

Tropical Halfspaces

MICHAEL JOSWIG

ABSTRACT. As a new concept tropical halfspaces are introduced to the (linear algebraic) geometry of the tropical semiring $(\mathbb{R}, \min, +)$. This yields exterior descriptions of the tropical polytopes that were recently studied by Develin and Sturmfels [2004] in a variety of contexts. The key tool to the understanding is a newly defined sign of the tropical determinant, which shares remarkably many properties with the ordinary sign of the determinant of a matrix. The methods are used to obtain an optimal tropical convex hull algorithm in two dimensions.

1. Introduction

The set \mathbb{R} of real numbers carries the structure of a semiring if equipped with the *tropical addition* $\lambda \oplus \mu = \min\{\lambda, \mu\}$ and the *tropical multiplication* $\lambda \odot \mu = \lambda + \mu$, where $+$ is the ordinary addition. We call the triplet $(\mathbb{R}, \oplus, \odot)$ the *tropical semiring*¹. It is an equally simple and important fact that the operations $\oplus, \odot : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous with respect to the standard topology of \mathbb{R} . So the tropical semiring is, in fact, a topological semiring. Considering the tropical scalar multiplication

$$\lambda \odot (\mu_0, \dots, \mu_d) = (\lambda + \mu_0, \dots, \lambda + \mu_d)$$

(and componentwise tropical addition) turns the set \mathbb{R}^{d+1} into a semimodule.

The study of the linear algebra of the tropical semiring and, more generally, of idempotent semirings, has a long tradition. Applications to combinatorial optimization, discrete event systems, functional analysis etc. abound. For an introduction see [Baccelli et al. 1992]. A recent contribution in the same vein, with many more references, is [Cohen et al. 2004].

This work has been carried out while visiting the Mathematical Sciences Research Institute in Berkeley for the special semester on Discrete and Computational Geometry.

¹Other authors reserve the name *tropical semiring* for $(\mathbb{N} \cup \{+\infty\}, \min, +)$ and call $(\mathbb{R} \cup \{+\infty\}, \min, +)$ the min-plus-semiring.

Convexity in the tropical world (and even in a more general setting) was first studied by Zimmermann [1977]. Following the approach of Develin and Sturmfels [2004] here we stress the point of view of discrete geometry. We recall some of the key definitions. A subset $S \subset \mathbb{R}^{d+1}$ is *tropically convex* if for any two points $x, y \in S$ the *tropical line segment*

$$[x, y] = \{ \lambda \odot x \oplus \mu \odot y \mid \lambda, \mu \in \mathbb{R} \}$$

is contained in S . The *tropical convex hull* of a set $S \subset \mathbb{R}^{d+1}$ is the smallest tropically convex set containing S ; it is denoted by $\text{tconv } S$. It is easy to see [Develin and Sturmfels 2004, Proposition 4] that

$$\text{tconv } S = \{ \lambda_1 \odot x_1 \oplus \cdots \oplus \lambda_n \odot x_n \mid \lambda_i \in \mathbb{R}, x_i \in S \}.$$

A *tropical polytope* is the tropical convex hull of finitely many points. Since any convex set in \mathbb{R}^{d+1} is closed under tropical multiplication with an arbitrary scalar, it is common to identify tropically convex sets with their respective images under the canonical projection onto the d -dimensional *tropical projective space*

$$\mathbb{TP}^d = \{ \mathbb{R} \odot x \mid x \in \mathbb{R}^{d+1} \} = \mathbb{R}^{d+1} / \mathbb{R}(1, \dots, 1).$$

In explicit computations we often choose *canonical coordinates* for a point $x \in \mathbb{TP}^d$, meaning the unique nonnegative vector in the class $\mathbb{R} \odot x$ which has at least one zero coordinate. For visualization purposes, however, we usually normalize the coordinates by choosing the first one to be zero (which can then be omitted): This identification $(\xi_0, \dots, \xi_d) \mapsto (\xi_1 - \xi_0, \dots, \xi_d - \xi_0) : \mathbb{TP}^d \rightarrow \mathbb{R}^d$ is a homeomorphism.

Develin and Sturmfels observed that *tropical simplices*, that is tropical convex hulls of $d+1$ points in \mathbb{TP}^d (in sufficiently general position), are related to Isbell's [1964] *injective envelope* of a finite metric space; see [Develin and Sturmfels 2004, Theorem 29] and the Erratum. Isbell's injective envelope in turn coincides with the *tight span* of a finite metric space that arose in the work of Dress and others; see [Dress et al. 2002] and its list of references. In a way, tropical simplices may be understood as nonsymmetric analogues of injective hulls or tight spans.

Additionally, tropical polytopes are interesting also from a purely combinatorial point of view: They bijectively correspond to the regular polyhedral subdivisions of products of simplices; see [Develin and Sturmfels 2004, Theorem 1].

The present paper studies tropical polytopes as geometric objects in their own right. It is shown that, at least to some extent, it is possible to develop a theory of tropical polytopes in a fashion similar to the theory of ordinary convex polytopes. The key concept introduced to this end is the notion of a tropical halfspace. One of our main results, Theorem 4.7, gives a characterization of tropical halfspaces in terms of the tropical determinant, which is the same as the min-plus-permanent already studied by Yoeli [1961] and others; see also [Richter-Gebert et al. 2005]. The proof leads to the definition of the *faces* of a tropical polytope in a natural way. In the investigation, in particular, we prove that the

faces form a distributive lattice; see Theorem 3.7. Moreover, as one would expect by analogy to ordinary convex polytopes, the tropical polytopes are precisely the bounded intersections of finitely many tropical halfspaces; see Theorem 3.6.

It is a further consequence of our results on tropical polytopes that some concepts and ideas from computational geometry can be carried over from ordinary convex polytopes to tropical polytopes. In Section 5 this leads us to a comprehensive solution of the convex hull problem in \mathbb{TP}^2 . The general tropical convex hull problem in arbitrary dimension is certainly interesting, but this is beyond our current scope.

The paper closes with a selection of open questions.

2. Hyperplanes and Halfspaces

We start this section with some observations concerning the topological aspects of tropical convexity. As already mentioned the tropical projective space \mathbb{TP}^d is homeomorphic to \mathbb{R}^d with the usual topology. Moreover, the space \mathbb{TP}^d carries a natural metric: For a point $x \in \mathbb{TP}^d$ with canonical coordinates (ξ_0, \dots, ξ_d) let

$$\|x\| = \max\{\xi_0, \dots, \xi_d\}$$

be the *tropical norm* of x . Equivalently, for arbitrary coordinates $(\xi'_0, \dots, \xi'_d) \in \mathbb{R} \odot x$ we have that $\|x\| = \max\{|\xi'_i - \xi'_j| \mid i \neq j\}$. We prove a special case of [Cohen et al. 2004, Theorem 17]:

LEMMA 2.1. *The map*

$$\mathbb{TP}^d \times \mathbb{TP}^d \rightarrow \mathbb{R} : (x, z) \mapsto \|x - z\|$$

is a metric.

PROOF. By definition the map is nonnegative. Moreover, it is clearly definite and symmetric. We prove the triangle inequality: Assume that $x = (\xi_0, \dots, \xi_d)$, $z = (\zeta_0, \dots, \zeta_d)$, and that $y = (\eta_0, \dots, \eta_d)$ be a third point. Then

$$\begin{aligned} \|x - z\| &= \max\{|\xi_i - \zeta_i| - |\xi_j - \zeta_j| \mid i \neq j\} \\ &= \max\{|\xi_i - \xi_j| - (\eta_i - \eta_j) + (\eta_i - \eta_j) - (\zeta_i - \zeta_j) \mid i \neq j\} \\ &\leq \max\{|\xi_i - \xi_j| - (\eta_i - \eta_j) + |(\eta_i - \eta_j) - (\zeta_i - \zeta_j)| \mid i \neq j\} \\ &\leq \max\{|\xi_i - \eta_i| - (\xi_j - \eta_j) \mid i \neq j\} + \max\{|(\eta_i - \zeta_i) - (\eta_j - \zeta_j)| \mid i \neq j\} \\ &= \|x - y\| + \|y - z\|. \quad \square \end{aligned}$$

The topology induced by this metric coincides with the quotient topology on \mathbb{TP}^d (and thus with the natural topology of \mathbb{R}^d). In particular, \mathbb{TP}^d is locally compact and a set $C \subset \mathbb{TP}^d$ is compact if and only if it is closed and bounded. Tacitly we will always assume that $d \geq 2$.

PROPOSITION 2.2. *The topological closure of a tropically convex set is tropically convex.*

PROOF. Let S be a tropically convex set with closure \bar{S} . Then, by [Develin and Sturmfels 2004, Proposition 4], $\text{tconv}(\bar{S})$ is the set of points in \mathbb{TP}^d which can be obtained as tropical linear combinations of points in \bar{S} . Now the claim follows from the fact that tropical addition and multiplication are continuous. \square

From the named paper by Develin and Sturmfels we quote a few results which will be useful in our investigation.

THEOREM 2.3 [Develin and Sturmfels 2004, Theorem 15]. *A tropical polytope has a canonical decomposition as a finite ordinary polytopal complex, where the cells are both ordinary and tropical polytopes.*

PROPOSITION 2.4 [Develin and Sturmfels 2004, Proposition 20]. *The intersection of two tropical polytopes is again a tropical polytope.*

PROPOSITION 2.5 [Develin and Sturmfels 2004, Proposition 21] (see also [Helbig 1988]). *For each tropical polytope P there is a unique minimal set $\text{Vert}(P) \subset P$ with $\text{tconv}(\text{Vert}(P)) = P$.*

The elements of $\text{Vert}(P)$ are called the *vertices* of P . The following is implied by Theorem 2.3. There is also an easy direct proof which we omit, however.

PROPOSITION 2.6. *A tropical polytope is compact.*

The *tropical hyperplane* defined by the *tropical linear form* $a = (\alpha_0, \dots, \alpha_d) \in \mathbb{R}^{d+1}$ is the set of points $(\xi_0, \dots, \xi_d) \in \mathbb{TP}^d$ such that the minimum

$$\min\{\alpha_0 + \xi_0, \dots, \alpha_d + \xi_d\} = \alpha_0 \odot \xi_0 \oplus \dots \oplus \alpha_d \odot \xi_d$$

is attained at least twice. The point $-a$ is contained in the tropical hyperplane defined by a , and it is called its *apex*. Note that any two tropical hyperplanes only differ by a translation.

PROPOSITION 2.7 [Develin and Sturmfels 2004, Proposition 6]. *Tropical hyperplanes are tropically convex.*

The complement of a tropical hyperplane \mathcal{H} in \mathbb{TP}^d has $d+1$ connected components corresponding to the facets of an ordinary d -simplex. We call each such connected component an *open sector* of \mathcal{H} . The topological closure of an open sector is a *closed sector*. It is easy to prove that each (open or closed) sector is convex both in the ordinary and in the tropical sense.

EXAMPLE 2.8. Consider the zero tropical linear form $0 \in \mathbb{R}^{d+1}$. The open sectors of the corresponding tropical hyperplane \mathcal{Z} are the sets S_0, \dots, S_d , where

$$S_i = \{(\xi_0, \dots, \xi_d) \mid \xi_i < \xi_j \text{ for all } j \neq i\}.$$

The closed sectors are the sets $\bar{S}_0, \dots, \bar{S}_d$, where

$$\bar{S}_i = \{(\xi_0, \dots, \xi_d) \mid \xi_i \leq \xi_j \text{ for all } j \neq i\}.$$

In canonical coordinates this can be expressed as follows:

$$S_i = \{(\xi_0, \dots, \xi_d) \mid \xi_i = 0 \text{ and } \xi_j > 0, \text{ for } j \neq i\}$$

and

$$\bar{S}_i = \{(\xi_0, \dots, \xi_d) \mid \xi_i = 0 \text{ and } \xi_j \geq 0, \text{ for } j \neq i\}.$$

Just as any two tropical hyperplanes are related by a translation, each translation of a sector is again a sector. We call such sectors *parallel*.

The following simple observation is one of the keys to the structural results on tropical polytopes in the subsequent sections. It characterizes the solvability of one tropical linear equation. For related results see [Akian et al. 2005].

PROPOSITION 2.9. *Let $x_1, \dots, x_n \in \mathbb{TP}^d$. Then $0 \in \text{tconv}\{x_1, \dots, x_n\}$ if and only if each closed sector \bar{S}_k of the zero tropical linear form contains at least one x_i .*

PROOF. We write ξ_{ij} for the canonical coordinates of the x_i in \mathbb{R}^{d+1} . Then all the $n(d+1)$ entries in the matrix

$$\begin{pmatrix} \xi_{10} & \cdots & \xi_{1d} \\ \vdots & \ddots & \vdots \\ \xi_{n0} & \cdots & \xi_{nd} \end{pmatrix}$$

are nonnegative. Hence

$$0 = \lambda_1 \odot x_1 \oplus \cdots \oplus \lambda_n \odot x_n$$

(with $\lambda_i \geq 0$, as we may assume without loss of generality) if and only if $\min\{\lambda_1 + \xi_{1k}, \dots, \lambda_n + \xi_{nk}\} = 0$ for all k . We conclude that zero is in the tropical convex hull of x_1, \dots, x_n if and only if for all k there is an i such that $\xi_{ik} = 0$ or, equivalently, $x_i \in \bar{S}_k$. \square

Throughout the following we abbreviate $[d+1] = \{0, \dots, d\}$, and we write $\text{Sym}(d+1)$ for the symmetric group of degree $d+1$ acting on the set $[d+1]$. Let e_i be the i -th unit vector of \mathbb{R}^{d+1} . Observe that under the natural action of $\text{Sym}(d+1)$ on \mathbb{TP}^d by permuting the unit vectors tropical convex sets get mapped to tropical convex sets. The set of all k -element subsets of a set Ω is denoted by $\binom{\Omega}{k}$.

We continue our investigation with the construction of a two-parameter family of tropical polytopes.

EXAMPLE 2.10. We define the k -th tropical hypersimplex in \mathbb{TP}^d as

$$\Delta_k^d = \text{tconv} \left\{ \sum_{i \in J} -e_i \mid J \in \binom{[d+1]}{k} \right\} \subset \mathbb{TP}^d.$$

It is essential that

$$\text{Vert}(\Delta_k^d) = \left\{ \sum_{i \in J} -e_i \mid J \in \binom{[d+1]}{k} \right\},$$

for all $k > 0$: This has to do with the fact that the symmetric group $\text{Sym}(d+1)$ acts on the set, due to which either all or none of the points $\sum_{i \in J} -e_i$ is a vertex. But from Proposition 2.5 we know that $\emptyset \neq \text{Vert}(\Delta_k^d) \subseteq \left\{ \sum_{i \in J} -e_i \mid J \in \binom{[d+1]}{k} \right\}$, and hence the claim follows. Develin, Santos, and Sturmfels [Develin et al. 2005] construct tropical polytopes from matroids. The tropical hypersimplices arise as the special case of uniform matroids.

It is worth-while to look at two special cases of the previous construction.

EXAMPLE 2.11. The first tropical hypersimplex in \mathbb{TP}^d is the d -dimensional *tropical standard simplex* $\Delta^d = \Delta_1^d = \text{tconv}\{-e_0, \dots, -e_d\}$. Note that Δ^d is a tropical polytope which at the same time is an ordinary polytope.

EXAMPLE 2.12. The second tropical hypersimplex $\Delta_2^d \subset \mathbb{TP}^d$ is the tropical convex hull of the $\binom{d+1}{2}$ vectors $-e_i - e_j$ for all pairs $i \neq j$. The tropical polytope Δ_2^d is not convex in the ordinary sense. It is contained in the tropical hyperplane \mathcal{Z} corresponding to the zero tropical linear form. For $d = 2$ see Figure 1 below.

PROPOSITION 2.13. *The second tropical hypersimplex $\Delta_2^d \subset \mathbb{TP}^d$ is the intersection of the tropical hyperplane \mathcal{Z} corresponding to the zero tropical linear form with the set of points whose tropical norm is bounded by 1.*

PROOF. Clearly, $-e_i - e_j \in \mathcal{Z}$ for $i \neq j$. We have to show that a point x with canonical coordinates (ξ_0, \dots, ξ_d) and $\|x\| \leq 1$ such that, e.g., $\xi_0 = \xi_1 = 0$, is a tropical linear combination of the $\binom{d+1}{2}$ vertices of Δ_2^d . We compute

$$x = (0, 0, 1, \dots, 1) \oplus \xi_2 \odot (0, 1, 0, 1, \dots, 1) \oplus \dots \oplus \xi_d \odot (0, 1, \dots, 1, 0),$$

and hence the claim. \square

In particular, this implies that Δ_2^d contains Δ_k^d , for all $k > 2$. A similar computation further shows that $\Delta_k^d \supseteq \Delta_{k+1}^d$, for all k .

EXAMPLE 2.14. The ordinary d -dimensional ± 1 -cube

$$C^d = \{(0, \xi_1, \dots, \xi_d) \mid -1 \leq \xi_i \leq 1\}$$

is a tropical polytope: $C^d = \text{tconv}\{-e_0 - 2e_1, \dots, -e_0 - 2e_d, e_1 + \dots + e_d\}$.

One way of reading Proposition 2.13 is that the intersection of the tropical hyperplane corresponding to the zero tropical linear form with the ordinary ± 1 -cube is a tropical polytope. An important consequence is the following.

COROLLARY 2.15. *The (nonempty) intersection of a tropical polytope with a tropical hyperplane is again a tropical polytope.*

PROOF. Let $P \subset \mathbb{TP}^d$ be a tropical polytope and \mathcal{H} a tropical hyperplane. As usual, up to a translation we can assume that $\mathcal{H} = \mathcal{Z}$ corresponds to the zero tropical linear form. By Proposition 2.13 the intersection $P \cap \mathcal{Z}$ is contained in a suitably scaled copy of the second tropical hypersimplex Δ_2^d . Now the claim follows from Proposition 2.4. \square

A *closed tropical halfspace* in \mathbb{TP}^d is the union of at least one and at most d closed sectors of a fixed tropical hyperplane. Hence it makes sense to talk about the *apex* of a tropical halfspace. An *open tropical halfspace* is the complement of a closed one. Clearly, the topological closure of an open tropical halfspace is a closed tropical halfspace. To each (open or closed) tropical halfspace \mathcal{H}^+ there is an *opposite* (open or closed) tropical halfspace \mathcal{H}^- formed by the sectors of the corresponding tropical hyperplane which are not contained in \mathcal{H}^+ . Two halfspaces are *parallel* if they are formed of parallel sectors.

LEMMA 2.16. *Let $a + \bar{S}_k \subset \mathbb{TP}^d$ be a closed sector, for some $k \in [d + 1]$, and $b \in a + \bar{S}_k$ a point inside. Then the parallel sector $b + \bar{S}_k$ is contained in $a + \bar{S}_k$.*

Note that this includes the case where b is a point in the boundary $a + (\bar{S}_k \setminus S_k)$. The proof of the lemma is omitted.

PROPOSITION 2.17. *Each closed tropical halfspace is tropically convex.*

PROOF. Let \mathcal{H}^+ be a closed tropical halfspace. Without loss of generality, we can assume that \mathcal{H}^+ is the union the of closed sectors $\bar{S}_{i_1}, \dots, \bar{S}_{i_l}$ of the tropical hyperplane \mathcal{Z} corresponding to the zero tropical linear form. Since we already know that each \bar{S}_{i_k} is tropically convex, it suffices to consider, e.g., $x \in \bar{S}_{i_1}$ and $y \in \bar{S}_{i_2}$ and to prove that $[x, y] \subset \mathcal{H}^+$. Let (ξ_0, \dots, ξ_d) and (η_0, \dots, η_d) be the canonical coordinates of $x, y \in \mathbb{TP}^d$, respectively. Since $x \in \bar{S}_{i_1}$ and $y \in \bar{S}_{i_2}$ we have that $\xi_{i_1} = 0$ and $\eta_{i_2} = 0$. Then the minimum

$$\min\{\lambda + \xi_0, \dots, \lambda + \xi_d, \mu + \eta_0, \dots, \mu + \eta_d\}$$

is $\lambda = \lambda + \xi_{i_1}$ or $\mu = \mu + \eta_{i_2}$, for arbitrary $\lambda, \mu \in \mathbb{R}$. This is equivalent to $\lambda \odot x + \mu \odot y \in \bar{S}_{i_1} \cup \bar{S}_{i_2}$, which implies the claim. \square

A similar argument shows that open tropical halfspaces are tropically convex.

COROLLARY 2.18. *The boundary of a tropical halfspace is tropically convex.*

PROOF. The boundary of a closed tropical halfspace \mathcal{H}^+ is the intersection of \mathcal{H}^+ with its opposite closed tropical halfspace \mathcal{H}^- . \square

TROPICAL SEPARATION THEOREM 2.19. Let P be tropical polytope, and $x \notin P$ a point outside. Then there is a closed tropical halfspace containing P but not x .

PROOF. From Proposition 2.9 we infer that there is a closed sector $x + \bar{S}_k$ of the tropical hyperplane with apex x which is disjoint from P . Now e_k is the unique coordinate vector such that $e_k \notin \bar{S}_k$. Since P is compact and \bar{S}_k is closed there

is some $\varepsilon > 0$ such that the closed sector $x + \varepsilon e_k + \bar{S}_k$ is disjoint from P . The complement of the open sector $x + \varepsilon e_k + S_k$ is a closed tropical halfspace of the desired kind. \square

Tropical halfspaces implicitly occur in [Cohen et al. 2004]. In particular, their results imply the Tropical Separation Theorem. In fact, a variation of this result already occurs in [Zimmermann 1977]. Another variant of the same is the Tropical Farkas Lemma [Develin and Sturmfels 2004, Proposition 9].

3. Exterior Descriptions of Tropical Polytopes

Throughout this section let $P \subset \mathbb{TP}^d$ be a tropical polytope. Like their ordinary counterparts tropical polytopes also have an exterior description.

LEMMA 3.1. *The tropical polytope P is the intersection of the closed tropical halfspaces that it is contained in.*

PROOF. Let P' be the intersection of all the tropical halfspaces which contain P . Clearly, P' contains P . Suppose that there is a point $x \in P' \setminus P$. Then, by the Tropical Separation Theorem, there is a closed tropical halfspace which contains P but not x . This contradicts the assumption that P' is the intersection of all such tropical halfspaces. \square

Of course, the set of closed tropical halfspaces that contain the given tropical polytope P is partially ordered by inclusion. A closed tropical halfspace is said to be *minimal* with respect to P if it is a minimal element in this partial order.

A key observation in what follows is that the minimal tropical halfspaces come from a small set of candidates only. For a given finite set of points $p_1, \dots, p_n \in \mathbb{TP}^d$ let the *standard affine hyperplane arrangement* be generated by the ordinary affine hyperplanes

$$p_i + \{(0, \xi_1, \dots, \xi_d) \in \mathbb{R}^{d+1} \mid \xi_j = 0\} \text{ and } p_i + \{(0, \xi_1, \dots, \xi_d) \in \mathbb{R}^{d+1} \mid \xi_j = \xi_k\}.$$

For an example illustration see Figure 3. A *pseudovertex* of P is a vertex of the standard affine hyperplane arrangement with respect to $\text{Vert}(P)$ which is contained in the boundary ∂P . In [Develin and Sturmfels 2004] our pseudovertrices are called the *vertices*.

Here is a special case of [Develin and Sturmfels 2004, Proposition 18].

PROPOSITION 3.2. *The bounded cells of the standard affine hyperplane arrangement are tropical polytopes which are at the same time ordinary convex polytopes.*

PROPOSITION 3.3. *The apex of a closed tropical halfspace that is minimal with respect to P is a pseudovertex of P .*

PROOF. Let \mathcal{H}^+ be a closed tropical halfspace, with apex a , which minimally contains P . Suppose that a is not a vertex of the standard affine hyperplane arrangement \mathfrak{A} generated by $\text{Vert}(P)$, but rather a is contained in the relative

interior of some cell C of \mathfrak{A} of dimension at least one. Now there is some $\varepsilon > 0$ such that for each point a' in C with $\|a' - a\| < \varepsilon$ the closed tropical halfplane with apex a' and parallel to \mathcal{H}^+ still contains P . For each $a' \in \mathcal{H}^+$ the corresponding translate is contained in \mathcal{H}^+ and hence \mathcal{H}^+ is not minimal. Contradiction.

It remains to show that $a \in P$. Again suppose the contrary. Then, by the Tropical Separation Theorem 2.19, there is a closed halfspace \mathcal{H}_1^+ containing P but not a . Now, since \mathcal{H}^+ is minimal, \mathcal{H}_1^+ is not contained in \mathcal{H}^+ and, in particular, \mathcal{H}_1^+ is not parallel to \mathcal{H}^+ . As $a \notin \mathcal{H}_1^+$ the closed tropical halfspace \mathcal{H}_2^+ with apex a which is parallel to \mathcal{H}_1^+ contains P . We infer that $\mathcal{H}^+ \cap \mathcal{H}_2^+ \subsetneq \mathcal{H}^+$ is a closed tropical halfspace (with apex a) which contains P . This contradicts the minimality of \mathcal{H}^+ . \square

COROLLARY 3.4. *There are only finitely many closed tropical halfspace which are minimal with respect to P .*

PROOF. The standard affine hyperplane arrangement generated by $\text{Vert}(P)$ is finite, and thus there are only finitely many pseudovertices. Since there are only $2^{d+1} - 2$ closed affine halfspaces with a given apex,² the claim now follows from Proposition 3.3. \square

removed redundant “This immediately gives the following.”

COROLLARY 3.5. *The tropical polytope P is the intersection of the (finitely many) minimal closed tropical halfspaces that it is contained in.*

We can now prove our first main result.

THEOREM 3.6. *The tropical polytopes are exactly the bounded intersections of finitely many tropical halfspaces.*

PROOF. Let P be the bounded intersection of finitely many tropical halfspaces $\mathcal{H}_1^+, \dots, \mathcal{H}_m^+$. Then P is the union of (finitely many) bounded cells of the standard affine hyperplane arrangement corresponding to the apices of $\mathcal{H}_1^+, \dots, \mathcal{H}_m^+$. By Proposition 3.2 each of those cells is the tropical convex hull of its pseudovertices. Since P is tropically convex, this property extends to P , and P is a tropical polytope.

The converse follows from Corollary 3.5. \square

Ordinary polytope theory is combinatorial to a large extent. This is due to the fact that many important properties of an ordinary polytope are encoded into its face lattice. While it is tempting to start a combinatorial theory of tropical polytopes from the results that we obtained so far, this turns out to be quite intricate. Here we give a brief sketch, while a more detailed discussion will be picked up in a forthcoming second paper.

A *boundary slice* of the tropical polytope P is the tropical convex hull of $\text{Vert}(P) \cap \partial\mathcal{H}^+$ where \mathcal{H}^+ is a closed tropical halfspace containing P . The

²The Example 3.9 shows that there may indeed be more than one minimal halfspace with a given apex.

boundary slices are partially ordered by inclusion; we call a maximal element of this finite partially ordered set a *facet* of P . Let F_1, \dots, F_m be facets of P . Then the set

$$F_1 \sqcap \dots \sqcap F_m = \text{tconv}(\text{Vert}(P) \cap F_1 \dots \cap F_m)$$

is called a *proper face* of P provided that $F_1 \sqcap \dots \sqcap F_m \neq \emptyset$. The sets \emptyset and P are the *nonproper faces*. The faces of a tropical polytope are partially ordered by inclusion, the maximal proper faces being the facets. Note that, by definition, faces of tropical polytopes are again tropical polytopes.

THEOREM 3.7. *The face poset of a tropical polytope is a finite distributive lattice.*

PROOF. We can extend the definition of \sqcap to arbitrary faces, this gives the *meet* operation. There is no choice for the *join* operation then: $G \sqcup H$ is the meet of all facets containing G and H , for arbitrary faces G and H . Denote the set of facets containing the face G by $\Phi(G)$, that is, $G = \sqcap \Phi(G)$. It is immediate from the definitions that $\Phi(G \sqcup H) = \Phi(G) \cap \Phi(H)$ and $\Phi(G \sqcap H) = \Phi(G) \cup \Phi(H)$. Hence the absorption and distributive laws are inherited from the boolean lattice of subsets of the set of all facets. \square

In order to simplify some of the discussion below we shall introduce a certain nondegeneracy condition: A set $S \subset \mathbb{TP}^d$ is called *full*, if it is not contained in the boundary of any tropical halfspace. If S is not contained in any tropical hyperplane, then, clearly, S is full. As the Example 3.9 below shows, however, the converse does not hold.

EXAMPLE 3.8. The tropical standard simplex

$$\Delta^2 = \text{tconv}\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$$

is not contained in a tropical hyperplane, and hence it is full; see Figure 1, left. It is the intersection of the three minimal closed tropical halfspaces $(1, 0, 0) + \bar{S}_0$, $(0, 1, 0) + \bar{S}_1$, and $(0, 0, 1) + \bar{S}_2$. The tropical line segments $[(0, 1, 1), (1, 0, 1)]$, $[(1, 0, 1), (1, 1, 0)]$, and $[(1, 1, 0), (0, 1, 1)]$ form the facets. The three vertices form the only other proper faces.

EXAMPLE 3.9. The second tropical hypersimplex

$$\Delta_2^2 = \text{tconv}\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$$

is a full tropical triangle in the tropical plane \mathbb{TP}^2 , which is contained in the tropical line \mathcal{Z} corresponding to the zero tropical linear form. There are *six* minimal closed tropical halfspaces: $\bar{S}_0 \cup \bar{S}_1$, $\bar{S}_1 \cup \bar{S}_2$, $\bar{S}_0 \cup \bar{S}_2$, $(1, 0, 0) + \bar{S}_0$, $(0, 1, 0) + \bar{S}_1$, and $(0, 0, 1) + \bar{S}_2$. Note that the three closed tropical halfspaces $\bar{S}_0 \cup \bar{S}_1$, $\bar{S}_1 \cup \bar{S}_2$, and $\bar{S}_0 \cup \bar{S}_2$ share the origin as their apex. The tropical line segments $[(1, 0, 0), (0, 1, 0)]$, $[(0, 1, 0), (0, 0, 1)]$, and $[(0, 0, 1), (1, 0, 0)]$ form the facets. As in the example above, the three vertices form the only other proper faces. The triangle is depicted in Figure 1, right.

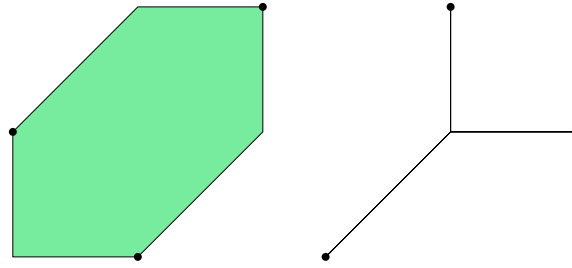


Figure 1. Two full tropical triangles in \mathbb{TP}^2 : the tropical standard simplex Δ^2 (left) and the second tropical hypersimplex $\Delta_2^2 \subset \mathbb{TP}^2$, which has collinear vertices. Both have the same face lattice as an ordinary triangle.

REMARK 3.10. It is a consequence of Proposition 2.4 and Corollary 2.15 that boundary slices of tropical polytopes are tropical polytopes, but they need not be faces: E.g., the intersection of Δ_2^2 with the boundary of the halfspace

$$(0, 0, 1/2) + (\bar{S}_1 \cup \bar{S}_2)$$

is the tropical (and at the same time ordinary) line segment $[(0, 0, 1/2), (0, 0, 1)]$, which contains the face $(0, 0, 1)$ and is properly contained in the intersection $[(1, 0, 0), (0, 0, 1)] \cap [(0, 1, 0), (0, 0, 1)]$ of two facets.

REMARK 3.11. It is easy to see that the face lattice of a tropical n -gon in \mathbb{TP}^2 (which is necessarily full) is always the same as that of an ordinary n -gon. This will play a role in the investigation of the algorithmic point of view in Section 5.

Minimal closed tropical halfspaces can be recognized by the intersection of their boundaries with P , provided that P is full.

PROPOSITION 3.12. *Assuming that P is full, let \mathcal{H}_1^+ and \mathcal{H}_2^+ be closed tropical halfspaces which are minimal with respect to P . If $\partial\mathcal{H}_1^+ \cap P = \partial\mathcal{H}_2^+ \cap P$ then $\mathcal{H}_1^+ = \mathcal{H}_2^+$.*

PROOF. If P is full then it is impossible that for any closed tropical halfspace \mathcal{H}^+ containing P the opposite closed halfspace \mathcal{H}^- also contains P .

Let a_1 and a_2 be the respective apices of \mathcal{H}_1^+ and \mathcal{H}_2^+ . By Lemma 3.3 the points a_1 and a_2 are contained in P and hence $a_1 \in \partial\mathcal{H}_2^+$ and $a_2 \in \partial\mathcal{H}_1^+$. In particular, $a_1 \in \mathcal{H}_2^-$. Therefore, the closed tropical halfspace $(a_1 - a_2) + \mathcal{H}_2^+$ with apex a_1 which is parallel to \mathcal{H}_2^+ is a closed tropical halfspace containing P . Since \mathcal{H}_1^+ is minimal, $\mathcal{H}_1^+ \subseteq (a_1 - a_2) + \mathcal{H}_2^+$ and hence $\mathcal{H}_1^+ \subseteq \mathcal{H}_2^+$. Symmetrically, $\mathcal{H}_2^+ \subseteq \mathcal{H}_1^+$, and the claim follows. \square

REMARK 3.13. The familiarity of the names for the objects defined could inspire the question whether tropical polytopes and, more generally, point configurations in the tropical projective space can be studied in the framework of oriented matroids. However, as the example in Figure 1 shows, the boundaries of tropical

halfspaces spanned by a given set of points do *not* form a pseudo-hyperplane arrangement, in general.

4. Tropical Determinants and Their Signs

For algorithmic approaches to ordinary polytopes it is crucial that the incidence of a point with an affine hyperplane can be characterized by the vanishing of a certain determinant expression. Moreover, by evaluating the sign of that same determinant, it is possible to distinguish between the two open affine halfspaces which jointly form the complement of the given affine hyperplane. This section is about the tropical analog.

Let $M = (m_{ij}) \in \mathbb{R}^{(d+1) \times (d+1)}$ be a matrix. Then the *tropical determinant* is defined as

$$\text{tdet } M = \bigoplus_{\sigma \in \text{Sym}(d+1)} \bigodot_{i=0}^d m_{i, \sigma(i)} = \min \{ m_{0, \sigma(0)} + \dots + m_{d, \sigma(d)} \mid \sigma \in \text{Sym}(d+1) \}.$$

Now M is *tropically singular* if the minimum is attained at least twice, otherwise it is *tropically regular*. Tropical regularity coincides with the strong regularity of a matrix studied by Butkovič [1994]; see also [Burkard and Butkovič 2003].

The following is proved in [Richter-Gebert et al. 2005, Lemma 5.1].

PROPOSITION 4.1. *The matrix M is tropically singular if and only if the $d+1$ points in \mathbb{TP}^d corresponding to the rows of M are contained in a tropical hyperplane.*

From the definition of tropical singularity it is immediate that M is tropically regular if and only if its transpose M^{tr} is. Hence the above proposition also applies to the columns of M .

The *tropical sign* of $\text{tdet } M$, denoted as $\text{tsgn } M$, is either 0 or ± 1 , and it is defined as follows. If M is singular, then $\text{tsgn } M = 0$. If M is regular, then there is a unique $\sigma \in \text{Sym}(d+1)$ such that $m_{0, \sigma(0)} + \dots + m_{d, \sigma(d)} = \text{tdet } M$. We let the tropical sign of M be the sign of this permutation σ . See also [Bacelli et al. 1992, § 3.5.1] and Remark 4.9 below.

As it turns out the tropical sign shares some key properties with the (sign of the) ordinary determinant.

PROPOSITION 4.2. *Let $M \in \mathbb{R}^{(d+1) \times (d+1)}$.*

- (1) *If M contains twice the same row (column), then $\text{tsgn } M = 0$.*
- (2) *If the matrix M' is obtained from M by exchanging two rows (columns), we have $\text{tsgn } M' = -\text{tsgn } M$.*
- (3) *$\text{tsgn } M^{\text{tr}} = \text{tsgn } M$.*

PROOF. The first property follows from Proposition 4.1. The second one is immediate from the definition of the tropical sign. And since permuting the

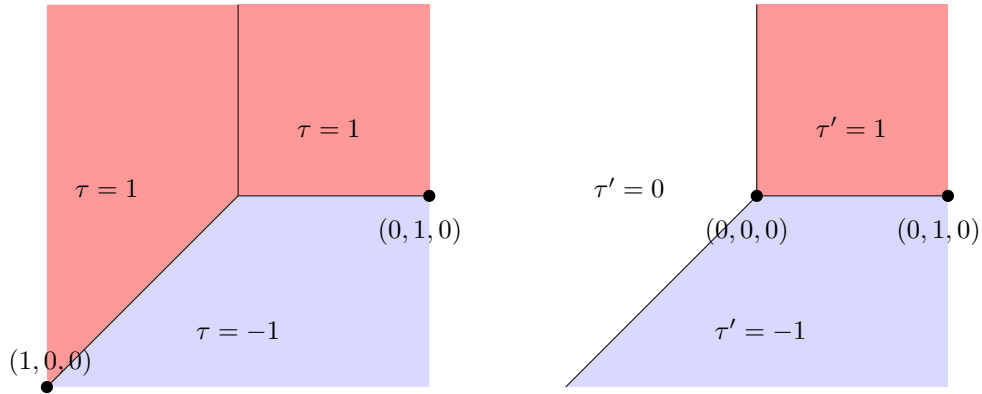


Figure 2. Values of $\tau_{p,q}$ in \mathbb{TP}^2 for two pairs of points: nondegenerate case for $\tau = \tau_{(1,0,0),(0,1,0)}$ (left) and degenerate case for $\tau' = \tau_{(0,0,0),(0,1,0)}$ (right). On the tropical line spanned by the two black points the values are zero in both cases.

rows of a matrix is the same as permuting the columns with the inverse, we conclude that $\text{tsgn } M^{\text{tr}} = \text{tsgn } M$. □

While the behavior of the sign of the ordinary determinant under scaling a row (column) by $\lambda \in \mathbb{R}$ depends on the sign of λ , the tropical sign is invariant under this operation. Given $v_0, \dots, v_d \in \mathbb{R}^{d+1}$ we write (v_0, \dots, v_d) for the $(d + 1) \times (d + 1)$ -matrix with row vectors v_0, \dots, v_d .

LEMMA 4.3. For $v_0, \dots, v_d \in \mathbb{R}^{d+1}$ and $\lambda_0, \dots, \lambda_d \in \mathbb{R}$ we have $\text{tsgn}(\lambda_0 \odot v_0, \dots, \lambda_d \odot v_d) = \text{tsgn}(v_0, \dots, v_d)$.

In fact, tsgn is a function on $(d + 1)$ -tuples of points in the tropical projective space \mathbb{TP}^d . For given p_1, \dots, p_d , consider the function

$$\tau_{p_1, \dots, p_d} : \mathbb{TP}^d \rightarrow \{-1, 0, 1\} : x \mapsto \text{tsgn}(x, p_1, \dots, p_d).$$

Note that we do allow the case where the points p_1, \dots, p_d are not in *general position*, that is, they may be contained in more than one tropical hyperplane; see the example in Figure 4, right.

EXAMPLE 4.4. Consider the real $(d + 1) \times (d + 1)$ -matrix formed of the vertices $-e_0, \dots, -e_d$ of the tropical standard simplex Δ^d . Then we have

$$\text{tdet}(-e_0, \dots, -e_d) = -d,$$

and the matrix is regular: The unique minimum is attained for the identity permutation, hence

$$\text{tsgn}(-e_0, \dots, -e_d) = 1,$$

or equivalently, $\tau_{-e_1, \dots, -e_d}(-e_0) = 1$.

PROPOSITION 4.5. *The function τ_{p_1, \dots, p_d} is constant on each connected component of the set $\mathbb{TP} \setminus \tau_{p_1, \dots, p_d}^{-1}(0)$.*

PROOF. Equip the set $\{-1, 0, 1\}$ with the discrete topology. Away from zero the function τ_{p_1, \dots, p_d} is continuous, and the result follows. \square

Throughout the following we keep a fixed sequence of points p_1, \dots, p_d , and we write π_{ij} for the j -th canonical coordinate of p_i . We frequently abbreviate $\tau = \tau_{p_1, \dots, p_d}$ as well as $p(\sigma) = \pi_{1, \sigma(1)} + \dots + \pi_{d, \sigma(d)}$ for $\sigma \in \text{Sym}(d+1)$.

REMARK 4.6. The points p_1, \dots, p_d are in general position if and only if no $d \times d$ -minor of the $d \times (d+1)$ -matrix with entries π_{ij} is tropically singular; see [Richter-Gebert et al. 2005, Theorem 5.3]. In the terminology of [Develin et al. 2005] this is equivalent to saying that the matrix (π_{ij}) has maximal tropical rank d .

THEOREM 4.7. *The set $\{x \in \mathbb{TP}^d \mid \tau(x) = 1\}$ is either empty or the union of at most d open sectors of a fixed tropical hyperplane. Conversely, each such union of open sectors arises in this way.*

PROOF. We can assume that $\tau(x) = 1$ for some $x \in \mathbb{TP}^d$, since otherwise there is nothing left to prove. From Proposition 4.1 we know that the $d+1$ points x, p_1, \dots, p_d are not contained in a tropical hyperplane, and hence they are the vertices of a full tropical d -simplex $\Delta = \text{tconv}\{x, p_1, \dots, p_d\}$. Consider the facet $F = \text{tconv}\{p_1, \dots, p_d\}$, and let \mathcal{H}^+ be the unique corresponding closed tropical halfspace which is minimal with respect to Δ and for which we have $\partial\mathcal{H}^+ \cap \Delta = F$. Let a be the apex of \mathcal{H}^+ , and let $a + S_k$ be the open sector containing x . By construction $a + S_k \subset \mathcal{H}^+$.

Assume that $\tau(y) \neq \tau(x)$ for some $y \in a + S_k$. Then there exists a point $z \in [x, y]$ with $\tau(z) = 0$. Since sectors are tropically convex, $z \in a + S_k$. By Proposition 4.1 there exists a tropical hyperplane \mathcal{K} which contains the points z, p_1, \dots, p_d . Let \mathcal{K}^+ be the minimal closed tropical halfspace of the tropical hyperplane \mathcal{K} containing x, p_1, \dots, p_d . As \mathcal{H}^+ and \mathcal{K}^+ are both minimal with respect to the tropical simplex Δ , the Proposition 3.12 says that $\mathcal{H}^+ = \mathcal{K}^+$, and in particular, $a + S_k \ni z$ is an open sector of \mathcal{K} . The latter contradicts, however, $z \in \mathcal{K}$.

For the converse, it surely suffices to consider the tropical hyperplane \mathcal{Z} corresponding to the zero tropical linear form, since otherwise we can translate. We have to prove that for each set $K \subset [d+1]$ with $1 \leq \#K \leq d$ there is a set of points $u_1, \dots, u_d \in \mathcal{Z}$ such that

$$\{x \in \mathbb{TP}^d \mid \tau_{u_1, \dots, u_d}(x) = 1\} = \bigcup \{S_k \mid k \in K\}.$$

More specifically, we will even show that, for arbitrary K , those d points can be chosen among the $\binom{d+1}{2}$ vertices of the second tropical hypersimplex Δ_2^d ; see Example 2.12. Since the symmetric group $\text{Sym}(d+1)$ acts on the set of open sectors of \mathcal{Z} as well as on the set $\text{Vert}(\Delta_2^d)$, it suffices to consider one set of sectors

for each possible cardinality $1, \dots, d$. Let us first consider the case where d is odd and $K = \{0, 2, 4, \dots, d - 1\}$ is the set of even indices, which has cardinality $(d + 1)/2$. We set

$$q_i = \begin{cases} -e_i - e_{i+1} & \text{for } i < d \\ -e_0 - e_d & \text{for } i = d. \end{cases}$$

If we want to evaluate $\tau_{q_1, \dots, q_d}(x)$ for some point $x \in \mathbb{TP}^d \setminus \mathcal{Z}$ with canonical coordinates (ξ_0, \dots, ξ_d) , we compute the tropical determinant and the tropical sign of $\text{tdet}(x, q_1, \dots, q_d)$, which in canonical row coordinates looks as follows:

$$Q_d = \begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \cdots & \xi_d \\ 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 1 & \cdots & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Since $x \notin \mathcal{Z}$, there is a unique permutation $\sigma_x \in \text{Sym}(d + 1)$ such that $\text{tdet}(Q) = \xi_{\sigma_x(0)} + q(\sigma_x)$. We can verify that $\text{tdet}(Q) = 0$ in all cases and that

$$\sigma_x = \begin{cases} (0) & \text{if } x \in S_0, \\ (0 \ k \ k + 1 \ \cdots \ d) & \text{if } x \in S_k \text{ for } k > 0. \end{cases}$$

Here we make use of the common cycle notation for permutations, and (0) denotes the identity. For $k > 0$ this means that σ_x is a $(d + 2 - k)$ -cycle, which is an even permutation if and only if k is even, since d is odd. We infer that $\tau_{q_1, \dots, q_d}(x) = 1$ if and only if $x \in S_k$ for k even.

We now discuss the case where $\#K \geq (d + 1)/2$ and d is arbitrary. As in the case above, by symmetry, we can assume that $K = \{0, 2, 4, \dots, 2(l - 1), 2l - 1, 2l, \dots, d\}$ for some $l < \lfloor d/2 \rfloor$. We define

$$q'_i = -e_0 - e_i,$$

for all $i \geq 2l + 1$, and we are concerned with the matrix $(x, q_1, \dots, q_l, q'_{l+1}, \dots, q'_d)$, which, in canonical row coordinates, looks like this:

$$Q_d^l = \begin{pmatrix} \xi_0 & \xi_1 & \xi_2 & \xi_3 & \xi_4 & \cdots & \xi_{2l-1} & \xi_{2l} & \cdots & \xi_d \\ 1 & 0 & 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & 0 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 & 1 & 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 0 & 1 \\ 0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix}.$$

Note that the upper left $2l \times 2l$ -submatrix is exactly Q_l . Hence the same reasoning now yields

$$\sigma_x = \begin{cases} (0) & \text{if } x \in S_0, \\ (0 \ k \ k+1 \ \dots \ 2l-1) & \text{if } x \in S_k \text{ for } k > 0, \end{cases}$$

and σ_x is an even permutation if and only if k is even or $k > 2l$.

Scrutinizing the matrices Q_d and Q_d^l yields that none of their $d \times (d+1)$ -submatrices consisting of all rows but the first contains a tropically singular minor. Equivalently, the points q_1, \dots, q_d as well as the points $q_1, \dots, q_l, q'_{l+1}, \dots, q'_d$ are in general position. But then the set

$$\{x \in \mathbb{TP}^d \mid \tau_{q_1, \dots, q_l, q'_{l+1}, \dots, q'_d}(x) = -1\}$$

is just the union of the sectors in the complement $[d+1] \setminus K$, and since further, according to Proposition 4.2, $\tau_{q_1, \dots, q_d} = -\tau_{q_2, q_1, q_3, \dots, q_l, q'_{l+1}, \dots, q'_d}$, the argument given so far already covers the remaining case of $\#K < (d+1)/2$. This completes the proof. \square

Now, for the fixed set of points p_1, \dots, p_d , we can glue together the connected components of $\mathbb{TP}^d \setminus \tau_{p_1, \dots, p_d}^{-1}(0)$ into two (if $\tau_{p_1, \dots, p_d} \not\equiv 0$) large chunks according to their tropical sign: To this end we define the *closure* of the function τ_{p_1, \dots, p_d} as follows. Let $\bar{\tau}_{p_1, \dots, p_d}(x) = \varepsilon$ if there is a neighborhood U of x such that τ_{p_1, \dots, p_d} restricted to $U \setminus \tau_{p_1, \dots, p_d}^{-1}(0)$ is identically ε ; otherwise let $\bar{\tau}_{p_1, \dots, p_d}(x) = 0$. Clearly, if $\tau_{p_1, \dots, p_d}(x) \neq 0$ then $\bar{\tau}_{p_1, \dots, p_d}(x) = \tau_{p_1, \dots, p_d}(x)$.

Theorem 4.7 then implies the following.

COROLLARY 4.8. *The set $\{x \in \mathbb{TP}^d \mid \bar{\tau}(x) = 1\}$ is empty or a closed tropical halfspace. Conversely, each closed tropical halfspace arises in this way.*

REMARK 4.9. One can show that $\bar{\tau}_{p_1, \dots, p_d}(x) = 1$ if and only if all optimal permutations, that is, all $\sigma \in \text{Sym}(d+1)$ with $\text{tdet}(x, p_1, \dots, p_d) = \xi_{\sigma(0)} + p(\sigma)$ are even. In this sense our function $\bar{\tau}$ captures the sign of the determinant in the symmetrized min-plus-algebra as defined in [Baccelli et al. 1992, § 3.5.1].

COROLLARY 4.10. *For each point $x = (\xi_0, \dots, \xi_d) \in \mathbb{TP}^d$ with $\bar{\tau}_{p_1, \dots, p_d}(x) = 0$ there are two permutations σ and σ' of opposite sign such that*

$$\text{tdet}(x, p_1, \dots, p_d) = \xi_{\sigma(0)} + p(\sigma) = \xi_{\sigma'(0)} + p(\sigma').$$

5. Convex Hull Algorithms in \mathbb{TP}^2

For points in the ordinary Euclidean plane the known algorithms can be phrased easily in terms of sign evaluations of certain determinants. It turns out that the results of the previous sections can be used to “tropify” many ordinary convex hull algorithms.

In this section we do not use canonical coordinates for points in the tropical projective space, but rather we normalize by setting the first coordinate to zero.

This way the description of the algorithms can be made in the ordinary affine geometry language more easily. In particular, a point in $(0, \xi_1, \xi_2) \in \mathbb{TP}^2$ is determined by its x -coordinate ξ_1 and its y -coordinate ξ_2 . We hope that this helps to see the strong relationship between the ordinary convex hull problem in \mathbb{R}^2 and the tropical convex hull problem in \mathbb{TP}^2 . Moreover, this way it may be easier to interpret the illustrations.

Consider a set $S = \{p_1, \dots, p_n\} \subset \mathbb{TP}^2$. Let $\text{bottom}(S)$ be the lowest point (least y -coordinate) of S with ties broken by taking the rightmost (highest x -coordinate) one. Similarly, let $\text{right}(S)$ be the rightmost one with ties broken by taking the highest, $\text{top}(S)$ the highest with ties broken by taking the rightmost, and $\text{left}(S)$ the leftmost one with ties broken by taking the highest. Clearly, some of the four points defined may coincide. If a set of points is in *general position*, that is, for any two points of the input their three rays are pairwise distinct, then there are unique points with minimum and maximum x - and y -coordinate respectively. In this case there are no ties.

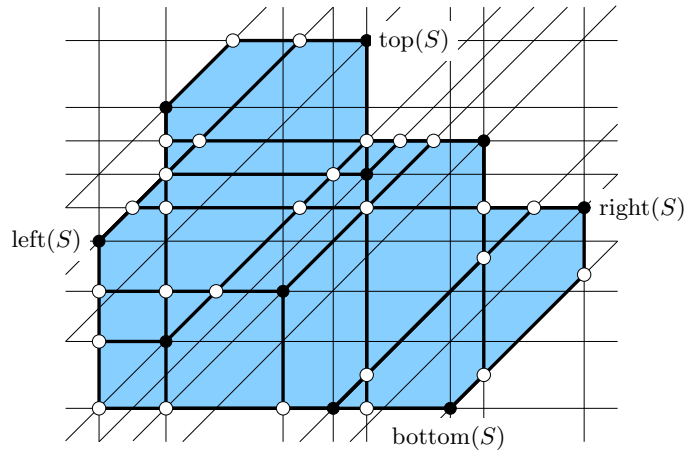


Figure 3. Standard affine line arrangement generated by a set of points $S \subset \mathbb{TP}^2$, displayed in black. The white points are the pseudoververtices on tropical line segments between any two points. Additionally, the tropical convex hull is marked.

LEMMA 5.1. *The points $\text{bottom}(S)$, $\text{right}(S)$, $\text{top}(S)$, $\text{left}(S)$ are vertices of the tropical polygon $\text{tconv}(S)$. Moreover, $[\text{bottom}(S), \text{left}(S)]$ is a facet provided that $\text{bottom}(S) \neq \text{left}(S)$.*

PROOF. By definition, the closed sector $\text{bottom}(S) + \bar{S}_1$ does not contain any point of S other than $\text{bottom}(S)$. This certifies that, indeed, the point $\text{bottom}(S)$ is a vertex because of Propositions 2.5 and 2.9. We omit the proofs of the remaining statements, which are similar. \square

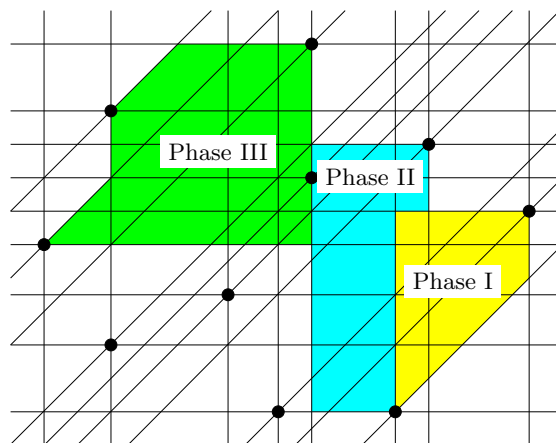


Figure 4. Three phases of Algorithm A.

Note that due to the special shape of the tropical lines, it is important how to break the ties. For instance, if two points have the same lowest y -coordinate, then the left one is also on the boundary, but not necessarily a vertex; see Figure 3.

A key difference between tropical versus ordinary polytopes is that the former only have few directions for their (half-)edges. This can be exploited to produce convex hull algorithms which do not have a directly corresponding ordinary version.

Through each point $p = (0, \xi, \eta) \in \mathbb{TP}^2$ there is a unique tropical line consisting of three ordinary rays emanating from p : We respectively call the sets $\{(0, \xi + \lambda, \eta) \mid \lambda \geq 0\}$, $\{(0, \xi, \eta + \lambda) \mid \lambda \geq 0\}$, and $\{(0, \xi - \lambda, \eta - \lambda) \mid \lambda \geq 0\}$ the *horizontal*, *vertical*, and *skew* ray through x . If we have a second point $p' = (0, \xi', \eta')$ then we can compare them according to the relative positions of their three rays. This way there is a natural notion of left and right, but there are two notions of above and below, which we wish to distinguish carefully: p' is *y-above* p if $\eta' > \eta$, and it is *skew-above* if $\eta' - \xi' > \eta - \xi$.

The introduction of the sign of the tropical determinant now clearly allows to take most ordinary 2-dimensional convex hull algorithms and produce a “tropified” version with little effort. For instance a suitable tropical version of Graham’s scan provides a worst-case optimal $O(n \log n)$ -algorithm. We omit the details since we describe a different algorithm with the same complexity. The commonly expected output of an ordinary convex hull algorithm in two dimensions is the list of vertices in counter-clockwise order. As the results in the previous section imply that the combinatorics of tropical polygons in \mathbb{TP}^2 does not differ from the ordinary, we adopt this output strategy.

The data structures for the Algorithm A below is are three doubly-linked lists L, Y, B such that each input point occurs exactly once in each list. It is important that all three lists can be accessed at their front and back with constant cost.

In order to obtain a concise description we assume that the input set S is in general position. For input not in general position the notions of left, right, above, and below have to be adapted as above. The complexity of the algorithm remains the same.

Input : $S \subset \mathbb{TP}^2$ finite
Output: list of vertices of $\text{tconv}(S)$ in counter-clockwise order

sort S from left to right and store the result in list L
 sort S from y -below to y -above and store the result in list Y
 sort S from skew-below to skew-above and store the result in list B

$H \leftarrow$ empty list; $v \leftarrow \text{front}(Y)$; $w \leftarrow \text{next}(v, Y)$
while w y -below $\text{back}(L)$ **do**
 | **if** w skew-below v **then**
 | | $v \leftarrow w$; append v to H
 | | $w \leftarrow \text{next}(w, Y)$
 $v \leftarrow \text{back}(L)$; append v to H
 $w \leftarrow \text{previous}(v, L)$
while w to the right of $\text{back}(Y)$ **do**
 | **if** w y -above v **then**
 | | $v \leftarrow w$; append v to H
 | | $w \leftarrow \text{previous}(w, L)$
 $v \leftarrow \text{back}(Y)$; append v to H ;
 $w \leftarrow \text{back}(B)$
while w skew-above $\text{front}(L)$ **do**
 | **if** w to the left of v **then**
 | | $v \leftarrow w$; append v to H
 | | $w \leftarrow \text{previous}(w, B)$
 $v \leftarrow \text{front}(L)$; append v to H
if $v \neq \text{front}(Y)$ **then**
 | append $\text{front}(Y)$ to H
return H

Algorithm A: Triple sorting algorithm.

PROPOSITION 5.2. *The Algorithm A correctly computes the vertices of the tropical convex hull in counter-clockwise order.*

PROOF. The algorithm has an initialization and three phases, where each phase corresponds to one of the three while-loops; for an illustration see Figure 4. In the first phase all the vertices between $\text{bottom}(S)$ and $\text{right}(S)$ are enumerated,

in the second phase the vertices between $\text{right}(S)$ and $\text{top}(S)$, and in the third phase the vertices between $\text{top}(S)$ and $\text{left}(S)$.

By Lemma 5.1 the point $\text{front}(Y) = \text{bottom}(S)$ is a vertex. Throughout the algorithm the following invariant is maintained: v is the last vertex found, and w is an input point not yet processed, which is a candidate for the next vertex in counter-clockwise order. We have a closer look at Phase I, the remaining being similar. If w is a vertex between $\text{bottom}(S)$ and $\text{right}(S)$ then it will be y -above of v , hence we process the points according to their order in the sorted list Y . However, none of those vertices can be y -above $\text{back}(L) = \text{right}(S)$, therefore the stop condition. Under these conditions w is a vertex if and only if w is skew-below the tropical line segment $[v, \text{right}(S)]$. \square

The worst-case complexity of the algorithm based on triple sorting is $O(n \log n)$. If, however, the points are uniformly distributed at random, say, in the unit square, then by applying Bucket Sort, we can sort the input in an expected number of $O(n)$ steps; see [Cormen et al. 2001].

If only few of the input points are actually vertices of the convex hull, then it is easy to beat an $O(n \log n)$ algorithm. For ordinary planar convex hulls the Jarvis' march algorithm is known as an easy-to-describe method which is output-sensitive in this sense. We sketch a "tropified" version, we will be instrumental later. Its complexity is $O(nh)$, where h is the number of vertices.

Input : $S \subset \mathbb{TP}^2$ finite

Output: list of vertices of $\text{tconv}(S)$ in counter-clockwise order

$v_0 \leftarrow \text{bottom}(S)$; $v \leftarrow v_0$; $H \leftarrow$ empty list

repeat

$w \leftarrow$ some point in S	
for $p \in S \setminus \{w\}$ do	
if $\bar{\tau}_{v,w}(p) = -1$ or $(\bar{\tau}_{v,w}(p) = 0$ and $\ p - v\ > \ w - v\)$ then	
\perp $w \leftarrow p$	
$v \leftarrow w$; $S \leftarrow S \setminus \{v\}$; append v to H	

until $v = v_0$

return H

Algorithm B: Tropical Jarvis' march algorithm.

In the ordinary case, Chan [1996] has given a worst-case optimal $O(n \log h)$ algorithm, based on a combination of Jarvis' march and a divide-and-conquer approach. We sketch how the same ideas can be used to obtain an $O(n \log h)$ convex hull algorithm in \mathbb{TP}^2 . If we split the input into $\lceil n/m \rceil$ parts of size at most m , then we can use our $O(n \log n)$ algorithm based on triple sorting to compute the $\lceil n/m \rceil$ hulls in $O((n/m)(m \log m)) = O(n \log m)$ time. Now we

use Jarvis' march to combine the $\lceil n/m \rceil$ tropical convex hulls into one. The crucial observation is that each vertex of the big tropical polygon is also a vertex of one of the $\lceil n/m \rceil$ small tropical polygons. Therefore, in order to compute the next vertex of the big tropical polygon in the counter clockwise order, we first compute the tropical tangent through the current vertex to each of the small tropical polygons. In each small tropical polygon this can be done by binary searching the vertices in their cyclic order; this requires $O(\log m)$ steps per small tropical polygon and per vertex of the big tropical polygon. Summing up this gives a total of $O(n \log m + h((n/m) \log m)) = O(n(1 + h/m) \log m)$ operations. That is to say, if we could know the number of vertices of the big tropical polygon beforehand, then we could split the input into portions of size at most h , thus arriving at a complexity bound of $O(n \log h)$. But this can be achieved by repeated guessing as has been suggested by Chazelle and Matoušek [1995].

We summarize our findings in the following result, which is identical to the ordinary case. Note that, as in the ordinary case, one has an $\Omega(n \log n)$ worst case lower bound for the complexity of the two-dimensional tropical convex hull problem which comes from sorting. In this sense our output-sensitive algorithm is optimal.

THEOREM 5.3. *The complexity of the problem to compute the tropical convex hull of n points in \mathbb{TP}^2 with h vertices is as follows:*

- (1) *There is an output-sensitive $O(n \log h)$ -algorithm.*
- (2) *There is an algorithm which requires expected linear time for random input.*

6. Concluding Remarks

One of the main messages of this paper is that, with suitably chosen definitions, it is possible to build up a theory of tropical polytopes quite similar to the one for ordinary convex polytopes. But, of course, very many items are still missing. We list a few open questions, and the reader will easily find more.

- (1) How are the face lattices of tropical polytopes related to the face lattices of ordinary convex polytopes? In particular, how do the face lattices of tropical polytopes in \mathbb{TP}^3 look alike?
- (2) It is known [Develin and Sturmfels 2004, Lemma 22] that the tropical convex hull of n points in \mathbb{TP}^d is the bounded subcomplex of some $(n + d)$ -dimensional unbounded ordinary convex polyhedron (defined in terms in inequalities). Hence the tropical convex hull problem can be reduced to solving a (dual) ordinary convex hull problem, followed by a search of the bounded faces in the face lattice. The question is: How does an intrinsic tropical convex hull algorithm look alike that works in arbitrary dimensions? Here *intrinsic* means that the algorithm should not take a detour via that face lattice computation in the realm of ordinary convex polytopes. While the complexity status of the ordinary convex

hull problem is notoriously open (output-sensitive with varying dimension) it is well conceivable that the tropical analog is, in fact, easier. An indication may be the easy to check certificate in Proposition 2.9 which leads to a simple and fast algorithm for discarding the nonvertices among the input points, a task which is polynomially solvable in the ordinary case, but which requires an LP-type oracle.

(3) What is the proper definition of a tropical triangulation? Such a definition would say that a tropical triangulation should be a subdivision into tropical simplices which meet properly. The precise notion of meeting is subtle, however. While it is obvious that the standard intersection as sets does not do any good, the more refined way by extending the \square operation also leads to surprising examples. A meaningful definition of a tropical triangulation should lead to one solution of the previous problem.

(4) Can the dimension of an arbitrary tropical polytope, which is not necessarily full, computed in polynomial time? Here *dimension* means the same as *tropical rank*. In fact, this is Question (1) at the end of the paper [Develin et al. 2005].

(5) As mentioned in Remark 3.13 point configurations in the tropical projective space do not generate an oriented matroid in the usual way. But does there exist a more general notion than an oriented matroid which encompasses the tropical case?

Acknowledgments

I am grateful to Julian Pfeifle and Francisco Santos for many helpful conversations on the contents of this paper and, especially, for suggesting better names for some of the notions defined than I could have come up with. I am indebted to Bernd Sturmfels for requiring the correction of a few minor errors. Finally, I would like to thank the anonymous referee who helped alleviate my ignorance of previously published results on max-plus-algebras.

References

- [Akian et al. 2005] M. Akian, S. Gaubert, and V. Kolokoltsov, “Set coverings and invertibility of functional Galois connections”, in *Idempotent mathematics and mathematical physics*, edited by G. L. Litvinov and V. P. Maslov, Contemp. Math. **377**, Amer. Math. Soc., Providence, RI, 2005. Available at arxiv.org/math.FA/0403441. to appear.
- [Baccelli et al. 1992] F. L. Baccelli, G. Cohen, G. J. Olsder, and J.-P. Quadrat, *Synchronization and linearity: An algebra for discrete event systems*, Wiley, Chichester, 1992.
- [Burkard and Butkovič 2003] R. E. Burkard and P. Butkovič, “Max algebra and the linear assignment problem”, *Math. Program.* **98**:1-3, Ser. B (2003), 415–429. Integer programming (Pittsburgh, PA, 2002).
- [Butkovič 1994] P. Butkovič, “Strong regularity of matrices—a survey of results”, *Discrete Appl. Math.* **48**:1 (1994), 45–68.

- [Chan 1996] T. M. Chan, “Optimal output-sensitive convex hull algorithms in two and three dimensions”, *Discrete Comput. Geom.* **16**:4 (1996), 361–368. Eleventh Annual Symposium on Computational Geometry (Vancouver, BC, 1995).
- [Chazelle and Matoušek 1995] B. Chazelle and J. Matoušek, “Derandomizing an output-sensitive convex hull algorithm in three dimensions”, *Comput. Geom.* **5**:1 (1995), 27–32.
- [Cohen et al. 2004] G. Cohen, S. Gaubert, and J.-P. Quadrat, “Duality and separation theorems in idempotent semimodules”, *Linear Algebra Appl.* **379** (2004), 395–422. Tenth Conference of the International Linear Algebra Society.
- [Cormen et al. 2001] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to algorithms*, Second ed., MIT Press, Cambridge, MA, 2001.
- [Develin and Sturmfels 2004] M. Develin and B. Sturmfels, “Tropical convexity”, *Doc. Math.* **9** (2004), 1–27. Erratum, *ibid.*, pp. 205–206.
- [Develin et al. 2005] M. Develin, B. Sturmfels, and F. Santos, “On the rank of a tropical matrix”, pp. 213–242 in *Combinatorial and computational geometry*, edited by J. E. Goodman et al., Math. Sci. Res. Inst. Publ. **52**, Cambridge U. Press, New York, 2005.
- [Dress et al. 2002] A. Dress, K. T. Huber, and V. Moulton, “An explicit computation of the injective hull of certain finite metric spaces in terms of their associated Buneman complex”, *Adv. Math.* **168**:1 (2002), 1–28.
- [Helbig 1988] S. Helbig, “On Carathéodory’s and Kreĭn-Milman’s theorems in fully ordered groups”, *Comment. Math. Univ. Carolin.* **29**:1 (1988), 157–167.
- [Isbell 1964] J. R. Isbell, “Six theorems about injective metric spaces”, *Comment. Math. Helv.* **39** (1964), 65–76.
- [Richter-Gebert et al. 2005] J. Richter-Gebert, B. Sturmfels, and T. Theobald, “First steps in tropical geometry”, in *Idempotent mathematics and mathematical physics*, edited by G. L. Litvinov and V. P. Maslov, Contemp. Math. **377**, Amer. Math. Soc., Providence, RI, 2005. Available at arxiv.org/math.FA/0403441. To appear.
- [Yoeli 1961] M. Yoeli, “A note on a generalization of Boolean matrix theory”, *Amer. Math. Monthly* **68** (1961), 552–557.
- [Zimmermann 1977] K. Zimmermann, “A general separation theorem in extremal algebras”, *Ekonom.-Mat. Obzor* **13**:2 (1977), 179–201.

MICHAEL JOSWIG
INSTITUT FÜR MATHEMATIK, MA 6-2
TECHNISCHE UNIVERSITÄT BERLIN
10623 BERLIN, GERMANY
joswig@math.tu-berlin.de

