a polyhedron of m edges. The polyhedron can easily be reconstructed from these diagonal lengths.

[12]Personal communication with Boris Aronov, June 1998. The essence of the argument is also present in [Alexandrov 1941].

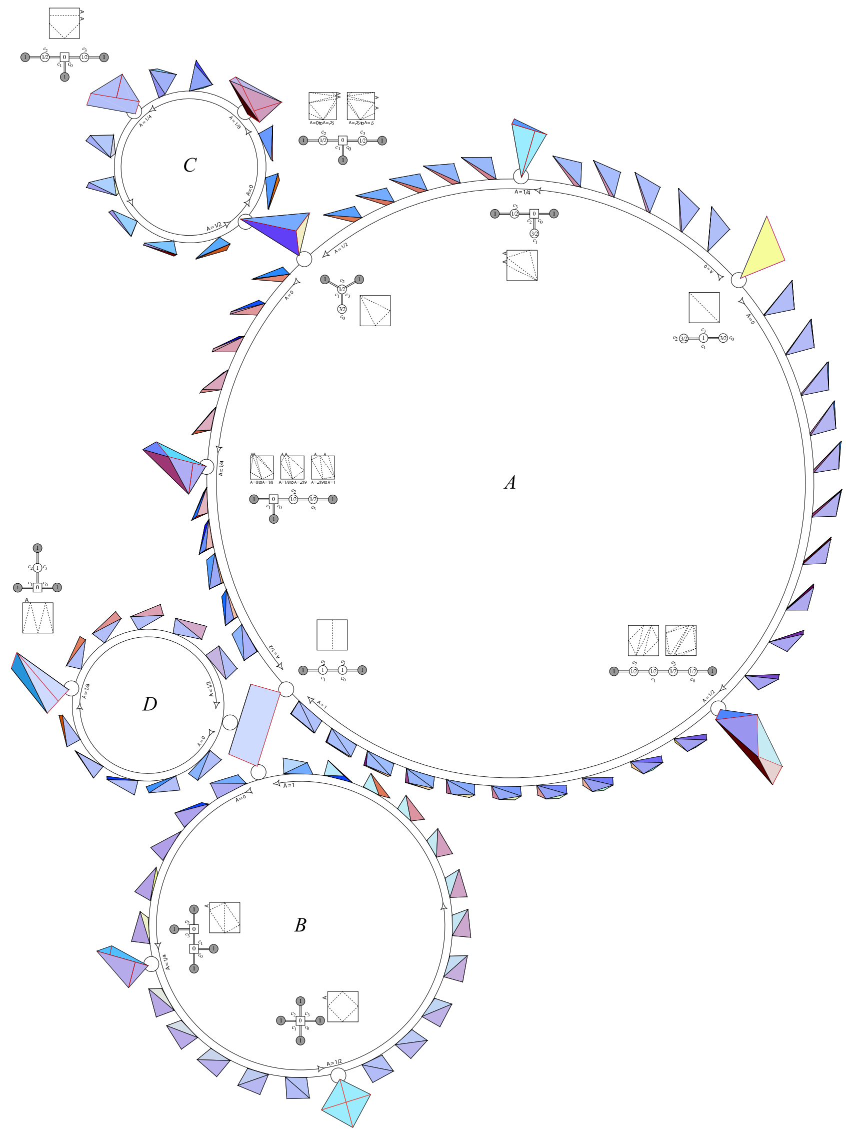


Figure 24. The continua of polyhedra foldable from a square. (Figure 2 of [Alexander et al. 2002].)

4.3. Folding nets into nonconvex polyhedra. Define a *net* to be a connected edge-to-edge gluing of polygons to form a tree structure, the edges shared by polygons denoting creases. An open problem mentioned in Section 4.2.3 is deciding whether a given net can be folded into a convex polyhedron using only the given creases. More generally, we can ask whether a given net folds into a nonconvex polyhedron. Now Aleksandrov's theorem and Cauchy's rigidity theorem do not apply, so for a given gluing we are no longer easily guaranteed existence or uniqueness.

Given the dihedral angles associated with creases in the net, it is easy to decide foldability in polynomial time [Biedl et al. 1999b; Sun 1999]: we only need to check that edges match up and no two faces cross. Without the dihedral angles, when does a given net fold into any polyhedron? Biedl, Lubiw, and Sun [Biedl et al. 1999b; Sun 1999] proved a closely related problem to be weakly NP-complete: does a given orthogonal net (each face is an orthogonal polygon) fold into an orthogonal polyhedron? The difference with this problem is that it constrains each dihedral angle to be $\pm 90°$. It remained open whether this constraint actually restricted what polyhedra could be folded, even for this particular reduction. More generally, is there a nonorthogonal polyhedron (i.e., one that has at least one dihedral angle not a multiple of $90°$) having orthogonal faces and that is homeomorphic to a sphere? The answer to this question (posed in [Biedl et al. 1999b]) turns out to be NO, as proved by Donoso and O'Rourke [Donoso and O'Rourke 2002]. The answer is YES, however, if the polyhedron is allowed to have genus 6 or larger; on the other hand, the answer remains NO for genus up to 2 [Biedl et al. 2002a]. It remains open whether such nonorthogonal polyhedra with orthogonal faces exist with genus 3, 4, or 5.

4.4. Continuously folding polyhedra. The results described so far for polyhedron folding and unfolding are essentially about folded or unfolded states, and not about the continuous process of reaching such states. In the context of paper folding, we saw in Section 3.3.3 that these two notions are largely equivalent. In the context of linkages, we saw that the two notions can differ, particularly in 3D. Relatively little has been studied in the context of polyhedron folding.

One special case that has been explored is orthogonal polyhedra. Specifically, Biedl, Lubiw, and Sun [Biedl et al. 1999b; Sun 1999] have proved that there is an edge-unfolding of an orthogonal polyhedron (which is an orthogonal net) that cannot be folded into the orthogonal polyhedron by a continuous motion that keeps the faces rigid and avoids self-intersection. The basis for their example is the locked polygonal arc in 3D (Figure 10), converted into an orthogonal locked polygonal arc in 3D, and then "thickened" into an orthogonal tube. A single chain of faces in the unfolding is what prevents the continuous foldability.

One would expect, analogous to the results described in Section 3.3.3 [Demaine and Mitchell 2001], that collections of polygons hinged together into a tree can be folded into all possible configurations if we allow additional creases during the motion. However, this extension (equivalent to a polygonal piece of paper) remains open. A particularly interesting version of this question, posed in [Biedl et al. 1999b], is whether a finite number of additional creases suffice.

An interesting collection of open questions arise when we consider polyhedron foldings with creases only at polyhedron edges. For example, do all convex

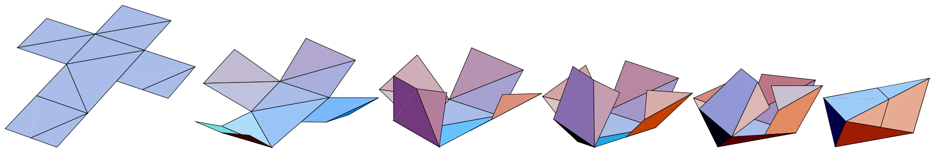


Figure 25. Folding the Latin cross into an octahedron, according to the crease pattern in Figure 23(e), by affinely interpolating all dihedral angles. (Figure 2 of [Demaine et al. 1999a].)

polyhedra have *continuous* edge-unfoldings? (This question may be easier to answer negatively than the classic edge-unfolding problem.) Figure 25 shows a simple example of such a folding, taken from a longer video [Demaine et al. 1999a], based on the simple rule of affinely interpolating each dihedral angle from start to finish. Connelly, as reported in [Miller and Pak 2003], asked whether the source unfolding can be *continuously bloomed*, i.e., unfolded so that all dihedral angles increase monotonically. Although an affirmative answer to this question has just been obtained,[13] it remains open whether every general unfolding can be executed continuously.

5. Conclusion and Higher Dimensions

Our goal has been to survey the results in the newly developing area of folding and unfolding, which offers many beautiful mathematical and computational problems. Much progress has been made recently in this area, but many important problems remain open. For example, most aspects of unfolding polyhedra remain unsolved, and we highlight two key problems in this context: can all convex polyhedra be edge-unfolded, and can all polyhedra be generally unfolded? Another exciting new direction is the developing connection between linkage folding and protein folding.

Finally, higher dimensions are just beginning to be explored. We mentioned in Section 2.4 that 1D (one-dimensional) linkages in higher dimensions have been explored. But 2D "linkages" in 4D — and higher-dimensional analogs — have received less attention. One model is 2D polygons hinged together at their edges to form a chain. Such a hinged chain has fewer degrees of freedom than a 1D linkage in 3D; for example, a hinged chain can be forced to fold like a planar linkage by extruding the linkage orthogonal to the plane. See Figure 26. Biedl, Lubiw, and Sun [Biedl et al. 1999b; Sun 1999] showed that even hinged chains of rectangles do not have connected configuration spaces, by considering an orthogonal version of Figure 10. It would be interesting to explore these chains of rectangles in 4D.

Turning to the origami context, one natural open problem is a generalization of the fold-and-cut problem: given a polyhedral complex drawn on a d-

[13] Personal communication with Stefan Langerman et al., February 2004.

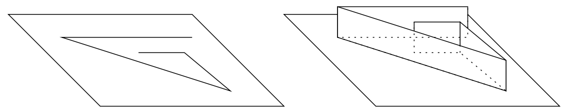


Figure 26. Extruding a linkage into an equivalent collection of polygons (rectangles) hinged together at their edges.

dimensional piece of paper, is it always possible to fold the paper flat (into d-space) while mapping the ($d-1$)-dimensional facets of the complex to a common ($d-1$)-dimensional hyperplane? What if our goal is to map all k-dimensional faces to a common k-dimensional flat, for all $k = 0, 1, \ldots, d$?

Salvador Dali's famous painting ("Christ") of Christ on an unfolded 4D hypercube suggests the possibilities for unfolding higher-dimensional polyhedra. All of the unsolved problems related to unfolding 3D to 2D are equally unsolved in their higher-dimensional analogs. We mentioned in Section 4.1 a rare exception: the vertex-unfolding algorithm generalizes to unfold simplicial manifolds without overlap in arbitrary dimensions. Miller and Pak [2003] have established that the source unfolding generalizes to higher dimensions to yield nonoverlapping unfoldings, but that the most natural generalization of the star unfolding does not even suffice to unfold, let alone without overlap. Nevertheless, with one general unfolding available, the natural analog of the edge-unfolding question remains: Does every convex d-polytope have a *ridge unfolding*, a cutting of ($d-2$)-dimensional faces that unfolds the polytope into $\mathbb{R}^{d-1}$ without overlap?

Acknowledgements

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References

[Agarwal et al. 1997] P. K. Agarwal, B. Aronov, J. O'Rourke, and C. A. Schevon, "Star unfolding of a polytope with applications", *SIAM J. Comput.* **26**:6 (1997), 1689–1713.

[Ahn et al. 2000] H.-K. Ahn, P. Bose, J. Czyzowicz, N. Hanusse, E. Kranakis, and P. Morin, "Flipping your lid", *Geombinatorics* **10**:2 (2000), 57–63.

[Aichholzer et al. 2001] O. Aichholzer, E. D. Demaine, J. Erickson, F. Hurtado, M. Overmars, M. A. Soss, and G. T. Toussaint, "Reconfiguring convex polygons", *Comput. Geom.* **20**:1–2 (2001), 85–95.

[Aichholzer et al. 2002] O. Aichholzer, C. Cortés, E. D. Demaine, V. Dujmović, J. Erickson, H. Meijer, M. Overmars, B. Palop, S. Ramaswami, and G. T. Toussaint, "Flipturning polygons", *Discrete Comput. Geom.* **28**:2 (2002), 231–253.

[Aichholzer et al. 2003] O. Aichholzer, D. Bremner, E. D. Demaine, H. Meijer, V. Sacristán, and M. Soss, "Long proteins with unique optimal foldings in the H-P model", *Comput. Geom.* **25**:1–2 (2003), 139–159.

[Alexander et al. 2002] R. Alexander, H. Dyson, and J. O'Rourke, "The convex polyhedra foldable from a square", pp. 31–32 in *Proceedings of the Japan Conference on Discrete and Computational Geometry* (Tokyo, 2002), 2002.

[Alexandrov 1941] A. D. Alexandrov, "Existence of a given polyhedron and of a convex surface with a given metric", *Doklady Acad. Sci. URSS* (*N.S.*) **30** (1941), 103–106. In Russian; translation as pp. 169–173 in *Selected Works*, Part I, edited by Yu. G. Reshetnyak and S. S. Kutateladze, Gordon and Breach, Amsterdam, 1996.

[Alexandrov 1950] A. D. Aleksandrov, *Vypuklye mnogogranniki*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950.

[Aloupis et al. 2002a] G. Aloupis, E. D. Demaine, V. Dujmović, J. Erickson, S. Langerman, H. Meijer, I. Streinu, J. O'Rourke, M. Overmars, M. Soss, and G. T. Toussaint, "Flat-state connectivity of linkages under dihedral motions", pp. 369–380 in *Proceedings of the 13th Annual International Symposium on Algorithms and Computation* (Vancouver, 2002), edited by P. Bose and P. Morin, Lecture Notes in Computer Science **2518**, 2002.

[Aloupis et al. 2002b] G. Aloupis, E. D. Demaine, H. Meijer, J. O'Rourke, I. Streinu, and G. Toussaint, "Flat-state connectedness of fixed-angle chains: Special acute chains", pp. 27–30 in *Proceedings of the 14th Canadian Conference on Computational Geometry* (Lethbridge, AB), 2002.

[Alt et al. 2004] H. Alt, C. Knauer, G. Rote, and S. Whitesides, "On the complexity of the linkage reconfiguration problem", pp. 1–13 in *Towards a theory of geometric graphs*, Contemp. Math. **342**, Amer. Math. Soc., Providence, RI, 2004.

[Arkin et al. 2003] E. M. Arkin, S. P. Fekete, and J. S. B. Mitchell, "An algorithmic study of manufacturing paperclips and other folded structures", *Comput. Geom.* **25**:1-2 (2003), 117–138.

[Arkin et al. 2004] E. M. Arkin, M. A. Bender, E. D. Demaine, M. L. Demaine, J. S. B. Mitchell, S. Sethia, and S. S. Skiena, "When can you fold a map?", *Comput. Geom.* **29**:1 (2004), 23–46.

[Aronov and O'Rourke 1992] B. Aronov and J. O'Rourke, "Nonoverlap of the star unfolding", *Discrete Comput. Geom.* **8**:3 (1992), 219–250.

[Aronov et al. 2002] B. Aronov, J. E. Goodman, and R. Pollack, "Convexification of planar polygons in $\mathbb{R}^3$", 2002.

[Berger and Leighton 1998] B. Berger and T. Leighton, "Protein folding in the hydrophobic-hydrophilic (HP) model is NP-complete", *Journal of Computational Biology* **5**:1 (1998), 27–40.

[Bern and Hayes 1996] M. Bern and B. Hayes, "The complexity of flat origami", pp. 175–183 in *Proceedings of the 7th Annual ACM–SIAM Symposium on Discrete Algorithms* (Atlanta, 1996), ACM, New York, 1996.

[Bern et al. 2002] M. Bern, E. Demaine, D. Eppstein, and B. Hayes, "A disk-packing algorithm for an origami magic trick", pp. 17–28 in *Origami*^{3}*: Proceedings of the 3rd International Meeting of Origami Science, Math, and Education* (Asilomar, CA, 2001), edited by T. Hull, A K Peters, Natick, MA, 2002.

[Bern et al. 2003] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, and J. Snoeyink, "Ununfoldable polyhedra with convex faces", *Comput. Geom.* **24**:2 (2003), 51–62.

[Biedl 2005] T. Biedl, "Polygons needing many flipturns", *Discrete Comput. Geom* (2005). To appear.

[Biedl et al. 1998] T. Biedl, E. Demaine, M. Demaine, A. Lubiw, M. Overmars, J. O'Rourke, S. Robbins, and S. Whitesides, "Unfolding some classes of orthogonal polyhedra", pp. 70–71 in *Proceedings of the 10th Canadian Conference on Computational Geometry* (Montreal), 1998. Available at http://cgm.cs.mcgill.ca/cccg98/proceedings/.

[Biedl et al. 1999a] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, M. Overmars, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides, "Locked and unlocked polygonal chains in 3D", Technical Report 060, Smith College, Northampton (MA), 1999. Available at arXiv:cs.CG/9910009.

[Biedl et al. 1999b] T. Biedl, A. Lubiw, and J. Sun, "When can a net fold to a polyhedron?", pp. 70–71 in *Proceedings of the 11th Canadian Conference on Computational Geometry* (Vancouver, 1999), 1999. Available at http://www.cs.ubc.ca/conferences/CCCG/elec_proc/fp16.ps.gz.

[Biedl et al. 2000] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides, "On reconfiguring tree linkages: Trees can lock", Technical Report SOCS-00.7, McGill University, Montreal, 2000. Available at arXiv:cs.CG/9910024.

[Biedl et al. 2001] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, M. Overmars, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides, "Locked and unlocked polygonal chains in three dimensions", *Discrete Comput. Geom.* **26**:3 (2001), 269–281. Full version: [Biedl et al. 1999a].

[Biedl et al. 2002a] T. Biedl, T. M. Chan, E. D. Demaine, M. L. Demaine, P. Nijjar, R. Uehara, and M. Wang, "Tighter bounds on the genus of nonorthogonal polyhedra built from rectangles", pp. 105–108 in *Proceedings of the 14th Canadian Conference on Computational Geometry* (Lethbridge, AB), 2002.

[Biedl et al. 2002b] T. Biedl, E. Demaine, M. Demaine, S. Lazard, A. Lubiw, J. O'Rourke, S. Robbins, I. Streinu, G. Toussaint, and S. Whitesides, "A note on reconfiguring tree linkages: trees can lock", *Discrete Appl. Math.* **117**:1-3 (2002), 293–297. Full version: [Biedl et al. 2000].

[Bose et al. 1999] P. Bose, F. Gomez, P. Ramos, and G. Toussaint, "Drawing nice projections of objects in space", *Journal of Visual Communication and Image Representation* **10**:2 (1999), 155–172.

[Calvo et al. 2001] J. A. Calvo, D. Krizanc, P. Morin, M. Soss, and G. Toussaint, "Convexifying polygons with simple projections", *Inform. Process. Lett.* **80**:2 (2001), 81–86.

[Cantarella and Johnston 1998] J. Cantarella and H. Johnston, "Nontrivial embeddings of polygonal intervals and unknots in 3-space", *J. Knot Theory Ramifications* **7**:8 (1998), 1027–1039.

[Cantarella et al. 2004] J. H. Cantarella, E. D. Demaine, H. N. Iben, and J. F. O'Brien, "An energy-driven approach to linkage unfolding", pp. 134–143 in *Proceedings of the*

20th Annual ACM Symposium on Computational Geometry (Brooklyn, NY, 2004), ACM, New York, 2004.

[Cauchy 1813] A. L. Cauchy, "Deuxième mémoire sur les polygones et polyèdres", *Journal de l'École Polytechnique* **9** (1813), 87–98.

[Chan and Dill 1990] H. S. Chan and K. A. Dill, "Origins of structure in globular proteins", *Proceedings of the National Academy of Sciences USA* **87** (1990), 6388–6392.

[Chan and Dill 1991] H. S. Chan and K. A. Dill, "Polymer principles in protein structure and stability", *Annual Reviews of Biophysics and Biophysical Chemistry* **20** (1991), 447–490.

[Chan and Dill 1993] H. S. Chan and K. A. Dill, "The protein folding problem", *Physics Today* **46**:2 (February 1993), 24–32.

[Chen and Han 1996] J. Chen and Y. Han, "Shortest paths on a polyhedron, I: Computing shortest paths", *Internat. J. Comput. Geom. Appl.* **6**:2 (1996), 127–144.

[Cocan and O'Rourke 2001] R. Cocan and J. O'Rourke, "Polygonal chains cannot lock in 4D", *Comput. Geom.* **20**:3 (2001), 105–129.

[Connelly 1980] R. Connelly, "The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces", *Adv. in Math.* **37**:3 (1980), 272–299.

[Connelly 1993] R. Connelly, "Rigidity", pp. 223–271 in *Handbook of convex geometry*, vol. A, North-Holland, Amsterdam, 1993.

[Connelly and Demaine 2004] R. Connelly and E. D. Demaine, "Geometry and topology of polygonal linkages", pp. 197–218 in *Handbook of discrete and computational geometry*, 2nd ed., edited by J. E. Goodman and J. O'Rourke, CRC, Boca Raton, FL, 2004.

[Connelly et al. 1997] R. Connelly, I. Sabitov, and A. Walz, "The bellows conjecture", *Beiträge Algebra Geom.* **38**:1 (1997), 1–10.

[Connelly et al. 2002] R. Connelly, E. D. Demaine, and G. Rote, "Infinitesimally locked self-touching linkages with applications to locked trees", pp. 287–311 in *Physical knots: knotting, linking, and folding geometric objects in* $\mathbb{R}^3$ (Las Vegas, 2001), edited by J. Calvo et al., Contemp. Math. **304**, Amer. Math. Soc., Providence, RI, 2002.

[Connelly et al. 2003] R. Connelly, E. D. Demaine, and G. Rote, "Straightening polygonal arcs and convexifying polygonal cycles", *Discrete Comput. Geom.* **30**:2 (2003), 205–239.

[Crescenzi et al. 1998] P. Crescenzi, D. Goldman, C. Papadimitriou, A. Piccolboni, and M. Yannakakis, "On the complexity of protein folding", *Journal of Computational Biology* **5**:3 (1998), 423–465.

[Cromwell 1997] P. R. Cromwell, *Polyhedra*, Cambridge University Press, Cambridge, 1997.

[Demaine 2001] E. D. Demaine, "Folding and unfolding linkages, paper and polyhedra", pp. 113–124 in *Discrete and computational geometry* (JCDCG, Tokyo, 2000: Revised papers), edited by M. U. Jin Akiyama, Mikio Kano, Lecture Notes in Comput. Sci. **2098**, Springer, Berlin, 2001.

[Demaine and Demaine 1997] E. D. Demaine and M. L. Demaine, "Computing extreme origami bases", Technical Report CS-97-22, University of Waterloo, Waterloo, ON, 1997.

[Demaine and Demaine 2002] E. D. Demaine and M. L. Demaine, "Recent results in computational origami", pp. 3–16 in *Origami^3: Proceedings of the 3rd International Meeting of Origami Science, Math, and Education* (Asilomar, CA, 2001), edited by T. Hull, A K Peters, Natick, MA, 2002.

[Demaine and Mitchell 2001] E. D. Demaine and J. S. B. Mitchell, "Reaching folded states of a rectangular piece of paper", pp. 73–75 in *Proceedings of the 13th Canadian Conference on Computational Geometry* (Waterloo, ON), 2001. Available at http://compgeo.math.uwaterloo.ca/~cccg01/proceedings/.

[Demaine and O'Rourke 2001] E. D. Demaine and J. O'Rourke, "Open problems from CCCG 2000", in *Proceedings of the 13th Canadian Conference on Computational Geometry* (Waterloo, ON), 2001. Available at http://compgeo.math.uwaterloo.ca/~cccg01/proceedings.

[Demaine and O'Rourke $\geq$ 2005] E. D. Demaine and J. O'Rourke, *Geometric folding algorithms: linkages, origami, and polyhedra*, Cambridge University Press, New York. To appear.

[Demaine et al. 1999a] E. Demaine, M. Demaine, A. Lubiw, J. O'Rourke, and I. Pashchenko, "Metamorphosis of the cube", pp. 409–410 in *8th Annual Video Review of Computational Geometry, Proceedings of the 15th Annual ACM Symposium on Computational Geometry* (Miami Beach, 1999), 1999. Available at http://theory.csail.mit.edu/~edemaine/metamorphosis/.

[Demaine et al. 1999b] E. D. Demaine, M. L. Demaine, and A. Lubiw, "Folding and one straight cut suffice", pp. 891–892 in *Proceedings of the 10th Annual ACM-SIAM Symposium on Discrete Algorithms* (Baltimore, MD), 1999.

[Demaine et al. 1999c] E. D. Demaine, M. L. Demaine, and J. S. B. Mitchell, "Folding flat silhouettes and wrapping polyhedral packages: new results in computational origami", pp. 105–114 in *Proceedings of the 15th Annual Symposium on Computational Geometry* (Miami Beach, 1999), ACM, New York, 1999.

[Demaine et al. 2000a] E. Demaine, M. Demaine, A. Lubiw, and J. O'Rourke, "The 85 foldings of the Latin cross", 2000. Available at http://theory.csail.mit.edu/~edemaine/aleksandrov/cross/.

[Demaine et al. 2000b] E. Demaine, M. Demaine, A. Lubiw, and J. O'Rourke, "Examples, counterexamples, and enumeration results for foldings and unfoldings between polygons and polytopes", Technical Report 069, Smith College, Northampton, MA, 2000. Available at arXiv:cs.CG/0007019.

[Demaine et al. 2000c] E. D. Demaine, M. L. Demaine, and A. Lubiw, "Folding and cutting paper", pp. 104–117 in *Discrete and computational geometry* (JCDCG, Tokyo, 1998: Revised papers), edited by M. K. J. Akiyama and M. Urabe, Lecture Notes in Comput. Sci. **1763**, Springer, Berlin, 2000.

[Demaine et al. 2000d] E. D. Demaine, M. L. Demaine, and J. S. B. Mitchell, "Folding flat silhouettes and wrapping polyhedral packages: new results in computational origami", *Comput. Geom.* **16**:1 (2000), 3–21.

[Demaine et al. 2002a] E. D. Demaine, M. L. Demaine, A. Lubiw, and J. O'Rourke, "Enumerating foldings and unfoldings between polygons and polytopes", *Graphs Combin.* **18**:1 (2002), 93–104. Available at arXiv:cs.CG/0107024.

[Demaine et al. 2002b] E. D. Demaine, S. Langerman, J. O'Rourke, and J. Snoeyink, "Interlocked open linkages with few joints", pp. 189–198 in *Proceedings of the 18th Annual ACM Symposium on Computational Geometry* (Barcelona), 2002.

[Demaine et al. 2003a] E. D. Demaine, D. Eppstein, J. Erickson, G. W. Hart, and J. O'Rourke, "Vertex-unfoldings of simplicial manifolds", pp. 215–228 in *Discrete geometry: in honor of W. Kuperberg's 60th birthday*, edited by A. Bezdek, Pure Appl. Math. **253**, Dekker, New York, 2003. Available at http://www.arXiv.org/abs/cs.CG/0110054.

[Demaine et al. 2003b] E. D. Demaine, S. Langerman, and J. O'Rourke, "Geometric restrictions on producible polygonal protein chains", pp. 395–404 in *Algorithms and computation* (14th Annual Symposium, Kyoto, 2003), edited by H. O. Toshihide Ibaraki, Naoki Katoh, Lecture Notes in Comput. Sci. **2906**, Springer, Berlin, 2003. To appear in *Algorithmica*.

[Demaine et al. 2003c] E. D. Demaine, S. Langerman, J. O'Rourke, and J. Snoeyink, "Interlocked open and closed linkages with few joints", *Comput. Geom.* **26**:1 (2003), 37–45.

[Demaine et al. 2004] E. D. Demaine, S. L. Devadoss, J. S. B. Mitchell, and J. O'Rourke, "Continuous foldability of polygonal paper", pp. 64–67 in *Proceedings of the 16th Canadian Conference on Computational Geometry* (Montreal), 2004.

[Dill 1990] K. A. Dill, "Dominant forces in protein folding", *Biochemistry* **29**:31 (1990), 7133–7155.

[Donoso and O'Rourke 2002] M. Donoso and J. O'Rourke, "Nonorthogonal polyhedra built from rectangles", 2002. Available at arXiv:cs.CG/0110059.

[Dürer 1525] A. Dürer, *Unterweysung der Messung mit dem Zirkel und Richtscheyt, in Linien Ebnen und gantzen Corporen*, 1525. Reprinted 2002, Verlag Alfons Uhl, Nördlingen; translated as *The Painter's Manual*, Abaris Books, New York, 1977.

[Erdős 1935] P. Erdős, "Problem 3763", *American Mathematical Monthly* **42** (1935).

[Fedorchuk and Pak 2004] M. Fedorchuk and I. Pak, "Rigidity and polynomial invariants of convex polytopes", preprint, 2004. Available at http://www-math.mit.edu/~pak/pp16.pdf.

[Fevens et al. 2001] T. Fevens, A. Hernandez, A. Mesa, P. Morin, M. Soss, and G. Toussaint, "Simple polygons with an infinite sequence of deflations", *Beiträge zur Algebra und Geometrie* **42**:2 (2001), 307–311.

[Gardner 1960] M. Gardner, "Mathematical games", *Scientific American* (June 1960). Reprinted as "Paper cutting", chapter 5, pp. 58–69 in *New mathematical diversions* (revised edition), Math. Assoc. of America, Washington, DC, 1995.

[Glass et al. 2004] J. Glass, S. Langerman, J. O'Rourke, J. Snoeyink, and J. K. Zhong, "A 2-chain can interlock with a *k*-chain", 2004. Available at www.arXiv.org/cs.CG/0410052.

[Grünbaum 1995] B. Grünbaum, "How to convexify a polygon", *Geombinatorics* **5** (1995), 24–30.

[Grünbaum and Zaks 1998] B. Grünbaum and J. Zaks, "Convexification of polygons by flips and by flipturns", Technical Report 6/4/98, Department of Mathematics, University of Washington, Seattle, 1998.

[Harper's 1873] Harper's, "National standards and emblems", *Harper's New Monthly Magazine* **47** (July 1873), 171–181. Available at http://memory.loc.gov/ammem/ndlpcoop/moahtml/title/lists/harp_V47I278.html.

[Hart and Istrail 1996] W. E. Hart and S. Istrail, "Fast protein folding in the hydrophobic-hydrophilic model within three-eighths of optimal", *Journal of Computational Biology* **3**:1 (1996), 53–96.

[Hayes 1998] B. Hayes, "Prototeins", *American Scientist* **86** (1998), 216–221.

[Hirata 2000] K. Hirata, "Polytope2", 2000. Available at http://weyl.ed.ehime-u.ac.jp/cgi-bin/WebObjects/Polytope2. Online software.

[Hopcroft et al. 1984] J. Hopcroft, D. Joseph, and S. Whitesides, "Movement problems for 2-dimensional linkages", *SIAM J. Comput.* **13**:3 (1984), 610–629.

[Houdini 1922] H. Houdini, *Paper magic*, Dutton, 1922.

[Hull 1994] T. Hull, "On the mathematics of flat origamis", *Congr. Numer.* **100** (1994), 215–224.

[Hull 2003] T. Hull, "Counting mountain-valley assignments for flat folds", *Ars Combin.* **67** (2003), 175–187.

[Jordan and Steiner 1999] D. Jordan and M. Steiner, "Configuration spaces of mechanical linkages", *Discrete Comput. Geom.* **22**:2 (1999), 297–315.

[Justin 1994] J. Justin, "Towards a mathematical theory of origami", pp. 15–29 in *Proceedings of the 2nd International Meeting of Origami Science and Scientific Origami* (Otsu, Japan, 1994), edited by K. Miura, 1994.

[Kaneva and O'Rourke 2000] B. Kaneva and J. O'Rourke, "An implementation of Chen & Han's shortest paths algorithm", pp. 139–146 in *Proceedings of the 12th Canadian Conference on Computational Geometry* (Fredericton, NB, 2000), 2000. Available at http://www.cs.unb.ca/conf/cccg/eProceedings/8.ps.gz.

[Kapoor 1999] S. Kapoor, "Efficient computation of geodesic shortest paths", pp. 770–779 in *Proceedings of the 32nd Annual ACM Symposium on Theory of Computing* (Atlanta, GA, 1999), ACM, New York, 1999.

[Kapovich and Millson 1995] M. Kapovich and J. Millson, "On the moduli space of polygons in the Euclidean plane", *J. Differential Geom.* **42**:1 (1995), 133–164.

[Kapovich and Millson 2002] M. Kapovich and J. J. Millson, "Universality theorems for configuration spaces of planar linkages", *Topology* **41**:6 (2002), 1051–1107.

[Kawasaki 1989] T. Kawasaki, "On the relation between mountain-creases and valley-creases of a flat origami", pp. 229–237 in *Proceedings of the 1st International Meeting of Origami Science and Technology* (Ferrara, Italy, 1989), edited by H. Huzita, 1989. An unabridged Japanese version appeared in *Sasebo College of Technology Report* **27** (1990), 153–157.

[Kempe 1876] A. B. Kempe, "On a general method of describing plane curves of the n^{th} degree by linkwork", *Proceedings of the London Mathematical Society* **7** (1876), 213–216.

[Lang 1994a] R. J. Lang, “Mathematical algorithms for origami design”, *Symmetry Cult. Sci.* **5**:2 (1994), 115–152.

[Lang 1994b] R. J. Lang, “The tree method of origami design”, pp. 73–82 in *Proceedings of the 2nd International Meeting of Origami Science and Scientific Origami* (Otsu, Japan, 1994), edited by K. Miura, 1994.

[Lang 1996] R. J. Lang, “A computational algorithm for origami design”, pp. 98–105 in *Proceedings of the 12th Annual ACM Symposium on Computational Geometry* (Philadelphia, 1996), ACM, New York, 1996.

[Lang 1998] R. J. Lang, “TreeMaker 4.0: A Program for Origami Design”, 1998. Available at http://origami.kvi.nl/programs/treemaker/trmkr40.pdf.

[Lang 2003] R. J. Lang, *Origami design secrets: Mathematical methods for an ancient art*, A K Peters, Natick, MA, 2003.

[Lau and Dill 1989] K. F. Lau and K. A. Dill, “A lattice statistical mechanics model of the conformation and sequence spaces of proteins”, *Macromolecules* **22** (1989), 3986–3997.

[Lau and Dill 1990] K. F. Lau and K. A. Dill, “Theory for protein mutability and biogenesis”, *Proceedings of the National Academy of Sciences USA* **87** (1990), 638–642.

[Lenhart and Whitesides 1995] W. J. Lenhart and S. H. Whitesides, “Reconfiguring closed polygonal chains in Euclidean d-space”, *Discrete Comput. Geom.* **13**:1 (1995), 123–140.

[Lipman and Wilber 1991] D. J. Lipman and W. J. Wilber, “Modelling neutral and selective evolution of protein folding”, *Proceedings of The Royal Society of London, Series B* **245**:1312 (1991), 7–11.

[Lubiw and O’Rourke 1996] A. Lubiw and J. O’Rourke, “When can a polygon fold to a polytope?”, Technical Report 048, Smith College, Northampton, MA, 1996.

[Lunnon 1968] W. F. Lunnon, “A map-folding problem”, *Math. Comp.* **22** (1968), 193–199.

[Lunnon 1971] W. F. Lunnon, “Multi-dimensional map-folding”, *Comput. J.* **14** (1971), 75–80.

[Merz and Le Grand 1994] K. M. Merz and S. M. Le Grand (editors), *The protein folding problem and tertiary structure prediction*, edited by K. M. Merz and S. M. Le Grand, Birkhäuser, Boston, 1994.

[Miller and Pak 2003] E. Miller and I. Pak, “Metric combinatorics of convex polyhedra: cut loci and nonoverlapping unfoldings”, preprint, 2003. Available at http://www.math.umn.edu/~ezra/Foldout/fold.pdf.

[Mitchell et al. 1987] J. S. B. Mitchell, D. M. Mount, and C. H. Papadimitriou, “The discrete geodesic problem”, *SIAM J. Comput.* **16**:4 (1987), 647–668.

[Montroll 1991] J. Montroll, *African animals in origami*, Dover, New York, 1991.

[Montroll 1993] J. Montroll, *Origami inside-out*, Dover, New York, 1993.

[Nagy 1939] B. Nagy, “Solution to problem 3763”, *American Mathematical Monthly* **46** (March 1939), 176–177.

[O’Rourke 2000] J. O’Rourke, “Folding and unfolding in computational geometry”, pp. 258–266 in *Discrete and computational geometry* (JCDCG, Tokyo, 1998: Revised papers), edited by M. U. Jin Akiyama, Mikio Kano, Lecture Notes in Comput. Sci. **1763**, Springer, Berlin, 2000.

[O'Rourke 2002] J. O'Rourke, "Computational geometry column 43", *Internat. J. Comput. Geom. Appl.* **12**:3 (2002), 263–265. Also in *SIGACT News* **33**:1 (2002), 58-60.

[Rote 2003] G. Rote, "Pseudotriangulations: A survey and recent results", 2003. Available at page.mi.fu-berlin.de/~rote/Papers/slides/Pseudotriangulations-JGA2003. Course notes, *Journées de Géométrie Algorithmique* 2003.

[Sabitov 1996] I. K. Sabitov, "The volume of a polyhedron as a function of its metric", *Fundam. Prikl. Mat.* **2**:4 (1996), 1235–1246. In Russian.

[Sabitov 1998] I. K. Sabitov, "The volume as a metric invariant of polyhedra", *Discrete Comput. Geom.* **20**:4 (1998), 405–425.

[Sallee 1973] G. T. Sallee, "Stretching chords of space curves", *Geometriae Dedicata* **2** (1973), 311–315.

[Schevon 1987] C. Schevon, "Unfolding polyhedra", Usenet article on sci.math, 1987. Available at http://www.ics.uci.edu/~eppstein/gina/unfold.html.

[Schevon 1989] C. Schevon, Ph.D. thesis, Johns Hopkins University, Baltimore, MD, 1989.

[Sen 1721] K. C. Sen, *Wakoku chiyekurabe* (*Mathematical contests*), 1721. Excerpts available from http://theory.csail.mit.edu/~edemaine/foldcut/sen_book.html.

[Shephard 1975] G. C. Shephard, "Convex polytopes with convex nets", *Math. Proc. Cambridge Philos. Soc.* **78**:3 (1975), 389–403.

[Smith 1976] J. S. Smith, "Origami profiles", *British Origami* **58** (June 1976).

[Smith 1980] J. S. Smith, *Pureland origami*, BOS **14**, British Origami Society, Stockport, 1980.

[Smith 1988] J. S. Smith, *Pureland origami* 2, BOS **29**, British Origami Society, Stockport, 1988.

[Smith 1993] J. S. Smith, *Pureland origami* 3, BOS **43**, British Origami Society, Stockport, 1993.

[Soss 2001] M. Soss, *Geometric and computational aspects of molecular reconfiguration*, Ph.D. thesis, School of Computer Science, McGill University, Montreal, 2001.

[Soss and Toussaint 2000] M. Soss and G. T. Toussaint, "Geometric and computational aspects of polymer reconfiguration", *J. Math. Chem.* **27**:4 (2000), 303–318.

[Soss et al. 2003] M. A. Soss, J. Erickson, and M. H. Overmars, "Preprocessing chains for fast dihedral rotations is hard or even impossible", *Comput. Geom.* **26**:3 (2003), 235–246.

[Streinu 2000] I. Streinu, "A combinatorial approach to planar non-colliding robot arm motion planning", pp. 443–453 in *41st Annual Symposium on Foundations of Computer Science* (Redondo Beach, CA, 2000), IEEE Comput. Soc. Press, Los Alamitos, CA, 2000.

[Sun 1999] J. Sun, *Folding orthogonal polyhedra*, Master's thesis, Department of Computer Science, University of Waterloo, Waterloo, ON, 1999.

[Toussaint 1999a] G. Toussaint, "Computational polygonal entanglement theory", pp. 269–278 in *Proceedings of the VIII Encuentros de Geometria Computacional* (Castellon, Spain, 1999), 1999.

[Toussaint 1999b] G. Toussaint, "The Erdős–Nagy theorem and its ramifications", pp. 1–13 in *Proceedings of the 11th Canadian Conference on Computational Geometry* (Vancouver, 1999), 1999. Available at http://www.cs.ubc.ca/conferences/CCCG/elec_proc/fp19.ps.gz.

[Toussaint 2001] G. Toussaint, "A new class of stuck unknots in Pol_6", *Beiträge Algebra Geom.* **42**:2 (2001), 301–306.

[Unger and Moult 1993a] R. Unger and J. Moult, "Finding the lowest free energy conformation of a protein is a NP-hard problem: Proof and implications", *Bulletin of Mathematical Biology* **55**:6 (1993), 1183–1198.

[Unger and Moult 1993b] R. Unger and J. Moult, "A genetic algorithm for 3D protein folding simulations", pp. 581–588 in *Proceedings of the 5th International Conference on Genetic Algorithms* (San Mateo, CA, 1993), 1993.

[Unger and Moult 1993c] R. Unger and J. Moult, "Genetic algorithms for protein folding simulations", *Journal of Molecular Biology* **231**:1 (1993), 75–81.

[Wang 1997] C.-H. Wang, *Manufacturability-driven decomposition of sheet metal*, Ph.D. thesis, Robotics Institute, Carnegie Mellon University, Pittsburgh, PA, 1997. Issued as technical report CMU-RI-TR-97-35.

[Wegner 1993] B. Wegner, "Partial inflation of closed polygons in the plane", *Beiträge Algebra Geom.* **34**:1 (1993), 77–85.

[Whitesides 1992] S. Whitesides, "Algorithmic issues in the geometry of planar linkage movement", *Australian Computer Journal* **24**:2 (1992), 42–50.

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